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ON THE POSSIBILITY TO USE THE SUBTRACTIVE CALCULUS FOR THE FORMALIZATION OF CONSTRUCTIVE THEORIES

1. It is known that the so-called *positive calculus* in symbolic logic can be characterized by the following axioms:

(1) $$a \to a$$

(2) $$\frac{a \to b \qquad b \to c}{a \to c}$$

(3a) $$\frac{a \to b \qquad a \to c}{a \to b \,\&\, c}$$ (3b) $$\frac{c \to a \qquad b \to a}{c \vee b \to a}$$

(4a) $$\frac{a \,\&\, b \to c}{a \to b \supset c}$$

(5a) $$\frac{a \to f(x)}{a \to (\forall x) f(x)} \quad *$$ (5b) $$\frac{f(x) \to a}{(\exists x) f(x) \to a} \quad *$$

The dotted lines denote that inference is allowed in both directions, and the asterisks indicate that a is supposed to be independent of x. A proposition U is *valid* in this calculus if the formula $\to U$ (with void left member) is deducible from the axioms; such formulae arise by (4a) when the place of the proposition a is void. There is an essential difference between the operation sign $\supset$ and the relation sign $\to$, because $a \supset b$ is a proposition belonging to the language of the calculus itself, whereas $a \to b$, as a relation between two propositions, belongs to its metalanguage.

2. The positive calculus should be very suitable for formalizing deductive theories, especially the constructive ones. For theories containing negation the following procedure could be recommended:

Certain propositions $T_1, T_2, \ldots, T_n$, assumed to be true, and a series of other propositions $F_1, F_2, \ldots, F_m$, assumed to be false, are formulated in the language of the positive calculus (accordingly without use of negation) enlarged with some fundamental terms of the theory. Using the abbreviations

$$T \underset{Df}{=} T_1 \,\&\, T_2 \,\&\, \ldots T_n, \qquad F \underset{Df}{=} F_1 \vee F_2 \vee \ldots F_m,$$

it is natural to say that the proposition P is *true* if

$$T \rightarrow P \vee F \tag{6}$$

can be deduced, and *false* if

$$T \,\&\, P \rightarrow F \tag{7}$$

is deducible. Further $\neg P$ (not-P, or P is absurd) is a convenient abbreviation for $P \supset F$.

3. In practice, however, it is difficult to carry out these principles in a satisfactory way; even if we have deduced a sequence of the form

$$T \,\&\, f(x) \rightarrow g(x) \vee F,$$

it is not allowed to infer

$$T \rightarrow (\forall x)[f(x) \supset g(x)] \vee F;$$

in general we have to be content with

$$T \rightarrow (\forall x)[f(x) \supset g(x) \vee F].$$

E.g., if we formalize the theory of separatedness and equivalence by setting up

$$T \underset{Df}{=} (\forall x)(\forall y)(\forall z)(x \neq y \supset x \neq z \vee y \neq z) \text{ and}$$

$$F \underset{Df}{=} (\exists x)(x \neq x),$$

we are able to prove the symmetry of our separatedness relation only in the form

$$T \rightarrow (\forall x)(\forall y)(x \neq y \supset y \neq x \vee F),$$

and, after defining equivalence by

$$x = y \underset{Df}{=} x \neq y \supset F,$$

we can prove the theorem "if $x = y$ and $y \neq z$ then $x \neq z$" only in the form

$$T \rightarrow (\forall x)(\forall y)(\forall z)(x = y \,\&\, y \neq z \supset x \neq z \vee F)$$

4. It is rather disturbing that the fundamental theorems T_i do not contain F, while the proved theorems (especially the most constructive ones) in general contain F one or more times, and it is also inconvenient that the set of theorems not containing F will be changed by adopting other theorems of the theory as fundamental. The inconvenience can cer-

tainly be formally overcome by hiding the F behind a new implication and a new quantifier

$$a \dot{\supset} b \underset{Df}{=} a \supset b \vee F, \qquad (\dot{\forall} x)\, f(x) \underset{Df}{=} (\forall x)\,[f(x) \vee F],$$

a procedure which is (in a certain sense) equivalent to adopting the Heyting calculus. Nevertheless I think it is worth while to discuss other schemes for formalizing constructive theories.

5. Instead of the positive calculus we are now going to adopt the so-called *subtractive calculus*, which is obtained by replacing (4a) by its dual

$$\text{(4b)} \qquad \frac{c \rightarrow b \vee a}{c - b \rightarrow a}$$

where $c - b$ can most conveniently be pronounced "c without b". (The word "without" should not be replaced by "and not", because the subtraction sign signifies a primitive concept, not a combination of conjunction with some sort of negation.)

As (1) and (2) are self-dual, and as (3a) and (5a) are dual to (3b) and (5b) respectively, we can obtain theorems valid in the subtractive calculus by simple translation from the positive calculus, e.g.

Positive calculus:	Subtractive calculus:
$a \,\&\, (a \supset b) \rightarrow b,$	$b \rightarrow (b - a) \vee a,$
$a \vee b \rightarrow (a \supset b) \supset b,$	$b - (b - a) \rightarrow b \,\&\, a,$
$a \,\&\, b \supset c \rightleftarrows a \supset (b \supset c),$	$(c - b) - a \rightleftarrows c - (a \vee b),$
$a \vee b \supset c \rightleftarrows (a \supset c) \,\&\, (b \supset c),$	$(c - b) \vee (c - a) \rightleftarrows c - (b \,\&\, a).$

It is obvious from the first formula in the right column that the disjunction has not in the subtractive calculus the same strictly constructive meaning as in the positive calculus. Consequently, if a constructive disjunction is needed, it has to find its place in the metatheory.

6. If, as before, we define the metatheoretical propositions "P is true" and "P is false" by (6) and (7) respectively, we shall (on account of the duality) certainly have the same trouble with T as formerly with F. But here is the essential point in this paper: *When the subtractive calculus is adopted, it will often be possible to use a void T.* The reason for this is primarily that every metatheoretical proposition of the form

$$(\forall x)\,[f(x) \supset g(x)] \text{ is true,}$$

in want of an implication, has to be replaced by

$$(\exists x)[f(x) - g(x)] \text{ is false.}$$

In other words, *every quantified implication has to be conceived as the impossibility of a certain construction:* in our case that is the construction of an x such that $f(x)$ without $g(x)$.

E.g., a formalization of the theory of separatedness can be achieved by choosing T void and $F \underset{Df}{=} F_1 \vee F_2$, where

$$F_1 \underset{Df}{=} (\exists x)\,(x \# x), \text{ and}$$

$$F_2 \underset{Df}{=} (\exists x)\,(\exists y)\,(\exists z)\,(x \# z - x \# y \vee z \# y).$$

We can deduce the theorem

$$(\exists x)\,(\exists y)\,(x \# y - y \# x) \rightarrow F,$$

which metatheoretically has to be conceived as "if $x \# y$ then $y \# x$". Further, if we define

$$(x = y) \underset{Df}{=} (x \# y \rightarrow F),$$

this *metatheoretical* equivalence relation can be proved to be reflexive, symmetric and transitive.

7. In order to illustrate the possibility of using a void T in a more complicated case, it may be mentioned that the formalization of a constructive group theory might be obtained by choosing

$$F \underset{Df}{=} F_1 \vee F_2 \vee \ldots \vee F_6,$$

where F_1 and F_2 are as before, and

$$F_3 \underset{Df}{=} (\exists x)\,(\exists y)\,(\exists x')\,(\exists y')\,(xy \# x'y' - x \# x' \vee y \# y'),$$

$$F_4 \underset{Df}{=} (\exists x)\,(\exists y)\,(\exists z)\,[(xy)\,z \# x\,(yz)],$$

$$F_5 \underset{Df}{=} (\exists x)\,(ex \# x \vee xe \# x), \text{ and}$$

$$F_6 \underset{Df}{=} (\exists x)\,(\quad y)\,(xy \# e).$$

These "anti-axiomes" may at first sight seem very strange when compared with the ordinary axiomes for group theory; but it cannot be denied that they have a constructive form, except possibly the last one, which might perhaps still more constructively be chosen as

$$F_6 \underset{Df}{=} (\exists x)\,[-(\exists y)\,(-xy \neq e)].$$

This is, however, not possible unless subtractions with void left members are permitted, and the use of such subtractions will be equivalent to the postulation of an "unity proposition" (such that $a \rightarrow 1$ for any a), which is in its turn dual to the postulation of a "zero proposition" (such that $\mathrm{o} \rightarrow a$ for any a) in the positive calculus.

8. It has been remarked that many important concepts, as implication, equivalence and constructive disjunction, are not to be found in our formalized theory, only in its metatheory, which has therefore to be studied thoroughly. Until it has been settled whether a satisfactory formalization of this metatheory is possible or not, and whether it corresponds to the needs of constructive theories, it will be too early to judge about the real value of adopting the subtractive calculus.

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