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DIFFERENTIAL GEOMETRY AND PHYSICS.

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Physics is connected with geometry since times of Archimedes and other ancient scientists and philosophers. This fact was poetically expressed by Plato who said that god makes always geometry. But if we think more specifically about differential non-Euclidean geometry as created by Riemann and others in the 19-th century, Einstein's general theory of relativity (1915), i.e., a theory of gravitational field, was its first important and fully successful application. By this fact differential geometry in this sense transgressed definitely the borders of physics as was expected by Riemann in his famous inaugural lecture (1854), but who said at its end that: "Here we stay on the border of a domain belonging to another science—physics, and the present day gives us no reason to transgress this border."

Innumerable, but rather unsuccessful attempts of further generalization of the Einstein relativity theory, in order to include all other physical fields and create the so-called unified field theory, filled a considerable part of history of physics and mathematics in the subsequent decades. Although these investigations gave some important impulses to development of differential geometry, their physical failure, or at least inconclusiveness, caused that they shifted more and more into the margin of physics and mathematics. One of the reasons of this failure was, perhaps, that such theories either completely neglected the quantum character of the physical phenomena or tried to take it into account in an artificial and unconnected with geometry manner. Actually, this problem is connected with an unsolved conceptual difficulty: if we try to quantize the geometry "itself", we come to a theory which is mathematically unclear and radically departs from the conventional differential geometry; if we quantize the field "in" a geometry, we depart from the Riemann-Einstein postulate of inner connection of physical and geometrical properties, cf. also [**1**].[1] We have to add that even the simplest quantum field theory of interacting fields, working in the flat space-time continuum of special relativity, is not yet mathematically completely clear, although the recent progress in a rigorous treatment of such non-linear theories creates some hope that this theory may be soon formulated. On the other hand, the experimental investigations of recent decades in high energy particle physics show that we are yet rather far from a closed and exhaustive picture of these phenomena, so time is not yet rape for speaking about the allcomprehending theory.

In such a situation the majority of modern theoretical and mathematical physicists is rather more interested in infinite-dimensional spaces of functional analysis and in abstract algebraic methods connected with the latter, than in finite-dimensional non-Euclidean differential geometries (except for the people working in general relativity, but they are not so numerous). In the meantime, the pure differential geometry changed

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1) Numbers in brackets refer to the references at the end of the paper.

considerably under the stronger influence of other parts of modern mathematics, especially topology, than of physical investigations. The local methods have been complemented by the global ones, and in such a way the modern concepts of differential manifold, diffeomorphism, fibre or vector bundles, etc., have been created, together with systematic applications of abstract Lie and holonomy groups. This development forms not only the qualitative methods of geometry, but also gives, so to say, a new quality to the differential geometry itself which has already a strong appeal to some physically minded mathematicians and mathematical physicists (among them, relativists, cf. [2]), if not yet to majority of theoretical physicists.

Before we show these new problems and perspectives, we would like to mention that the failure of the Einstein concept of unified field theory does not mean that the general Einstein idea of covariance in physics is wrong. On the contrary, independent of relativity, it appeared to be exceptionally successful in all parts of physics, making the tensor and group concepts extraordinary popular among physicists. The excellent series of textbooks on theoretical physics by Landau and Lifshitz shows how many-sided and deep the tensorial methods are and how groups appear in all parts of physics. A considerable contribution to this development has been given by Japanese, let us mention only the Tensor Society and Journal founded by A. Kawaguchi and the exceptional school of mathematical engineering founded by K. Kondo. These investigations also showed how many-sided applications of differential-geometrical spaces are possible in physics and engineering, completely independently of such physical ideology as Einstein's unified field theory.

On the other hand, we should like to mention that it is not excluded, rather very probable, that in the further development of mathematics some connections between differential geometry and functional analysis will be found. Also in this problem the Japanese mathematicians are pioneers: A. Kawaguchi yet 46 years ago (cf. [3]-[6]) and Y. Ichijo recently. May be, some new, probably much more abstract, connection between the ideas of Riemann and Banach will give a mathematical clue for the future physics. However, "the present day gives us no resason to transgress its border". Therefore, in the following we can only give some typical examples of less known applications of modern differential geometry to physics of today, claiming that perhaps in some cases these applications may influence again the main trend of pure differential geometry.

1. Electrodynamics. Finsler geometry can be applied to the classical (non-quantum) theory of motion of a charged particle in an electromagnetic field (e.g., in an electron microscope or other electronic device or a particle accelarator), if only the magnetic field is present (cf. [7] and independent [8]). Confining ourselves for simplicity only to the static electromagnetic field in vacuum, it can be described by 4 harmonic functions of space coordinates x^k ($k = 1, 2, 3$): the electric or scalar potential $\varphi(x^k)$ and the magnetic or vector potential $A_i(x^k)$ ($i = (1, 2, 3)$). The charged (point) particle in point x^k is described by its charge e and mass m (for simplicity we here assume that the electric potential is gauged so that the total energy of the particle $E = 0$ and that the units are such that the velocity of light $c = 1$). Then we obtain a special Finsler space called by me a *Randers space* ([8]-[9]) with metric

$$ds = (a_{ij}(x^k)dx^i dx^j)^{1/2} + a_i(x^k)dx^i \tag{1}$$

(we assume the Einstein summation convention), where

(2) $a_{ij}(x^k) = (e^2\varphi^2(x^k) - m^2)\delta_{ij}$, $a_i(x^k) = eA_i(x^k)$ $(i, j = 1, 2, 3)$

and (Δ = the Laplace operator)

(3) $\Delta\varphi(x^k) = 0$, $\Delta A_i(x^k) = 0$ $(i = 1, 2, 3)$.

Such a theory has been applied to the electron microscope by the present author in his doctor thesis done in 1948, but published in 1957 [**8**]. The actual problem of this paper was the investigation of not so much detail quantitative and local properties of such spaces, as of qualitative and global characteristics of the motion, namely, the question if the absolute point representation of space is possible in the electron microscope (as it is possible in the optical case of inhomogeneous medium known as the "fish eye" example of Maxwell). For this purpose the author solved the problem of imbedding of a Finsler space in a Minkowski space [**10**]. The author's answer to the central question was in the negative, but under the conjecture that the necessary condition of the absolute representation is that the space (1) is of a constant Berwald curvature. It seems that this conjecture is not proved (or disproved) up to now. Since then, however, the general theory of Randers spaces, especially of constant curvature (in different senses), have been much developed in Japan by M. Matsumoto with his pupils ([**11**]-[**15**]) and H. Yasuda [**16**]. It would be desirable, therefore, to continue this investigation in collaboration between mathematicians and physicists.

On the other hand, at the previous (1974) and the present (1975) Ohmihachiman conference T. Kawaguchi presented papers concerning a generalization of the metric (1)-(3) for an electromagnetic field not in vacuum, but in a ferromagnetic material with hysteresis, in view of applications to electric machines. Actually, he used only one branch of the hysteresis loop, but it is obvious that such a theory can be generalized leading to a completely new sort of spaces, not yet defined. In such spaces the indicatrix has many sheets and the geodesic motion depends on its history. Since in the electric machines we have periodic demagnetization processes (changes of direction of magnetization by 180°), the use of the full hysteresis loop is necessary. In such a way we see that also in this field there is a lot of unsolved difficult problems which have a great fundamental and practical importance. Also here the discussion of global properties is necessary.

2. Mechanics. The above problem belongs also to mechanics, but the latter discusses the problem of motion more generally, not only in arbitrary potential fields, but also for forces without potential, and for the case when the potential V depends not only on coordinates (positions), but also on velocities (momenta), i.e., when we cannot speak about a "field". In the latter case mechanics is called Lagrangian since it has the Lagrangian function $L = T - V$ (T = the kinetic energy). Recently for such a case J. Kern [**17**] formulated a generalization of the Finsler space which he called a *Lagrange space* and which differs from the former by the possibility of its Lagrangian function not being homogeneous in directions (velocities). This opens new interesting possibilities for the global properties of the space and the motion.

Recent decades are characteristic by very deep and extensive investigations of the qualitative and global, in particular ergodic, properties of (classical) mechanical motion. These investigations were inaugurated by H. Poincaré, A. M. Liapunov, G. D. Birkhoff, E. Hopf (in particular in connection with the problems of astronomy), and then intensely developed by a lot of excellent mathematicians, as C. L. Siegel, A. N. Kolmogorov, Ya. G. Sinai, P. R. Halmos, J. Moser, M. M. Peixeto, R. Thom, S. Smale, V. I. Arnold,

R. F. Arestorf, C. Pugh, A. Kelley, R. Abraham, J. E. Marsden and others, cf. the excellent book of the two latter [18] on the foundations of mechanics, where the connection between these problems and global differential geometry is explicitly shown. But the connection between analytic mechanics and differential geometry has been particularly investigated in France where the tradition of E. Cartan persists and develops (cf. [19]), we may mention the names of A. Lichnérowicz, F. Gallissot, J. Klein, C. Godbillon. This school has explicitly shown that classical mechanics has two aspects, the Hamiltonian and the Lagrangian ones, Hamiltonian equations of motion being covariant, while Lagrangian contravariant, the two theories being dually connected by the Legendre transformation. The Hamiltonian equations are now interpreted as a dynamical system on a cotangent bundle of the configuration space, while the Lagrangian equations as a dynamical system on a tangent bundle of the configuration space (cf. [19]–[20]). The Hamiltonian aspect is connected with a canonical simplectic structure on the cotangent bundle, while the Lagrangian aspect is associated with a more rich differential calculus on the tangent bundle, also connected with the technique of the simplectic geometry.

We have to point out that the differential-geometric spaces used in (1)–(3) and in the mentioned more general mechanical theories have nothing to do with the geometric structure of the usual space-time, as in the general relativity or the unified theories. They represent rather the geometry of forces or potentials acting on a particle. This is, in particular, seen in (1)–(3) where the considered geometry is not gauge invariant, while the electromagnetic field and the trajectories of motion are gauge invariant in the usual space-time. Further, the field equations (3), although expressed *in* the geometry, have no direct geometrical meaning, as Einstein's equations of the gravitational field which are expressed by the curvature tensor of the Riemannian geometry. Finally, the geometry depends on the parameters of the particle (as e, m, and in general E), so for each sort of particles we have a different space (this is avoided only in the gravitational field, where the geodesic inertial motion is independent of the mass of the particle). When we have many interacting particles, the respective spaces are many-dimensional, much more dimensional than the usual space-time. Such situation is similar to that of Hilbert or Banach spaces used in quantum theories, where we cannot speak of any direct connection between these spaces and the usual space-time. Only topological and global properties of trajectories and of the electromagnetic Randers space are gauge invariant and have a direct physical interpretation in space-time. This is in contrast, e.g., to the point of view of Y. Takano [21] who considers the field theory in Finsler space rather in the spirit of the unified field theory.

In this situation one may ask the question which is the purpose of discussing the differential-geometric spaces in physics and engineering, where most practical problems has been already solved by traditional methods without direct application of differential geometry. The answer is that just for solving qualitative global problems which cannot be even formulated without using this modern mathematical approach. One of the methods of getting a geometrical insight into the nature of the global problems is the method of imbedding of more complicated spaces into a simpler one with a known topological structure. Only when the global problem is solved, we can say that we understand completely a given physical or engineering system.

Methods of differential geometry can be also applied, as we mentioned, to the non-potential case, in particular, to the so-called non-holonomic and rheonomic systems. Such investigations have been started yet before the last war by a Polish mathematician A. Wundheiler ([22]-[23]) and others (cf. [24]), and are continued, among others, in Japan (T. Kawaguchi-Kanai (Mrs.) in Sapporo). On the other hand, the modern methods of groups of diffeomorphisms have been recently successfully applied to hydrodynamics of incompressible fluids by D. G. Ebin and J. E. Marsden [25] and others. There is also a lot of applications of differential geometry to elastomechanics and plasticity, and to the solid state in general (e.g., using of Finsler geometry for anisotropic media by K. Kondo's school in Japan), but we do not like to go into details of these questions here.

3. Thermodynamics. We would like to develop this point in a little more detail in connection with the recent investigations of the author. The last items mentioned in the previous point are already connected with macroscopic (phenomenological) thermodynamics. Here we shall try to connect the macroscopical theory with the microscopical one in its quantum form, for the first time in this paper involving quantum mechanics into our considerations. In such a way we come to the point where, as if, on the background of infinite-dimensional functional spaces (Hilbert or Banach spaces, cf. [26]) there appear finite-dimensional spaces of differential geometry, Riemann, Finsler, Kawaguchi or even, possibly, more general ones.

The application of differential-geometrical methods to thermodynamics (especially of differential forms) has been started by C. Carathéodory [27] who created the so-called axiomatic phenomonological thermodynamics (for modern presentation cf. [28], Chap. 6, we also recommend the other books by R. Hermann for exposition of applications of modern differential geometry to physics, in particular, [29]). In statistical physics A. A. Vlasov [30] can be considered as a pioneer of using differential-geometric spaces, the Finsler space including. Here we use, however, a more general point of view, that of information thermodynamics, i.e., thermodynamics based on information-theoretical estimation (cf. [26], [31]-[34]).

Let us consider a quantum system of f degrees of freedom (it is not necessary that f is large, we may have, e.g., $f=1$ as in the case of one linear oscillator, but we consider the system as contained in a "heath bath", i.e., influenced by outer stochastic disturbances, cf. [26]). The system has $2f$ independent observables: positions Q_r and momenta P_r $(r=1, \ldots, f)$, fulfilling the well-known commutation relations

$$(4)\qquad [Q_r, Q_s]=0\,,\qquad [Q_r, P_s]=i\delta_{rs}\,,\qquad [P_r, P_s]=0\qquad (r, s=1, \ldots, f)\,,$$

and independent in the sense that any other observable on the system is an (operator) function of these observables. We denote

$$(5)\qquad (A_i)=(Q_r, P_r)\qquad (i=1, \ldots, 2f, r=1, \ldots, f)\,,$$

$$(6)\qquad A_{ij}=\tfrac{1}{2}(A_iA_j+A_jA_i), \ldots, A_{i_1\ldots i_n}=\mathrm{Sym}\,(A_{i_1}\ldots A_{i_n})\,,$$

where Sym denotes the symmetrization operation. We now assume that we have the n-th order macroscopic information about the system ($n=1, 2, \ldots$ if the system is enclosed in a finite volume, $n=2, 4, 6, \ldots$ if the volume is infinite), i.e., that we know all statistical moments (correlations) defined as mean values of observables (5) and (6) up to order n:

$$(7)\qquad \alpha_i=\mathrm{Tr}\,(A_i\rho)\,,\qquad \alpha_{ij}=\mathrm{Tr}\,(A_{ij}\rho), \ldots,\qquad \alpha_{i_1\ldots i_n}=\mathrm{Tr}\,(A_{i_1\ldots i_n})\,,$$

where ρ is an (unknown) state of the system (described by a density operator). We call the set M_n of all density operators fulfilling (7) a *macrostate of order n*. Maximizing the entropy (information) of the state of the system

$$H(\rho) = -\mathrm{Tr}\,(\rho \ln \rho) \tag{8}$$

on the macrostate M_n and denoting

$$S = S(\alpha_i, \alpha_{ij}, \ldots, \alpha_{i_1 \ldots i_n}) = \sup_{\rho \in M_n} H(\rho)\ , \tag{9}$$

we obtain (if $\alpha_i, \ldots, \alpha_{i_1 \ldots i_n}$ fulfill some conditions (cf. [**35**]–[**36**]) of uniqueness of the presented problem called by us the *truncated problem of moments*) a unique estimation for the state of the system in the form

$$\rho = Z^{-1} \exp\,(-\beta^i A_i - \beta^{ij} A_{ij} - \cdots - \beta^{i_1 \cdots i_n} A_{i_1 \ldots i_n})\ , \tag{10}$$

where

$$Z = Z(\beta^i, \beta^{ij}, \ldots, \beta^{i_1 \cdots i_n}) = \mathrm{Tr}\,(\exp\,(-\beta^i A_i - \cdots - \beta^{i_1 \cdots i_n} A_{i_1 \ldots i_n}))\ , \tag{11}$$

$$\beta^i = \partial S / \partial \alpha_i\ , \qquad \beta^{ij} = \partial S / \partial \alpha_{ij}, \ \ldots\ , \qquad \beta^{i_1 \cdots i_n} = \partial S / \partial \alpha_{i_1 \ldots i_n}\ , \tag{12}$$

while

$$\alpha_i = -\partial (\ln Z) / \partial \beta^i,\ \ldots\ , \qquad \alpha_{i_1 \ldots i_n} = -\partial (\ln Z) / \partial \beta^{i_1 \cdots i_n}\ . \tag{13}$$

Statistical parameters $\beta^i, \ldots, \beta^{i_1 \cdots i_n}$ may be called *temperature coefficients* or *inverse temperatures* (*of higher order*, in general) associated with statistical moments or correlations $\alpha_i, \ldots, \alpha_{i_1 \ldots i_n}$. As is shown by our notation, β's are in some sense contravariant when α's are covariant (and we sum over repeated co- and contravariant indices). Anyway we have two dual spaces of α's and β's and, although the mathematical meaning of this duality is rigorously defined by the above formulae, we do not know if there exists any of the known spaces of differential geometry (Riemann, Finsler, Cartan, Kawaguchi) in which this duality can be interpreted.

In order to answer the last question we have to fix a group of transformation with respect to which the truncated problem of moments is invariant. We see that in order to preserve the sense of summations in (10), the transformation group should not destroy the concept of order of observable (then of moments and temperatures) described by the number of indices. In other words, from the "simple" power statistical moments with respect to $A_i = (Q_r, P_r)$ we can go over to any other "combined" power moment which have the order (the highest power with respect to A_i) as a property, as central moments, cumulants, etc., but not, e.g., to "transcendental" moments, as trigonometric, hyperbolic, etc., which have no order in this sense. The most general group with this property is

$$A'_i = f_i(A_k)\ , \quad A'_{ij} = f_{ij}(A_k, A_{lm}), \ \ldots\ , \quad A'_{i_1 \ldots i_n} = f_{i_1 \ldots i_n}(A_k, \ldots, A_{l_1 \ldots l_n})\ , \tag{14}$$

where $f_i, f_{ij}, \cdots, f_{i_1 \ldots i_n}$ are some polynomials of the 1-st, 2-nd, ..., n-th order, symmetrized with respect to indices, and with real coefficients. Which sort of geometry can be obtained in this manner is an open problem at the moment. It seems that it will be a generalization of the concept of Finsler space, may be also a generalization of Kawaguchi space. On the other hand, it is obvious that we obtain a generalization of the geometrical construction of mechanics mentioned above. Thus also here we obtain a very rich and interesting domain of open problems for geometrical investigations. We may remark that the above problem, formulated quantum-mechanically, can be also formulated for classical mechanics, (q_r, p_r) taken as a point of a classical phase space.

To obtain a correspondence with the Carathéodory approach to thermodynamics we have to consider not two dual spaces of α's and β's but one space of α's, β's, of S, and some other macroscopical parameters, as volume of the system or shape parameters of boundary conditions, parameters of external fields, etc., which all together define a given thermodynamic problem. Since all processes develop only in some directions in this space (some surfaces or more general conditions on differentials), we naturally come to a theory of differential forms and a sort of simplectic geometry. Also this more general approach (it may be called an approach of *thermodynamical space*) is not yet sufficiently investigated mathematically, so it may be an invitation for mathematicians or mathematical physicists.

On the other hand, we obtain a hydrodynamical case (or its generalizations) when instead of numerical conditions (7) we consider some functional conditions, e.g., for mean distribution of mass and velocities in the configuration space.

4. Optics. We discuss this point only very shortly to show that also here there are interesting, although very difficult, problems for mathematicians. In electron optics such a sort of problem has been actually already mentioned in point 1, namely, the problem of hysteresis recently investigated by T. Kawaguchi. In this question we have a possibility of not uniquely defined indicatrix of a Finsler space (going over from one to another branch of the hysteresis loop). In light optics we also have a possibility of two-valued indicatrices (birefrigence of light). In such a way we have in physics not only a case of non-positive definite indicatrix (as in special and general relativity of 4-dimensional space-time), but also of many-valued, as a Riemann surface. In other words, the indicatrix of a Minkowski or Finsler space can be considered, in a generalized theory, as not topologically equivalent with a sphere. E.g., we may think about an indicatrix of a torus shape (I do not say that this shape has just a direct application to physics). It seems that such a generalization may lead to interesting global properties of Finsler or more general spaces and the geodesic motions in them.

In our lecture we have purposely concentrated ourselves on problems laying outside of the general relativity theory which, of course, have also many interesting and important problems, especially those connected with cosmology and astrophysics (many of them of global and qualitative character), which may present a useful inspiration for mathematicians working in differential geometry. The latter problems concern, however, only the Riemannian geometry, while our conference is chiefly devoted to the Finsler geometry.

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REFERENCES

[1] M. A. Markov: On possible conceptual difficulties of quantum field theories involving gravitation, *Joint Inst. Nucl. Res. Dubna, Communication* E2-8838, (1975).

[2] A. Trautman: Fibre bundles associated with space-time, *Rep. Math. Phys.*, **1** (1970), 29–62.

[3] A. Kawaguchi: Sur les différentes connexions de l'espace fonctionnel, *Comp. Rend. Paris*, **189** (1929), 189–191.

[4] A. Kawaguchi: Über Übertragungen im Funktionenraume, *Comp. Rend. du* 1*er Congrès des Math. des Pays Slaves, Warszawa*, (1929), 329–334.

[5] A. Kawaguchi: Die Differentialgeometrie in den verschiedenen Funktionalräumen, I. Vektorialen und Tensorialen, *J. Fac. Sci. Hokkaido Imp. Univ., Ser.* I, **3** (1935), 43–106.

[6] A. Kawaguchi: The foundation of the theory of displacements, II (Application to the functional manifold), *Proc. Imp. Academy, Tokyo*, **10** (1934), 45–48.

[7] A. Lichnérowicz: Théories relativistes de la gravitation et de l'électromagnétisme, *Masson, Paris*, (1955).

[8] R. S. Ingarden: On the geometrically absolute optical representation in the electron microscope, *Prace Wrocławskiego Tow. Nauk., Wrocław, Ser.* **B**, **45** (1957).

[9] G. Randers: On the asymmetrical metric in the four-space of general relativity, *Phys. Rev.*, **59** (1941), 195–199.

[10] R. S. Ingarden: Über die Einbettung eines Finslerschen Raumes in einem Minkowskischen Raum, *Bull. Acad. Polon. Sci., Cl.* III, **2** (1954), 305–308.

[11] M. Matsumoto: On Finsler spaces with Randers' metric and special forms of important tensors, *J. Math. Kyoto Univ.*, **14** (1974), 477–498.

[12] M. Matsumoto: On C-reducible Finsler spaces, *Tensor, N. S.*, **24** (1972), 29–37.

[13] M. Matsumoto: On three-dimensional Finsler spaces satisfying the T-and B^p-conditions, *Tensor, N. S.*, **29** (1975), 13–20.

[14] M. Hashiguchi, S. Hojo and M. Matsumoto: On Landsberg spaces of two dimensions with (α, β)-metric, *J. Korean Math. Soc.*, **10** (1973), 17–26.

[15] M. Matsumoto: Foundation of Finsler geometry and special Finsler spaces, to be published (*VEB Deutscher Verl. Wiss., Berlin*).

[16] H. Yasuda: On extended Lie systems. III (Finsler spaces), *Tensor, N. S.*, **23** (1972), 115–130.

[17] J. Kern: Lagrange geometry, *Arch. der Math., Basel*, **25** (1974), 438–443.

[18] R. Abraham and J. E. Marsden: Foundations of mechanics, *Benjamin, New York*, (1967).

[19] G. Godbillon: Géométrie différentielle et méchanique analytique, *Hermann, Paris*, (1964).

[20] K. Yano and S. Ishihara: Tangent and cotangent bundles, differential geometry, *Dekker, New York*, (1973).

[21] Y. Takano: Variational principle in Finsler spaces, *Lett. Nuovo Cimento*, **11** (1974), 486–490.

[22] A. Wundheiler: Über die Variationsgleichungen für affine geodätische Linien und nichtholonome, nichtkonservative dynamische Systeme, *Prace Mat. Fiz., Warszawa*, **38** (1931), 129–147.

[23] A. Wundheiler: Rheonome Geometrie. Absolute Mechanik, *Prace Mat. Fiz., Warszawa,*, **40** (1932), 97–142.

[24] J. L. Synge: Tensorial methods in dynamics, *Univ. of Toronto Studies, Applied Math. Ser., Toronto*, **2**, (1936).

[25] D. G. Ebin and J. E. Marsden: Groups of diffeomorphisms and the motion of an incompressible fluid, *Annals of Math.*, **92** (1970), 102–163.

[26] R. S. Ingarden and A. Kossakowski: On the connection of non-equilibrium information thermodynamics with non-hamiltonian quantum mechanics of open systems, *Annals of Phys., New York*, **89** (1975), 451–485.

[27] C. Carathéodory: Untersuchungen über die Grundlagen der Thermodynamik, *Math. Ann.*, **67** (1909), 355–386.

[28] R. Hermann: Geometry, physics, and systems, *Dekker, New York*, (1973).

[29] R. Hermann: Vector bundles in mathematical physics, Vol. I and II, *Benjamin, New York*, (1970).

[30] A. A. Vlasov: Statistical distribution functions, *Nauka, Moscow*, (1966) (in Russian).

[31] E. T. Jaynes: Information theory and statistical mechanics, *Phys. Rev.*, **106** (1957), 620–630.

[32] R. S. Ingarden: Information theory and variational principles in statistical theories, *Bull. Acad. Polon. Sci., Ser. math.*, **11** (1963), 541–547.

[33] R. S. Ingarden: Information theory and thermodynamics of light, Part I. Foundations of information theory, *Fortschr. Phys.*, **12** (1964), 567–594.

[34] R. S. Ingarden: Information theory and thermodynamics of light, Part II. Principles of information thermodynamics, *Fortschr. Phys.*, **13** (1965), 755–805.

[35] W. Bayer and W. Ochs: Quantum states with maximum information entropy I, *Z. für Naturforsch.*, **28a** (1973), 693–701.

[36] W. Ochs and W. Bayer: Quantum states with maximum information entropy II, *Z. für Naturforsch.*, **28a** (1973), 1571–1585.