Formal systems for topos-theoretic modalities.

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Introduction

In Reyes [6], a topos-theoretic approach to the logic of reference is developped which shows that modal operators of possibility and necessity may be canonically defined in the context of a topos $\mathbb{E}$ (thought of as "variable sets") over a base topos $\mathbb{S}$ (thought of as "constant sets"), i.e., in the context of a geometric morphism $\mathbb{E} \to \mathbb{S}$.

In this paper we shall concentrate on the case that $\mathbb{S}$ is the category of Sets and we shall prove soundness and completeness for the resulting first-order modal logic, IBM (for "Intuitionistic with Boolean modalities").

In the first Section, we briefly review the introduction of modal operators in the topos-theoretic context just mentioned. In Section 2, we specialize these results to the case that $\mathbb{S}$ is a Boolean topos (e.g. Sets). The system IBM is described in the next section, together with two other systems which will play an auxiliary, but important rôle in the sequel : IS4 (for "Intuitionistic S4"), and MAO (for "Modal adjoint operators"). Some obvious soundness theorems are stated. In section 4, we prove soundness and completeness for an extension of IBM for the case that $\mathbb{E}$ is a topos of Kripke trees, i.e., a topos of presheaves over a pre-ordered set $\mathbb{P}$. Our proof of the completeness theorem for IBM is rather indirect and proceeds via completeness for IS4. A system of sequents for IS4 is presented in section 5, together with a proof of cut elimination for this system. For our purposes, however, a variant of this system is more adequate for the completeness theorem for IS4. We consider section 6 as the most important one of our paper. We introduce Beth trees, forests

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and grafts to define a new semantics : graft semantics. Graft semantics is *not* the topos-theoretic semantics described in §1,§2. It is rather non-standard since it uses the presentation of a topos as sheaves over a forest of Beth trees with grafts to define the modal operators. Two soundness and completeness theorems are proved for this semantics : for IS4 and for IBM. It turns out, however, that in the case of IBM, the axioms on Boolean modalities reduce the graft semantics to the topos-theoretical one. Thus, we have proved completeness of IBM for the topos semantics.

§1. Modal operators on a topos over a base.

In this section we shall briefly review the introduction of modal operators on predicates of constant sheaves of a topos $\mathcal{E}$ (thought of as "variable sets") over a base topos $\mathcal{S}$ (thought of as "constant sets"). For motivation and proofs, see Reyes [6].

Given a topos $\mathcal{E}$ over a base topos $\mathcal{S}$, i.e., a geometric morphism $\mathcal{E} \to \mathcal{S}$ with finitely left exact inverse image Δ and direct image Γ ($\Delta \dashv \Gamma$), we define maps

$$\Omega_{\mathcal{S}} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Gamma(\Omega_{\mathcal{E}})$$

as follows : given a generalized element $X \overset{p}{\to} \Omega_{\mathcal{S}}$, let $P \rightarrowtail X$ be the subobject classified by p. Applying Δ, we obtain a subobject $\Delta P \rightarrowtail \Delta X$ which is classified by a map $\Delta X \to \Omega_{\mathcal{E}}$. We define $\delta(p) : X \to \Gamma(\Omega_{\mathcal{E}})$ to be the transpose of this map.

On the other hand, given a generalized element $X \overset{K}{\to} \Gamma(\Omega_{\mathcal{E}})$, its transpose $\Delta X \to \Omega_{\mathcal{E}}$ classifies a subobject $\tilde{K} \rightarrowtail \Delta X$. Applying Γ, we obtain $\Gamma(\tilde{K}) \rightarrowtail \Gamma\Delta X$. We define $\gamma(K) : X \to \Omega_{\mathcal{S}}$ to be the map which classifies the subobject $N(K) \rightarrowtail X$ given by the pull-back diagram

$$\begin{array}{ccc} \Gamma(\tilde{K}) & \to & \Gamma\Delta X \\ \uparrow & & \uparrow \eta_X \\ N(K) & \to & X \end{array}$$

where $\eta : Id \to \Gamma\Delta$ is the unit of the adjunction $\Delta \dashv \Gamma$.

Using the fact that $\Delta \dashv \Gamma$ and Δ is finitely left exact, it is easy to check that :

$$\delta \dashv \gamma ;$$
$$\delta T = T ;$$
$$\delta(p \wedge q) = \delta(p) \wedge \delta(q).$$

Definition 1. *Let $\mathcal{E} \to \mathcal{S}$ be a geometric morphism. The necessity operator $\Box : \Gamma(\Omega_{\mathcal{E}}) \to \Gamma(\Omega_{\mathcal{E}})$ is defined to be $\Box = \delta\gamma$.*

Proposition 2. *The necessity operator $\Box$ has the following properties :*
(1) $\Box \leq \mathrm{Id}$,
(2) $\Box^2 = \Box$,
(3) $\Box T = T$,
(4) $\Box (K_1 \wedge K_2) = \Box K_1 \wedge \Box K_2$.

Proof. Just notice that $\Box = \delta\gamma$ is a lex cotriple. (Alternatively, a direct check is possible).

We can now define the action of $\Box$ on predicates of constant sheaves as follows : if $\Delta S \xrightarrow{\varphi} \Omega_{\mathcal{E}}$ is a predicate of ΔS, we define $\Delta S \xrightarrow{\Box_S \varphi} \Omega_{\mathcal{E}}$ as the transpose of $S \xrightarrow{\Box \circ \iota(\varphi)} \Gamma(\Omega_{\mathcal{E}})$ where $S \xrightarrow{\iota(\varphi)} \Gamma(\Omega_{\mathcal{E}})$ is the transpose of φ.

Remark : We may also observe, and this will be used in the case of predicate logic, that $\Box D_x = D_x$, where D_x is the transpose of $\Delta X \times \Delta X \xrightarrow{\mathrm{diag}} \Omega_{\mathcal{E}}$.

We will repeat here the three main examples of Reyes [6] to see what $\Box$ is in particular cases.

Example 1.

$$\mathbb{S} = \text{Sets} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \text{Sets}^I = \mathbb{E}$$

In this case, $\Delta_S = (S)_{i\in I}$, $\Gamma((X_i)_{i\in I}) = \prod_{i\in I} X_i$. Furthermore, $\Omega_{\mathbb{S}} = 2 \underset{\gamma}{\overset{\delta}{\rightleftarrows}} 2^I = \Gamma(\Omega_\varepsilon)$ are given by $\delta(p) = \{i \in I \mid p\}$, $\gamma(K) = [\![\forall i \in I\ (i \in K)]\!]$. The action of $\Box$ on predicates may be described (using the "forcing" relation) as follows :

$$i \Vdash \Box\varphi\,[s] \quad \text{iff} \quad \forall j \in I \qquad j \Vdash \varphi[s],$$

for all $s \in S = \Delta S(i) = \Delta S(j)$.

Example 2.

$$\mathbb{S} = \text{Sets} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \text{Sets}^{\mathbb{P}^{op}} = \mathbb{E}$$

where $\mathbb{P} = (P,\leq)$ is a pre-ordered set.
In this example, $\Delta S(U) = S$ for all $U \in P$ and $\Gamma(F) = \varprojlim_{\mathbb{P}^{op}} F$. The maps $\Omega_{\mathbb{S}} = 2 \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Omega(1) = \Gamma(\Omega_{\mathbb{E}})$, where $\Omega(1) = \{K \subseteq P \mid K \text{ is downwards closed}\}$, are given by $\delta(p) = \{U \in P \mid p\}$ and $\gamma(K) = [\![\forall U \in P\ (U \in K)]\!]$.

Once again, the action of $\Box$ on predicates of constant sheaves (or rather presheaves in this case) may be described by

$$U \Vdash \Box\varphi\,[s] \qquad \text{iff} \qquad \forall V \in P \qquad V \Vdash \varphi[s],$$

for all $s \in S = \Delta S(U) = \Delta S(V)$.

Example 3.

$\mathbb{E} \to \mathbb{S}$ is bounded.

This means that $\mathbb{E}$ may be presented as the category of sheaves over a site $\mathbb{C}$ in $\mathbb{S}$:

$$\begin{array}{ccc} \mathbb{S} & \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} & Sh_{\mathbb{S}}(\mathbb{C}) \\ \Gamma_0 \nwarrow & \searrow \Delta_0 & a \downarrow \uparrow i \\ & & \mathbb{S}^{\mathbb{C}^{op}} \end{array}$$

where i is the inclusion of presheaves in sheaves, a, the associated sheaf functor, $\Delta_0(S)(C) = S$ for all $C \in \mathbb{C}$, $\Gamma(F) = \varprojlim_{\mathbb{C}^{op}} F$, $\Delta = a\Delta_0$ and $\Gamma = \Gamma_0\, i$.

The maps $\Omega_{\mathbb{S}} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Gamma(\Omega_{\mathbb{E}})$ may be described as follows :

$\delta(p)$ = closure of $\{C \in \mathbb{C} \mid p\}$,

$\gamma(K) = [\![\forall c \in \mathbb{C}\ (c \in K)]\!]$,

where $\Gamma(\Omega_{\mathbb{E}})$ has been identified with the set (in the sense of $\mathbb{S}$) of closed sieves of $\mathbb{C}$. We recall that a sieve on $\mathbb{C}$ is a set K such that $C \in K$ and $C' \to C \in \mathbb{C}$ implies $C' \in K$. If K is a sieve, closure of $K = \{C \in \mathbb{C} \mid \exists\, \{C_i \to C\}_{i \in I} \in Cov(C)\ \forall i \in I\ C_i \in K\}$. A sieve is closed if it coincides with its closure.

The action of $\Box$ on precidates may be described as

$$C \Vdash \Box\varphi\,[(\ell_S)_C(s)] \quad \text{iff} \quad \exists\, \{C_i \to C\}_{i \in I} \in Cov(C)\ \forall i \in I\ \ \forall C'\ C' \Vdash \varphi\,[(\ell_S)_{C'}(s)]$$

where $S \overset{(\ell_S)_C}{\to} i\Delta S(C)$ is the C-component of the unit $\Delta_0 S \overset{\ell_S}{\to} ia\,\Delta_0 S = i\Delta S$ of the adjunction $a \dashv i$.

Notice that this gives the required action for C-elements of $i\Delta S$ of the form $(\ell_S)_C(s)$ (i.e., constant ones) only. The general case, however, follows easily from this :

if $\xi \in i\Delta S(C)$, $C \Vdash \Box\varphi\,[\xi]$ iff $\exists\ \{C_i \to C\}_{i\in I} \in \mathrm{Cov}(C)$ $\forall i \in I\ (\exists s_i \in S\ \ \xi \restriction C_i = s_i \wedge \exists\ \{C_{ij} \to C_i\}_j \in \mathrm{Cov}(C_i)$ $\forall j \in I\, \forall C' \in \mathbb{C}\ \ C' \Vdash \varphi\,[(\ell S)_{C'}(s_i)])$. (Indeed, ξ is locally constant).

In the rest of this paper, we will be specially interested in cases in which $\mathcal{E} \to \mathcal{S}$ satisfies some further properties. First, some definitions.

Definition 3. (Johnstone [3], Joyal and Tierney [4]) *A geometric morphism $\mathcal{E} \to \mathcal{S}$ is open iff the map $\delta : \Omega_{\mathcal{S}} \to \Gamma(\Omega_{\mathcal{E}})$ has a left adjoint $\lambda \dashv \delta$.*

Remark 4. The map λ automatically satisfies the Frobenius conditions $\lambda(K \wedge \delta(p)) = \lambda(K) \wedge p$.

For us, the important fact about open map is that the possibility operator is definable.

Definition 5. *Let $\mathcal{E} \to \mathcal{S}$ be an open geometric morphism. Let us define the possibility operator $\Diamond : \Gamma(\Omega_{\mathcal{E}}) \to \Gamma(\Omega_{\mathcal{E}})$ to be $\Diamond = \delta\lambda$.*

Proposition 6. *Let $\mathcal{E} \to \mathcal{S}$ be an open geometric morphism and let $\Diamond, \Box : \Gamma(\Omega_{\mathcal{E}}) \to \Gamma(\Omega_{\mathcal{E}})$ defined by $\Diamond = \delta\lambda$ and $\Box = \delta\gamma$. Then the couple $(\Diamond, \Box)$ has the following properties :*

(1) $\Diamond \dashv \Box$;

(2) $\Box \leq \mathrm{Id} \leq \Diamond$;

(3) $\Box^2 = \Box$, $\Diamond^2 = \Diamond$;

(4) $\Diamond(K_1 \wedge \Box K_2) = \Diamond K_1 \wedge \Box K_2$.

Proof. Straightforward computation. For (4), use the Remark 4.

Definition 7. *A MAO couple on $\Gamma(\Omega_{\mathcal{E}})$ is a couple of operators $\Diamond, \Box : \Gamma(\Omega_{\mathcal{E}}) \to \Gamma(\Omega_{\mathcal{E}})$ satisfying (1) - (4).*

We may define the action of $\Diamond$ on predicates of constant sheaves in the same way as for $\Box$: if $\Delta S \overset{\varphi}{\to} \Omega_{\mathcal{E}}$ is an arbitrary predicate of ΔS, we define $\Delta S \overset{\Diamond_S \varphi}{\to} \Omega_{\mathcal{E}}$ to be the transpose of $S \overset{\Diamond\sigma t(\varphi)}{\to} \Gamma\Omega_{\mathcal{E}}$, where $S \overset{t(\varphi)}{\to} \Gamma(\Omega_{\mathcal{E}})$ is the transpose of φ.

We shall return to our examples to study the action of $\Diamond$ on these particular cases.

<u>Example 1</u> (bis)

$$\mathcal{S} = \text{Sets} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \text{Sets}^I = \mathcal{E}.$$

In this case, we may check directly that $\lambda : 2^I \to 2$ defined by $\lambda(K) = [\![\exists i \in I\ (i \in K)]\!]$ is the left adjoint of $\delta : 2 \to 2^I$.
The action of $\Diamond = \delta\,\lambda$ on predicates is given by $i \Vdash \Diamond\,\varphi\,[s]$ iff $\exists j \in I\ j \Vdash \varphi[s]$ for all $s \in S = \Delta S(i) = \Delta S(j)$.

Therefore, this example is the well-known "possible worlds" semantics.

<u>Example 2</u> (bis)

$$\mathcal{S} = \text{Sets} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \text{Sets}^{\mathbb{P}^{op}} = \mathcal{E}.$$

Once again, we may check directly that $\lambda : \Omega(1) \to 2$ defined by $\lambda(K) = [\![\exists\ U \in P\ (U \in K)\ ]\!]$ is the left adjoint of $\delta : 2 \to \Omega(1)$.
The action of $\Diamond = \delta\,\lambda$ on predicates is given by $U \Vdash \Diamond\,\varphi\,[s]$ iff $\exists V \in P\ V \Vdash \varphi\,[s]$ for all $s \in S = \Delta S(U) = \Delta S(V)$.

<u>Example 3</u> (bis)

$\mathcal{E} \to \mathcal{S}$ is bounded and open.

In this case, $\mathcal{E}$ may be presented as the category of sheaves over an open site of $\mathbb{C}$, i.e., a site such that every cover is inhabited (cf Joyal and Tierney [4]).

The action of $\Box$ on predicates may be simplified as follows :

$C \Vdash \Box_S \varphi[(\ell_S)_C(s)]$ iff $\forall C' \in \mathbb{C}$ $C' \Vdash \varphi[(\ell_S)_{C'}(s)]$ for all $s \in S$.

From the adjunction $\Diamond \dashv \Box$ we conclude $C \Vdash \Diamond_S \varphi[(\ell_S)_C(s)]$ iff $\exists C' \in \mathbb{C}$ $C' \Vdash \varphi[(\ell_S)_{C'}(s)]$ for all $s \in S$.

Let us spell the forcing clause for $\Box$ and $\Diamond$ in the general case of $\xi \in i\Delta S(C)$:

$$C \Vdash \Box_S \varphi[\xi] \text{ iff } \exists \{C_i \to C\}_{i\in I} \in Cov(C) \ \forall i \in I$$
$$(\exists s_i \in S \quad \xi\lceil C_i = s_i \wedge \forall C' \in \mathbb{C} \quad \mathbb{C}' \Vdash \varphi \ [(\ell_S)_{C'}(s_i)]).$$
$$C \Vdash \Diamond_S \varphi[\xi] \text{ iff } \exists \{C_i \to C\}_{i\in I} \in Cov(C) \ \forall i \in I$$
$$(\exists s_i \in S \quad \xi\lceil C_i = s_i \wedge \exists C' \in \mathbb{C} \quad \mathbb{C}' \Vdash \varphi \ [(\ell_S)_{C'}(s_i)]).$$

Before leaving this section, let us remark that open maps are ubiquitous, as the following shows :

Proposition 8

(i) *Assume that* $\mathbb{E} \xrightarrow{\Gamma} \mathbb{S}$ *is a geometric morphism such that its inverse image* Δ *has a left adjoint* $\Pi_0 \dashv \Delta$. *Then* Γ *is open.*

(ii) Assume that $\mathbb{E} \xrightarrow{\Gamma} \mathbb{S}$ *is a geometric morphism such that* $\mathbb{S}$ *is a Boolean topos. Then* Γ *is open.*

Proof: For (i), see Reyes [6] ; for (ii) see Johnstone [3]. These results, however, may be proved rather straightforwardly.

As a consequence of (ii), every Grothendieck topos $\mathbb{E}$ is open over Sets and hence $\Gamma(\Omega_{\mathbb{E}})$ has a canonical couple $(\Diamond, \Box)$ of MAO operators.

§2. Modal operators on a topos over a Boolean base.

In this section we shall study the particular case of a geometric morphism $\mathbb{E} \to \mathbb{S}$ with $\mathbb{S}$ a Boolean topos. We shall see that this hypothesis implies some special properties about $\Diamond, \Box$.

Proposition 1. *Assume that* $\mathbf{E} \to \mathbf{S}$ *is a geometric morphism and let* $\mathbf{S}$ *be Boolean. Then* $\Box$ *satisfies the "modal excluded middle"* MEM : $\Box K \vee \neg \Box K = T$, *for all* $K \in \Gamma(\Omega_{\mathbf{E}})$ *and* $\Diamond$ *is definable in terms of* $\Box$: $\Diamond K = \neg \Box \neg K$, *for all* $K \in \Gamma(\Omega_{\mathbf{E}})$.

Proof : Let us check MEM. Let $K \in \Gamma(\Omega_{\mathbf{E}})$ be given. Then $\gamma(K) \in \Omega_{\mathbf{S}}$ has a Boolean complement $\neg\gamma(K)$ which satisfies

$$\gamma(K) \vee \neg \ \gamma(K) = T$$
$$\gamma(K) \wedge \neg \ \gamma(K) = \bot.$$

Since δ preserves $\wedge$ and T as well as $\vee$ and $\bot$ (recall that $\delta \dashv \gamma$), we conclude that

$$\delta\gamma K \vee \delta(\neg\gamma K) = T$$
$$\delta\gamma K \wedge \delta(\neg\gamma K) = \bot.$$

This shows that the complement $\neg\delta\gamma K$ is the Boolean complement of $\delta\gamma K$, i.e. (since $\Box = \delta\gamma$)

$$\Box K \vee \neg \ \Box \ K = T.$$

To show $\Diamond = \neg \Box \neg$, we first check that δ preserves $\neg$ (and, in fact $\to$) ; we have the following equivalences :

$$\frac{K \leq \delta(p \to q)}{\lambda K \leq (p \to q)} \ (\lambda \dashv \delta)$$

$$\frac{}{\lambda K \wedge p \leq q}$$

$$\frac{}{\lambda(K \wedge \delta p) \leq q} \ \text{(Frobenius)}$$

$$\frac{}{K \wedge \delta p \leq \delta q} \ (\lambda \dashv \delta)$$

$$\frac{}{K \leq \delta p \to \delta q.}$$

Furthermore, the equivalences

$$\frac{\gamma\neg K \leq \gamma\neg K}{} \ (\delta \dashv \gamma)$$

$$\frac{\dfrac{\dfrac{\dfrac{\delta\gamma\neg K \le \neg K}{K \wedge \delta\gamma(\neg K) \le \bot}}{K \le \neg\delta\gamma(\neg K)}}{K \le \delta\neg\gamma\neg K}\ (\delta \text{ preserves } \neg)}{\lambda K \le \neg\gamma\neg K}$$

show that $\lambda K \le \neg\gamma\neg K$.

On the other hand, $\neg\gamma\neg K \le \lambda K$ follows from

$$\frac{\lambda K \le \lambda K}{K \le \delta\lambda K}.$$

From here, $\neg\delta\lambda K \le \neg K$. The equivalences

$$\frac{\dfrac{\neg\delta\lambda K \le \neg K}{\delta(\neg\lambda K) \le \neg K}\ (\delta \text{ preserves } \neg)}{\neg\lambda K \le \gamma\neg K}$$

show that $\neg\gamma\neg K \le \neg\neg\lambda K = \lambda K$ since Ω_S is Boolean. We have proved that $\lambda K = \neg\gamma\neg K$ and this implies (since δ preserves $\neg$) that

$$\delta\lambda K = \neg\delta\gamma\neg K$$

i.e.,

$$\Diamond K = \neg\Box\neg K.$$

<u>Remark</u>: Adapting the arguments of lemmas 3 and 4 of §4, we can conclude that $\Diamond = \neg\Box\neg$ from the fact that $\Box K \vee \neg\Box K = T$.

§3. Formal systems for modal adjoint operators.

In this section we describe three formal systems of first-order modal logic for the necessity operator $\Box$, MAO couples $(\Diamond,\Box)$ and MAO couples $(\Diamond,\Box)$ over a Boolean base. We shall call these systems IS4 ("intuitionistic S4"), MAO ("modal adjoint operators") and IBM ("intuitionistic with Boolean modality").

We first introduce a first-order modal language L and define an interpretation of L into a topos $\mathcal{E}$ over $\mathcal{S} : \mathcal{E} \overset{\Gamma}{\to} \mathcal{S}$.

To simplify the exposition, we shall assume that L has only one sort. As usual, L has for each n a set F_n of n-ary function symbols and a set R_n of n-ary relation symbols. Terms and formulas are constructed from these primitives by using the logical symbols $=, \neg, \wedge, \vee, \to, \perp, \forall, \exists, \Box$ and $\Diamond$ in the usual way.

An *interpretation* is the association of a constant sheaf ΔS to the unique sort of L, a morphism $\|f\| : (\Delta S)^n \to \Delta S$ to each $f \in F_n$, a predicate $\|r\| : (\Delta S)^n \to \Omega_{\mathcal{E}}$ to each $r \in R_n$. This interpretation is extended by induction to terms and formulas as usual in topos theory (Cf. Makkai-Reyes [5]). The only clauses which are new are those for $\Box, \Diamond$ and we shall give them in detail. Assume that the formula $\varphi(x_1,\dots,x_n)$ has been interpreted as $[\![\vec{x} : \varphi]\!] : \Delta S_1 \times \dots \times \Delta S_n \to \Omega_{\mathcal{E}}$. We define

$$[\![\vec{x} : \Box\varphi]\!] : \Delta S_1 \times \dots \times \Delta S_n \to \Omega_{\mathcal{E}}$$

as the transpose of $\Box \circ t([\![\vec{x} : \varphi]\!])$, where

$$t([\![\vec{x} : \varphi]\!]) : S_1 \times \dots \times S_n \to \Gamma(\Omega_{\mathcal{E}})$$

is the transpose of $[\![\vec{x} : \varphi]\!]$. Similarly for $\Diamond$. We say that σ is *valid* for the interpretation (when σ is a sentence) if $[\![: \sigma]\!] : \Delta 1 \to \Omega_{\mathcal{E}}$ is T.

Furthermore, a rule of inference is *sound* if it yields valid conclusions from valid premises.

All the systems to be described will contain IL, i.e., a formal system for intuitionistic logic. We shall not specify IL further ; the reader may take his/her favorite system or consult for example, Dummett [2].

(1) The *language of IS4* is L, but without the possibility symbol $\Diamond$. The *system IS4* consists of IL together with the following axioms and rules of inference

$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

$\Box\varphi \to \varphi$

$$\Box\varphi \to \Box\Box\varphi$$
$$x = y \to \Box\,(x = y)$$
$$\frac{\varphi}{\Box\varphi}.$$

In §5 we shall present a sequent calculus for IS4.

(2) The *language of MAO* is L. The *system MAO* (with=) consists of IL together with the following axioms and rules of inference

$$\Box\varphi \to \varphi,\ \varphi \to \Diamond\varphi$$
$$\Box\varphi \to \Box\Box\varphi,\ \Diamond\Diamond\varphi \to \Diamond\varphi$$
$$\varphi \to \Box\Diamond\varphi,\ \Diamond\Box\varphi \to \varphi$$
$$\Diamond\varphi \wedge \Box\psi \to \Diamond(\varphi \wedge \Box\psi)$$
$$x = y \to \Box\,(x = y)$$
$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \qquad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}.$$

(3) The language of IBM is L. The system IBM consists of MAO together with the following axiom (MEM) $\Box\varphi \vee \neg\Box\varphi$.
An equivalent system to IBM will be presented in §5.

Remark. The system IBM is not new (at least as far as the propositional analogue is concerned). In fact, it was introduced by Bull [1] as a system of modal logic "acceptable to intuitionists". Needless to say, our motivation is different.

The considerations of §1,2 yield the following soundness theorems :

Proposition 1. *Let* $\mathcal{E} \to \mathcal{S}$ *be an arbitrary geometric morphism and let* I *be any interpretation of* L. *Then all axioms of IS4 are valid and all rules are sound.*

Proposition 2. *Let* $\mathcal{E} \to \mathcal{S}$ *be an open geometric morphism and let* I *be an intepretation of* L. *Then all axioms of MAO are valid and all rules are sound.*

Proposition 3. *Let* $\mathcal{E} \to \mathcal{S}$ *be a geometric morphism and let* $\mathcal{S}$ *be a Boolean topos. Then all axioms of IBM are valid and all rules are sound.*

§4. The modal logic of presheaves.

In this paragraph, we develop with more detail the example of presheaves over a preordered set $(\mathbb{P},\leq)$ (cfr example 2 of §1)

$$\mathbf{S} = \mathrm{Set} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathrm{Set}^{\mathbb{P}^{op}} = \mathbf{E}.$$

Let us fix an object S of $\mathbf{E}$. Consider a first-order modal language L and an interpretation of L in $\mathbf{E} \overset{\Gamma}{\rightarrow} \mathbf{S}$. We already know that the formal system IBM is valid in this interpretation (§3, proposition 2). But the fact that we interpret in the constant presheaf ΔS has another effect : if x has no free occurrence in φ, all formulas

$$\text{(CD)}\ \forall x\ (\varphi \vee \psi(x)) \rightarrow \varphi \vee \forall x \psi(x)$$

(CD for "constant domain") are also valid.

Proof : (We omit the mention of irrelevant variables).
Let $U \in P$, $U \Vdash \forall x\ (\varphi \vee \psi(x))$, $U \not\Vdash \varphi$ and prove $U \Vdash \forall x \psi(x)$; to do this, let $V \leq U$, let $a \in \Delta S(V)$ and prove $V \Vdash \psi(a)$; this is clear since $a \in \Delta S(V) = S = \Delta S(U)$ and $U \Vdash \psi(a)$.

Let us call IBM* the system formed by IBM plus the extra scheme (CD) . We have just proved that IBM* is valid for any S-interpretation in $\mathbf{S}^{\mathbb{P}^{op}}$. We want to prove a converse of this, i.e. a completeness theorem showing that for any theory Γ and sentence φ such that $\Gamma \nvdash \varphi$ in IBM*, there exists a preordered set $\mathbb{P} = (P,\leq)$, there exists a set S and an S-interpretation which is a model of Γ and does not force φ. The proof is a combination of completeness proofs by Kripke models for intuitionistic logic + CD and similar proofs for modal logic. Since the proofs involving CD are not too well-known, we give a sufficiently detailed sketch.

We assume the language L we start with is denumerable (this restriction should not come as a surprise : the proof of completeness for CD involves a kind of omitting types argument). We add to L a denumerable set C of new constants thus forming the language L + C.

Definition 1. *Let* Γ *be a set of sentences in* $L + C$.

(1) Γ is $\vdash$ *-closed if* $\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma$.

(2) Γ *is* $\vee$*-rich if* $\Gamma \vdash \varphi \vee \psi \Rightarrow \Gamma \vdash \varphi$ or $\Gamma \vdash \psi$.

(3) Γ *is* $C,\exists$*-rich if* $\Gamma \vdash \exists x\varphi(x) \Rightarrow \exists c \in C \quad \Gamma \vdash \varphi(c)$.

(4) Γ *is* $C, \forall$*-rich if* $\forall c \in C \quad \Gamma \vdash \varphi(c) \Rightarrow \Gamma \vdash \forall x\varphi(x)$.

(5) Γ *is* C*-rich if* Γ *is* $\vdash$*-closed,* $\vee$*-rich,* $C, \exists$*-rich and* $C,\forall$*-rich..*

Lemma 2. *Let* Γ *be a set of sentences of* $L + C$ *and* φ *and* ψ *be sentences of* $L + C$ *such that*

(1) $\Gamma \nvdash \varphi \rightarrow \psi$

(2) Γ *is* $C, \forall$*-rich.*

Then there exists a C*-rich* Δ *containing* Γ *such that* $\varphi \in \Delta$ *and* $\psi \notin \Delta$.

Proof : We enumerate the sentences of $L + C$ as a sequence $(\omega_n)_{n\in\omega}$ and construct a sequence $(\Delta_n,\varepsilon_n)_{n\in\omega}$ of sets Δ_n of sentences and sentences ε_n in such a way that for all $n \in \omega$

(3) $\Delta_n \subseteq \Delta_{n+1}$,

(4) Δ_n is a finite extension of Γ,

(5) ε_{n+1} is of the form $\varepsilon_n \vee \beta$ for some sentence β,

(6) $\Delta_n \nvdash \varepsilon_n$.

For $n = 0$, we let $\Delta_0 = \Gamma \cup \{\varphi\}$ and $\varepsilon_0 = \psi$.

To define Δ_{n+1} and ε_{n+1} we examine ω_n and distinguish four cases according to its form :

(Case 1) $\omega_n = \forall x\ \omega(x)$.

(Case 11) $\Delta_n \nvdash \varepsilon_n \vee \forall x\ \omega(x) \quad (7)$.

Since Δ_n is a finite extension of Γ, we may write it as $\Gamma \cup \{\alpha_1,...,\alpha_k\}$ and letting $\alpha = \bigwedge_{i=1}^{k} \alpha_i$, (7) implies successively

$$\Gamma \cup \{\alpha\} \nvdash \varepsilon_n \vee \forall x\ \psi(x),$$

$$\Gamma \nvdash \alpha \rightarrow \varepsilon_n \vee \forall x\ \psi(x),$$

$$\Gamma \nvdash \alpha \rightarrow \forall x\ (\varepsilon_n \vee \psi(x)) \qquad \text{(by CD)},$$

$$\Gamma \nvdash \forall x\ (\alpha \rightarrow \varepsilon_n \vee \psi(x))$$

and by the C, $\forall$-richness of Γ :

$$\text{for some } c_n \in C \quad \Gamma \nvdash \alpha \to \varepsilon_n \vee \psi(c_n).$$

We therefore let $\Delta_{n+1} = \Delta_n$ and $\varepsilon_{n+1} = \varepsilon_n \vee \psi(c_n)$.

(Case 12) $\Delta_n \vdash \varepsilon_n \vee \forall x\ \omega(x)$.

In this case, we let $\Delta_{n+1} = \Delta_n \cup \{\forall x\omega(x)\}$ and $\varepsilon_{n+1} = \varepsilon_n$.

(Case 2) $\omega_n = \exists x\ \omega(x)$.

(Case 21) $\Delta \cup \{\exists x\ \omega(x)\} \nvdash \varepsilon_n$ (8).

As in case 11, we may replace Δ_n by $\Gamma \cup \{\alpha\}$ and (8) implies successively :

$$\Gamma \cup \{\alpha\} \nvdash \exists x\ \omega(x) \to \varepsilon_n,$$

$$\Gamma \nvdash \alpha \wedge \exists x\ \omega(x) \to \varepsilon_n,$$

$$\Gamma \nvdash \exists x\ (\alpha \wedge \omega(x)) \to \varepsilon_n,$$

$$\Gamma \nvdash \forall x\ (\alpha \wedge \omega(x)) \to \varepsilon_n)$$

and by the C, $\forall$-richness of Γ :

$$\text{for some } c_n \in C \quad \Gamma \nvdash \alpha \wedge \omega(c_n) \to \varepsilon_n.$$

We therefore let $\Delta_{n+1} = \Delta_n \cup \{\exists x\psi(x), \psi(c_n)\}$ and $\varepsilon_{n+1} = \varepsilon_n$.

(Case 22) $\Delta_n \cup \{\exists x\omega(x)\} \vdash \varepsilon_n$.

In this case, we let $\Delta_{n+1} = \Delta_n$ and $\varepsilon_{n+1} = \varepsilon_n$.

(Case 3) $\omega_n = \gamma \vee \delta$.

(Case 31) $\Delta_n \cup \{\gamma \vee \delta\} \nvdash \varepsilon_n$.

Then clearly $\Delta_n \cup \{\gamma\} \nvdash \varepsilon_n$ or $\Delta_n \cup \{\delta\} \nvdash \varepsilon_n$.

In the first case we let $\Delta_{n+1} = \Delta_n \cup \{\beta \vee \gamma, \gamma\}$, $\varepsilon_{n+1} = \varepsilon_n$. The second case is symmetric.

(Case 32) $\Delta_n \cup \{\gamma \vee \delta\} \vdash \varepsilon_n$.

We let $\Delta_{n+1} = \Delta_n$ and $\varepsilon_{n+1} = \varepsilon_n$.

(Case 4) ω_n is not as in cases 1 to 3.

(Case 41) $\Delta_n \cup \{\omega_n\} \nvdash \varepsilon_n$.

Let $\Delta_{n+1} = \Delta_n \cup \{\omega_n\}$ and $\varepsilon_{n+1} = \varepsilon_n$.

(Case 42) $\Delta_n \cup \{\omega_n\} \nvdash \varepsilon_n$.

Let $\Delta_{n+1} = \Delta_n$ and $\varepsilon_{n+1} = \varepsilon_n$.
This achieves the construction of $(\Delta_n, \varepsilon_n)$. One easily verifies conditions (3)-(6) in each case. Arguments typical of this kind of construction will show that $\Delta = \bigcup_{n \in \omega} \Delta_n$ is the required C-rich set.

Turning to the modal aspects, we begin by giving a sample of theorems of MAO and its extensions.

Lemma 3. *The following are rules or theorems of MAO :*

(1) $\dfrac{\varphi \rightarrow \Box \psi}{\Diamond \varphi \rightarrow \psi}$ and $\dfrac{\Diamond \varphi \rightarrow \psi}{\varphi \rightarrow \Box \psi}$;

(2) $\Box T$; $\Box \varphi \wedge \Box \psi \leftrightarrow \Box(\varphi \wedge \psi)$;

(3) $\Diamond \varphi \leftrightarrow \Box \Diamond \varphi$;

(4) $\Diamond(\varphi \wedge \neg \varphi) \rightarrow \bot$;

(5) $\Diamond \varphi \wedge \Box \psi \rightarrow \Diamond(\varphi \wedge \psi)$;

(6) $\Diamond \varphi \rightarrow \neg \Box \neg \varphi$;

(7) $\Box \varphi \rightarrow \neg \Diamond \neg \varphi$.

Proof of (1) .

$$\dfrac{\dfrac{\varphi \rightarrow \Box \psi}{\Diamond \varphi \rightarrow \Diamond \Box \psi} \qquad \Diamond \Box \psi \rightarrow \psi}{\Diamond \varphi \rightarrow \psi}$$

and similarly in the other direction.

Proof of (2-4) . These are all easy consequences of (1), for example

$$\dfrac{\dfrac{\dfrac{\Box \varphi \wedge \Box \psi \rightarrow \Box \varphi}{\Diamond(\Box \varphi \wedge \Box \psi) \rightarrow \varphi} \qquad \dfrac{\Box \varphi \wedge \Box \psi \rightarrow \Box \psi}{\Diamond(\Box \varphi \wedge \Box \psi) \rightarrow \psi}}{\Diamond(\Box \varphi \wedge \Box \psi) \rightarrow \varphi \wedge \psi}}{\Box \varphi \wedge \Box \psi \rightarrow \Box(\varphi \wedge \psi).}$$

Proof of (5). Use the axiom $\Diamond \varphi \wedge \Box \psi \rightarrow \Diamond(\varphi \wedge \Box \psi)$.

Proof of (6). From (4) and (5), one derives $\Diamond\varphi \wedge \Box\neg\varphi \rightarrow \bot$, from which (6) follows.

Proof of (7). Similar to the proof of (6).

Lemma 4. *The following are theorems of IBM*

(1) $\neg\neg\Box\varphi \leftrightarrow \Box\varphi$

(2) $\Diamond\varphi \vee \neg\Diamond\varphi$

(3) $\neg\neg\Diamond\varphi \leftrightarrow \Diamond\varphi$

(4) $\neg\Diamond\varphi \rightarrow \Box\neg\varphi$

(5) $\neg\Box\neg\varphi \leftrightarrow \Diamond\varphi$

(6) $\Diamond\neg\Box\varphi \rightarrow \neg\Box\varphi$.

Proof of (1). Immediate from (MEM).

Proof of (2). Apply lemma 3(3) and (MEM).

Proof of (3). Immediate from (2).

Proof of (4). Use lemma 3(7), point (3) of the present lemma and lemma 3(1)

$$\frac{\dfrac{\varphi \rightarrow \Box\Diamond\varphi \qquad \Box\Diamond\varphi \rightarrow \neg\Diamond\neg\Diamond\varphi}{\varphi \rightarrow \neg\Diamond\neg\Diamond\varphi}}{}$$

$$\frac{\Diamond\neg\Diamond\varphi \rightarrow \neg\neg\Diamond\neg\Diamond\varphi \qquad \neg\neg\Diamond\neg\Diamond\varphi \rightarrow \neg\varphi}{\dfrac{\Diamond\neg\Diamond\varphi \rightarrow \neg\varphi}{\neg\Diamond\varphi \rightarrow \Box\neg\varphi.}}$$

Proof of (5) : One implication is given by lemma 3(6) and the other follows from the present lemma points (3) and (4).

Proof of (6) : Use points (5) and (1) of the present lemma.

Lemma 5. *The following is a theorem of* MAO :

$$\forall x\,\Box\varphi \rightarrow \Box\forall x\varphi.$$

Proof : Use lemma 3(1) as follows :

$$\frac{\dfrac{\dfrac{\forall x\, \Box\varphi \to \Box\varphi}{\Diamond\forall x\, \Box\varphi \to \varphi}}{\Diamond\forall x\, \Box\varphi \to \forall x\varphi}}{\forall x\, \Box\varphi \to \Box\forall x\varphi.}$$

Definition 6. *Let* Γ *be a set of sentences of* L + C. *The necessity kernel* $N\Gamma$ *of* Γ *is defined by* :

$$N\Gamma = \{\varphi \mid \Gamma \vdash \Box\varphi\}.$$

Lemma 7. $N\Gamma \vdash \varphi$ iff $\Gamma \vdash \Box\varphi$.

Proof : One implication is obvious. In the other direction, let $N\Gamma \vdash \varphi$; then, for some $\psi_1,\dots,\psi_k \in N\Gamma$, $\vdash \psi_1 \wedge \dots \wedge \psi_k \to \varphi$, $\vdash \Box(\psi_1 \wedge \dots \wedge \psi_k) \to \Box\varphi$, $\vdash \Box\psi_1 \wedge \dots \wedge \Box\psi_k \to \Box\varphi$ by lemma 3(2), and $\Gamma \vdash \Box\varphi$ since $\Gamma \vdash \Box\psi_1,\dots,\Gamma \vdash \Box\psi_k$.

We introduce now a canonical "accessibility relation" R between C-rich sets.

Definition 8. *Let* Γ *and* Δ *be* C*-rich sets.*
Define R *by* :

$$\Gamma R \Delta \Leftrightarrow N\Delta \subseteq \Gamma.$$

Proposition 9. $\Gamma R \Delta \Leftrightarrow$ *for all sentences* φ *of* L + C

$$(\varphi \in \Gamma \Rightarrow \Diamond\varphi \in \Delta).$$

Proof : (From left to right). Let $\Gamma R \Delta$, let $\varphi \in \Gamma$ and assume $\Diamond\varphi \notin \Delta$; then $\neg\Diamond\varphi \in \Delta$ by lemma 4(2), $\Box\neg\varphi \in \Delta$ by lemma 4(4), $\neg\varphi \in \Gamma$ by the condition $\Gamma R \Delta$ and Γ would be inconsistent. (From right to left). Assume the condition on the right, let $\Box\varphi \in \Delta$ and prove $\varphi \in \Gamma$.

In fact $\Box\varphi \in \Gamma$ for otherwise $\neg\Box\varphi \in \Gamma$ by (MEM), $\Diamond\neg\Box\varphi \in \Delta$ by the condition, $\neg\Box\varphi \in \Delta$ by lemma 4(6) and Δ would be inconsistent.

Proposition 10. R *is an equivalence relation.*

Proof. Reflexivity clearly follows from the scheme $\Box\varphi \to \varphi$, transitivity from $\Box\varphi \to \Box\Box\varphi$ and symmetry from $\Diamond\Box\varphi \to \varphi$ (or $\varphi \to \Box\Diamond\varphi$) by proposition 9.

Proposition 11 *If* Γ *is* C, $\forall$*-rich then* $N\Gamma$ *is* C, $\forall$*-rich..*

Proof : Let $N\Gamma \vdash \Box\varphi(c)$ for all $c \in C$. Then for all $c \in C$, $\Gamma \vdash \Box\varphi(c)$ by lemma 7, $\Gamma \vdash \forall x \Box\varphi(x)$ by the C, $\forall$-richness of Γ, $\Gamma \vdash \Box\forall x\varphi(x)$ by lemma 5 and $\forall x\varphi(x) \in N\Gamma$by definition of $N\Gamma$.

We are now ready to prove the completeness theorem. Let Γ and φ in L be such that $\Gamma \nvdash \varphi$. Add C to L as described before. Clearly Γ is C, $\forall$-rich : if $\Gamma \vdash \psi(c)$ for all $c \in C$, then $\Gamma \vdash \forall x\, \psi(x)$ by the theorem on constants. We can therefore apply lemma 2 to the data $\Gamma \nvdash T \to \varphi$ and Γ C, $\forall$-rich to find a C-rich Γ_0 containing Γ and such that $T \in \Gamma_0$ (a trivial information) and $\varphi \notin \Gamma_0$. We therefore define our model by taking

$$P = \{\Delta | \Delta \text{ C-rich and } \Gamma_0 R \Delta\}$$

$$\Delta \leq \Delta' \text{ iff } \Delta' \subseteq \Delta$$

and

$$S = \{t/\sim | \text{ t closed term of L + C}\}$$

where ~ is defined by

$$t \sim t' \text{ iff } t = t' \in N\Gamma_0$$

(~ is clearly an equivalence relation). Of course we interpret $f \in F_n$ by $|f|$ defined at level Δ by

$$|f|^\Delta (t_1/\sim,\ldots,t_n/\sim) = ft_1\ldots t_n/\sim$$

and $r \in R_n$ by $|r|$ defined at level Δ by

$$|r|^\Delta = \{< t_1/\sim,\ldots t_n/\sim > | rt_1\ldots t_n \in \Delta\}.$$

Note that it is not necessary to define P by the set of *consistent* C-rich sets Δ such that $\Gamma_0 R\Delta$: Γ_0 is consistent and in general Δ_1 consistent and $N\Delta_2 \subseteq \Delta_1$ together imply that Δ_2 is consistent.

The fundamental lemma of completeness proofs by canonical models holds :

Lemma 12. *For every* $\Delta \in P$, *closed term* t *and closed formula* φ *in* L + C

(1) $|t|^{\Delta} = t/\sim$,

(2) $\Delta \Vdash \varphi$ iff $\varphi \in \Delta$.

Proof of (1). By induction on the form of t.

Proof of (2). By induction on the form of φ. The atomic case is easy, but it should be remarked that the axiom $x = y \rightarrow \Box(x = y)$ is used for equalities :

$$t_1 = t_2$$

iff $|t_1|^{\Delta} = |t_2|^{\Delta}$

iff $t_1/\sim \; = t_2/\sim$ (by point (1))

iff $t_1 = t_2 \in N\Gamma_0$

iff $t_1 = t_2 \in \Delta$,

this last equivalence because, $t_1 = t_2 \in \Delta$ implies $\Box(t_1 = t_2) \in \Delta$ by the axiom, hence $t_1 = t_2 \in N\Delta$ and $t_1 = t_2 \in N\Gamma_0$ ($N\Delta = N\Gamma_0$ by proposition 9).

For the connectors $\neg, \wedge, \vee, \rightarrow, \bot, \forall, \exists$, the proof runs as usual, using the C-richness ; note however that for implication, lemma 2 is used in its full strength to prove

$$\varphi \rightarrow \psi \notin \Delta \Rightarrow \exists \Delta' \leq \Delta \; (\varphi \in \Delta' \text{ and } \psi \notin \Delta').$$

Let us check the case of $\Box$ (we omit the mention of variables). If $\Box\varphi \in \Delta$ and $\Delta' \in P$ then $\varphi \in N\Delta = N\Gamma_0 \subseteq \Delta'$; $\varphi \in \Delta'$ and $\Delta' \Vdash \varphi$ by the induction hypothesis ; consequently $\Box\varphi \in \Delta$ implies $\Delta \Vdash \Box\varphi$. In the other direction, assume $\Box\varphi \notin \Delta$; then $\varphi \notin N\Delta$; but $N\Delta$ is C, $\forall$-rich by proposition 11 ; consequently, by lemma 2, $N\Delta$ is included in some C-rich Δ' which does not contain φ ; hence by the induction hypothesis, $\Delta' \nVdash \varphi$, which proves that $\Delta \not\Vdash \Box\varphi$. The case of $\Diamond$ may be reduced to the case of $\Box$ via the use of lemma 4(2).

Returning to the original Γ and φ such that $\Gamma \nvdash \varphi$, it remains to observe that the canonical model we constructed forces every sentence ψ of Γ at level Γ_0 :

$$\begin{aligned} \psi \in \Gamma &\Rightarrow \psi \in \Gamma_0 \\ &\Rightarrow \Gamma_0 \Vdash \psi \text{ (by lemma 12)} \end{aligned}$$

and does not force φ, since $\varphi \notin \Gamma_0$ and $\Gamma_0 \not\Vdash \varphi$, again by lemma 12.

We have thus proved :

Theorem 13. *Let* $\mathbb{L}$ *be a denumerable first-order modal language. Let* Γ *be a set of sentences of* $\mathbb{L}$ *and* φ *be a sentence of* $\mathbb{L}$ *such that, in IBM**, $\Gamma \nvdash \varphi$. *Then there exists an ordered set* $\mathbb{P} = (P,\leq)$, *there exists a set* S *and an* S*-interpretation of* $\mathbb{L}$ *in* Sets $^{\mathbb{P}^{op}}$ *which is a countermodel to* $\Gamma \vdash \varphi$.

<u>Additional completeness theorems</u> (1) Let us add to IBM* the scheme $\Diamond\varphi\wedge\Diamond\psi \rightarrow \Diamond(\varphi\wedge\psi)$ and consider the semantic condition : $\forall U, V \in P$, $\exists W \leq U, W \leq V$. The soundness theorem is clear. To prove the completeness, prove that the canonical model satisfies the semantic condition ; it suffices to show that for Δ, Δ' in the canonical P, $\Delta\cup\Delta'$ is consistent ; this is so, because $\Delta\cup\Delta'$ inconsistent implies successively :

$$\begin{aligned} &\vdash \delta \wedge \delta' \rightarrow \bot \text{ for some } \delta \in \Delta \text{ and } \delta' \in \Delta', \\ &\vdash \Diamond(\delta\wedge\delta') \rightarrow \Diamond\bot, \\ &\vdash \Diamond(\delta\wedge\delta') \rightarrow \bot \quad \text{by lemma 2(3)}, \\ &\vdash \Diamond\delta \wedge \Diamond\delta' \rightarrow \bot \text{ by the scheme considered}, \\ &(\Diamond\delta \wedge \Diamond\delta' \rightarrow \bot) \in \Delta, \\ &\bot \in \Delta, \end{aligned}$$

a contradiction ; the last step is justified by the fact that $\delta \in \Delta$, $\delta' \in \Delta'$ and $N\Delta = N\Delta'$ all together imply $\Diamond\delta \in \Delta$ and $\Diamond\delta' \in \Delta$.

(2) Let us add to IBM* the scheme $\Box(\varphi \vee \psi) \wedge \neg\Box\varphi \rightarrow \Box\psi$ and consider the semantic condition : $\exists V \in P\ \forall W \in P, W \leq V$. Again, soundness is

clear. To prove the completeness, prove that the common $N\Gamma_0$ is in fact C-rich, hence belongs to the canonical P and is the greatest element sought for.

(3) One falls back on classical S5 by adding to IBM* the scheme $\varphi \vee \neg\varphi$. The canonical model is then the traditional one : an equivalence class of maximal consistent C-rich sets. This is in fact "the" first-order modal logic of the example Set^I.

(4) Adding $\varphi \to \Box\varphi$ to IBM* gives classical logic with the trivial necessity operator. Since $N\Gamma = \Gamma$ in that case the canonical model is the traditional one : a unique maximal consistent C-rich set.

(5) It remains for us an open problem whether one can characterize the semantic condition of connection : $\forall U, V \in P, \exists n\, \exists U_0,\ldots,U_n \in P$ $(U = U_0 \leq U_1 \geq U_2 \leq U_3 \geq \ldots \leq U_{n-1} \geq U_n = V)$. The axiom scheme $\Box(\varphi \vee \psi) \wedge \Diamond\varphi \wedge \Diamond\psi \to \Diamond(\varphi \wedge \psi)$ is clearly satisfied in constant presheaves over such a P, but we have no completeness proof for that scheme.

§5. A calculus of sequents for IS4.

Although our main objective is a completeness proof for IBM, the system IS4 is a naturel step to consider and has interest in itself.

We first note that IBM may be obtained from IS4 by adding the axiom scheme

$$\text{(SMEM)}\ \Box\varphi \vee \Box\neg\Box\varphi$$

(SMEM for "strong modal excluded middle") and the definition $\Diamond\varphi \equiv \neg\Box\neg\varphi$.

Proposition 1. *The system* IBM *is equivalent to* IS4 + (SMEM).

Proof : IBM proves (SMEM) because $\neg\Box\neg\Box\varphi \leftrightarrow \Diamond\Box\varphi$ and $\Diamond\Box\varphi \to \Box\varphi$ are theorems of IBM and (SMEM) follows by (MEM). It is also easy to show that IS4 + (SMEM) proves MAO. Let us show for example the adjunction $\Diamond\Box\varphi \to \varphi$:

$$\Diamond\Box\varphi \leftrightarrow \neg\Box\neg\Box\varphi$$
$$\leftrightarrow \neg\Box\neg\Box\varphi \wedge (\Box\varphi \vee \Box\neg\Box\varphi)$$
$$\leftrightarrow \neg\Box\neg\Box\varphi \wedge \Box\varphi$$

which implies $\Box\varphi$, hence φ.

We now present a system of sequents, to be denoted by $IS4_{Sq}$, which has the same strength as IS4. A sequent is here an ordered pair $\Gamma : \varphi$ where Γ is a finite set of sentences and φ is either the empty set or a sentence. We consider a system of sequents IL_{Sq} for intuitionistic logic and add two rules for the modal operator of necessity. To be specific, we adopt Dummett's system ([2], p. 133) and add :

$$(\Box :)\ \frac{\Gamma,\alpha:\gamma}{\Gamma,\Box\alpha:\gamma} \qquad (:\Box)\ \frac{\Gamma:\alpha}{\Gamma:\Box\alpha}$$

with the restriction on $(:\Box)$ that all sentences of Γ are *necessary*, i.e. of the form $\Box\xi$. To simplify the exposition, we will also neglect equality and function symbols, for which the reader may easily make the adaptations.

Proposition 2. *The calculus of sequents* $IS4_{Sq}$ *has the cut elimination property.*

Proof : The proof is similar to the proof of cut elimination for IL_{Sq} : it suffices to examine when necessary the two new rules. Let $IS4^{+}_{Sq}$ be the system $IS4_{Sq}$ augmented with the cut rule :

$$(\text{Cut})\ \frac{\Gamma:\alpha \quad \Delta,\alpha:\gamma}{\Gamma,\Delta:\gamma}$$

One considers an initial use of (Cut), i.e. a proof $\mathfrak{D}$ of the form

$$(\text{Cut})\ \frac{\begin{matrix}\Downarrow \mathfrak{D}_1 \\ \Gamma_1 : \alpha\end{matrix} \qquad \begin{matrix}\Downarrow \mathfrak{D}_2 \\ \Gamma_2,\alpha : \gamma\end{matrix}}{\Gamma_1,\Gamma_2 : \gamma}$$

where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ do not use (Cut). It suffices to eliminate this use of (Cut) or to import it in $\mathfrak{D}_1$ or $\mathfrak{D}_2$, or to diminish the complexity of α in such a way that a finite induction will give the desired result. We concentrate only on the modal steps of the procedure.

(A) Suppose first that $\mathfrak{D}_1$ does not end with the introduction of α.
(A1) If $\Gamma : \alpha$ is axiomatic, (Cut) may be trivially eliminated.
(A2) If $\mathfrak{D}_1$ ends with the introduction on the left of a sentence β in Γ_1, one examines the different possibilities. The case of the intuitionistic rules is well-known and the case of ($\Box$:) is easy : one transforms

$$\dfrac{\dfrac{\Gamma'_1,\beta_1 : \alpha}{\Gamma'_1,\Box\beta_1 : \alpha}(\Box:) \quad \Gamma_2,\alpha : \gamma}{\Gamma'_1,\Box\beta_1,\Gamma_2 : \gamma}(\text{Cut})$$

into

$$\dfrac{\dfrac{\Gamma'_1,\beta_1 : \alpha \qquad \Gamma_2,\alpha : \gamma}{\Gamma'_1,\beta_1,\Gamma_2 : \gamma}(\text{Cut})}{\Gamma'_1,\Box\beta_1,\Gamma_2 : \gamma}(\Box:)$$

and (Cut) has now been imported in $\mathfrak{D}_1$. (In the sequel, we refer to this straightforward transformation as a commutation of ($\Box$:) and (Cut)).
(B) Suppose now that $\mathfrak{D}_1$ ends with the introduction of α.
(B_1) If α is introduced by (Thin), one trivially substitutes to the proof a proof which uses (Thin) only.
(B_2) If α has not been introduced by (Thin), one examines $\mathfrak{D}_2$. Three cases are possible.
(B21) If $\mathfrak{D}_2$ ends with an introduction in Γ_2, one proceeds as in (A) to import (Cut) in $\mathfrak{D}_2$.
(B22) If $\mathfrak{D}_2$ ends with an introduction of γ on the right, the only non-intuitionistic case is when γ is of the form $\Box\gamma_1$ and is therefore introduced by (:$\Box$). The proof has the following shape :

$$\dfrac{\begin{array}{c}\Psi\,\mathfrak{D}_1\\ \Gamma_1 : \alpha\end{array} \qquad (:\Box)\,\dfrac{\begin{array}{c}\Psi\\ \Gamma_2,\alpha : \gamma_1\end{array}}{\Gamma_2,\alpha : \Box\gamma_1}}{\Gamma_1,\Gamma_2 : \Box\gamma_1}$$

and the restriction on $(:\Box)$ shows that α and the sentences of Γ_2 are necessary. This means that the last step of $\mathfrak{D}_1$ is the introduction of $\alpha \equiv \Box\alpha_1$ by

$$(:\Box)\,\dfrac{\Gamma_1 : \alpha_1}{\Gamma_1 : \Box\alpha_1}$$

and that the sentences of Γ_1 are necessary. We can therefore legitimately transform the given proof into

$$(:\Box)\,\dfrac{(\text{Cut})\,\dfrac{\Gamma_1 : \Box\alpha_1 \quad \Gamma_2\,,\Box\alpha_1 : \gamma_1}{\Gamma_1,\Gamma_2 : \gamma_1}}{\Gamma_1,\Gamma_2 : \Box\gamma_1\,,}$$

where the use of (Cut) has been imported in $\mathfrak{D}_2$.

(B23) It remains to examine the case where $\mathfrak{D}_2$ (as $\mathfrak{D}_1$) ends with the introduction of α. In this case, the induction is on the complexity of the sentence α. Leaving aside the intuitionistic cases, we consider the casewhere α is $\Box\alpha_1$. The given proof ends with

$$(\text{Cut})\,\dfrac{(:\Box)\,\dfrac{\Gamma_1 : \alpha_1}{\Gamma_1 : \Box\alpha_1} \qquad (\Box:)\,\dfrac{\Gamma_2,\alpha_1 : \gamma}{\Gamma_2,\Box\alpha_1 : \gamma}}{\Gamma_1,\Gamma_2 : \gamma}$$

and this is transformed into

$$\frac{\Gamma_1 : \alpha_1 \qquad \Gamma_2, \alpha_1 : \gamma}{\Gamma_1, \Gamma_2 : \gamma}$$

This finishes the proof of proposition 2.

The cut elimination property is the only non trivial verification leading to the following corollary :

Corollary 3. *The sequent* $\Gamma : \alpha$ *is provable in* $IS4_{Sq}$ *iff*

$$IS4 \vdash \wedge\Gamma \rightarrow \alpha.$$

To prove the completeness theorem for IL and Beth trees, M. Dummett considers (loc. cit. p. 229) a variant of IL_{Sq} admitting sequents of the form $\Gamma : \Delta$ where Δ (as Γ) is a finite set of sentences. We denote by IL_{MSq} (for 'multiple conclusion sequent') Dummett's system and by $IS4_{MSq}$ the same system together with the rules

$$(:\Box)\ \frac{\Gamma, \Box\alpha, \alpha : \Delta}{\Gamma, \Box\alpha : \Delta} \qquad (\Box:)\ \frac{N\Gamma : \alpha}{\Gamma : \Box\alpha, \Delta}$$

where $N\Gamma = \{\Box\xi \mid \Box\xi \in \Gamma\}$.

An essential feature of $IS4_{MSq}$ is that it has no explicit rule of thinning. To show the equivalence of $IS4_{Sq}$ and $IS4_{Msq}$, it is therefore necessary to prove first the elimination of the rules of thinning. Let $IS4^{+}_{MSq}$ be the system $IS4_{MSq}$ together with

$$(\text{Thin:})\ \frac{\Gamma : \Delta}{\Gamma, \alpha : \Delta} \qquad (\text{:Thin})\ \frac{\Gamma : \Delta}{\Gamma : \alpha, \Delta}.$$

Proposition 4. *The systems* $IS4_{MSq}$ *and* $IS4^{+}_{MSq}$ *are equivalent.*

Proof : Let us consider an initial use of one of the rules of thinning and let $\mathfrak{D}$ be the proof which is strictly above that occurrence ($\mathfrak{D}$ is therefore in $IS4_{MSq}$). If $\mathfrak{D}$ is an axiomatic sequent, thinning is useless because the consequence is again axiomatic. In the other cases, we import the use of thinning in $\mathfrak{D}$. Suppose, for example that $\mathfrak{D}$ ends with $(:\Box)$:

$$\dfrac{\dfrac{N\Gamma : \alpha}{\Gamma : \Box\alpha, \Delta}\ (:\Box)}{\Gamma, \xi : \Box\alpha, \Delta.}\ (\text{Thin:})$$

If ξ is a necessary sentence, we can commute $(:\Box)$ and (Thin:). If ξ is not necessary, then $N(\Gamma,\xi) = N(\Gamma)$ and the proof may be transformed into a proof without thinning :

$$(:\Box)\ \dfrac{N\Gamma \equiv N(\Gamma,\xi) : \alpha}{\Gamma,\xi : \alpha,\Delta}\ .$$

The preceding proposition makes it easy to establish

Corollary 5. *The systems* $IS4_{Sq}$ *and* $IS4_{MSq}$ *are equivalent.*

§6. Beth models for IS4 and IBM.

6.1. Description of the semantics.

Let F be a *forest*. By 'forest', we mean here a countable set of countable trees. We also suppose that trees have a root, viewed as a maximum. The nodes of a tree are indexed by finite sequences of natural numbers, the ordering being the opposite of the initial inclusion of sequences, the root corresponding to the empty sequence.

Let A be a tree and i a node of A. A set of nodes of A *bars* i (or is a *bar for* i) if every path of A containing i also contains a node of S. A set K of nodes is *saturated* if with a node it also contains all smaller nodes. It is easy to see that the bars for i which are saturated form the covering sieves of a Grothendieck topology on A. Since all these covers are inhabited, A becomes an open site and we call that particular topology the *Beth topology* on A. The Beth topology on a forest is given by the Beth topology on every tree of the forest ; we then speak of a *Beth forest.*

The semantics which we want to describe reduces in the case of IBM (at least for the canonical model constructed in the completeness proof) to the standard topos-theoretic semantics of example 3 but contains in the case of IS4 a new ingredient (grafts) which is not present in that standard topos-theoretic semantics.

To cover both cases and show the connections between them and the presentation of [2], we repeat with some precision the elementary concepts of interpretation and forcing (or satisfaction).

Let F be a Beth forest and E be a non empty set. An *interpretation* $\mathcal{M}$ over (F,E) associates with every n-ary relation symbol r and element $\underline{c} \in E^n$ a set $N(r,\underline{c})$ of nodes of F in such a way that the following two clauses of functoriality and localization be satisfied :

(1) if $j \leq i$ and $i \in N(r,\underline{c})$, then $j \in N(r,\underline{c})$;

(2) if S is a sieve covering i and $S \subseteq N(r,\underline{c})$, then $i \in N(r,\underline{c})$.

If $i \in N(r,\underline{c})$, we say that $r(\underline{c})$ is forced or satisfied in i and write $\mathcal{M} \Vdash_i r(\underline{x})\ [\underline{c}]$ or more simply $\mathcal{M} \Vdash_i r(\underline{c})$. This notion of forcing is extended to all formulas in the usual sheaf-theoretic manner, e.g. :

(a) $\mathcal{M} \Vdash_i \varphi \vee \psi\ [\underline{c}]$ iff there exists a sieve S covering i such that for every $j \in S$, $\mathcal{M} \Vdash_j \varphi\ [\underline{c}]$ or $\mathcal{M} \Vdash_j \psi\ [\underline{c}]$;

(b) $\mathcal{M} \Vdash_i \neg\varphi\ [\underline{c}]$ iff for every $j \leq i$, $\mathcal{M} \not\Vdash_j \varphi\ [\underline{c}]$;

(c) $\mathcal{M} \Vdash_i \exists x\ \varphi(x,\underline{y})\ [\underline{d}]$ iff there exists a sieve S covering i and, for every $j \in S$, an element $c_j \in E$ such that $\mathcal{M} \Vdash_j \varphi\ (x,\underline{y})\ [c_j,\underline{d}]$.

To connect this with the introduction, we consider the sheaf ΔE associated to the constant presheaf $\Delta_0 E$ defined by $\Delta_0 E(i) = E$ for all $i \in$

F. Given the conditions on r, it is clear that r defines a subsheaf Δr of ΔE in such a way that

$$(\ell_E)_i(\underline{c}) \in \Delta r(i) \text{ iff } \quad i \in N(r,\underline{c}).$$

In other words, by the usual sheaf-theoretic definition of forcing,

$$i \Vdash r(\underline{x})\,[(\ell_E)_i(\underline{c})] \quad \text{iff} \quad \mathcal{M} \underset{i}{\Vdash} r(\underline{x})\,[\underline{c}].$$

Clearly, that is all what is needed to prove by induction on non-modal φ :

$$i \Vdash \varphi(\underline{x})\,[(\ell_E)_i(\underline{c})] \quad \text{iff} \quad \mathcal{M} \underset{i}{\Vdash} r(\underline{x})\,[\underline{c}].$$

Turning now to the modal aspects, we distinguish two semantics for the modal operators, one for IBM, the other for IS4.

A. <u>The case of IBM</u>. It is a consequence of the first paragraphs that since we are dealing with an open site,

$$i \Vdash \Box\varphi(\underline{x})\,[(\ell_E)_i(\underline{c})] \text{ iff } \forall j \in F,\, j \Vdash \varphi(\underline{x})\,[(\ell_E)_j(\underline{c})].$$

Preferring here the simpler notation of [2], we could as well complete (a), (b), (c) and the like by :

$$\mathcal{M} \underset{i}{\Vdash} \Box\varphi(x)[\underline{c}] \text{ iff } \quad \forall j \in F, \mathcal{M} \underset{j}{\Vdash} \varphi(x)\,[\underline{c}].$$

B. <u>The case of IS4.</u> To give the semantics to IS4, we introduce <u>grafts</u>. Let F be a forest and $(A_s)_{s\in\omega}$ be an enumeration of its trees. We often designate a node of A_s by (s,i) and speak of the tree s instead of A_s. Let us be given a mapping associating to each $s > 0$ a node (s_1,j) with $s_1 < s$. We say that the tree s has been *grafted* in the node (s_1,j) and write $(s,\phi) \prec (s_1,j)$ (where (s,ϕ) is the root of the tree s). This relation allows us to associate to the forest F a new tree, by taking the ordering generated by $\prec$ and the orderings of the trees of F. We still denote by $\prec$ this new ordering. By <u>graft semantics</u>, we understand the use of $\prec$ to interpret necessity :

$(s,i) \Vdash \Box\varphi\ [(\mathcal{L}_E)_{(s,i)}\ (\underline{c})]$ iff there exists a sieve S covering i in A_s such that for every $(s,j) \in S$ and every $(s',i') \prec (s,j)$, $(s',i') \Vdash \varphi\ [(\mathcal{L}_E)_{(s',i')}\ (\underline{c})]$. We could as well complete (a), (b), (c) and the like by :

$\mathcal{M} \Vdash_{(s,j)} \Box\varphi\ [\underline{c}]$ iff there exists a sieve S covering i in A_s such that for every $(s,j) \in S$ and every $(s',i') \prec (s,j)$, $\mathcal{M} \Vdash_{(s',i')} \varphi\ [\underline{c}]$.

In both cases (IBM and IS4), these definitions are extended to a general $\xi \in \Delta E(i)$ as is explained in the first paragraphs. We will henceforth use the notation $\mathcal{M} \Vdash_{(s,i)} \varphi\ [\underline{c}]$ or $\mathcal{M} \Vdash_{i} \varphi[\underline{c}]$ which the topos-oriented reader my consider as a shorthand for the more precise $(s,i) \Vdash \varphi\ [(\mathcal{L}_E)_{(s,i)}\ (\underline{c})]$ and for which he may extend the result to the general case $\xi \in \Delta E((s,i))$.

In both cases (IBM and IS4), we verify by induction the functorial and local character of forcing :

Proposition 1 *For every formula* $\varphi(x)$ *and every* $\underline{c}$:

(1) *if* $\mathcal{M} \Vdash_{i} \varphi\ [\underline{c}]$ *and* $j \leq i$, *then* $\mathcal{M} \Vdash_{j} \varphi\ [\underline{c}]$;

(2) *if there exists a sieve* S *covering* i *such that for every* $j \in S$, $\mathcal{M} \Vdash_{j} \varphi\ [\underline{c}]$, *then* $\mathcal{M} \Vdash_{i} \varphi\ [\underline{c}]$.

We also verify that the semantics is sound :

Proposition 2. IBM *is sound for the topos-theoretic semantics on a Beth forest and* IS4 *is sound for the graft semantics on a Beth forest with grafts.*

Proof : For the intuitionistic part, one has the usual computations. For the modal part, let us check for example : (A) the validity of $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ for Beth forests with grafts and (B) the validity of (SMEM) for Beth forests.

(A) Assume $\mathcal{M} \Vdash_{(s,i)} \Box(\varphi \to \psi)$ and $\mathcal{M} \Vdash_{(s,i)} \Box\varphi$.

Then there exists a sieve S_1 covering i in s such that for every $(s,j_1) \in S_1$ and $(s',i') \prec (s,j_1)$, $\mathcal{M} \underset{(s',i')}{\Vdash} \varphi \to \psi$. Similarly, there exists a sieve S_2 covering i in s such that for every $(s',j_2) \in S_2$ and $(s',i') \prec (s,j_2)$, $\mathcal{M} \underset{(s',i')}{\Vdash} \varphi$. Clearly, $S = S_1 \cap S_2$ will show that $\mathcal{M} \underset{(s,i)}{\Vdash} \Box\psi$.

(B) We want to show that $\mathcal{M} \underset{(s,i)}{\Vdash} \Box\varphi \vee \Box\neg\Box\varphi$. As sieve covering i in s, we consider the total sieve $\{j \in A_s | j \leq i\}$. It suffices to show that $\mathcal{M} \underset{(s,i)}{\Vdash} \Box\varphi$ or $\mathcal{M} \underset{(s,i)}{\not\Vdash} \Box\neg\Box\varphi$. Suppose $\mathcal{M} \underset{(s,i)}{\not\Vdash} \Box\varphi$ and show that for every $(r,k) \in M$, $\mathcal{M} \underset{(r,k)}{\Vdash} \neg\Box\varphi$. It suffices to take $k' \leq k$ in A_r and to show that $\mathcal{M} \underset{(r,k')}{\not\Vdash} \Box\varphi$. But this is immediate since if $\mathcal{M} \underset{(r,k')}{\Vdash} \Box\varphi$, then $\mathcal{M} \underset{(s,i)}{\Vdash} \Box\varphi$.

6.2. Planting a Beth forest with grafts.

In completeness proofs, one often begins by enriching the language with new constants. Limiting ourselves to the denumerable case, we suppose that this has been done and we denote by L this enriched language which has thus a denumerable number of constants. Let $(\Gamma : \Delta)$ be a sequent of L. We associate to it a Beth forest with grafts.

Let $\mathcal{F}$ be the (denumerable) set of sentences of L and $\mathcal{F}_\omega$ be the disjoint union of a denumerable set of copies of $\mathcal{F}$. Let also $(\alpha_k)_{k \in \omega}$ be an enumeration of $\mathcal{F}_\omega$. We construct by induction on k finite Beth forest (with grafts) F_k and in each node (s,i) of F_k attach a sequent $(\Gamma_{(s,i),k} : \Delta_{(s,i),k})$.

For k = 0, F_0 has only one node (say $(0,\phi)$) and the sequent attached to that node is the initial $(\Gamma : \Delta)$.

For k > 0, the induction hypothesis is that we have a finite Beth forest (with grafts) F_k and its attached sequents. Consider the (k+1)-th sentence α of the enumeration of $\mathcal{F}_\omega$. For each node (s,i) of F_k, we make the following construction. If $(\Gamma_{(s,i),k} : \Delta_{(s,i),k})$ is axiomatic or does not contain α, we leave it unmodified. Otherwise, we examine the form of α. If α is not necessary, we proceed as in [2], pp. 233-235. We simply recall the case $\alpha \equiv (\beta \rightarrow \gamma)$ to fix our terminology :

(Case i) $(\beta \rightarrow \gamma)$ is in $\Gamma_{(s,i),k}$; if β is in $\Delta_{(s,i),k}$ or γ in $\Gamma_{(s,i),k}$, we leave the sequent unmodified ; otherwise, we <u>choose</u> to add β to $\Delta_{(s,i),k}$ or γ to $\Gamma_{(s,i),k}$.

(Case ii) $(\beta \rightarrow \gamma)$ is in $\Delta_{(s,i),k}$; if there exists in A^k_s (the tree with index s already constructed in F_k) a node $i' \leq i$ with $\beta \in \Gamma_{(s,i'),k}$ and $\gamma \in \Delta_{(s,i'),k}$, we leave the sequent unmodified ; otherwise, we add new nodes to the tree A^k_s as described below :

(a) We choose a new i' below i and on the right in the sense that if i is the sequence of natural numbers $(n_1,...,n_r)$, then i' is a sequence $(n_1,...,n_r,n')$ with n' strictly greater than any n such that $(n_1,...,n_r,n)$ is already in A^k_s. The new attached sequents are

- in (s,i') : $(\Gamma_{(s,i'),k+1} : \Delta_{(s,i'),k+1}) \equiv (\Gamma_{(s,i),k} \cup \{\beta\} : \{\gamma\})$,
- in (s,i) : $(\Gamma_{(s,i),k+1} : \Delta_{(s,i),k+1}) \equiv (\Gamma_{(s,i),k} : \Delta_{(s,i),k+1})$.

(b) Moreover, if i is a minimal node in A^k_s, we also introduce a new node i'' below i but 'on the left' in the sense that if $i = (n_1,...,n_r)$ then $i'' = (n_1,...,n_r,0)$. The new sequents are then given by :

$$i(\Gamma_{(s,i),k+1} : \Delta_{(s,i),k+1}) \equiv (\Gamma_{(s,i),k} : \Delta_{(s,i),k})$$

i'' $\qquad\qquad$ i'

$(\Gamma_{(s,i''),k+1} : \Delta_{(s,i''),k+1})$ $\qquad$ $(\Gamma_{(s,i'),k+1} : \Delta_{(s,i'),k+1})$
$\equiv (\Gamma_{(s,i),k} : \Delta_{(s,i),k})$ $\qquad$ $\equiv (\Gamma_{(s,i),k} \cup \{\beta\} : \{\gamma\})$.

Considering the modal case $\alpha \equiv \Box\beta$, we proceed as follows :

(Case i) $\Box\beta$ is in $\Gamma_{(s,i),k}$; in this case, we let $\Gamma_{(s,i),k+1} \equiv \Gamma_{(s,i),k} \cup \{\beta\}$ and leave $\Delta_{(s,i),k}$ unmodified.

(Case ii) $\Box\beta$ is in $\Delta_{(s,i),k}$; if there exists (s',i') in F_k such that $(s',i') \prec (s,i)$ and $\beta \in \Delta_{(s',i'),k}$, we leave the sequent unmodified ; otherwise, we add a new tree with a new index s' and graft it on the node (s,i) : $(s',\phi) \prec (s,i)$; in this case the sequent in (s,i) is left unmodified and the sequent attached to (s',ϕ) is

$$(\Gamma_{(s',\phi),k+1} : \Delta_{(s',\phi'),k+1}) \equiv (N\Gamma_{(s,i),k} : \{\beta\}).$$

This finishes the construction of F_{k+1}. We define F to be the union of the F_k's.

6.3. Completeness of IS4.

We extend here to IS4 the completeness proof for IL by Beth trees given in [2].

Let (F_k) be a sequence of finite forests with grafts having in each node an attached sequent, as has been constructed from a sequent $(\Gamma : \Delta)$ in the preceding section. We say that (F_k) is a *refutation* of $(\Gamma : \Delta)$ if no sequent is axiomatic in any node.

It is easy to adapt [2] to establish the following result :

Proposition 3. *If* $(\Gamma : \Delta)$ *has no proof of length* $\leq r$, *the sequence* (F_k) *may be chosen in such a way that no sequent in any node for* F_r *be axiomatic.*

The essential point in the proof of proposition 3 is that the construction of the preceding section inverts one of the rules of deduction at each step and in each new node. Considering the case of modal rules, we observe that if a new tree has been grafted, this is due to the inversion of rule $(:\Box)$. The idea is then to construct (F_k) in such a way that for each k, $0 \leq k \leq r$, and for every node (s,i) of F there is no proof of $\Gamma_{(s,i),k} : \Delta_{(s,i),k}$ of length r - k. To do this, the modal rules do not complicate matters, since the only choices in the construction of F appear in three cases :

a) α_{k+1} is $(\beta \wedge \gamma)$ and is in $\Delta_{(s,i),k}$; in this case, one has to add β or γ to $\Delta_{(s,i),k}$;

b) α_{k+1} is $(\beta \vee \gamma)$ and is in $\Gamma_{(s,i),k}$; in this case, one has to add β or γ to $\Gamma_{(s,i),k}$;

c) α_{k+1} is $(\beta \rightarrow \gamma)$ and is in $\Gamma_{(s,i),k}$; in this case, one has to add β to $\Delta_{(s,i),k}$ or γ to $\Gamma_{(s,i),k}$.

None of these cases is directly modal.

König's lemma applied to proposition 3 gives the main corollary :

Corollary 4. *Every sequent* $(\Gamma : \Delta)$ *has a proof or a refutation.*

Let $(\Gamma : \Delta)$ be a non-provable sequent. According to corollary 4, it has a refutation $(F_k)_{k\in\omega}$. We associate to this refutation an interpretation $\mathcal{M}$ over (F,E), where the forest F is the union of the F_k's and E is the set of closed terms of L (divided by the equivalence induced by equality as in § 4 lemma 12 if we want to consider it) and decide for each node of each tree s of F which atomic sentences are satisfied.

Let i be a node of the tree s and let $\Gamma_{(s,i)} : \Delta_{(s,i)}$ be the infinite sequent defined by $\Gamma_{(s,i)} = \bigcup_k \Gamma_{(s,i),k}$ and $\Delta_{(s,i)} = \bigcup_k \Delta_{(s,i),k}$. We say that the node i of the tree s is in $N(r,\underline{c})$ (or that $\mathcal{M} \Vdash_{(s,i)} r(\underline{c})$) if the set of nodes j of the tree s such that $r(c) \in \Gamma_{(s,j)}$ is a bar for i in the tree s.

Proposition 5. *The definition of* $N(r,\underline{c})$ *determines an interpretation over* (F,E).

Proof : The functoriality condition is easy : if $r(\underline{c}) \in \Gamma_{(s,i),k}$ and if $i' \leq i$ in s, the construction of section 6.2 ensures that $r(\underline{c}) \in \Gamma_{(s,i),k'}$ for all $k' \geq k$. The locality condition is trivially ensured by the definition of $N(r,\underline{c})$.

It remains to show that $\mathcal{M}$ is a counter-model of $(\Gamma : \Delta)$:

Proposition 6. *Let* $F = (F_k)$ *be a refutation of the sequent* $(\Gamma : \Delta)$ *and* $\mathcal{M}$ *be the corresponding interpretation. For all nodes* (s,i) *of* F, *every sentence of* $\Gamma_{(s,i)}$ *is satisfied in* (s,i) *(in the sense of grafts semantics) and no sentence of* $\Delta_{(s,i)}$ *is satisfied in* (s,i)*(in the sense of grafts semantics).*

Proof : Let (s,i) be a node of F, $\alpha(\underline{x})$ a formula and $\underline{c}$ closed terms. We prove by induction on the form of α that if $\alpha(\underline{c}) \in \Gamma_{(s,i)}$, then $\mathcal{M} \Vdash_{(s,i)} \alpha(\underline{c})$ and if $\alpha(\underline{c}) \in \Delta_{(s,i)}$, then $\mathcal{M} \nVdash_{(s,i)} \alpha(\underline{c})$. The atomic case and the non-modal inductive cases are as in [2], pp. 241-243. Let us examine the modal case.

(A) Let α be of the form $\Box\beta$ in $\Gamma_{(s,i)}$. Let k be the stage where $\Box\beta$ has been introduced in $\Gamma_{(s,i),k}$ and let S_1 be the set of terminal nodes of the tree A^k_s which are below i. As A^k_s is finite, S_1 is a bar for i in A^k_s. Let $S_2 = \{(s,i') \mid i' \leq i, (s,i') \notin A^k_s\}$. It is clear that $S = S_1 \cup S_2$ is a bar for i in A_s. Moreover, for every $k' \geq k$ and $(s,i') \in S \cap A_s^{k'}$, $\Box\beta \in \Gamma_{(s,i'),k'}$. A graft $(s',\phi) \prec (s,i')$ for $(s,i') \in S$ can only be introduced at a stage $k' \geq k$ and then $\Gamma_{(s',\phi,)k'+1} \supseteq N\Gamma_{(s,i'),k'}$ and $\Box\beta \in \Gamma_{(s',\phi),k'+1}$. When $\Box\beta$ is examined at a further stage, β is introduced on the left and hence, for every $(s',j) \prec (s,i') \in S$, $\beta \in \Gamma_{(s',j)}$ and β is satisfied in (s',j) by the inductive hypothesis. All this proves that $\Box\beta$ is satisfied in (s,i).

(B) Let α be of the form $\Box\beta$ in $\Delta_{(s,i)}$. Let k be the stage where $\Box\beta \vee \Box\neg\Box\beta$ has been examined say at stage k. It has therefore been introduced in $\Delta_{(s,i),k}$. Suppose $\Box\beta$ is satisfied in (s,i). Then there exists a bar S for i in A_s such that for all $(s,i') \in S$ and for all $(s',j) \prec (s,i')$, $\mathcal{M} \Vdash_{(s',j)} \beta$.(1). But then the leftmost path from (s,i) crosses S, say at node (s,i_1), in which one has $\mathcal{M} \Vdash_{(s,i_1)} \beta$. It is the key property of the leftmost path that sequents are repeated. From this, $\Box\beta \in \Delta_{(s,i_1)}$ and at some stage a graft $(s',\phi) \prec (s,i_1)$ is introduced with $\beta \in \Delta_{(s',\phi)}$. By induction, $\mathcal{M} \nVdash_{(s',\phi)} \beta$. This contradicts (1) and shows that $\Box\beta$ is not satisfied in (s,i).

Theorem 7. *If the sequent* $(\Gamma : \Delta)$ *is not provable in* IS4, *then it has a counter-model in graft semantics.*

This follows from corollary 4 and proposition 6.

6.4. Completeness of IBM.

What has been proved of IS4 may be trivially extended to IS4-theories, i.e. to the system IS4 with a set $\mathcal{A}$ of axioms added. If $\mathcal{A}$ is finite, the result is already contained in proposition 6 and theorem 7. If $\mathcal{A}$ is infinite, one can adapt the planting procedure of forests (with grafts) of section 6.2. Let α be the (k+1)-th sentence in the enumeration of $\mathcal{F}_\omega$. If α is not in $\mathcal{A}$, proceed as indicated ; if α is an axiom, begin by adding

If α is not in $\mathcal{A}$, proceed as indicated ; if α is an axiom, begin by adding it to the $\Gamma_{(s,i),k}$ for all $(s,i) \in F_k$ and examine it according to the procedure. This way of proceeding preserves the finite character of sequents, which is an advantage when treating quantifiers.

The preceding observation is commonplace for non logical axioms, but in the case of IBM (which is the IS4-theory having as a set of axioms all sentences of the form $\Box\varphi \vee \Box\neg\Box\varphi$), we wish a modification of the semantics of IS4 which interprets $\Box$ by "for all nodes of the forest".

Proposition 8. *In models of IBM constructed as in proposition* 6, $\mathcal{M} \underset{(s,i)}{\Vdash} \Box\beta$ *iff for all* (s',i') *of* $F, \mathcal{M} \underset{(s',i')}{\Vdash} \beta$.

Proof : It clearly suffices to revise point (A) of the proof of proposition 6 and observe that by the very construction of the counter-model the relation $\prec$ is connected. Hence there are essentially two things to prove :

(1) if $(s',i') \prec (s,i)$ and $\Box\beta \in \Gamma_{(s,i)}$, then $\Box\beta \in \Gamma_{(s',i')}$;

(2) if $(s,i) \prec (s'',i'')$ and $\Box\beta \in \Gamma_{(s,i)}$, then $\Box\beta \in \Gamma_{(s'',i'')}$.

For (1), we recall that the construction ensures that $(s,i') \leq (s,i)$ implies $\Gamma_{(s,i)} \subseteq \Gamma_{(s,i')}$ and $(s',\phi) \prec (s,i)$ implies $N\Gamma_{(s',\phi)} \subseteq \Gamma_{(s,i)}$; it follows that $(s',i') \prec (s,i)$ implies $\Gamma_{(s,i)} \subseteq \Gamma_{(s',i')}$.

For (2), we observe that at node (s'',i''), $\Box\beta\vee\Box\neg\Box\beta$ has been examined say at stage k. It has therefore been introduced in $\Gamma_{(s'',i''),k}$ and one of the terms of the disjunction has been added to $\Gamma_{(s'',i''),k}$. Assume $\Box\neg\Box\beta$ has been chosen ; in that case, $\Box\neg\Box\beta \in \Gamma_{(s'',i'')}$, $\Box\neg\Box\beta \in \Gamma_{(s,i)}$ (since $(s,i) \prec (s'',i'')$) and $\Box\neg\Box\beta \in \Gamma_{(s,i)}$. It would follow that $\Box\beta$ and $\neg\Box\beta$

would be in $\Gamma_{(s,i)}$, and that F would not be a refutation. Consequently, one has to choose $\Box\beta$, i.e. $\Box\beta \in \Gamma_{(s'',i'')}$.

As a consequence of proposition 8, the method of section 6.3 modified as suggested above establishes the completeness of IBM in constant sheaves for the usual topos-theoretic semantics : just forget the relation $\prec$ at the end.

To summarize, we have proven :

Theorem 9. *Let* L *be a denumerable first-order modal language. Let* $\Gamma : \Delta$ *be a sequent of* L *which is not provable in the calculus of sequents with multiple conclusions corresponding to* IBM. *Then* $\Gamma : \Delta$ *has a counter-model in constant sheaf for the usual topos-theoretic semantics.*

Applying our previous remarks on axioms, we may extend this theorem to calculi of sequents with multiple conclusions corresponding to IBM with (non-logical) axioms. Using the equivalence of IS4MSq and IS4, we may return to IBM itself to obtain.

Theorem 10. *Let* L *be a denumerable first-order modal language . Let* Γ *be a set of sentences of* L *and* φ *be a sentence of* L *such that in* IBM, $\Gamma \nVdash \varphi$. *Then* $\Gamma \vdash \varphi$ *has a counter-model in constant sheaf for the usual topos-theoretic semantics.*

6.5. An open problem.

We conclude paragraph 6 by mentioning an open problem.

In IS4, one can obviously define $\Diamond\varphi$ by $\neg\Box\neg\varphi$, but this does not preclude the question of having at hand a less boolean notion of possibility. Let us call IIS4 the system with two modalities $\Box$ and $\Diamond$ given by IS4 and the axioms

$$\Diamond\Diamond\varphi \to \Diamond\varphi,$$
$$\varphi \to \Diamond\varphi,$$
$$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi),$$
$$\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi).$$

A calculus of sequents equivalent to IIS4 may be obtained by adding to $IS4_{Sq}$ two rules :

$$(\Diamond:)\quad \frac{\Gamma,\alpha:\gamma}{\Gamma,\Diamond\alpha:\Delta} \qquad (:\Diamond)\quad \frac{\Gamma:\alpha}{\Gamma:\Diamond\alpha}$$

the first rule being subject to the condition that all sentences of Γ be necessary and that γ be empty or a finite disjunction of possible sentences (i.e. of the form $\mathrm{V}\Diamond\xi$). The proof of equivalence of the two systems may be done by cut elimination as in proposition 2 of § 5.

The open problem is that of completeness of IIS4 for grafts semantics. We think that to take into account the specific role of $\Diamond$, distinguished grafts should be introduced.

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Added in proof. An alternative proof of Cut Elimination for a calculus of sequents for IS4 has been given in M. OKADA "A Heyting algebra for normalization theorems", to appear.

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