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An Axiomatic Definition of Information in Quantum Mechanics

by

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Gleason [1] gave a quantum generalization of the classical theory of probability formulated axiomatically by Kolmogorov [2] by means of the measure theory. Gleason defined a measure on the closed subspaces of a Hilbert space as "a function μ which assigns to every closed subspace a non-negative real number such that if $\{A_i\}$ is a countable collection of mutually orthogonal subspaces having closed linear span B, then

$$\mu(B) = \sum \mu(A_i)\text{''}, \tag{1}$$

(cf. [1] p. 885). Since in this definition the mutual orthogonality of A_i's is essential, it seems that Gleason's generalization cannot be considered as based in a natural way on the so-called quantum logic of events introduced by Birkhoff and von Neumann ([3], cf. also [4]), where the linear addition is considered for any set of closed subspaces, not necessarily of mutually orthogonal ones. For a quantum generalization of the concept of information as defined axiomatically for the classical case in [5]—[10] it would be yet much more artificial, if at all possible, to use the Birkhoff—von Neumann approach. In the present note we show that, alternatively, for both problems the concept of a partial Boolean algebra as defined by Specker and Kochen [11]—[14] is an appropriate and natural base.

By a *partial Boolean ring* (PBR), in symbols

$$(A; \perp\!\!\perp; +, \cdot; 0, 1), \tag{2}$$

we mean a set of elements $a, b, c, \ldots$ containing the zero element 0 and the unit element 1, in which there are defined: a binary relation $\perp\!\!\perp$ (commeasurability) and two binary partial operations: $+$ (addition) and $\cdot$ (multiplication) defined only for pairs of commeasurable elements, and such that:

($P1$) the relation $\perp\!\!\perp$ is reflexive and symmetric,

($P2$) for all $a \in A$, $a \perp\!\!\perp 0$, $a \perp\!\!\perp 1$, $a+0 = a$, $1 \cdot a = a$,

($P3$) the relation $\perp\!\!\perp$ is closed under the operations, i.e. for $a_i \perp\!\!\perp a_j$ $(i, j = 1, 2, 3)$ $(a_1+a_2) \perp\!\!\perp a_3$, $a_1 \cdot a_2 \perp\!\!\perp a_3$, (*P*4) any three mutually commeasurable elements generate a Boolean ring with respect to operations + and ·.

An element a of a PBR A is said to be *included* in an element b of A, in symbols $a \leqslant b$, iff $a \perp b$ and $ab = a$. An *atom* of a PBR A is such and only such an element a of A that $a \neq 0$ and from $b \leqslant a$ and $a \neq b$ it follows that $b = 0$. A PBR A is *atomic* iff each non-zero element of A contains at least one atom. Two elements, a and b, of a PBR are said to be *disjoint* iff $a \perp\!\!\!\perp b$ and $ab = 0$. We call a *measure* on a PBR A a non-negative and non-identically infinite function $f(a)$ defined on elements of A and such that if $a_1, a_2, \ldots$ is a sequence of mutually disjoint elements of A, then

$$f\left(\sum_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} f(a_i), \tag{3}$$

provided that the sum $a_1 + a_2 + \ldots$ exists in A. The latter is always true iff A is *complete*, i.e. if for any set of mutually commeasurable elements of A there exist in A their sum and their product. A measure is called a *probability measure* iff it is normalized, i.e. $f(1) = 1$. Iff besides of that from $f(a) = 0$ it follows that $a = 0$, a measure is called a *strict probability measure*.

We call a PBR A a *dimensional* PBR (DPBR) iff A is atomic, complete and there exists a measure $d(a)$ on A, called the *dimension* of a, such that

$$d(a) = 1 \quad \text{iff} \quad a = \text{atom}. \tag{4}$$

(E.g., each finite Boolean ring is dimensional, but not each finite PBR).

The number

$$d(1) = d(A) = n \qquad (n = 0, 1, 2, \ldots) \tag{5}$$

is called the *dimension* of A. Iff $n < \infty$, A is *finite-dimensional.*

For a DPBR A we denote by S_A the set of all the Boolean rings X which may be formed by elements of A with respect to operations + and · such that $1_X = 1_A$. (1_A is the unit element of A). A mapping g of a DPBR A onto a DPBR B is called a *homomorphism* iff $g(1_A) = 1_B$ and for all $a, b \in A$ such that $a \perp\!\!\!\perp b$ we have

$$g(a) \perp\!\!\!\perp g(b), \qquad g(a+b) = g(a) + g(b), \qquad g(ab) = g(a)\, g(b). \tag{6}$$

A homomorphism is an *isomorphism* iff the mapping is one-to-one. An isomorphism is an *automorphism* iff $A = B$.

A subset B of a DPBR is called a *dimensional partial Boolean subring* (DPBS) iff B is a DPBR with respect to the same operations and the same dimension function. A DPBS B of A is called an *inner* DPBS (IDPBS) iff there exists such an element a of A that B is the set of all elements of A included in a, in symbols $B = \{a\}_A$. A DPBS B of A is called an *external* DPBS (EDPBS) iff there exist such commeasurable elements $a_1, \ldots, a_m$ of A that B is the least DPBS of A containing $a_1, \ldots, a_m$, in symbols $B = [a_1, \ldots, a_m]_A$. Each EDPBS is a Boolean ring.

A set Ω of finite-dimensional DPBR's $A, B, C, \ldots$ is said to be a *dimensional partial Boolean ladder* (DPBL) iff

($L1$) if $A \in \Omega$ and B is an IDPBS of A, then $B \in \Omega$,

($L2$) for each $A \in \Omega$ it exists such a $C \in \Omega$ that $A \neq C$ and A is an IDPBS of C,

($L3$) if $A, B \in \Omega$ and $d(A) = d(B)$, then A and B are isomorphic.

Putting in correspondence to any $A \in \Omega$ and to any Boolean ring X belonging to S_A a real number $F(A, X)$ we get a real function defined on a part of Ω^2. An A is called *F-homogeneous* iff for each automorphism φ of A, each IDPBS B of A, and each $X \in S_B$ we have

$$F(B, X) = F(\varphi(B), \varphi(X)). \tag{7}$$

We call a *maximum Boolean ring of a* DPBR A any element $\tilde{X}$ of S_A such that all atoms of $\tilde{X}$ are atoms of A.

We define a *pseudometric* on Ω by means of a real function $F(A, X)$ as follows

$$\delta_F(A, B) = \begin{cases} \min\limits_{\varphi} \max\limits_{C} \max\limits_{Z} \left|F(C, Z) - F\left(\varphi^{-1}(C), \varphi^{-1}(Z)\right)\right| & \text{if } d(A) > d(B), \\ \min\limits_{\psi} \max\limits_{D} \max\limits_{V} \left|F(D, V) - F\left(\psi^{-1}(D), \psi^{-1}(V)\right)\right| & \text{if } d(A) \leqslant d(B), \end{cases} \tag{8}$$

where $A, B \in \Omega$, C is any IDPBS of B, D is any IDPBS of A, $Z \in S_C$, $V \in S_D$, φ is a homomorphism of A onto B, and ψ is a homomorphism of B onto A. Two BPBR's $A, B \in \Omega$ are *F-comparable* iff $\delta_F(A, B) = 0$.

A function $F(A, X)$ $(A \in \Omega, X \in S_A)$ is called *regular* in Ω iff for each $A \in \Omega$ there exists a sequence $A_1, A_2, \ldots \in \Omega$ of F-homogeneous DPBR's such that $d(A) \leqslant d(A_1) \leqslant d(A_2) \leqslant \ldots$ and

$$\lim_{n \to \infty} \delta_F(A_n, A) = 0. \tag{9}$$

A regular function $H(A, X)$ $(A \in \Omega, X \in S_A)$ is called a *relative information* of A with respect to X iff

(H1) *Axiom of monotony*: for any $X \in S_A$, $Y \in S_B$, $A, B \in \Omega$ such that A is an IDPBS of B, $X \perp\!\!\!\perp Y$, $X \neq Y$ and X is a Boolean subring of Y it holds

$$H(A, X) < H(B, Y),$$

(H2) *Axiom of additivity*: if $A \in \Omega$ is H-homogeneous, then for each maximum Boolean ring $\tilde{X}$ of A and for any non-zero disjoint elements $x_1, \ldots x_m \in \tilde{X}$ such that $x_1 + \ldots + x_m = 1_A$:

$$H(A, \tilde{X}) = H\left(A, [x_1, \ldots, x_m]_{\tilde{X}}\right) + \sum_{i=1}^{m} \frac{d(x_i)}{x(A)} H\left(\{x_i\}_A, \{x_i\}_{\tilde{X}}\right).$$

(H3) *Axiom of indistinguishability*: isomorphic H-homogeneous DPBR's of Ω ore H-comparable.

THEOREM. *Let $H(A, X)$ be a relative information on Ω. Then there exists a positive number k and for each $A \in \Omega$ a unique strict probability measure $p_A(a)$ $(a \in A)$ such that*

(10) $$p_B(b) = \frac{p_A(b)}{p_A(1_B)}, \quad (b \in B),$$

where B is any IDPBS of A, and

(11) $$H(A, X) = -k \sum_{i=1}^{n} p_A(x_i) \log p_A(x_i),$$

where $x_1, \ldots, x_n$ $(n = d(X))$ are all the atoms of X.

The Theorem can be proved in a strict analogy to the proof given in [8] or [9] with the correction in [10].

Except of the relative information $H(A, X)$ we may define the *absolute information* of A as

(12) $$H(A) = \inf_{\tilde{X}} H(A, \tilde{X}).$$

If we define a real function $E(x_i) = E_i$ $(i = 1, \ldots, n)$ on all atoms of some $\tilde{X} \in S_A$ $(n = d(\tilde{X}))$ we get a *random variable* (*observable*) E. Denoting by E_α different values of E_i $(\alpha = 1, \ldots, m \leqslant n)$ we define elements

(13) $$x_\alpha = \sum_{i:E(x_i)=E_\alpha} x_i \quad (\alpha = 1, \ldots, m).$$

Elements x_α's generate the Boolean subring $[x_1, \ldots, x_m]_{\tilde{X}} \in S_A$. Since each element X of S_A can be constructed in such a way, we may interpret $H(A, X)$ as the relative informaiton of A with respect to some observable.

The set π_n of all projection operators on the closed subspaces of an n-dimensional Hilbert space is a DPBR of dimension n if we interpret the relation $\perp\!\!\!\perp$ as the commutability of operators, the operation $\cdot$ as the usual multiplication of operators and the operation $\dotplus$ as the symmetric addition of projection operators (i.e., if $+$ denotes the usual addition of operators P_1, P_2, then $P_1 \dotplus P_2 = P_1 + P_2 - P_1 P_2$). But not every DPBR of dimension n is isomorphic with π_n. We call any DPBR isomorphic with a π_n a *Hilbert* PBR (HPBR). Only HPBR's have meaning in quantum mechanics. A measure in HPBR corresponds then to the Gleason one (1).

In a HPBR A, according to the Gleason theorem [1], any probability measure $p_A(a)$ can be presented in the form

(14) $$p_A(a) = \mathrm{Tr}(\varrho_A P_a) \quad (a \in A),$$

where ϱ_A is a density operator (i.e. a positive definite Hermitian operator with unit trace) and P_a is the projection operator corresponding to a.

Using (14) and Klein's theorem [15] we get for the absolute information (12) the von Neumann formula [16]

(15) $$H(A) = -k\mathrm{Tr}\,(\varrho_A \log \varrho_A).$$

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