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MATHEMATICS
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Axioms for Birkhoff — v. Neumann Quantum Logic

by

J. KOTAS

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1. G. Birkhoff and J. v. Neumann [1] have given the following definition of the quantum logic: the quantum logic is a system $\mathfrak{M} = \langle M: \cup, \cap, ' \rangle$, where M is a set of propositions which is a modular ortocomplementary lattice [2] with respect to the binary operations $\cup$ and $\cap$ called the alternative and conjunction, respectively, and the unary operation $'$ called the negation. In the Birkhoff—v. Neumann quantum logic the distributive law is abandoned. This law is replaced by a weaker law called the modular identity: $(a \cap b) \cup (a \cap c) = a \cap (b \cup (a \cap c))$.

From the definition of quantum logic it follows that there are two fixed propositions which are denoted by 0 and 1. 0 is the false proposition and 1 is the true proposition.

The quantum logic has to be an ortocomplementary lattice, because in the modular not distributive lattices the complementary operation is not an one-to-one operation, and the ortocomplementary operation is such an operation [3].

There is no implication operation in the Birkhoff—v. Neumann quantum logic, although there is a relation implication. We call implication any relation of partial order type. In a Boolean algebra, a implies b, if $a \cup b = b$. The implication in a Boolean algebra is unique and is defined as the operation $a' \cup b$. In the Birkhoff—v. Neumann quantum logic this operation has few properties of ordinary implication, e.g. $a' \cup b = 1$ does not imply the implication relation $a \cup b = b$.

We show that in the Birkhoff—v. Neumann quantum logic it is possible to define six operations which in the Boolean algebra are identical with implication operation. These operations we shall call implications and they will be denoted by $\to_i$, $i = 0, 1, 2, 3, 4, 5$. Moreover, the relations "$a \to_i b = 1$" for $i = 0, 1, 2, 3, 4$ are equal and each of them is equal to the implication relation defined by Birkhoff — v. Neumann.

It turns out that one can define two implication-negation logic systems which are equivalent to Birkhoff—v. Neumann quantum logic.

2. DEFINITION 1. If Φ is a connective of the classical propositional calculus, then the connective Ψ of the quantum logic corresponding to Φ is any operation of the same number of variables as Φ which satisfies the matrix of Φ for 0 and 1.

We shall use the same name for Φ as for corresponding Ψ. For example, we call an implication any binary operation satisfying the matrix

	0	1
0	1	1
1	0	1

Let $\Phi(p, q)$ be a binary connective of the two-valued propositional calculus and let $\Phi_0(p, q)$ be a formula equivalent to $\Phi(p, q)$ and of the form $\Phi_0(p, q) = x_1 \cup x_2 \cup x_3 \cup x_4$, where $x_1 = 0$ or $x_1 = p \cap q$, $x_2 = 0$ or $x_2 = p \cap q$, $x_3 = 0$ or $x_3 = p \cap q$, $x_4 = 0$ or $x_4 = p' \cap q'$.

LEMMA 1. *There exist six binary falsum operations*

$F_0(p, q) = 0$,
$F_1(p, q) = p \cap (p' \cup q) \cap (p' \cup q')$,
$F_2(p, q) = q \cap (p \cup q') \cap (p' \cup q')$,
$F_3(p, q) = F_1'(p, q) \cap (F_1(p, q) \cup F_2(p, q))$,
$F_4(p, q) = F_2'(p, q) \cap (F_1(p, q) \cup F_2(p, q))$,
$F_5(p, q) = F_1(p, q) \cup F_2(p, q)$.

LEMMA 2. *The connectives of the quantum logic*

$$\Psi_i(p, q) = \Phi_0(p, q) \cup F_i(p, q), \quad i = 0, 1, 2, 3, 4, 5$$

are all connectives that correspond to the connective $\Phi(p, q)$.

Lemmas 1 and 2 follow by results of [4].

THEOREM 1. *The following equalities hold*

(i) $p \to_0 q = (p \cap q) \cup (p' \cap q) \cup (p' \cap q')$,
(ii) $p \to_1 q = (p' \cup q) \cap (p \cup (p' \cap q) \cup (p' \cap q'))$,
(iii) $p \to_2 q = (p' \cap q') \cup q$,
(iv) $p \to_3 q = p' \cup (p \cap q)$,
(v) $p \to_4 q = (p' \cup q) \cap (q' \cup (p' \cap q) \cup (p \cap q))$,
(vi) $p \to_5 q = p' \cup q$.

THEOREM 2. *For* $i = 0, 1, 2, 3, 4$: *the equality* $p \to_i q = 1$ *holds if and only if* $p \cup q = q$.

DEFINITION 2. A formula of the quantum logic is called a tautology, if an arbitrary substitution for variables in that formula, of elements from an arbitrary model of the quantum logic gives the value 1 of this model.

Now we consider the following expressions:

(i) $(p \to_i q) \to_5 (q' \to_i p')$,
(ii) $(p \to_i q) \to_5 (p \to_i p \cap q)$,
(iii) $(p \to_i q) \to_5 (p \cup q \to_i q)$,
(iv) $(p \to_i q) \to_5 (p \cup r \to_i q \cup r)$,
(v) $(p \to_i q) \to_5 (p \cap r \to_i q \cap r)$.

LEMMA 3. *For* $i = 2$ *and* $i = 3$ *the formulas* (i)—(v) *are tautologies of the quantum logic*; $i \neq 2$ *or* $i \neq 3$ *at least two among the formulas* (i)—(v) *are not tautologies of the quantum logic.*

3. By Lemma 3 the implications $\rightarrow_2$ and $\rightarrow_3$ have the most similar properties to those of the implication of classical propositional calculus.

It may be seen that any law of hypothetical syllogism in which only one sign of implication appears, is not a tautology of Birkhoff—v. Neumann quantum logic, and expression

$$(p \rightarrow_i q) \rightarrow_5 ((p \rightarrow_i r) \rightarrow_5 (p \rightarrow_i r)) \quad \text{for} \quad i = 2, 3$$

is a tautology of this system. Then if we wish the law of hypothetical syllogism to be an axiom for quantum logic, we must take as primitive notions the implications $\rightarrow_2$ and $\rightarrow_5$ and negation $'$ or the implications $\rightarrow_3$ and $\rightarrow_5$ and negation $'$.

In the following we shall write $\supset$, $>$ and $\rightarrow$ for $\rightarrow_2$, $\rightarrow_3$ and $\rightarrow_5$, respectively.

We define the system $\mathfrak{S}_1$ in the following way:

1) the primitive symbols of the system $\mathfrak{S}_i$ are $\rightarrow$, $\supset$, $'$. ("$a \rightarrow b$", "$a \supset b$", "a'" we read: "if a, then b", "a implies b", "not-a", respectively);

2) the axioms are:

I $p \supset (q \rightarrow p)$,

II $p \rightarrow (q \supset p)$,

III $(p' \rightarrow q) \supset (q' \rightarrow p)$,

IV $(p \supset q) \rightarrow (q' \supset p')$,

V $(p \rightarrow (q \rightarrow r)) \supset (q \rightarrow (p \rightarrow r))$,

VI $(p \supset q) \supset ((p' \rightarrow q) \rightarrow q)$,

VII $((p \rightarrow q) \rightarrow q) \supset (p' \supset q)$,

IX $(p \supset q) \rightarrow ((p' \rightarrow r) \supset (q' \rightarrow r))$,

X $p'' \supset p$,

XI $p \supset p''$,

XII $(((p \rightarrow q) \rightarrow r) \rightarrow p')' \supset ((p \rightarrow q) \rightarrow (p \rightarrow r')')$,

XIII $(p \supset q) \rightarrow ((p' \rightarrow q) \supset q)$,

XIV $(p \rightarrow q) \supset ((p \rightarrow q)' \supset p')$;

3) we assume as the rules of deduction: the rule of substitution for propositional variables, the rules of modus ponens for the implications $\rightarrow$ and $\supset$.

We define the alternative, the conjunction and the equivalence in the system $\mathfrak{S}_1$ as follows:

Definition 3. $p \cup q \underset{\text{df}}{=} p' \rightarrow q$.

Definition 4. $p \cap q \underset{\text{df}}{=} (p' \cup q')'$.

Definition 5. $p \supset\subset q \underset{\text{df}}{=} (p \supset q) \cap (q \supset p)$.

The symbol "$\vdash P$" means that the formula P may be proved in the system $\mathfrak{S}_1$.

We introduce following equivalence relation between formulas of the system $\mathfrak{S}_1$: $P = Q$ if and only if $\vdash (P \supset\subset Q)$.

We define the system $\mathfrak{S}_2$ in the following way:

a) the primitive symbols of the system $\mathfrak{S}_2$ are $\rightarrow$, $>$, $'$. ("$a \rightarrow b$", "$a > b$", "a'" we read: "if a, then b", "a implies b", "not-a", respectively).

b) the axioms are:

I $p > (q \to p)$,

II $p \to (q > p)$,

III $(p' \to q) > (q' \to p)$,

IV $(p > q) \to (q' > p')$,

V $\big(p \to (q \to r)\big) > \big(q \to (p \to r)\big)$,

VI $(p > q) > \big((p \to q') \to p'\big)$,

VII $\big((p \to q) \to p'\big) > (p > q')$,

VIII $(p > q) \to \big((q > r) \to (p > r)\big)$,

IX $(p > q) \to \big((p' \to r) > (q' \to r)\big)$,

X $p'' > p$,

XI $p > p''$,

XII $\big(((p \to q) \to r) \to p'\big)' > \big((p \to q) \to (p \to r')'\big)$,

XIII $(p > q) \to \big(p > (p \to q')'\big)$,

XIV $(p \to q) > \big(p > (p \to q)\big)$.

c) we assume as the rules of deduction: the rule of substitution for propositional variables, the rules of modus ponens for the implications $\to$ and $>$.

We define the alternative and the conjunction like in $\mathfrak{S}_1$. The equivalence is defined by:

DEFINITION 6. $p \times q \underset{\mathrm{df}}{=} (p > q) \cap (q > p)$.

The symbol "$\vdash P$" means that formula P may be proved in the system $\mathfrak{S}_2$. We introduce following equivalence relation between formulas of the system $\mathfrak{S}_2$: $P = Q$ if and only if $\vdash (P \times Q)$.

4. It is not difficult to prove that systems $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are equivalent in the following sense: replacing in any formula provable in $\mathfrak{S}_1$ every expression of the form $P \supset Q$ by the expression $Q' > P'$ we obtain the formula provable in $\mathfrak{S}_2$ and vice versa.

THEOREM 3. *The system $\mathfrak{S}_1$ (and $\mathfrak{S}_2$) is complete with respect to the quantum logic i.e. any formula A is a tautology of Birkhoff*—v. *Neumann quantum logic if and only if A is provable in $\mathfrak{S}_1$ (respectively in $\mathfrak{S}_2$).*

DEPARTMENT OF MATHEMATICS, N. COPERNICUS UNIVERSITY, TORUŃ
(KATEDRA MATEMATYKI, UNIWERSYTET IM. M. KOPERNIKA, TORUŃ)

REFERENCES

[1] G. Birkhoff and J. von Neumann, *The logic of quantum mechanics*, Annals of Mathematics, **37** (1936), 823—843.

[2] G. Birkhoff, *Lattice theory*, New York, 1948, pp. XIII+283.

[3] Orrin Frink, *New algebras of logic*, The American Mathematical Monthly, **45**, 1938, 210—219.

[4] J. Kotas, *On the decomposition of modular ortocomplementary finite-generated lattice*, [this issue], p. 639.