DUAL STRUCTURES IN JBW-ALGEBRAS

by

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Vitae

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Abstract

Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ with a cyclic and separating vector. From the theory of Tomita and Takesaki it is known that there exists a $*$-antiisomorphism of $\mathfrak{M}$ onto its commutant $\mathfrak{M}'$ in $\mathcal{B}(\mathfrak{H})$. We extend this result into the realm of JBW-algebras.

We prove:

(i) If A is an atomic JBW-algebra and ϕ a faithful state in E then V_ϕ is order isomorphic to A.

(ii) If A is a JBW-algebra possessing a faithful trace τ in E, then V_τ is order isomorphic to A.

As a corollary to (ii) we also show

(iii) For each positive $\sigma \in V_\tau$ there exists a positive a_0 in A such that

$$\langle b, \sigma \rangle = \langle 2a_0 \circ (a_0 \circ b) - a_0^2 \circ b, \tau \rangle$$

Now suppose that ϕ is a faithful state in the predual, E, of any JBW-algebra A. Let $a \in A$; then

the linear functional $\psi_a: A \to R$ defined by $\langle b, \psi_a \rangle = \langle U_a b, \phi \rangle$ is in E. Suppose that a in A is a square and ψ_a is faithful on A. Then

(iv) V_{ψ_a} is order isomorphic to V_ϕ

Combining (ii), (iii) and (iv) we have

(v) Let A be a JBW-algebra with a faithful trace τ in E. Then A is order isomorphic to V_σ for any faithful state σ in E dominated by x.

We say a JBW-algebra A posseses the Radon-Nikodym property if, for any pair ψ, ϕ in the predual with

$$0 \leq \langle a^2, \psi \rangle \leq \langle a^2, \phi \rangle$$

for every a in A, there exists a square b in A such that $\langle a, \psi \rangle = \langle U_b a, \phi \rangle$ for every a in A. It is known that the self-adjoint part of a von Neumann algebra possesses the Radon - Nikodym property.

(vi) Let ψ and ϕ be two faithful states in the predual of a JWB-algebra which has the Radon - Nikodym property. Then V_ϕ and V_ψ are order isomorphic.

Table of Contents

Introduction In their paper of 1936,([16]) Jordan, von Neumann and Wigner introduced the concept of an "r - number system" and showed its connection to the formulation of quantum theory. The notion of an r - number system was generalized and this generalization has been extensively studied by various authors. Specifically let V be a vector space over some field with a bilinear, commutative multiplication $(v, w) \mapsto v \circ w$ which satisfies the following identity for every pair of elements v, w in V:

$$v^2 \circ (v \circ w) = v \circ (v^2 \circ w)$$

V is called a Jordan algebra, a name apparently introduced by Albert in 1946. For the algebraic study of Jordan algebras the reader is referred to [2] and [15].

Alfsen and Shultz recently gave an infinite dimensional generalization of r - number systems; a JB-algebra is a real Jordan algebra, A, which is also a Banach space whose norm and multiplication are related by

(i) $\|a \circ b\| \leq \|a\| \|b\|$

(ii) $\|a^2\| = \|a\|^2$

(iii) $\|a^2\| \leq \|a^2 + b^2\|$

for every pair $a,b, \in A$.

In a series of papers, [4], [5], [6], and [7], Alfsen and Shultz have given a complete geometric characterization of the state space of a C*-algebra. In doing so they developed the theory of JB-algebras and introduced certain ideas and concepts that seem of relevance for axiomatic quantum theory. This connection has been further studied by Araki in [9].

The class of JB-algebras contains the self-adjoint parts of C*-algebras when equipped with the symmetrized product $x \circ y = 2^{-1}(xy + yx)$. An important part of the theory of von Neumann algebras, which has become crucial in the application of operator algebras to quantum theory, is Tomita-Takesaki theory (see [19]). We have studied the problem of generalizing this theory in the JB-algebra context.

The JB-algebra generalization of a von Neumann algebra is a JBW-algebra; this we introduce in Chapter I, § 1 and prove a technical theorem we will need throughout. In § 2 we study the order ideal V_ϕ associated with a state ϕ; for the special case of a von Neumann algebra we indicate how V_ϕ can be identified with the commutant of the G N S representation with respect to ϕ. At the end of Chapter I we state and discuss our main conjecture: V_ϕ is order isomorphic to the JBW-algebra when ϕ is a faithful normal state.

An atom of a JBW-algebra is a minimal idempotent. A JBW-algebra is atomic if every idempotent is the least upper bound of orthogonal atoms. In Chapter II we show our conjecture to be true for atomic JBW-algebras.

Chapter III begins with a consideration of a JBW-algebra, A, with a faithful normal trace, τ, and we show that V_τ is isomorphic to A. As a corollary we prove a Radon-Nikodym theorem for traces. Then under the hypothesis of a Radon-Nikodym property we show that the order ideals associated with any two faithful normal states are order isomorphic and consequently order isomorphic to A.

Chapter I Dual structures in JBW-algebras

In this chapter we introduce the basic mathematical structures involved in this work. §1 is devoted to introducing JBW-algebras and recalling previously known results that will be pertinent to our studies. Nevertheless Theorem 1.7 is new. In §2 we define the object V_ϕ and establish a number of results that set the scene for the rest of the thesis. At the end of §2 we present and discuss our main conjecture.

§1 JBW-algebras

A real Jordan algebra, A, is a real vector space equipped with a commutative, but not necessarily associative, bilinear multiplication

$$(a,b) \in A \times A \to a \circ b \in A$$

which satisfies the following identity

$$a^2 \circ (b \circ a) = a \circ (b \circ a^2)$$

valid for all pairs of elements a,b of A. (Here, and in the sequel, $a^2 = a \circ a$.)

Given any real associative algebra R with multiplication $(a,b) \to a\,b$, we can make R a Jordan algebra by forming the symmetrised product.

$$a \circ b \equiv 2^{-1}(a\,b + b\,a)$$

A Jordan algebra A is called special if it can be embedded as a Jordan sub-algebra of an associative algebra equipped with this symmetrized product. We note immediately that there exist Jordan algebras that are not special; such algebras are called exceptional. Indeed Albert showed in [1] that M_3^8 - the hermitian 3×3 matrices over the Cayley numbers - is an exceptional Jordan algebra.

An important composition in the study of Jordan algebras is the Jordan triple product

$$\{a\,b\,c\} \equiv (a \circ b) \circ c - (c \circ a) \circ b + (b \circ c) \circ a$$

We will be particularly interested in the following linear maps $U_a : A \to A$ defined for every a in A by

$$U_a : x \in A \to \{a\,x\,a\} \in A$$

These maps will be used throughout our work.

The algebraic theory of Jordan algebras has been studied extensively. The reader is referred to [15] for example for an exposition of this theory. An important result to be found in [15] is Macdonald's Theorem. Here and in the sequel we will assume that all Jordan algebras considered possess an identity, 1.

<u>Theorem</u> (Macdonald) Every polynomial Jordan identity in three variables and 1, which is of degree at most one in one of these and which holds for all special Jordan algebras, is valid for all Jordan algebras.

<u>Definition 1.1</u> ([8]) A JB-algebra is a real Jordan algebra, A, which is also a Banach space whose norm, $\|\cdot\|$, satisfies the following axioms

(i) $\|a \circ b\| \leq \|a\| \, \|b\|$

(ii) $\|a^2\| = \|a\|^2$

(iii) $\|a^2\| \leq \|a^2 + b^2\|$

for all pairs of elements of a,b in A.

Theorem 2.1 of [8] tells that the set $A^+ = \{a^2 | a \in A\}$ is a proper convex cone in A that organizes A into a complete order unit space whose distinguished order unit is 1, and such that the order unit norm coincides with the given one. (See Appendix B.)

We also note that Proposition 2.7 of [8] states that the linear maps $U_a : x \to \{a\,x\,a\}$ are positive $(U_a(A^+) \subset A^+)$ for every $a \in A$.

If A is a JB-algebra, we say A is monotone complete if, whenever $\{a_\alpha\} \subset A$ is an increasing net bounded above, the least upper bound of $\{a_\alpha\}$ exists. A bounded linear functional, p, on A is called normal if $\lim\langle a_\alpha, p\rangle = \langle a, p\rangle$ for every increasing net $\{a_\alpha\} \subset A$ with $\text{l.u.b.}\,a_\alpha = a$.

<u>Definition 1.2</u> A JB-algebra which is also a Banach dual space is called a JBW-algebra.

Let a be a JBW-algebra with predual E (i.e. $E^* = A$). In [18] Shultz shows that A is monotone complete; that E is essentially unique, in the sense that it consists of the normal bounded linear functionals on A; and that E is complete as a Banach space.

Let

$$E^+ \equiv \{p \in E \mid \langle a^2, p\rangle \geq 0 \;\; \forall a \in A\}$$

and

$$K \equiv \{p \in E^+ \mid \langle 1, p\rangle = 1\}$$

Then E^+ is a proper convex cone with base K and (E,K) is a base-norm space (see Appendix B) whose base-norm coincides with the original norm.

In the course of this work we will have occasion to use the concepts and results of the non-commutative spectral theory developed by Alfsen and Shultz in [4]. The main defintions and results pertinent to our work are collected in Appendix C.

In [4], Alfsen and Shultz proved that the pair $(A,1)$ and (E,K) are in spectral duality, and that A has a well defined functional calculus for bounded real valued Borel functions on $\mathbb{R}$.

We now give four examples to illustrate the concepts of JB- and JBW-algebras

(i) Let $\mathfrak{H}$ be an infinite dimensional Hilbert space and let $C(\mathfrak{H})$ be the algebra of compact operators acting on $\mathfrak{H}$. With the operator norm $C(\mathfrak{H})$ is known to be a C*-algebra (see for example [17], P. 46). However $C(\mathfrak{H})$ is not a Banach dual space. Let A be the self-adjoint part of $C(\mathfrak{H})$. With the symmetrized product and given norm A is seen to be a JB-algebra; A is not a JBW-algebra.

(ii) Let $\mathfrak{M}$ be a von Neumann algebra (see Appendix A) and let A be the real linear space of all self-adjoint elements of $\mathfrak{M}$. With the symmetrized multiplication

$$x \circ y \equiv {}^{1}/_{2}\,(xy + yx) \;;\; x, y \in A$$

and the norm inherited from $\mathfrak{M}$, A is a JBW-algebra. The normal linear functionals of A are the restriction to A of the normal self-adjoint linear functional on $\mathfrak{M}$.

(iii) M_3^8 can be normed in such a way as to make it a JBW-algebra. In fact M_3^8 is one of the "r-number systems" introduced by Jordan, von Neumann and Wigner, [16]. JB-algebras thus appear as a generalization of r-number systems to arbitrary dimension. The JBW-structure allows generalization of the functional calculus developed by Jordan, von Neuman and Wigner for finite dimensional r-number systems.

(iv) Let H be a real Hilbert space of dimension at least three and let e be a distinguished unit vector of H. Let $N = \{e\}^{\perp}$, so that $H = N \oplus \mathbb{R}e$. Define the following Jordan multiplication on H:

$$(a+\alpha e) \circ (b+\beta e) \equiv [\alpha\beta + (a|b)]e + \alpha b + \beta a$$

$$a,b \in N, \quad \alpha,\beta \in R.$$

The set of squares for this multiplication is a proper convex cone in H with order unit e. The order-unit norm is equivalent to the inner product norm on H. With this order unit norm and multiplication, H is a JBW-algebra. (See [20].)

We will make frequent use of the $\sigma(A,E)$- and $\sigma(E,A)$-topologies on A and E respectively. They will be refered to as the weak topologies (on A and E respectively), and such statements as "weak convergence in A" will mean "$\sigma(A,E)$-convergence in A".

The strong topology on A is the topology defined by the family of semi-norms:

$$a \to \langle a^2, p\rangle^{1/2}, \quad p \in K.$$

This topology is a locally convex Hausdorff topology. From spectral theory (See Appendix C and [4], P. 68) we know that

$$\langle a,p\rangle^2 \leq \langle a^2,p\rangle, \quad a \in A,\ p \in k$$

so that strong convergence implies weak convergence. On the other hand

$$\langle a^2,p\rangle \leq \sup_{p \in K} \langle a^2, p\rangle = \|a^2\| = \|a\|^2$$

so norm convergence implies strong convergence.

The proof of [18, Lemma 2.2] shows that the dual U_a^* of the map U_a defined on P. 2 maps E into E; hence U_a is weakly continuous. One verifies that A satisfies the assumptions of [8, §4] to prove the following two preliminary results.

<u>Lemma 1.3</u> [8, Lemma 4.1] For monotone nets in A, strong and weak convergence coincide. If the net is increasing, and its weak/strong limit exists, this limit is the least upper bound; conversely, its least upper bound will be its weak/strong limit. (If the net is decreasing similar conclusions hold.)

<u>Lemma 1.4</u> [8, Lemma 4.1] Multiplication in A is separately weakly continuous in each variable, and it is jointly strongly continuous on bounded subsets.

<u>Lemma 1.5</u> Suppose $\{a_\alpha\} \subset A$ is a increasing net with least upper bound a. Then, for each $b \in A$, $U_{a_\alpha} b$ converges weakly to $U_a b$.

<u>Proof</u> By assumption $a_\alpha \leq a$ for each α. We can assume, without loss of generality that $a_\alpha \geq 0$ for all α. Then $\|a_\alpha\| \leq \|a\|$. Hence $a_\alpha{}^2$ converges strongly to a^2 and $a_\alpha \circ b$ converges strongly to $a \circ b$. Thus

$$U_{a_\alpha} b = 2a_\alpha \circ (a_\alpha \circ b) - a_\alpha{}^2 \circ b$$

converges strongly (and so weakly) to $U_a b$.

q.e.d. (Lemma 1.5).

<u>Lemma 1.6</u> The following two Jordan identities

(i) $\{a\{c\{c\,b\,c\}c\}a\} = \{a\{c^2 b\,c^2\}a\}$

(ii) $\{c\{a\{c\,b\,c\}a\}c\} = \{\{c\,a\,c\}b\{c\,a\,c\}\}$

hold in all Jordan algebras.

<u>Proof</u> Let A be a special Jordan algebra, with multiplication

$$x \circ y = {}^1/_2 (xy + yx).$$

Then

$$\{xyx\} = 2x \circ (x \circ y) - x^2 \circ y$$
$$= xyx$$

Therefore

$$\{a\{c\{cbc\}c\}a\} = a(c(cbc)c)a$$
$$= ac^2bc^2a$$
$$= \{a\{c^2bc^2\}a\}$$

By Macdonald's theorem, this identity is valid in all Jordan algebras.

(ii) is proved in a similar manner.

q.e.d. Lemma 1.6

An element $u \in A$ is called an idempotent if $u^2 = u$. If A is a JBW-algebra, we know from the spectral theorem in A(see Appendix C) that A is generated by its idempotents and that the set, $\mathcal{U}$, of idempotents forms a complete orthomodular lattice. We denote by $\vee$ and $\wedge$ the lattice operation of taking least upper bound and greatest lower bound of subsets of $\mathcal{U}$. Note that for every idempotent u, U_u is a positive projection on A. (See Appendix C.)

The support $r(a)$ of a positive element, a, of A is defined to be

$$r(a) \equiv \wedge \{ u \mid u^2 = u, \ U_u a = a \} .$$

<u>Theorem 1.7</u> Let A be a JBW-algebra and let $a_0 \in A$ be positive. Then, for every $a \in A$ with $0 \leq a \leq a_0^2$ there is a positive $b \in A$ such that $U_{a_0} b = a$.

<u>Proof</u> Firstly we notice that if the result is true when $r(a_0) = 1$ it is true for every positive $a_0 \in A$. Indeed suppose $r(a_0) \neq 1$; then $a \in \operatorname{im} U_{r(a_0)}$. But, by

Proposition 2.3 of [5], im $U_{r(a_0)}$ is a JBW-algebra with identity $r(a_0)$, so if the result is true in this relativised setting it is clearly true in A.

Therefore suppose that $r(a_0) = 1$. Consider the sequence of functions

$$\phi_n(t) = \begin{cases} 0, & 0 \leq t < n^{-1}, \\ t^{-1}, & n^{-1} \leq t \leq \|a_0\| . \end{cases}$$

For each n, $\phi_n(t)$ is a bounded Borel function on $[0, \|a_0\|]$. Let

$$a_0 = \int_0^{\|a_0\|} \lambda \, de_\lambda$$

be the spectral decomposition of a_0. (See Appendix C.) A is closed under the functional calculus of bounded Borel functions, so for each n, there exists a unique c_n in A such that

$$c_n = \int_0^{\|a_0\|} \phi_n(\lambda) \, de_\lambda$$

Each c_n is contained in the weak closure, W, of the norm closed associative subalgebra of A generated by a_0 and 1 [8, P. 31]. W is also an associative subalgebra of A [8, Lemma 4.2] and therefore contains the positive square root of c_n for each n. ($c_n^{1/2}$ is contained in the norm closed associative subalgebra of A generated by c_n and 1. Since $c_n \in W$, this subalgebra is a subalgebra of W.)

Therefore

$$\{c_n^{1/2} \; a \quad c_n^{1/2}\} = 2c_n^{1/2} \circ (c_n^{1/2} \circ a_0) - c_n \circ a_0$$
$$= c_n \circ a_0$$
$$= \int_0^{\|a_0\|} \phi_n(\lambda) \; de_\lambda$$
$$= \int_{n^{-1}}^{\|a_0\|} de_\lambda$$

By proposition 8.2 of [4],

$$\int_{n^{-1}}^{\|a_0\|} de_\lambda$$

is an idempotent of A, which we denote by u_n. It is clear that $u_n \leq u_m$ when $n < m$. Let $u = \vee_n u_n$. u is the least upper bound of the monotone increasing sequence u_n, so $\{u_n\}_{n=1}^{\infty}$ has strong limit u (Lemma 1.3). By Lemma 1.5 $U_{u_n} b$ converges weakly to $U_u b$ for each b in A. In particular, $U_{u_n} a_0$ converges weakly to $U_u a_0$. On the other hand,

$$\begin{aligned} U_{u_n} a_0 &= \{u_n a_0 u_n\} \\ &= u_n \circ a_0 \\ &= \int_{n^{-1}}^{\|a_0\|} \lambda \, de_\lambda \end{aligned}$$

since $u_n \in W$ for each n, and W is associative. For each $p \in K$,

$$\begin{aligned} \langle U_{u_n} a_0, p \rangle &= \int_{n^{-1}}^{\|a_0\|} \lambda \, d\langle e_\lambda , p \rangle \\ &\to \int_0^{\|a_0\|} \lambda \, d\langle e_\lambda , p\rangle \quad \text{as} \quad u \to \infty \\ &= \langle a_0, p \rangle \end{aligned}$$

by the spectral theory. Hence $U_{u_n} a_0$, which we have already seen to converge weakly to $U_u a_0$, also converges weakly to a_0; therefore $U_u a_0 = a_0$ and as the support of a_0 is 1, $u = 1$.

Next, we notice that, by virtue of the comments following Lemma 4.4 of [8] that $L_{c_n^{1/2}} L_{a_0} = L_{a_0} L_{c_n^{1/2}}$ for each integer n, where $L_x : A \to A$ is the linear mapping $L_x : y \to x \circ y$ defined for element x of A. From the fact that $U_x = 2L_x^2 - L_{x^2}$, for each x in A, it follows easily that $U_{c_n^{1/2}} U_{a_0} = U_{a_0} U_{c_n^{1/2}}$.

Using this observation and Lemma 1.6, we obtain for arbitrary $b \in A$:

$$
\begin{aligned}
\{c_n \{a_0 b\, a_0\} c_n\} &= \{a_0 \{c_n b\, c_n\} a_0\} \\
&= \{a_0 \{c_n^{1/2} \{c_n^{1/2} b\, c_n^{1/2}\} c_n^{1/2}\} a\} \\
&= \{c_n^{1/2} \{a_0 \{c_n^{1/2} b\, c_n^{1/2}\} a_0\} c_n^{1/2}\} \\
&= \{\{c_n^{1/2} a_0\, c_n^{1/2}\} b \{c_n^{1/2} a_0\, c_n^{1/2}\}\} \\
&= \{u_n b\, u_n\}.
\end{aligned}
$$

As noted above $\{u_n \; b u_n\}$ converges weakly to $\{u b u\} = b$; that is $\{c_n\{a_0 b a_0\} c_n\}$ converges weakly to b for every $b \in A$.

Now let $a \in A$ with $0 \leq a \leq a_0^2$. Since U_{c_n} is positive for each n:

$$0 \leq U_{c_n} a \leq U_{c_n} a_0^2 = c_n^2 \circ a_0^2 = u_n \leq 1$$

i.e. $\{U_{c_n} a\}_{n=1}^{\infty}$ is a sequence contained in the order interval $[0,1] = \{a \in A \mid 0 \leq a \leq 1\}$. $[0,1]$ is weakly compact, so we may choose a weakly convergent subsequence $\{U_{c_{n_k}} a\}_{k=1}^{\infty}$ of $\{U_{c_n} a\}_{n=1}^{\infty}$ with limit point b in $[0,1]$.

We claim that $U_{a_0} b = a$. Indeed, since U_{a_0} is weakly continuous, $\{U_{a_0} U_{c_{n_k}} a\}_{k=1}^{\infty}$ converges weakly to $U_{a_0} b$. But by what was shown above the weak limit of $\{U_{a_0} U_{c_n} a\}_{n=1}^{\infty}$ is a. That is $U_{a_0} b = a$.

q.e.d. Theorem 1.7

Theorem 1.7 provides us with an important tool, that we use in both Chapter II and Chapter III. It allows to show that when $r(a_0) = 1$ the linear mapping U_{a_0} of A into the order ideal generated by ${a_0}^2$ is in fact a bijection onto (see Theorems 2.6 and 3.8).

§ 2 The order ideal associated with a state.

In this section we turn our attention to the study of the states on a JB-algebra. Let A be a JB-algebra, and let A^* be the Banach dual space of A. The proper cone in A^* defined as

$$(A^*)^+ \equiv \{\phi \in A^* \mid \langle a^2, \phi \rangle \geq 0 \ \forall a \in A\}$$

will play an important role in the sequel.

A bounded linear functional, ϕ, is said to be a state if $\phi \in (A^*)^+$ and $\langle 1, \phi \rangle = 1$. The set of states on A is a compact convex set denoted by K^*.

<u>Definition 1.8</u> Let ϕ be a state on A; then the order ideal generated by ϕ in A^* is the linear subspace of A^*

$$V_\phi \equiv \{\sigma \in A^* \mid \exists \lambda \in R^+ \text{ s.t. } -\lambda\phi \leq \sigma \leq \lambda\phi\}$$

Notice that $V_\phi^+ \equiv V_\phi \cap (A^*)^+$ is the smallest face of $(A^*)^+$ containing ϕ (see Appendix B for the definition of a face), and

$$V_\phi = \text{lin.span}\,[V_\phi^+]$$

V_ϕ is an order-unit space ([3, P. 173]) and has an order-unit norm:

$$\|\sigma\|_\phi \equiv \inf\{\lambda \in R^+ \mid -\lambda\phi \leq \sigma \leq \lambda\,\phi\}.$$

Our study of V_ϕ is motivated by the following theorem familiar in the theory of W*- and C*- algebras. (See [13] and Appendix A).

<u>Theorem 1.9</u> Let $\mathfrak{M}$ be a von Neumann algebra and ϕ a state on $\mathfrak{M}$. Let $\{\pi_\phi, \mathfrak{H}_\phi\}$ be the G.N.S. representation $\mathfrak{M}$ canonically associated to ϕ. Then there is an order isomorphism, $\hat{\pi}_\phi$, of V_ϕ onto $\pi_\phi(\mathfrak{M})'_{s.a.}$, where $\pi_\phi(\mathfrak{M})'$ is the commutant of $\pi_\phi(\mathfrak{M})$ in $\mathcal{B}(\mathfrak{H}_\phi)$ and $\pi(\mathfrak{M})'_{s.a.}$ is its self-adjoint part.

In [14, P. 86], Emch shows there is a one-to-one correspondence between the positive functionals in V_ϕ

and the positive operators in $\pi_\phi(\mathfrak{M})'$. The theorem itself is stated (without proof) in [3].

When we generalize from C*-algebras to JB-algebras, it is not possible in general to represent a JB-algebra over a Hilbert space in the nice canonical way it is with a C*-algebra using the G.N.S. construction. (Indeed as is the case with M_3^8, it may not be possible to find any such representation at all.) Consequently we have no immediate notion of a commutant. However Theorem 1.9 suggests we have a suitable replacement at hand, namely V_ϕ. The work of this thesis is intended to support this claim, in a way which will be made more explicit later.

Let A be a JB-algebra with state space K*. From [4, §12] we know that K* is strongly spectral (see Appendix C for the definition of strongly spectral convex sets). Therefore each a in A has a unique decomposition $a = a^+ - a^-$ with a^+, a^- positive orthogonal elements of A. (See Appendix C.)

For $\sigma \in K^*$, we define the following semi-norm on A:

$$p_\sigma(a) = \langle a^+, \sigma \rangle + \langle a^-, \sigma \rangle .$$

By the uniqueness of the decomposition $a = a^+ - a^-$, p_σ is well-defined, and it is clear that p_σ is homogeneous. To see that p_σ is also subadditive, let $a, b \in A$ have orthogonal decompositions $a = a^+ - a^-$, $b = b^+ - b^-$, and $c = a + b$ have orthogonal decomposition $c = c^+ - c^-$. Now $a^+ \geq a$ and $b^+ \geq b$ so $a^+ + b^+ \geq a + b = c$, and by Proposition 9.3 of [4], $a^+ + b^+ \geq c^+$. In a similar way c^- is seen to be less than $a^- + b^-$; hence we see that

$$\begin{aligned} p_\sigma(a+b) &= \langle c^+, \sigma\rangle + \langle c^-, \sigma\rangle \\ &\leq \langle a^+ + b^+, \sigma\rangle + \langle a^- + b^-, \sigma\rangle \\ &= p_\sigma(a) + p_\sigma(b). \end{aligned}$$

<u>Definition 1.10</u> A state $\phi \in K^*$ is said to be faithful if $\langle a^2, \phi\rangle = 0$ implies $a = 0$ $(a \in A)$.

Let $\phi \in K^*$ be faithful. Then it is seen that p_ϕ is a norm on A, which we will denote by $\| \cdot \|_\phi$. Let

$$(A_\phi)_1 \equiv \{a \in A \mid \|a\|_\phi \leq 1\}$$

$$C_\phi \equiv \{a \in A^+ \mid \langle a, \phi \rangle = 1\}$$

$$S_\phi \equiv co\{C_\phi \cup (-C_\phi)\}.$$

Since ϕ is faithful on A, C_ϕ is a base for the cone A^+ and the Minkowski functional, q_ϕ, associated with S_ϕ is a semi-norm on A. (See Appendix B.) In fact it follows from the next lemma that $q_\phi(a) = \|a\|_\phi$ for every $a \in A$:

<u>Lemma 1.11</u> $S_\phi = (A_\phi)_1$

<u>Proof</u> Let $a \in S_\phi$; then there exist $\lambda, \mu \in R^+$ with $\lambda + \mu = 1$, and $a_1, a_2 \in C_\phi$ such that $a = \lambda a_1 - \mu a_2$. If $a = a^+ - a^-$ is the unique orthogonal decomposition of a, then as was seen above, $\lambda a_1 \geq a^+$ and $\mu a_2 \geq a^-$. Hence

$$1 = \lambda + \mu = \langle \lambda a_1, \phi \rangle + \langle \mu a_2, \phi \rangle$$

$$\geq \langle a^+, \phi \rangle + \langle a^-, \phi \rangle = \|a\|_\phi$$

That is $S_\phi \subset (A_\phi)_1$

Conversely, since S_ϕ is absorbing, it suffices to show $\|a\|_\phi = 1$ implies $a \in S_\phi$. Suppose, therefore, that $\|a\|_\phi = 1$ and let $\mu = \langle a^+, \phi\rangle$ and $\lambda = \langle a^-, \phi\rangle$ where $a = a^+ - a^-$ is the orthogonal decomposition of a. Notice that $\mu + \lambda = \langle a^+, \phi\rangle + \langle a^-, \phi\rangle = 1$. Without loss of generality we can assume that neither μ or λ is zero, (if either is $a \in C_\phi$), and define $a_1 = \mu^{-1}a^+$ and $a_2 \equiv \lambda^{-1}a^-$. Then $a_1, a_2 \in C_\phi$, and $a = \mu a_1 - \lambda a_2 \in S_\phi$

q.e.d. Lemma 1.11

For the next theorem we denote by A_ϕ, A endowed with the norm $\|\cdot\|_\phi$ (ϕ faithful) and assume that V_ϕ is endowed with the order-unit norm mentioned earlier.

<u>Theorem 1.12</u> $A_\phi^* = V_\phi$, algebraically and topologically.

<u>Proof</u> Firstly, we note that, from Lemma 1.11, A_ϕ is a base-norm space whose base is $C_\phi = A^+ \cap \phi^{-1}(1)$. It follows therefore, from Theorem B4 of Appendix B, that A_ϕ^* is an order unit space with distinguished order unit ϕ. Furthermore we have

$$(A_\phi^*)^+ = \{\tau \in A_\phi^* \mid \tau(a) \geq 0 \quad a \in A\}$$

Let τ be an element of $(A_\phi^*)^+$ and define $|\!|\!|\tau|\!|\!| = \inf\{\lambda \in R^+ \mid 0 \leq \tau \leq \lambda\phi\}$ to be the norm of τ in A_ϕ^*. Then for every $a \in A$ (= A_ϕ, algebraically):

$$\begin{aligned} |\tau(a)| &\le \max[\tau(a^+), \tau(a^-)] \\ &\le \max[\, |||\tau|||\ \phi(a^+),\ |||\tau|||\ \phi(a^-)] \\ &= |||\tau|||\ \max[\, \|a^+\|_\phi\ , \|a^-\|_\phi] \\ &\le |||\tau|||\ \max[\, \|a^+\|\ , \|a^-\|\,] \end{aligned}$$

since $\|b\| = \sup\{ |\langle b, \sigma \rangle| \mid \sigma \in K^*\}$ for every $b \in A$. Therefore

$$|\tau(a)| \le |||\tau|||\ \ \|a\|$$

and we see that τ is in $(A^*)^+$, where A is equipped with its original norm. Furthermore $0 \le \tau(a) \le |||\tau|||\ \langle a, \phi \rangle$ for every positive a in A; and since $V_\phi^{\ +}$ is a face of $(A^*)^+$, τ is a member of $V_\phi^{\ +}$.

On the other hand if $\sigma \in V_\phi^{\ +}$ there exists $\lambda \in R^+$ such that $0 \le \sigma \le \lambda\,\phi$. Let $a = a^+ - a^- \in A_\phi$. Then

$$|\langle a, \sigma \rangle| \le \langle a^+ + a^-, \sigma \rangle \le \lambda \langle a^+ + a^-, \phi \rangle = \lambda \|a\|_\phi$$

so $\sigma \in A_\phi^{\ *}$. But $\langle a, \sigma \rangle \ge 0$ for all $a \in A_\phi^{\ +}$; therefore $\sigma \in (A_\phi^{\ *})^+$.

Hence we have shown that $V_\phi^{\ +} = (A_\phi^{\ *})^+$. Using the fact that V_ϕ and A_ϕ are both linear spans of their positive parts we conclude that, as vector spaces, $V_\phi = A_\phi^*$.

To prove the topological part of the theorem it is sufficient to show $|||\tau||| = \|\tau\|_\phi$ for every τ in V_ϕ. To this end we use Lemma 1.11 in the following computation:

$$\begin{aligned}
|||\tau||| &= \sup_{\|a\|_\phi \leq 1} |\langle a, \tau\rangle| \\
&= \sup\{|\langle a, \tau\rangle| \mid a \in \mathrm{Co}\,(C_\phi \cup - C_\phi)\} \\
&= \sup\{|\langle a, \tau\rangle| \mid a \in C_\phi\} \\
&= \inf\{\lambda \in R^+ \mid -\lambda\langle a, \phi\rangle \leq \langle a, \tau\rangle \leq \langle a, \phi\rangle,\ a \in C_\phi\} \\
&= \inf\{\lambda \in R^+ \mid -\lambda\,\phi \leq \sigma \leq \lambda\,\phi\} \\
&= \|\tau\|_\phi
\end{aligned}$$

q.e.d. Theorem 1.12

We now turn to the normal states on a JBW-algebra. Let A be a JBW-algebra with predual E. Recall that E consists of all the normal functionals on A. Let K^* be the state space of A, and

$$K = \{p \in K^* \mid p \text{ is normal}\}.$$

It is clear that K is a convex subset of K^*. In fact, K is a face of K^*, as will be seen as a consequence of the following result.

<u>Lemma 1.13</u> E^+ is a face of $(A^*)^+$

<u>Proof</u> It is clear that if $\lambda \in R^+$ and σ, $p \in E^+$, then $\lambda\sigma \in E^+$ and $\sigma + p \in E^+$. To show that E^+ is a face of $(A^*)^+$ we must show that $p \in E^+$ and $0 \le \sigma \le p$ implies that $\sigma \in E^+$; i.e. we must show that σ is normal. To do so, it is clearly sufficient to show that, whenever $\{a_\alpha\} \subset A$ is a monotonically decreasing net with greatest lower bound 0, g.l.b.$\langle a_\alpha, \sigma \rangle = 0$.

So let $\{a_\alpha\} \subset A$ be as described. Then g.l.b.$\langle a_\alpha, p\rangle = 0$, and for each α

$$0 \le \langle a_\alpha, \sigma \rangle \le \langle a_\alpha, p\rangle$$

Therefore g.l.b.$\langle a_\alpha, \sigma \rangle = 0$ and σ is normal.

q.e.d. Lemma 1.13

As was remarked after the definition of V_ϕ, V_ϕ^+ is the smallest face of $(A^*)^+$ containing ϕ. Therefore we see from the preceding Lemma, that, when ϕ is a normal state, V_ϕ is a linear subspace of E.

Definition 1.14 Let A be a JBW-algebra and let ψ be a normal state on A. Let

$$v = \vee \{v' \in A \mid v'^2 = v', \langle v', \psi \rangle = 0\}$$

and $s(\psi) = 1 - v$ $s(\psi)$ is called the support of ψ.

Proposition 1.15 Let ϕ be a normal state on a JBW-algebra, A, with predual E. The following statements are equivalent:

(i) ϕ is faithful

(ii) $s(\phi) = 1$

(iii) V_ϕ is norm dense in E

Proof From the definition of $s(\phi)$, it follows that $\langle s(\phi), \phi \rangle = 1$, and so $\langle 1 - s(\phi), \phi \rangle = 0$. Therefore, if ϕ is faithful, $s(\phi) = 1$.

Conversely, if ϕ is not faithful then $H \equiv \{a \in A^+ \mid \langle a, \phi \rangle = 0\}$ is seen to be a weakly closed proper face of A^+. By [4, Theorem 12.3] H is of the form $\text{im}^+ U_v$ for some idempotent, v, different from zero. Therefore $\langle v, \phi \rangle = 0$ and $s(\phi) \leq 1 - v$;

i.e. $s(\phi) \neq 1$. Thus we have shown (i) and (ii) are equivalent.

To show that (i) and (iii) are equivalent, we remark that V_ϕ is norm dense in E if and only if

$$(V_\phi^+)_* \equiv \{a \in A \mid \langle a,\sigma \rangle \geq 0 \; \forall \sigma \in V_\phi^+\}$$

is a proper cone in A. ([13], Lemma 6).

Now if ϕ is faithful, by Theorem 1.12 $V_\phi = A_\phi^*$, so $(V_\phi^+)_* = A^+$ which is a proper cone of A.

On the other hand, if ϕ is not faithful, there exists $a \in A^+$ such that $\langle a, \phi \rangle = 0$. Then $\langle a, \sigma\rangle = 0$ for every $\sigma \in V_\phi^+$ and $\langle -a, \sigma\rangle = 0$ for every $\sigma \in V_\phi^+$. Therefore both a and $-a$ belong to $(V_\phi^+)_*$ and $(V_\phi^+)_*$. So V_ϕ is not a proper cone of A.

q.e.d. Proposition 1.15

We now present the main conjecture to which this work is intended to lend support.

<u>Conjecture 1.6</u> Let A be a JBW-algebra and ϕ a faithful normal state on A. Let V_ϕ be the order ideal generated by ϕ. Then there exists an order isomorphism, Λ, of V_ϕ onto A with $\Lambda(\phi) = 1$.

In the case where A is the self-adjoint part of a von Neumann algebra in standard form with respect to a faithful normal state ϕ, Theorem 1.9 shows that the conjecture is true, via the Tomita-Takesaki isomorphism theorem between the von Neumann algebra and its commutant. (See Appendix A.) The purpose of this thesis is to show that when presented in the above form the Tomita-Takesaki isomorphism theorem extends into the realm of JBW-algebras. The tools we use should throw some light on some hitherto neglected aspects of the structures involved in the Tomita-Takesaki theory.

Chapter II: The isomorphism theorem for atomic JBW-algebras

In this chapter we prove Conjecture 1.16 for an atomic JBW-algebra with a faithful normal state. The chapter is in two sections: § 1 gives the basic definitions, and recalls some relevant results we need from the literature in § 2, the desired isomorphism is proved as Theorem 2.7. The generalization is genuine since, in particular it covers the exceptional Jordan algebra M_3^8.

§ 1 Basic definitions and results

Let A be a JBW-algebra with predual E and normal state space K. An idempotent, u, of A is said to be an atom if $0 \leq v \leq u$, v an idempotent, implies either $v = 0$ or $v = u$. Recall from Appendix C that a projective face of K is a face of form $(\mathrm{im}\, U_v^*) \cap K$ for some idempotent v in A, and that two idempotents v_1 amd v_2 are said to be orthogonal if $v_1 + v_2 \leq 1$.

<u>Proposition 2.1</u> ([5, Proposition 1.13]) Let u be an atom of A. Then U_u has one dimensional range (i.e. $\mathrm{im} U_u \simeq \mathbb{R}$) and the corresponding projective face is a singleton subset of K consisting of an extreme point.

Moreover, there is a one-to-one correspondence between atoms in A and extreme points of K.

<u>Definition 2.2</u> A JBW-algebra is said to be atomic if every idempotent is the least upper bound of orthogonal atoms.

We give two examples and a counter-example.

(i) Any JBW-algebra whose dimension, as a vector space, is finite is atomic. In particular, M_3^8 is atomic.

(ii) Let $\mathfrak{H}$ be a complex Hilbert space. The bounded self-adjoint operators, $\mathcal{B}_s(\mathfrak{H})$, form a JBW-algebra with the operator norm and Jordan product given by

$$R \circ S = 1/2(RS + SR)$$

$$R, S \in \mathcal{B}_s(\mathfrak{H}).$$

$\mathcal{B}_s(\mathfrak{H})$ is an atomic JBW-algebra. The atoms in $\mathcal{B}_s(\mathfrak{H})$ are the orthogonal projections onto one dimensional subspaces of $\mathfrak{H}$.

(iii) Let $I = [0,1] \subseteq \mathbb{R}$. Then $\mathcal{L}^\infty(I, \mu)$, where μ is Lebesgue measure on I, is an associative JBW-algebra. $\mathcal{L}^\infty(I, \mu)$ is not atomic; indeed, it does not contain any atoms.

The following two results, due to Alfsen and Shultz, will be crucially important in the rest of this chapter.

Theorem 2.3 ([6, Lemma 5.5]) Let A be an atomic JBW-algebra with predual E and normal state space K. Then there exists a bipositive map $\psi : E \to A$ which is injective, satisfies $\|\psi\| \leq 1$ and maps the extreme points of K onto the corresponding atoms of A.

Corollary 2.4 ([6, Proposition 5.6]) Every $\rho \in K$ can be written in the form $\rho = \sum_{i=1}^{\infty} \lambda_i \rho_i$ where $\{\rho_i\}_{i=1}^{\infty}$ are pairwise orthogonal extreme points of K and $\{\lambda_i\}_{i=1}^{\infty}$ are positive numbers with $\sum_{i=1}^{\infty} \lambda_i = 1$ and $\psi(\rho) = \sum_{i=1}^{\infty} \lambda_i u_i$ where u_i is the atom corresponding to ρ_i.

§ 2 The isomorphism theorem

The main result of this section is Theorem 2.7. We decompose its proof into two steps: Proposition 2.5 and Theorem 2.6.

<u>Proposition 2.5</u> Let A be an atomic JBW-algebra with predual E, and let ϕ be a normal state on A. Let $\psi : E \to A$ be the map of Theorem 2.3; $\hat{\phi} = \psi(\phi)$ and $J(\hat{\phi})$ be the order ideal of A generated by $\hat{\phi}$. Furthermore let V_ϕ be the order ideal of Definition 1.8. Then the restriction of ψ to V_ϕ is an order isomorphism of V_ϕ onto $J(\hat{\phi})$.

<u>Proof</u> Let ψ_o be the restriction of ψ to V_ϕ. By definition we have

$$V_\phi = \{\sigma \in E \mid \exists \lambda \in R^+ \text{ s.t.} -\lambda\phi \leq \sigma \leq \lambda\phi\}$$

$$J(\hat{\phi}) = \{a \in A \mid \exists \lambda \in R^+ \text{ s.t. } -\lambda\hat{\phi} \leq a \leq \lambda\hat{\phi}\}$$

Clearly, then, the positivity of ψ_o implies that $\psi_o(V_\phi) \subseteq J(\hat{\phi})$. ψ_o is injective and bipositive so it is only necessary to show ψ_o is surjective.

We must show that for each $a \in J(\hat{\phi})$ there exists $\sigma \in V_\phi$ such that $\psi_o(\sigma) = a$. Without loss of generality we can assume that $0 \leq a \leq \hat{\phi} = \psi(\phi)$. Let $a = \int_{R^+} \lambda \, de_\lambda$ be the spectral decomposition of a and $\alpha \in R^+$, $\alpha \neq 0$. By the definition of the spectral decomposition

$$\alpha(1 - e_\alpha) \leq a$$

Let $u_1, \ldots, u_n$ be orthogonal atoms below $1 - e_\alpha$; $\hat{u}_1, \ldots, \hat{u}_n$ be the corresponding extreme points of K; and $\rho = \alpha \sum_{i=1}^{n} \hat{u}_i$

Then

$$\alpha \, \psi \left(\sum_{i=1}^{n} \hat{u}_i \right) = \alpha \sum_{i=1}^{n} u_i \leq \psi(\phi)$$

Therefore $\alpha \sum_{i=1}^{n} u_i \leq \psi(\phi)$, since ψ is bipositive. Upon using the fact that $\|\psi\| \leq 1$, we find:

$$\|\rho \phi\|^2 \geq \langle \psi(\phi), \rho \rangle \geq \alpha^2 \langle \sum_{i=1}^{n} u_i, \sum_{i=1}^{n} \hat{u}_i \rangle = n\alpha^2$$

Thus we have shown that there are only finitely many atoms below $1 - e_\alpha$. We can therefore conclude that $\alpha = \sum_{i=1}^{\infty} \mu_i u_i$ where $\{u_i\}_{i=1}^{\infty}$ consists of orthogonal atoms and $\mu_i \geq 0$.

Now

$$1 = \langle 1, \phi \rangle \geq \sum_{i=1}^{n} \langle 1, \mu_i \hat{u}_i \rangle = \sum_{i=1}^{n} \mu_i$$

so $\sum_{i=1}^{\infty} \mu_i \leq 1$, and we see that $\sum_{i=1}^{\infty} \mu_i \hat{u}_i$ is

a norm convergent series in E. Denote this sum by σ.

$$\psi(\sigma) = \psi\Big(\sum_{i=1}^{\infty} \mu_i \hat{u}_i\Big) = \lim_{n\to\infty} \psi\Big(\sum_{i=1}^{n} \mu_i \hat{u}_i\Big)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu_i u_i$$

$$= \hat{a}.$$

We use the bipositivity of ψ again to see that

$0 \leq \sigma \leq \phi$. i.e. $\sigma \in V_\phi$.

q.e.d. Proposition 2.5

<u>Theorem 2.6</u> With the assumption of proposition 2.5, assume further that ϕ is faithful. Then $U_{\hat{\phi}^2}{}^{-1}$: $A \to J(\hat{\phi})$ is an order isomorphism of A onto $J(\hat{\phi})$ with $U_{\hat{\phi}^2}{}^{-1}(1) = \hat{\phi}$.

<u>Proof</u> To simplify notation we write T for $U_{\hat{\phi}^{2^{-1}}}$. $T(1) = \{\hat{\phi}^{2^{-1}}\ 1\ \hat{\phi}^{2^{-1}}\} = \hat{\phi}$ and T is positive so $T(A) \subset J(\hat{\phi})$.

We begin by showing that the support, $r(\hat{\phi}^{2^{-1}})$, of $\hat{\phi}^{2^{-1}}$ in A is 1. Using Corollary 2.4 we can write $\hat{\phi} = \sum_{i=1}^{\infty} \lambda_i u_i$ where the u_i are orthogonal atoms and $\lambda_i \in R$ are positive with $\sum_{i=1}^{\infty} \lambda_i = 1$.

For each i let $\hat{u}_i$ be the extreme point of K corresponding to u_i. Suppose that $\sum_{i=1}^{\infty} u_i \neq 1$. Then $\sum_{i=1}^{\infty} u_i$ is an idempotent less then 1 and $u_0 = 1 - \sum_{i=1}^{\infty} u_i$ is orthogonal to u_i, $i = 1,2, \cdots$. Evaluating ϕ on u_o we get

$$\begin{aligned}\langle u_0, \phi \rangle &= \langle u_0, \sum_{i=1}^{\infty} \lambda_i \hat{u}_i \rangle \\ &= \sum_{i=1}^{\infty} \lambda_i \langle u_0, \hat{u}_i \rangle \\ &= 0\end{aligned}$$

a contradiction, since ϕ is faithful. Hence $u_0 = 0$, and $\sum_{i=1}^{\infty} u_i = 1$.

Multiplication shows that $\sum_{i=1}^{\infty} \lambda_i^{2^{-1}} u_i$ is the positive square root of $\hat{\phi}$ and we claim $\sum_{n=1}^{\infty} u_i$ is the support of $\hat{\phi}^{2^{-1}}$. Indeed, suppose u is an idempotent such that $U_u \hat{\phi}^{2^{-1}} = \hat{\phi}^{2^{-1}}$. For each i, $0 \leq \lambda_i^{2^{-1}} u_i \leq \hat{\phi}^{2^{-1}}$; therefore, since $\mathrm{im}^+ U_u$ is a face of A^+ (see Appendix C), it follows from the definition of a face (See Appendix B) that $u_i \in \mathrm{im}^+ U_u$. Thus $u_i \leq u$ for every i and $1 \geq u \geq \sum_{i=1}^{\infty} u_i = 1$. Therefore we have shown that $r(\hat{\phi}^{2^{-1}}) = \sum_{i=1}^{\infty} u_i = 1$.

We can now use Theorem 1.7 and the fact that $J(\hat{\phi})$ is the linear span of the order interval $[0, \hat{\phi}]$ to conclude that $T : A \to J(\hat{\phi})$ is onto.

Next we show that T is bipositive: Let $a \in J(\hat{\phi})$ be positive and suppose $a = Tb$, $b \in A$. Without loss of generality we may assume that $0 \leq a \leq \hat{\phi}$. By Theorem 1.7 there exists $c \in A^+$ such that $Tc = a$. Let $\{c_n\}_{n=1}^{\infty}$ be the sequence of elements constructed in the proof of Theorem 1.7 with $a_0 = \hat{\phi}^{2^{-1}}$, and recall that $U_{c_n} Tc$

converges weakly to c. Therefore $U_{c_n} a = U_{c_n} Tb = U_{c_n} Tc$ converges weakly to c. But $U_{c_n} Tb$ is weakly convergent to b, and by the uniqueness of weak limits in A, b = c. That is b is positive, and T is bipositive.

To show that T is injective we must show that Ta = 0 implies that a = 0. By the bipositivity of T, a must be positive so the proof will be complete if we can show that a > 0, Ta = 0 implies a = 0.

Thus let a be positive and $Ta = \{\hat{\phi}^{2^{-1}} \; a \; \hat{\phi}^{2^{-1}}\} = 0$. From [8, Proposition 2.8] $\{a \hat{\phi} a\} = 0$ also. Using the norm continuity of the map $b \to \{a b a\}$ we have

$$\begin{aligned} 0 = \{a \hat{\phi} a\} &= \lim_{n \to \infty} \{a \, (\sum_{i=1}^{n} \lambda_i u_i) \, a\} \\ &= \lim_{n \to \infty} \sum_{i=1}^{n} \lambda_i \{a u_i a\} \\ &= \sum_{i=1}^{\infty} \lambda_i \{a u_i a\} \end{aligned}$$

From [8, Corollary 2.2] we know that A is formally real; i.e. $\sum_{i=1}^{m} b_j^{\,2} = 0$ implies $b_j = 0$ for $j = 1, --, m$ for any $\{b_j\}_{j=1}^{m} \subset A$. Therefore, since $\{a u_i a\}$ is positive for each i, we will have $\{a u_i a\} = 0$ for every i and $0 = \sum_{i=1}^{\infty} \{a u_i a\} = \{a \sum_{i=1}^{\infty} u_i a\}$ $= \{a\, 1\, a\} = a^2$. Hence a is zero.

q.e.d. Theorem 2.6

Combining Theorem 2.6 and Proposition 2.5 we have the main theorem of this chapter:

> <u>Theorem 2.7</u> Let A be an atomic JBW-algebra and let ϕ be a faithful normal state on A. Then there exists an order isomorphism $N: V_\phi \to A$ with $N(\phi) = 1$.

<u>Proof</u> By Proposition 2.5 we know that there exists an order isomorphism $\psi_0 = \psi|_{V_\phi}$ of V_ϕ onto $J(\phi)$ with $\psi_0(\phi) = \phi$. Let $a = \hat{\phi}^{1/2}$. Then Theorem 2.6 shows that U_a is an order isomorphism of A onto $J(\phi)$ with $U_a 1 = \phi$. Therefore the map $N = \psi_0 \circ U_a^{-1}$ is an order isomorphism of V_ϕ onto A with $N(\phi) = 1$.

q.e.d. Theorem 2.7.

Chapter III The isomorphism theorem for finite JBW-algebras

A JBW-algebra is said to be finite when it has a faithful normal trace. In § 1 we define the notion of a trace for a JBW-algebra, A, and prove that when τ is a faithful normal trace on A then V_τ is order isomorphic to A. We also prove in this section a Radon-Nikodym theorem for faithful normal traces. In § 2 we discuss a property we call the Radon-Nikodym property. This property is known to hold for all von Neumann algebras We extend the isomorphism theorem to those normal states on a finite JBW-algebra dominated by a faithful normal trace, and also prove that for any JBW-algebra satisfying the Radon-Nikodym property V_ϕ and V_ψ are order isomorphic for any pair of faithful normal states, ϕ and ψ.

§ 1 The order ideal associated with a trace

The definition of a trace is based on the following result.

<u>Lemma 3.1</u> Let A be a JBW-algebra with normal state space K. Then for each τ in K the following are equivalent:

(i) $U^*_{1-v}\tau + U^*_v\tau = \tau$ for every idempotent v in A.

(ii) $\langle u, U^*_v\tau\rangle = \langle v, U^*_u\tau\rangle$ for every pair of idempotents u and v in A.

<u>Proof</u> To show (i) implies (ii) we notice that for each idempotent w in A, from [Theorem 12.2, 4]. we have

$$w \circ a = 2^{-1}(I + U_w - U_{1-w})a$$

for each element a in A.

Therefore given idempotents $u, v \in A$ and using (i)

$$\begin{aligned}\langle u, U^*_v\tau\rangle &= \langle u, 2^{-1}(I + U^*_v - U^*_{1-v})\tau\rangle \\ &= \langle u \circ v, \tau\rangle \\ &= \langle v, 2^{-1}(I + U^*_u - U^*_{1-u})\tau\rangle \\ &= \langle v, U^*_u\tau\rangle\end{aligned}$$

We now show (ii) implies (i). It follows from the spectral theory (See Appendix C) that the subspace of A consisting of elements of the form $\sum_{i=1}^{n} \lambda_i u_i$ where n is finite, $\lambda_i \in R$ and u_i are idempotents, is norm dense in A. Assuming (ii), on this subspace we have

$$\langle \sum_{i=1}^{n} \lambda_i u_i, \; U_v^* \tau + U_{1-v}^* \tau \rangle$$

$$= \sum_{i=1}^{n} \lambda_i \langle u_i, U_v^* \tau + U_{1-v}^* \tau \rangle$$

$$= \sum_{i=1}^{n} \lambda_i \langle v + 1 - v, \; U_{u_i}^* \tau \rangle$$

$$= \langle \sum_{i=1}^{n} \lambda_i u_i, \tau \rangle.$$

for any idempotent $v \in A$. Therefore the continuous functions τ and $U_v^* \tau + U_{1-v}^* \tau$ agree on a norm dense subspace of A and so must be equal.

q.e.d. Lemma 3.1

<u>Definition 3.2</u> A normal state satisfying the equivalent conditons of Lemma 3.1 will be called a trace. A JBW-algebra which has a faithful normal trace is said to be finite.

We give three examples and a counterexample.

(i) Let $I = [0,1]$ and let $\mathcal{L}^{\infty}(I,\mu)$ be the space of (equivalence classes of) essentially bounded Lebesgue measurable function on I. $\mathcal{L}^{\infty}(I,\mu)$ is an associative JBW-algebra and every normal state on $\mathcal{L}^{\infty}(I,\mu)$ is a trace.

(ii) M_3^8 is known to possess a trace. Indeed this is a particular case of the easily verifiable fact that every JBW-algebra whose vector space dimension is finite, is finite in the sense of definition 3.2

(iii) If A is a non-associative JBW-algebra (even if it is finite dimensional), then A possesses normal states that are not traces. As an example let A be the JBW-algebra consisting of the 2×2 hermitian matrices over $\mathbb{C}$ with symmetrized product

$$a \circ b = 2^{-1}(ab + ba)$$

(a, b in A). Consider the faithful normal state, ϕ, on A defined by

$$\langle a, \phi \rangle \equiv tr(\Phi a)$$

where Φ is the matrix:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \qquad (0 < \lambda < 1)$$

If $\lambda \neq 2^{-1}$, ϕ is not a trace.

(iv) We give an example of a finite JBW-algebra which is not finite dimensional. The reader is referred to [10, Chapter 1, § 9] for the details of the construction.

Let Z be the one dimensional torus group and G the discrete subgroup $G \equiv \{ n\theta \pmod 1 \mid n \in N, \theta$ irrational$\}$. Let G act on Z by $\xi \to s\xi$, $s \in G$, $\xi \in Z$. Let μ be the usual Haar measure on Z. Then for each $s \in G$, the action $\xi \to s\,\xi$, defines a measure μ_s on Z by

$$\mu_s(E) \equiv \mu(sE), \text{ for each Borel subset of } Z.$$

μ_s is equivalent to μ; let r_s be the Radon-Nikodyn derivative of μ_s with respect to μ:

$$d\mu_s(\xi) = r_s(\xi)\ d\mu(\xi) \quad (\text{note: } r_s(\xi) = 1)$$

We define the following unitary operator on $\mathfrak{H} \equiv L^2(Z, \mu)$:

$$(U_s k)(\xi) \equiv r_{s^{-1}}(\xi)^{1/2} \; k(s^{-1}\xi).$$

$s \to U_s$ is a unitary representation of G in $\mathcal{B}(\mathfrak{H})$.

For each $s \in G$ let $\mathfrak{H}_s$ be a copy of $\mathfrak{H}$ and $J_s : \mathfrak{H}_s \to \mathfrak{H}_s$ an isometry. Let $\mathcal{H}$ be the Hilbert space direct sum of $(\mathfrak{H}_s)_{s \in G}$. Let $R \in \mathcal{B}(\mathcal{H})$; then R has a matrix representation $(R_{s,t})_{s,\,t \in G}$ where

$$R_{s\,t} = J_s^* R J_t \in \mathcal{B}(\mathfrak{H}).$$

Let $\mathfrak{M} = L^\infty(Z, \mu)$ and for each $f \in \mathfrak{M}$ define

$(f) \equiv (R_{s\,t})_{s,\,t \in G}$ where

$$R_{s\,t} = \begin{cases} 0, & s \neq t \\ f, & s = t \end{cases}$$

and for each $y \in G$ define $\tilde{U}_y \in \mathcal{B}(\mathcal{H})$ by $\tilde{U}_y \equiv (R_{s\,t})_{s,\,t \in G}$

$$R_{s\,t} = \begin{cases} 0, & s\,t^{-1} \neq y \\ U_y, & s\,t^{-1} = y \end{cases}$$

Let $\mathfrak{N}$ be the von Neumann algebra generated by

$$(f_1)\,\tilde{U}_{y_1} + \dots + (f_n)\,\tilde{U}_{y_n}$$

$f_1, \dots, f_n \in \mathfrak{M}$, in $\mathcal{B}(\mathcal{H})$. Then $\mathfrak{N}$ is a finite von Neumann algebra. Indeed it can be shown that x in $\mathfrak{N}$ is represented by a matrix of the form

$$(T_{st^{-1}}\ \ U_{st^{-1}})$$

where T_y is in $\mathfrak{M}$ for each y in G. Further, if x is positive

$$T_e = J_e^* \, x \, J_e \in \mathfrak{M}$$

where e is the identity of G. Therefore let $\tau(x)$ be defined by

$$\tau(x) = \int_Z T_e(s)\, d\mu(s)$$

τ is a faithful normal trace on $\mathfrak{N}$ with $\tau(\ (f)) = \int_Z f(s)\, d\mu(s)$ for each f in $\mathfrak{M}^+$.

The self-adjoint part of $\mathfrak{M}$ with symmetrized product is a finite JBW-algebra.

For the rest of this section τ will be a faithful normal trace on a JBW-algebra, A. Let V_τ be the order generated by τ in the predual of A. For each idempotent, v, in A, the map $U_v : A \to A$ is a positive projection. In the language of non-commutative spectral theory U_v is a P-projection and it has a "quasi complementary" P-projection, U_{1-v}, such that

$$\ker^+ U_v = \mathrm{im}^+ U_{1-v}, \quad \ker^+ U_{1-v} = \mathrm{im}^+ U_v .$$

The other defining characteristics of P-projections are (See Appendix C):

$$\|U_v\| \leq 1, \quad \|U_{1-v}\| \leq 1$$

and for every positive normal linear functional, σ,

$$\| U_v^* \sigma\| = \|\sigma\| \Rightarrow \sigma \in \mathrm{im}^+ U_v^*$$

$$\| U_{1-v}^* \sigma\| = \|\sigma\| \Rightarrow \sigma \in \mathrm{im}^+ U_{1-v}^*$$

By Lemma 3.1, $0 \leq U_v^* \tau \leq \tau$ for each idempotent $v \in A$. Therefore $U_v^* |_{V_\tau} : V_\tau \to V_\tau$ and since U_v^* is weakly continuous, $U_v^* |_{V_\tau}$ is $\sigma(V_\tau, A)$ - continuous.

We assume in the sequel that A_τ^* and V_τ are equipped with the norms in Theorem 1.12, and we thus have $A_\tau^* = V_\tau$ algebraically and topologically. Hence V_τ

is endowed with its order unit norm and A_τ is A with the norm $\|a\|_\tau = \langle a^+, \tau\rangle + \langle a^-, \tau\rangle$, $a = a^+ - a^-$ being the orthogonal decomposition of a in A. Let $C_\tau = \{a \in A_\tau^+ \mid \langle a, \tau\rangle = 1\}$.

The notion of weak spectral duality is defined in Appendix C. The structures involved in this definition will be illustrated in the course of the proof of the following result.

<u>Theorem 3.3</u> (A_τ, C_τ) and (V_τ, τ) are in weak spectral duality.

<u>Proof</u> Let v be an idempotent of A and denote by Ω_v the restriction of U_v^* to V_τ. As was noted above Ω_v is a weakly (that is $\sigma(V_\tau, A_\tau)$-) continuous positive projection on V_τ. That $\|\Omega_v\| \leq 1$ follows since Ω_v is positive and $0 \leq \Omega_v \tau \leq \tau$. Furthermore

$$\operatorname{im}^+ \Omega_v^* = \operatorname{im}^+ U_v = \ker^+ U_{1-v} = \ker^+ \Omega_{1-v}^*$$

follows from the facts that V_τ, when regarded as a subspace of E, is norm dense, and Ω_v is $\sigma(V_\tau, A_\tau)$-continuous. Therefore to show that Ω_v and Ω_{1-v} are quasi-complementary P-projections on V_τ, it is necessary to show that

(i) $\|\Omega_v^* a\|_\tau = \|a\|_\tau \Rightarrow a \in \operatorname{im}^+ \Omega_v^*$

(ii) $\| \Omega^*_{1-v} a \|_\tau = \| a \|_\tau \Rightarrow a \in \mathrm{im}^+ \Omega^*_{1-v}$

when $a \in A_\tau$ is positive.

The proof of (i) is as follows. If $a \in A$ is positive, so is $\Omega^*_v a$. Therefore, assuming

$$\| \Omega^*_v a \|_\tau = \| a \|_\tau$$

we have

$$\langle a, U_v^* \tau \rangle = \langle \Omega^*_v a, \tau \rangle = \| \Omega^*_v a \|_\tau = \| a \|_\tau$$

$$= \langle a, \tau \rangle = \langle a, U^*_v \tau + U_{1-v} \tau \rangle$$

and thus $\langle a, U_{1-v}^* \tau \rangle = 0$. By the fàithfulness of τ, $0 = U_{1-v} a = \Omega_{1-v}^* a$, and $a \in \mathrm{ker}^+ \Omega_{1-v}^*$ $= \mathrm{im}^+ \Omega^*_v$.

(ii) is proved in a similar manner.

Having that each idempotent $v \in A$ gives rise to a P-projection Ω_v on V_τ we will show that every V_τ-exposed face of C_τ (See Appendix C for the definition of exposed face) is of theform $(\mathrm{im}^+ \Omega^*_v) \cap C_\tau$ for some idempotent v in A. Indeed if F is a V_τ-exposed face of C_τ, then, by definition, there exists $\sigma \in V_\tau$, $\sigma \geq 0$ with $\sigma = 0$ on F and $\sigma > 0$ on $C_\tau \setminus F$. Let $H^+ = \bigcup_{\lambda > 0} (\lambda F)$. Then $H^+ = \{a \in A^+ \mid \langle a,\sigma \rangle = 0\}$ and we see that by considering σ to be an element of E, H^+ is a weakly (i.e. $\sigma(A,E)$-) closed face of A^+. Now [4, Theorem 12.3] tells us that there exists an idempotent v of A such that $H^+ = \mathrm{im}^+ U_v$, and we see that $F = (\mathrm{im}^+ \Omega_v^*) \cap C_\tau$.

Since V_τ is a Banach dual space V_τ is monotone complete, and to complete our proof it remains to show that every $\sigma \in V_\tau$ has an orthogonal decomposition with respect to the duality (V_τ, A_τ) (See Appendix C, Theorem C 2) Let $\sigma \in V_\tau$; by [4, Theorem 12.6] there exists an idempotent $v \in A$ such that $\sigma = U_v^* \sigma + U_{1-v}^* \sigma$, with $U_v^* \sigma$ and $-U_{1-v}^* \sigma$ in E^+. Recall that Ω_v and Ω_{1-v} are restrictions of U_v^*, U_{1-v}^* to V_ϕ; therefore $\sigma = \sigma^+ - \sigma^-$, where

$$\sigma^+ = \Omega_v \sigma \in E^+, \quad \sigma^- = -\Omega_{1-v} \sigma \in E^+$$

is the orthogonal decomposition we want.

Thus V_τ and A_τ are in weak spectral duality.

q.e.d. Theorem 3.3

Corollary 3.4 There is a one-to-one correspondence between the lattice of idempotents in A and the lattice of projective units in V_τ. This correspondence preserves the lattice operations.

Proof If v is an idempotent of A the corresponding projective unit in V_τ is $\Omega_v \tau$. On the other hand from the proof of Theorem 3.3 we know all projective units are of the form $\Omega_u \tau$ for some idempotent $u \in A$. Furthermore we know that the quasi-complement of $\Omega_u \tau$ is $\Omega_{1-u}\tau$, so the correspondence preserves orthocomplementation in the lattices.

Suppose u and v are idempotents of A with $u \leq v$. Then $\mathrm{im}^+ U_u^* \subset \mathrm{im}^+ U_v^*$, and we see that $\mathrm{im}^+ \Omega_v \subset \mathrm{im}^+ \Omega_v$; i.e. $\Omega_u \tau \leq \Omega_v \tau$. Similarly we see that the converse is also true.

q.e.d. Corollary 3.4

With Corollary 3.4 at hand we are able to construct the following map $k : A \rightarrow V_\tau$:

Let $a = \int_R \lambda \, de_\lambda$ be an element of A; for each $\lambda \in R$, let $\tau_\lambda = \Omega_{e_\lambda} \tau$. Corollary 3.4 shows that $\{\tau_\lambda\}_{\lambda \in R}$ is a spectral family in V_τ. Define $k(a) = \int_R \lambda \, d\tau_\lambda$; by [4, Corollary 6.10] this is a well-defined element of V_τ; the weak spectral duality of V_τ and A_τ implies that $k: A \to V_\tau$ is surjective. Clearly, also $k(a) = 0$ if and only if $a = 0$.

As yet we do not know if k is linear: this is proved in the following theorem.

<u>Theorem 3.5</u> The map $k: A \to V_\tau$ just constructed is a linear order isomorphism of A onto V_τ with $k(1) = \tau$.

<u>Proof</u> Let A_0 be the subspace of A consisting of those elements of the form $\sum_{i=1}^{n} \lambda_i v_i$, where n is finite, λ_i is in R and v_i is an idempotent, $i = 1, \ldots, n$. For $a = \sum_{i=1}^{n} \lambda_i v_i$ in A_0 and for each idempotent, u, we have

$$\langle u, \sum_{i=1}^{n} \lambda_i \Omega_{v_i} \tau \rangle = \langle u, \sum_{i=1}^{n} \lambda_i U_{v_i}^{*} \tau \rangle = \sum_{i=1}^{n} \langle u, U_{v_i}^{*} \tau \rangle$$

and thus, by Lemma 3.1:

$$\langle u, \sum_{i=1}^{n} \lambda_i \Omega_{v_i} \tau \rangle = \sum_{i=1}^{n} \lambda_i \langle v_i, U_u^* \tau \rangle = \langle a, U_u^* \tau \rangle .$$

Now the idempotents of A are precisely the extreme points of the order interval $[0,1]$([4, Proposition 8.7]). Thus, if a in A_0, $a = \sum_{i=1}^{n} \lambda_i v_i$, is positive, by the Krein-Milman Theorem $\sum_{i=1}^{n} \lambda_i \Omega_{v_i} \tau$ is a positive element of V_τ. Therefore if we define $k_0 \colon A_0 \to V_\tau$ by:

$$k_0(b) \equiv \sum_{i=1}^{n} \mu_i \Omega_{u_i} \tau , \quad b = \sum_{i=1}^{m} \mu_i u_i \in A_0$$

we see that k_0 is a well-defined, positive, linear map, and since $k_0(1) = \tau$, of norm less than or equal to one.

A_0 is a norm dense subspace of A by the spectral theorem so k_0 has a bounded linear extension to A. On the other hand, again by the spectral theorem, the set of elements with finite spectrum is norm dense in A, and clearly for this set k_0 and k coincide. Thus the unique extension of k_0 is in fact k.

Thus $k \colon A \to V_\tau$ is a bounded linear positive map with $k(1) = \tau$ and k is surjective. We have already remarked that $k(a) = 0$ if and only if $a = 0$; thus

k is bijective.

To conclude we need to show that k is bipositive. This will clearly follow if we can establish the formula

$$\langle a, k(b) \rangle = \langle b, k(a) \rangle \qquad (*)$$

for every pair $a,b \in A$.

To do this we first suppose that $a = \sum_{i=1}^{n} \lambda_i v_i$, $b = \sum_{j=1}^{m} \mu_j u_j \in A$.

Again using Lemma 3.1, we obtain

$$\begin{aligned}
\langle a, k(b)\rangle &= \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j \langle v_i, \Omega_{u_j} \cdot \tau\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j \langle v_i, U_{u_j}^* \tau\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j \langle u_j, U_{v_i}^* \tau\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j \langle u_j, \Omega_{v_i} \tau\rangle \\
&= \langle b, k(a)\rangle
\end{aligned}$$

By the continuity of k and the norm denseness of A_0 in A, this extends to A giving (*).

q.e.d. Theorem 3.5

<u>Remark</u> This theorem proves Conjecture 1.16 is true in the case of a faithful normal trace.

We close the section with a Radon-Nikodym theorem for faithful normal traces; we will need the following lemma.

<u>Lemma 3.6</u> Let τ be a normal trace on a JBW-algebra, and let u and v be orthogonal idempotents; then for every $a \in A$:

(i) $\langle v \circ (v \circ a), \tau \rangle = \langle U_v a, \tau \rangle = \langle v \circ a, \tau \rangle$

(ii) $\langle u \circ (v \circ a), \tau \rangle = 0$

<u>Proof</u> (i) $\langle v \circ (v \circ a), \tau \rangle$

$$= \langle 2^{-1}(I + U_v - U_{1-v})\, 2^{-1}(I + U_v - U_{1-v})a, \tau \rangle$$

$$= 4^{-1}\langle (I - U_{1-v} + 3U_v)\, a, \tau \rangle$$

$$= 4^{-1}\langle a, (I - U_{1-v})^* \tau + 3U_v^* \tau \rangle$$

and upon using Lemma 3.1

$$\langle v \circ (v \circ a), \tau \rangle = \langle a, U_v^* \tau \rangle$$

$$= \langle U_v a, \tau \rangle$$

$$= \langle a, 2^{-1}(I + U_v - U_{1-v})^* \tau \rangle$$

$$= \langle v \circ a, \tau \rangle$$

(ii) $$\langle u \circ (v \circ a), \tau \rangle = \langle 2^{-1}(I + U_v - U_{1-v})a, U_u^* \tau \rangle$$

$$= 2^{-1}[\langle a, U_u^* \tau \rangle + \langle a, U_v^* U_u^* \tau \rangle - \langle a, U_{1-v}^* U_u^* \tau \rangle]$$

Since u and v are orthogonal $U_v^* U_u^* = 0$ and $U_{1-v}^* U_u^* = U_u^*$. Therefore $\langle u \circ (v \cdot a), \tau \rangle = 0$ indeed.

q.e.d. Lemma 3.5

<u>Proposition 3.7</u> Let τ be a faithful normal trace on a JBW-algebra A. Then for each positive $\sigma \in V_\tau$ there exists a positive $b \in A$ such that $\langle a, \sigma \rangle = \langle \{b a b\}, \tau \rangle$ for every $a \in A$.

<u>Proof</u> Firstly suppose that $b \in A$ is positive and has finite spectrum; that is $b = \sum_{i=1}^{n} \lambda_i v_i$ where $\lambda_i \geq 0$ and v_i are pairwise orthogonal idempotents. We have for every a in A $\langle a, k(b) \rangle = \langle a, \sum_{i=1}^{n} \lambda_i U_{v_i}^* \tau \rangle$

$$= \sum_{i=1}^{n} \lambda_i \langle U_{v_i} a, \tau \rangle = \sum_{i=1}^{n} \lambda_i \langle 2v_i \circ (v_i \circ a) - v_i^2 \circ a, \tau \rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle 2(\lambda_i^{2^{-1}} v_i) \circ (\lambda_j^{2^{-1}} v_j \circ a, \tau \rangle$$

$$- \sum_{i=1}^{n} \langle (\lambda_i v_i^2) \circ a, \tau \rangle$$

$$= \langle 2 \sum_{i=1}^{n} \lambda_i^{2^{-1}} v_i \circ \left(\left(\sum_{i=1}^{n} \lambda_j^{2^{-1}} v_j \right) \circ a \right), \tau \rangle$$

$$- \langle \left(\sum_{i=1}^{n} \lambda_i^{2^{-1}} v_i \right)^2 \circ a, \tau \rangle$$

$$= \langle \{ b^{2^{-1}} \; a \; b^{2^{-1}} \}, \tau \rangle.$$

Now let $\sigma \in V_\tau$ be positive; then by Theorem 3.5 there exists a positive $b \in A$ such that $k(b) = \sigma$. By spectral theory there exists a sequence $\{b_n\}_{n=1}^{\infty}$ of positive elements with finite spectrum in the weakly closed associative subalgebra of A generated by a and 1, converging to b in norm. This subalgebra is isometrically isomorphic to the continuous functions on some compact Hausdorff space. Therefore $b_n^{2^{-1}}$ will converge in norm to $b^{2^{-1}}$, and for each $a \in A$, $\{b_n^{2^{-1}} \; a \; b_n^{2^{-1}}\}$ will converge weakly to $\{b\,a\,b\}$ (Lemma 1.4). Thus we have

$$
\begin{aligned}
\langle a, \sigma \rangle &= \langle a, k(b) \rangle \\
&= \langle b, k(a) \rangle \\
&= \lim_{n \to \infty} \langle b_n, k(a) \rangle \\
&= \lim_{n \to \infty} \langle a, k(b_n) \rangle
\end{aligned}
$$

and thus by the previous argument

$$
\begin{aligned}
\langle a, \sigma \rangle &= \lim_{n \to \infty} \langle \{ b_n^{2^{-1}} \; a \; b_n^{2^{-1}} \}, \tau \rangle \\
&= \langle \{ b\, a\, b \}, \tau \rangle
\end{aligned}
$$

for every a in A.

q.e.d. Proposition 3.7

§ 2 The isomorphism theorem and the Radon-Nikodym Property

<u>Theorem 3.8</u> Let A be a JBW-algebra with predual E and let ϕ and ψ be two faithful normal linear functionals on A. Suppose that there exists a positive $a_0 \in A$ such that $\langle \{ a_0\, b a_0 \}, \phi \rangle = \langle b, \psi \rangle$

for every $b \in A$. Then there exists an order isomorphism $\Lambda : V_\phi \to V_\psi$ such that $\Lambda(\phi) = \psi$.

<u>Proof</u> By assumption $\langle b, \psi \rangle = \langle U_{a_0} b, \phi \rangle$ for every $b \in A$; that is $\psi = U^*_{a_0} \phi$. $U^*_{a_0}$ is positive, so $U^*_{a_0} |_{V_\phi} : V_\phi \to V_\psi$. Let $\Lambda = U^*_{a_0} |_{V_\phi}$. Obviously $\Lambda(\phi) = \psi$ and Λ is positive.

To continue the proof we will need three lemmas.

<u>Lemma 3.9</u> The support of a_0, $r(a_0)$, is the identity in A.

<u>Proof</u> Let $u \in A$ be an idempotent such that $U_u a_0 = a_0$. Then $U_{1-u} a_0 = 0$ and by [8, Proposition 2.8] $U_{a_0}(1-u) = 0$. Therefore $\langle 1-u, \psi \rangle = \langle U_{a_0}(1-u), \phi \rangle = 0$ and by the faithfulness of ψ, $u = 1$. Thus $r(a_0) = 1$.

q.e.d. Lemma 3.9

Let $J(a_0^2)$ be the order ideal of A generated by a_0^2. It follows from Theorem 1.7 that $U_{a_0}(A) = J(a^2)$.

<u>Lemma 3.10</u> $U_{a_0}: A \to J(a_0^2)$ is injective.

<u>Proof</u> The reader will observe in the proof Theorem 2.6 that atomicity was used only to show that the support of the element of A considered there was the identity. Therefore that proof applies in the present case as well.

q.e.d. Lemma 3.10

<u>Lemma 3.11</u> $\wedge$ is injective.

<u>Proof</u> It follows from a result of Edwards ([12, Lemma 3.1]) that $U_{a_0}(A) = J(a_0^2)$ is weakly dense in A. By a standard argument [11, VI] $U_{a_0}^* : E \to E$ is injective, and thus so is $\wedge$.

q.e.d. Lemma 3.11

We are now ready to complete the proof of Theorem 3.8. It remains to show that $\wedge$ is surjective and bipositive. To this end, we recall the following notation from I, § 1 .

$$\phi_n(t) = \begin{cases} 0, & 0 \leq t < n^{-1} \\ t^{-1}, & n^{-1} \leq t \leq \| a_0 \| \end{cases}$$

$$c_n = \int_0^{\| a_n \|} \phi_n(\lambda)\, d e_\lambda$$

where $a_0 = \int_0^{\|a_n\|} \lambda \, d e_\lambda$ is the spectral decomposition of a_0. For each $b \in J(a_0^2)$ the weak limit of $\{c_n b c_n\}_{n=1}^{\infty}$ is the element $a \in A$ such that $U_{a_0} a = b$.

Also

$$u_n \equiv \int_{n^{-1}}^{\|a_0\|} d e_\lambda$$

is a monotone increasing sequence of idempotents with least upper bound 1. (See Theorem 1.7.)

Let ρ be an element of V_ψ such that $0 \le \rho \le \psi$. For each n define $\rho_n \equiv U_{c_n}^* \rho$. ρ_n is positive for for every n. Let $a \in A$ and suppose $a = a^+ - a^-$ is its orthogonal decomposition; then

$$\begin{aligned}
|\langle a, \rho_n\rangle| &\le \max\{\langle a^+, \rho_n\rangle, \langle a^-, \rho_n\rangle\} \\
&\le \max\{\langle a^+, U_{c_n}^* \psi\rangle, \langle a^+, U_{c_n}^* \psi\rangle\} \\
&= \max\{\langle U_{a_0} U_{c_n} a^+, \phi\rangle, \langle U_{a_0} U_{c_n} a^-, \phi\rangle\} \\
&= \max\{\langle U_{u_n} a^+, \phi\rangle, \langle U_{u_n} a^-, \phi\rangle\} \\
&\le \max\{\|a^+\| \, \|\phi\|, \|a^-\| \, \|\phi\|\} \\
&= \|a\| \, \|\phi\|.
\end{aligned}$$

By the principle of uniform boundedness

$$\lim_{\|a\|\to 0} \langle a, \rho_n \rangle = 0 \text{ uniformly in } n \qquad (*)$$

For each b in $J(a_0^2)$, $\lim_{n\to\infty} \langle b, \rho_n \rangle$ exists; in fact, by Lemma 3.9 and the remark following it, there exists a in A^+ such $b = U_{a_0} a$ and

$$\begin{aligned} \lim_{n\to\infty} \langle b, \rho_n \rangle &= \lim_{n\to\infty} \langle U_{c_n} b, \rho \rangle \\ &= \lim_{n\to\infty} \langle U_{c_n} U_{a_0} a, \rho \rangle \\ &= \lim_{n\to\infty} \langle U_{u_n} a, \rho \rangle \\ &= \langle a, \rho \rangle . \end{aligned}$$

Therefore we define

$$\sigma : J(a_0^2) \cap A^+ \to R$$

by $\sigma(b) = \lim_{n\to\infty} \langle b, \rho_n \rangle$. By ([12], Lemma 3.3) the norm closure of $J(a_0^2) \cap A^+$ is A^+, and using (*) we can extend σ to al of A^+. Notice that for $a \in A^+$,

$\sigma(a)$ will be $\lim_{n\to\infty} \langle a, \rho_n \rangle$.

To extend σ to all of A, we write $a \in A$ as $a^+ - a^-$ (its orthogonal decomposition) and define $\sigma(a) = \sigma(a^+) - \sigma(a^-)$. Now $\sigma : A \to R$ is well-defined and $\sigma(a) = \lim_{n\to\infty} \langle a, \rho_n \rangle$ for each $a \in A$. Each ρ_n is linear, so σ will be. Moreover for a positive

$$
\begin{aligned}
0 \leq [\sigma(a)] &= \lim_{n\to\infty} \langle a, \rho_n \rangle \\
&= \lim_{n\to\infty} \langle U_{c_n} a, \rho \rangle \\
&\leq \lim_{n\to\infty} \langle U_{c_n} a, \psi \rangle \\
&= \lim_{n\to\infty} \langle U_{a_0} U_{c_n} a, \phi \rangle \\
&= \lim_{n\to\infty} \langle U_{u_n} a, \phi \rangle \\
&= \langle a, \phi \rangle.
\end{aligned}
$$

Hence, σ is a positive, continuous linear functional on A dominated by ϕ; i.e. $\sigma \in V_\phi$. It remains to show $U_{a_0}^* \sigma = \rho$. For each a in A we have:

$$\langle a, U_{a_0}^* \sigma \rangle = \langle U_{a_0} a, \sigma \rangle$$

$$= \lim_{n\to\infty} \langle U_{a_0} a, U^*_{c_n} \rho \rangle$$

$$= \lim_{n\to\infty} \langle U_{c_n} U_{a_0} a, \rho \rangle$$

$$= \langle a, \rho \rangle.$$

Hence Λ is surjective and bipositive, and the proof is complete.

q.e.d. Theorem 3.8

<u>Corollary 3.12</u> Let A be a JBW-algebra with a faithful normal trace τ. Suppose that ϕ is a normal state on A which is dominated by τ. Then there is an order isomorphism of A onto V_ϕ.

<u>Proof</u> From Proposition 3.7 we know that there exists an element $b \in A^+$ such that $\langle a, \phi \rangle = \langle \{b\,a\,b\}, \tau \rangle$ for every $a \in A$. We now apply Theorem 3.8 to find an order isomorphism of V_τ onto V_ϕ. It follows that since A is order isomorphism to V_τ, it is also order isomorphic to V_ϕ.

q.e.d. Corollary 3.12

Let A be a JBW-algebra. We will say that A has the Radon-Nikodym Property if given a positive normal linear functional ϕ on A and any positive normal linear functional ψ on A dominated by ϕ there exists a positive element b in A such that

$$\langle a, \psi\rangle = \langle\{b\,a\,b\}, \phi\rangle$$

for every a in A. In view of Theorem 3.8 we will study the class $\mathfrak{N}$ of JBW-algebras satisfying the Radon-Nikodym Property.

We note immediately that the JBW-algebras that are the self-adjoint part of a von Neumann belong to the class $\mathfrak{N}$ ([17 , Theorem 1.24.3]). We also remark that the property above is a quadratic Radon-Nikodym Property; we draw the reader's attention to Appendix D, where it is shown that all JBW-algebras satisfy a linear Radon-Nikodym property.

<u>Theorem 3.13</u> Let A be a JBW-algebra of class $\mathfrak{N}$, and let ϕ and ψ be faithful normal states on A. Then V_ϕ and V_ψ are order isomorphic.

<u>Proof</u> Consider the faithful normal linear functional $\rho = \phi + \psi$. ρ dominates both ϕ and ψ. Therefore since A is of class $\mathfrak{N}$, we can use Theorem 3.8 to see that $V_\phi \simeq V_\rho \simeq V_\psi$, where $\simeq$ indicates order isomorphism.

<u>Corollary 3.14</u> Let A be a JBW-algebra of class $\mathfrak{N}$, and let ϕ and ψ be normal states on A,

with supports $s(\phi)$ and $s(\psi)$ respectively. Then if $s(\phi) = s(\psi)$, V_ϕ is order isomorphic to V_ψ.

<u>Proof</u> Let $v = s(\phi) = s(\psi)$ and $B = \mathrm{im}U_v$. B is a JBW-algebra which clearly is of class $\mathfrak{N}$, and ϕ and ψ are faithful normal states on B. Furthermore, both V_ϕ and V_ψ are subspaces of $\mathrm{im}U_v^*$, the predual of B. Hence by the above theorem, V_ϕ and V_ψ are order isomorphic.

<u>Corollary 3.15</u> Suppose that A is a finite JBW-algebra, and further that A is of class $\mathfrak{N}$. Let ϕ be a faithful normal state on A. Then V_ϕ is order isomorphic to A.

<u>Proof</u> A is finite so there exists a faithful normal trace, τ, on A. By Theorem 3.5, V_τ and A are order isomorphic. Applying Theorem 3.13 to ϕ and τ, we get the desired result.

q.e.d. Corollary 3.15

<u>Remark</u> Corollary 3.15 proves Conjecture 1.16 for a finite JBW-algebra of class $\mathfrak{N}$.

Chapter IV Conclusion

In their study of JB-algebras, Alfsen and Shultz have developed axioms satisfied by the lattice of P-projections of a JBW-algebra. As they pointed out, these axioms have an interpretation in the study of axiomatic quantum theory. In [9], Araki made this connection even more concrete by giving a characterization of the state space of a quantum system in terms of a lattice of filters with axioms closely resembling those of Alfsen and Shultz for P-projections.

On the other hand, the Tomita-Takesaki theory of modular algebras has proven, in the last ten years, to be of fundamental importance in quantum statistical mechanics. One aspect of this theory is to establish under favorable circumstances an isomorphism between a von Neumann algebra and its commutant.

The first aim of this thesis was to find a generalization of the commutant of a von Neumann algebra in the more general setting of JBW-algebras. In connection with the above remarks our formulation seems to have the advantage, in the study of the foundations of quantum theory, of only involving objects and operations that have physical interpretation. Indeed we were initially motivated in our study by a problem in non-commutative ergodic theory.

In general, a JBW-algebra does not have a concrete realization as an operator algebra acting on some Hilbert space; the notion of commutant therefore has to be revised from the start. We achieved this in the following manner. Let A be a JBW-algebra and ϕ a normal state; we have studied the order ideal generated by ϕ in the dual of A. We denote this order ideal by V_ϕ. Via Theorem 1.9 we have seen that when A is the self-adjoint part of a von Neumann algebra $\mathfrak{R}$, then in a natural manner V_ϕ can be identified with the self-adjoint part of the commutant of the G.N.S. representation of $\mathfrak{R}$ with respect to ϕ.

Our main conjecture is that when ϕ is a faithful normal state, there exists an order isomorphism of A onto V_ϕ.

In the case where A is an atomic JBW-algebra it was shown that indeed there does exist an order isomorphism of A onto V_ϕ for every faithful normal state ϕ, and the order isomorphism was explicitly constructed. We point out that this case includes the algebras $\mathcal{B}(\mathfrak{H})_{s.a}$, where $\mathfrak{H}$ is a complex Hilbert space and the exceptional algebra, M_3^8. Furthermore, we remark that this case covers the quantum systems investigated in [9], as the systems studied there are finite dimensional.

A JBW-algebra is said to be finite if it possesses a faithful normal trace τ. The example (iv) constructed on P.43 shows there exist finite JBW-algebras that have no atoms. We have also proved the conjecture to be true for a faithful normal trace, and as an application proved a Radon-Nikodym theorem for normal traces.

In Chapter III, § 2 it was shown that if two faithful normal states ϕ and ψ on any JBW-algebra A are related by

$$\langle a, \psi \rangle = \langle \{ b\,a\,b \}, \phi \rangle, \quad a \in A$$

for some fixed positive b in A, then V_ϕ and V_ψ are order isomorphic.

A JBW-algebra, A, is said to satisfy the Radon-Nikodym Property if given any normal state, ϕ, on A and any positive normal linear functional ψ dominated by ϕ there exists an element b of A such that

$$\langle a, \psi \rangle = \langle \{ b a b \}, \phi \rangle, \quad a \in A.$$

The class of JBW-algebras satisfying this property is denoted by $\mathfrak{N}$.

Let A be a JBW-algebra in $\mathfrak{N}$. Further, let ϕ be a faithful normal state on A. Then we proved that in this case also A and V_ϕ are order isomorphic.

The results we have obtained are reminiscent of those parts of the Tomita-Takesaki theory obtained by Dixmier, [10, Chapter III, § 1]. On the other hand our aim has been to extend Tomita-Takesaki theory from operator algebras to abstract JBW-algebras using methods taken from the theory of partially ordered vector spaces and non-commutative spectral theory. In addition to providing a genuine generalization, our methods also throw light into the basic structures involved in the isomorphism theorem, not just for the generalization, but also for operator algebras themselves.

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Appendix A Tomita-Takesaki Theory

The general references for this appendix are: Dixmier [10], Sakai [17] and Takesaki [19].

Let $\mathfrak{H}$ be a Complex Hilbert space and let $\mathfrak{M}$ be a subalgebra of $\mathcal{B}(\mathfrak{H})$. $\mathfrak{M}$ is said to be a *-subalgebra if whenever $x \in \mathfrak{M}$, $x^* \in \mathfrak{M}$ also. Let $\mathfrak{M}'$ be the commutant of $\mathfrak{M}$ in $\mathcal{B}(\mathfrak{H})$; i.e.

$$\mathfrak{M}' = \{x' \in \mathcal{B}(\mathfrak{H}) \mid x'x = xx' \quad \forall x \in \mathfrak{M}\}.$$

In general we will have

$$\mathfrak{M} \subset \mathfrak{M}'' = \mathfrak{M}^{(iv)} = \ldots$$

$$\mathfrak{M}' = \mathfrak{M}''' = \mathfrak{M}^{(v)} = \ldots$$

where $\mathfrak{M}'' = (\mathfrak{M}')'$, etc.

<u>Definition A1</u> A von Neumann algebra acting on $\mathfrak{H}$ is a *-subalgebra $\mathfrak{M}$ of $\mathcal{B}(\mathfrak{H})$ which coincides with its bicommutant $\mathfrak{M}'$.

It can be shown that a von Neumann algebra $\mathfrak{M}$ is a Banach dual space whose self-adjoint part, $\mathfrak{M}_{s.a.}$ $= \{x \in \mathfrak{M} \mid x = x^*\}$, is a JB-algebra with product $x \circ y$ $\equiv 2^{-1}(xy + yx)$, $x, y \in \mathfrak{M}_{s.a.}$

Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$. A vector $\xi \in \mathfrak{H}$ is said to be cyclic for $\mathfrak{M}$ if the closure in $\mathfrak{H}$ of the subspace

$$\mathfrak{M}\xi. \equiv \{x\xi \mid x \in \mathfrak{M}\}$$

is $\mathfrak{H}$ itself. $\eta \in \mathfrak{H}$ is called separating if $x \in \mathfrak{M}$, $x\eta = 0$ implies $x = 0$. It is known that if $\eta \in \mathfrak{H}$ is separating for $\mathfrak{M}$, η is cyclic for $\mathfrak{M}'$. If there exists a vector $\xi \in \mathfrak{H}$ which is both cyclic and separating for $\mathfrak{M}$, $\mathfrak{M}$ is said to be in standard form. (Via the Gelfand-Naimark-Segal representation it is always possible to represent a von Neumann algebra on a Hilbert space in such a way that the representation is in standard from.)

<u>Theorem A 2</u> Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$, and let $\xi \in \mathfrak{H}$ be a cyclic and separating vector for $\mathfrak{M}$. Then there exist a unique antilinear isometry $J: \mathfrak{H} \to \mathfrak{H}$ with $J^2 = I$ and $J\xi = \xi$ and a positive, self-adjoint (in general unbounded but densely defined and invertible) $\Delta: \mathfrak{H} \to \mathfrak{H}$ such that

$$\mathfrak{M} = J\mathfrak{M}'J, \quad \Delta^{it}\mathfrak{M}\Delta^{it} = \mathfrak{M}$$

for every $t \in R$. The map $x \to Jx^*J$ is a *-anti-isomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$ and for every $t \in R$, the map $x \to \Delta^{it} x \Delta^{it}$ is a *-automorphism of $\mathfrak{M}$.

This is the main theorem in Tomita-Takesaki theory (see [19]). We give here only a sketch of the construction of the operators J and Δ .

One begins by defining two densely defined antilinear operators S_0: $\mathfrak{M}\xi \to \mathfrak{H}$ and F_0: $\mathfrak{M}'\xi \to \mathfrak{H}$ by $S_0 : x\xi \to$ $\to x^*\xi$ and F_0: $x'\xi \to x'^*\xi$. One then shows that S_0 is preclosed and its adjoint F is an extension of F_0. Further the minimal closed extension, S, of S_0 satisfies $S^* = F$ and $S^2\xi = \xi$, $F^2\eta = \eta$ for ξ and η in the domains of F and S respectively.

Let $S = J\Delta^{2^{-1}}$ be the polar decomposition of S. Using the above relations between S and F - $(\Delta = S^*S = FF^*)$ - one shows that J is an anti-linear isometry with $J^2 = I$; $J\Delta^{it} = \Delta^{it}J$ for every $t \in R$; and $J\xi = \xi$ $= \xi$.

The construction of J and Δ emphasizes that the proof of the theorem, as its traditional statement given above, depends in an essential way on the fact that $\mathfrak{M}$ is an algebra of operators acting on a Hilbert space. Moreover the details of the proof involve an amazing amount of tricky analysis of one complex variable, a field of mathematics one would hardly expect to be called upon in the von Neumann theory of rings of operators.

Appendix B Partially ordered vector spaces

The general references for this appendix are Alfsen [3] and Ellis [13].

Let X be a real vector space. A non-empty subset X^+ of X is called a cone if whenever $x, y \in X^+$, $x + y \in X^+$ and $\lambda x \in X^+$ for every non-negative real λ, and $-x \in X^+$ implies that $x = 0$. Let X^+ be a cone in X. X is said to be directed if $X = X^+ - X^+$. We define a partial ordering on X by writing $y \leq x$ if $x - y \in X^+$ $(x, y \in X)$. This ordering is said to be archimedian if $y \leq \lambda x$ for some $x \geq 0$ and for all positive real λ implies $y \leq 0$. For a given $x \in X$ the set

$$J(x) = \{ y \in X \mid \exists \lambda \in R^+ \text{ s.t. } -\lambda x \leq y \leq \lambda x \}$$

is called the order ideal of X generated by x. If $J(x) = X$, x is called an order unit. A convex subset, K, of X^+ is called a base of X^+ if $X^+ = \bigcup_{\lambda \in K} (\lambda K)$. A face of X^+ is a subset of X^+ of the form $F = X^+ \cap H$, where $X^+ \setminus F$ is convex, for some subspace H of X. This is equivalent to F being a subcone of X^+ with the property that $0 \leq y \leq x$ and $x \in F$ implies $y \in F$.

<u>Definition B 1</u> Let A be a real linear space partially ordered by a positive cone A^+ such that A is directed. Let e be a distinguished order unit in A and consider the seminorm

$$\| a \| = \inf\{ \lambda \in R^+ \mid -\lambda e \leq a \leq \lambda e \} .$$

If A is archimedian ordered then $\| \cdot \|$ is a norm, and (A,e) is called an order unit space.

<u>Proposition B 2</u> Let (A,e) and (A',e') be two order unit spaces. Suppose that $k: A \to A'$ is a vector space homomorphism such that $k(e) = e'$. Then k is continuous with $\| k \| = 1$ if and only if $k(A^+) \subseteq A'^+$ (i.e. k is positive). If in fact k is a bijective then K is an isometry if and only if both k and k^{-1} are positive (i.e. k is bipositive).

<u>Definition B 3</u> Let E be a partially ordered vector space with positive cone E^+ and suppose that E is directed. Suppose further that E^+ has a base K. Let p be the Minkowski functional of $co(K \cup -K)$. If p is a norm, (E,K) is said to be a base norm space.

<u>Theorem B 4</u> Let (E,K) be a base norm space, and let E^* be its Banach dual space. Define

$$(E^*)^+ = \{ f \in E^* \mid f(x) \geq 0 \ \forall \ x \in E^+ \}$$

Then there exists a unique $e \in (E^*)^+$ such that $e(x) = 1$ for every $x \in K$ and $(E^*)^+$ is a cone in E^* which induces an ordering on E^* in such a way that $(E^*;e)$ is an order unit space and the order unit norm is the dual norm.

In this case E^* can be identified with the linear space of bounded affine functions on K, $A^b(K)$, with norm

$$\| f \| = \max_{x \in K} |f(x)|$$

$$f \in A^b(K).$$

Appendix C Non-commutative spectral theory

In this appendix we summarize the main concepts and results of non-commutative spectral theory developed by E.M. Alfsen and F. W. Shultz in [4] and [5]. Throughout this section we will assume that (E,K) is a base norm space and (A,e) is an order unit space with $E^* = A$. We will refer to the $\sigma(E,A)$ and $\sigma(A,E)$ topologies as the weak topologies on E and A respectively.

Definitions Let P be a weakly continuous projection on A. We write

$$\mathrm{im}^+ P = (\mathrm{im}\, P) \cap A^+ , \quad \mathrm{ker}^+ P = (\mathrm{ker}\, P) \cap A^+$$

P will have a weakly continuous adjoint, $P^*: E \to E$; we define $\mathrm{im}^+ P^*$ and $\mathrm{ker}^+ P^*$ in a similar manner to the above. P is said to be neutral if P has norm at most one, and the following implication is valid for $\rho \in E^+$:

$$\|P^*\rho\| = \|\rho\| \Rightarrow \rho \in \mathrm{im}^+ P^*$$

Now consider another weakly continuous projection $Q: A \to A$. P and Q are said to be quasi-complementary if

$$\mathrm{im}^+ P = \mathrm{ker}^+ Q, \quad \mathrm{im}^+ Q = \mathrm{ker}^+ P.$$

Q is referred to as the quasi-complement of P. P is said to be a P-projection if it has a quasi-complement Q, and both P and Q are neutral (see [4, §1, §2] for equivalent formulations of this concept.)

Remark In [4] it was pointed out that the concept of a P-projection has possible interpretations in quantum theory. This idea has been further developed in [9].

To each P-projection P on A we associate the element $Pe \in A$; Pe is called a projective unit and is an extreme point of $[0,e] = \{a \in A \mid 0 \leq a \leq e\}$. If P' is the quasi-complement of P, P'e is also a projective unit and $P'e = e - Pe$.

A convex subset F of K is called a face of K if whenever $\lambda\rho + (1-\lambda)\sigma \in F$, $\rho, \sigma \in K$, $\lambda \in [0,1]$, then $\rho, \sigma \in F$. The faces of K are in one-to-one correspondence with the faces of E^+. A face F is said to be A-exposed if there exists a positive $a \in A$ such that $a = 0$ on F and $a > 0$ on K/F. F is called a projective face if F is of the form $K \cap (\mathrm{im}^+ P^*)$ for some P-projection P on A. For the rest of this Appendix we will assume that every A-exposed face of F is in fact a projective face. (This is a strong assumption, necessary for the development of the spectral theory; see [4, §4].)

Let U be the set of projective units in A and order U with the ordering induced by A. We point out that $Pe \leq Qe$ if and only if $\mathrm{im}^+ P \subset \mathrm{im}^+ Q$, if and only if $PQ = QP = P$. We introduce the following orthogonality on U: $Pe \perp Qe \iff Pe + Qe \leq e$. $Pe \perp Qe$ implies $Pe \leq Q'e$ which in turn implies $PQ = QP = 0$.

<u>Proposition C.1</u> U is a complete orthomodular lattice with the operations $\perp$ and $\leq$. ([4], Theorem 4.5.)

Let $\{e_\lambda\}_{\lambda \in R}$ be a set of projective units. $\{e_\lambda\}_{\lambda \in R}$ is called a spectral family if it satisfies the following conditions:

(i) $e_\lambda \leq e_\mu$ for $\lambda < \mu$

(ii) $e_\lambda = \bigwedge_{\mu > \lambda} e_\mu$

(iii) $\bigwedge_R e_\lambda = 0$, $\bigvee_R e_\lambda = e$.

We say that a projective unit Pe is compatible with $a \in A$ if $Pa + P'a = a$, and that E and A are in weak spectral duality if for every $a \in A$ and every $\lambda \in R$ there exists a P-projection P_λ such that

$$P_\lambda a \leq \lambda P_\lambda e$$

$$P'_\lambda a \geq \lambda P'_\lambda e$$

and $P_\lambda e$ is compatible with a.

Two elements $a, b \in A^+$ are said to be orthogonal if there exists a P-projection, P, such that

$$a \in \text{im}^+ P \quad \text{and} \quad b \in \text{ker}^+ P.$$

Theorem C 2 Suppose that every A- exposed face of K is projective. Then a necessary and sufficient condition for E and A to be in weak spectral duality is that every $a \in A$ has a decomposition $a = a^+ - a^-$ where $a^+, a^- \in A^+$ are orthogonal. In this case to every $a \in A$ is a spectral family $\{e_\lambda\}_{\lambda \in R}$ such that

$$a = \int_R \lambda \, d\, e_\lambda$$

where the integral is a norm convergent abstract Riemann-Stiltjes integral. Further $e_\lambda = e$ for $\lambda \geq \|a\|$ and $e_\lambda = 0$ for $\lambda \leq -\|a\|$.

We note that if E and A are in weak spectral duality then each spectral family $\{e_\lambda\}_{\lambda \in R}$ with the property that there exist $\alpha, \beta \in R$ such that $e_\lambda = 0$ for $\lambda \leq \beta < \alpha$ and $e_\lambda = e$ for $\lambda \geq \alpha$ determines a unique $a \in A$ via formula $\int_\alpha^\beta \lambda \, de_\lambda$. However it is not known known in general whether weak spectral duality uniquely determines the spectral family of a given $a \in A$.

Theorem C 3 Let E and A be in weak spectral duality. Then each $a \in A$ has a unique spectral decomposition $a = \int_R \lambda \, d e_\lambda$ if and only if the orthogonal decomposition $a = a^+ - a^-$ in Theorem C 2 is unique. In this case we say that E and A are in spectral duality.

Theorem C 4 Let E and A be in spectral duality. Then for each $a \in A$ and each bounded Borel function $f: R \to R$ there exists a unique element in A, denoted by $f(a)$, such that

$$\langle f(a), \sigma \rangle = \int_R f(\lambda) \, d\langle e_\lambda, \sigma \rangle$$

for every $\sigma \in K$. Furthermore if f is the characteristic function of a Borel set, then $f(a)$ is a projective unit in A.

Now suppose that K is compact in some locally convex Hausdorff topology. Since we may associate A with $A^b(K)$, the bounded affine functions on K, we see that $A(K) \subseteq A^b(K)$, where $A(K)$ is the space of continuous affine real-valued functions on K, and ask when is $A(K)$ closed under functional calculus of continuous functions from R to R. In fact we have

$$A(K)^* \simeq E, \quad E^* \simeq A^b(K)$$

and the question is answered by the following theorem.

Theorem C 5 Let K be compact in some locally convex Hausdorff topology, and suppose E and A are in spectral duality. Then A(K) is closed under the functional calculus of continuous functions if and only if whenever $a \in A(K)$, a has orthogonal decomposition $a = a^+ - a^-$ with $a^+, a^- \in A(K)$.

From now on we assume that (E,K) and (A,e) are in spectral duality. The spectral theory allows us to form squares in A: $a^2 \equiv \int \lambda^2 d\, e_\lambda$, where $a = \int \lambda \, d\, e_\lambda$, and we may consider the product

$$a \circ b = 2^{-1}[(a+b)^2 - a^2 - b^2] \qquad (*)$$

In general this product can fail to be bilinear, but when it is, A is a JBW-algebra, as was shown in

[4, § 12]. In [2] the following was shown to be necessary and sufficient for (*) to be a bilinear product.

Theorem C 6 (*) is a bilinear product making A a JBW-algebra if and only if

$$[P, Q]\, e = [Q', P']\, e$$

for every pair of P-projections P and Q on A. (Here $[\cdot, \cdot]$ denotes the Lie bracket: $[P,Q] \equiv 2^{-1}(PQ - QP)$).

If the product (*) is bilinear then the projective units of A are in fact the idempotents of A and for each idempotent, u, the corresponding P-projection is given by $a \to \{u\, a\, u\}$, $a \in A$, where $\{\ \ \}$ is the Jordan triple product.

Appendix D A linear Radon-Nikodym theorem

We show that, with minor modifications, the proof of Sakai's linear Radon-Nikodym theorem goes through in the context of JBW-algebras.

Theorem (See [17, 1.24.4]) Let ϕ be a positive normal linear functional on a JBW-algebra A. Then for each normal linear functional ψ such that $0 \leq \psi \leq \phi$ there exists a positive element $a \in A$ such that $\langle b, \psi \rangle = \langle a \circ b, \phi \rangle$ for every $b \in A$.

Proof Firstly assume that ϕ is faithful on A, and consider the linear map $a \to \phi_a$ where ϕ_a is the normal linear functional defined by $\langle b, \phi_a \rangle = \langle a \circ b, \phi \rangle$; $a, b \in A$. This map is continuous with respect to the weak topologies on A and its predual, E. Indeed suppose that $\{a_\alpha\}$ is a directed net in A converging weakly to zero. The mapping $c \to d \circ c$ is weakly continuous [8, Lemma 4.1], so for each $b \in A$, $\{a_\alpha \circ b\}$ converges weakly to zero, and $\langle b, \phi_{a_\alpha} \rangle = \langle a_\alpha \circ b, \phi \rangle \to 0$. Therefore $\{\phi_{a_\alpha}\}$ converges weakly to zero in E.

Let T be the image of $[-1,1] = \{a \in A \mid -1 \leq a \leq 1\}$ under $a \to \phi_a$. $[-1,1]$ is weakly compact, so using the continuity of $a \to \phi_a$, T is seen to be weakly compact in E. Furthermore T is convex.

Now suppose that $0 \leq \psi \leq \phi$ and $\psi \notin T$. Then there exists $b_0 \in A$ such that $\langle b_0, \psi \rangle > 1$ and $|\langle b_0 , \sigma \rangle| \leq 1$ for every $\sigma \in T$. Let $b_0 = b_0^+ - b_0^-$ be the decomposition of b_0 into positive and negative parts, and let v be an idempotent of A such that $U_v b_0 = b_0^+$, $U_{1-v} b_0 = -b_0^+$. Let $h = 2v - 1$; $h \in [-1,1]$ and $h \circ b_0 = 2(v \circ b_0) - b_0 = 2 U_v b_0 - b_0^+ + b_0^-$

$$= b_0^+ + b_0^-$$

Therefore, since $h \in [-1,1]$, $\phi_h \in T$ and

$$1 \geq \langle b_0 \, \phi_h \rangle = \langle h \circ b_0, \phi \rangle$$
$$= \langle b_0^+ + b_0^- , \phi \rangle$$
$$\geq \langle b_0^+ + b_0^- , \psi \rangle$$

Multiplying the last inequality by -1 we also have $-1 \leq \langle b_0^+ + b_0^- , -\psi \rangle$ and

$$-1 \leq \langle b_0^+ + b_0^- , -\psi \rangle$$
$$\leq \langle b_0^+ - b_0^- , \psi \rangle$$
$$= \langle b_0 , \psi \rangle$$

$$\leq \langle b_0^+ + b_0^-, \psi \rangle \leq 1.$$

This contradicts the assumption that $\langle b_0, \psi \rangle > 1$ and we conclude $\psi \in T$; i.e. there exists $a \in [-1,1]$ such that $\langle b, \psi \rangle = \langle a \circ b, \phi \rangle$ for every $b \in A$.

Let $a = a^+ - a^-$ be the orthogonal decomposition of a, and let v be an idempotent in A such that $U_{1-v} a^+ = 0$, $U_v a^- = a^-$. Then

$$0 \leq \langle v, \psi \rangle = \langle v \circ a, \phi \rangle$$

$$= \langle U_v a, \phi \rangle$$

$$= \langle a^-, \phi \rangle \leq 0.$$

implies by the positivity and faithfulness of ϕ that $a^- = 0$.

Hence we have shown that if ϕ is faithful, and $0 \leq \psi \leq \phi$ there exists a positive element $a \in A$ such that $\langle b, \psi \rangle = \langle a \circ b, \psi \rangle$ for every $b \in A$.

We complete the proof by showing that ϕ does not need to be faithful. If ϕ is not faithful, let $s(\phi)$ be the support of ϕ. Then ϕ is faithful on the JBW-algebra $\mathrm{im}^+ U_{s(\phi)}$, so by what we have already proved there exists $a \in \mathrm{im}^+ U_{s(\phi)}$ such that $\langle b, \psi \rangle = \langle a \circ b, \phi \rangle$,

$0 \leq \psi \leq \phi$, for every $b \in \mathrm{im}^+ U_{s(\phi)}$. Now since $a \in \mathrm{im}^+ U_{s(\phi)}$, and both ψ and ϕ are in $\mathrm{im}^+ U^*_{s(\phi)}$, for any $b \in A$ we have

$$\begin{aligned}
\langle b,\psi \rangle &= \langle U_{s(\phi)} b, \psi \rangle \\
&= \langle a \circ (U_{s(\phi)} b), \phi \rangle \\
&= \langle U_{s(\phi)} (a \circ b), \phi \rangle \\
&= \langle a \circ b, \phi \rangle .
\end{aligned}$$

(Recall from [8, Lemma 2.11] that since $U_{s(\phi)} a = a$, we have

$$\begin{aligned}
L_a U_{s(\phi)} &= L_a (2 L^2_{s(\phi)} - L_{s(\phi)}) \\
&= (2 L^2_{s(\phi)} - L_{s(\phi)}) L_a \\
&= U_{s(\phi)} L_a
\end{aligned}$$

where L_c denotes the map $L_c d = c \circ d$.)