BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys.— Vol. XI, No. 10, 1963

MATHEMATICS
(ALGEBRA)

On Decomposition of the Modular Ortocomplementary Finite-generated Lattice

by

J. KOTAS

Presented by A. MOSTOWSKI on August 24, 1963

G. Birkhoff and J. von Neumann [1], give the following definition of the quantum logic:

The quantum logic is a system $\mathfrak{M} = \langle M; \cup, \cap, ' \rangle$, where M is a set of propositions which is a modular ortocomplementary lattice [2] with respect to the binary operations $\cup$ and $\cap$ which are called the alternative and conjuntion respectively, and the unary operation $'$ which is called the negation. Thus every model of quantum logic is a modular ortocomplementary lattice. The most important model of quantum logic is the ortocomplementary lattice of linear subspaces of a linear space. A formula of $\mathfrak{M}$ is called a tautology of quantum logic, if an arbitrary substitution, for variables in that formula, of elements from an arbitrary model gives 1 of this model. An arbitrary formula with n variables may be identified with a term of modular ortocomplementary lattice generated by n elements. The problem of deciding whether a formula is a tautology of quantum logic is much more simple if there is given a decomposition of modular ortocomplementary lattice onto the direct product of sublattices.

The aim of this note is to prove that every modular ortocomplementary finite-generated lattice M may be decomposed onto the direct sum of two sublattices M^0 and M^*. In this decomposition M^* is a distributive lattice.

Let $M[g, h]$ be a modular ortocomplementary lattice generated by g and h. We shall use the following notations:

$$g_1 = g \cap (g' \cup h) \cap (g' \cup h'),$$

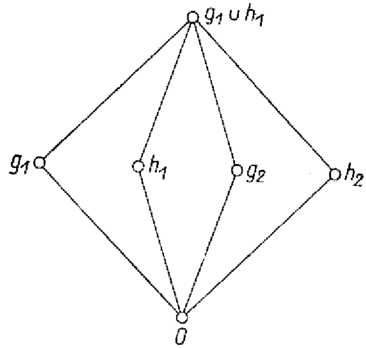
$$h_1 = h \cap (g \cup h') \cap (g' \cup h'),$$

$$g_2 = g_1' \cap (g_1 \cup h_1),$$

$$h_2 = h_1' \cap (g_1 \cup h_1).$$

The following two lemmas are easy consequences of the above definitions:

LEMMA 1. *Either equalities hold* $g_1 = h_1 = g_2 = h_2 = 0$ *or the elements* $0, g_1, h_1, g_2, h_2, g_1 \cup h_1$ *constitute a sublattice* $M^0[g, h]$ *of* $M[g, h]$ *which has the following diagram*:



LEMMA 2. *For every* $a \in M[g, h]$ *we have the following unique representation*:

$$a = s \cup t_1 \cup t_2 \cup t_3 \cup t_4,$$

where $s \in M^0[g, h]$, $t_1 = 0$ *or* $t_1 = g \cap h$, $t_2 = 0$ *or* $t_2 = g \cap h'$, $t_3 = 0$ *or* $t_3 = g' \cap h$, $t_4 = 0$ *or* $t_4 = g' \cap h'$.

As a consequence we obtain that $M[g, h]$ has at most 96 elements.

Note the following equalities which are true in an arbitrary modular ortocomplementary lattice:

(i) $g = g_1 \cup (g \cap h) \cup (g \cap h')$,

(ii) $h = h_1 \cup (g \cap h) \cup (g' \cap h)$,

(iii) if $(g \cap h) \cup (g \cap h') \cup (g' \cap h) \cup (g' \cap h') = 1$, then $g_1 = h_1 = 0$.

We shall use the following notation:

$$\|(x, y) = (x \cap y) \cup (x \cap y') \cup (x' \cap y) \cup (x' \cap y').$$

LEMMA 3. *If* a, b, c *are arbitrary elements of a modular ortocomplementary lattice and* $\|(a, b) = \|(b, c) = \|(c, a) = 1$, *then the triple* a, b, c *is distributive.*

Proof. Observe that it is sufficient to prove the equality $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$. Since $\|(a, b) = \|(b, c) = \|(c, a) = 1$, we have the following equalities:

$$a = (a \cap b) \cup (a \cap b'), \qquad a = (a \cap c) \cup (a \cap c'),$$
$$b = (a \cap b) \cup (a' \cap b), \qquad b = (b \cap c) \cup (b \cap c'),$$
$$c = (a \cap c) \cup (a' \cap c), \qquad c = (b \cap c) \cup (b' \cap c).$$

Thus,

$$a \cap (b \cup c) = (a \cap b) \cup [(a' \cap b') \cap (a \cap c)],$$
$$(a \cap b) \cup (a \cap c) = (a \cap b) \cup [(a' \cup b') \cap ((a \cap b) \cup (a \cap c))].$$

Hence,

$$a \cap (b \cup c) \subset (a \cap b) \cup b) \cup (a \cap c).$$

The inverse inclusion is trivial.

COROLLARY 1. *Let A be a modular ortocomplementary lattice. If for every pair $a, b \in A$ $\|(a, b) = 1$, then A is a distributive lattice.*

Let $M = M[p_1, p_2, \ldots, p_n]$ be a modular ortocomplementary lattice generated by $p_1, p_2, \ldots, p_n$.

Put $q_j^0 = p_j$, $q_j^1 = p_j$, $j = 1, 2, \ldots, n$.

Let $t_1, t_2, \ldots, t_n$ be an arbitrary sequence such that $t_i = 0, 1$. We shall use the following notations:

$$a_{(t_1, t_2, \ldots, t_n)} = q_1^{t_1} \cap q_2^{t_2} \cap \ldots \cap q_n^{t_n},$$

$$b_i = \bigcup \{a_{(t_1, t_2, \ldots, t_n)} : t_1 + t_2 + \ldots + t_n = i\}, \quad i = 0, \ldots, n \text{ *)},$$

$$b_i^k = \bigcup \{a_{(t_1, t_2, \ldots, t_n)} : t_1 + t_2 + \ldots + t_n = i,\ t_k = 0\},$$

$$i = 0, \ldots, n-1, \quad k = 1, 2, \ldots, n,$$

$$p_k^* = \bigcup_{i=0}^{n-1} b_i^k, \quad k = 1, 2, \ldots, n,$$

$$p_k^0 = p_k \cap (p_k^*)', \quad k = 1, 2, \ldots, n,$$

$$1^* = \bigcup_{i=1}^{n} b_i,$$

$$1^0 = (1^*)'.$$

LEMMA 4. *There are the following equalities:*

(i) $p_k^0 = p_k \cap 1^0$,

(ii) $\bigcup_{k=1}^{n} p_k = 1^0$,

(iii) $(p_k^0)' \cap 1^* = 1^*$,

(iv) $(p_k^*)' \cap 1^0 = 1^0$.

For an arbitrary elements $x \in M$ let us put

$$(x)'_0 = x' \cap 1^0, \quad (x)'_* = x' \cap 1^*.$$

Let M^0 and M^* be subsets consisting of all elements of M which may be obtained from $p_1^0, p_2^0, \ldots, p_n^0$ and $p_1^*, p_2^*, \ldots, p_n^*$ by use of $\cup, \cap, {}'_0$ and $\cup, \cap, {}'_*$ respectively. It is easy to see that M^0 and M^* are modular ortocomplementary lattices with respect to $\cup, \cap, {}'_*$ and $\cup, \cap, {}'_*$ respectively.

LEMMA 5. *M^* is a distributive lattice.*

LEMMA 6. *M^* is equal to D, where $D = D[p_1, p_2, \ldots, p_n]$ is a distributive complementary lattice generated by $p_1, p_2, \ldots, p_n$.*

THEOREM. *M is the direct sum of M^0 and M^*.*

---

*) Here $\bigcup A = \bigcup_{x \in A} x$.

Proof. Since $p_k^0 \in M^0$ and $p_k^* \in M^*$, then it is sufficient to prove that:

(i) $p_k = p_k^0 \cup p_k^*$,

(ii) if $a = a_1 \cup a_2$ and $a_1 \in M^0$, $a_2 \in M^*$, then $a' = (a_1)'_0 \cup (a_2)'_*$ where $(a_1)'_0 \in M^0$, $(a_2)'_* \in M^*$,

(iii) if $a = a_1 \cup a_2$, $b = b_1 \cup b_2$ and $a_1, b_1 \in M^0$, $a_2, b_2 \in M^*$

then a) $a \cup b = (a_1 \cup b_1) \cup (a_2 \cup b_2)$,

b) $a \cap b = (a_1 \cap b_1) \cup (a_2 \cap b_2)$,

where $a_1 \cup b_1$, $a_1 \cap b_1 \in M^0$, $a_2 \cup b_2$, $a_2 \cap b_2 \in M^*$.

(i) is obvious. The proof of (ii) is as follows:

$$a' = (a_1 \cup a_2)' = a_1' \cap a_2' = a_1' \cap (1^0 \cup 1^*) \cap a_2' = \big((a_1' \cap 1^0) \cup 1^*\big) \cap a_2' = (a_1' \cap 1^0) \cup (a_2' \cap 1^*) = (a_1)'_0 \cup (a_2)'_*.$$

Part a) of (iii) is obvious, the proof of b) is as follows:

$$a \cap b = (a' \cup b')' = \big(((a_1)'_0 \cup (b_1)'_0) \cup ((a_2)'_* \cup (b_2)'_*)\big)' = \big((a_1)'_0 \cup (b_1)'_0\big)'_0 \cup \big((a_2)'_* \cup (b_2)'_*\big)'_* = (a_1 \cap b_1) \cup (a_2 \cap b_2).$$

COROLLARY. *M is distributive if and only if*

$$p_1^0 = p_2^0 = \ldots = p_n^0 = 0.$$

DEPARTMENT OF MATHEMATICS, N. COPERNICUS UNIVERSITY, TORUŃ
(KATEDRA MATEMATYKI, UNIWERSYTET M. KOPERNIKA, TORUŃ)

REFERENCES

[1] G. Birkhoff and J. von Neumann, *The logic of quantum mechanics*, Annals of Mathematics, **37** (1936), 823—843.

[2] G. Birkhoff, *Lattice theory*, New York, 1948, pp. XIII+283.