BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 7, 1960

MATHEMATICS

On Certain Properties of φ-functions

by

W. MATUSZEWSKA and W. ORLICZ

Presented by W. ORLICZ on May 5, 1960

1. In this paper we use the same definitions and notations as in [3]. Given a φ-function let us define for $\lambda > 0$ the following extended-value functions

$$(\circ) \quad \underline{h}_\varphi(\lambda) = \varliminf_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)}, \qquad (\circ\circ) \quad \overline{h}_\varphi(\lambda) = \varlimsup_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)},$$

$$(\circ') \quad \underline{h}^a_\varphi(\lambda) = \inf_{u>0} \frac{\varphi(u)}{\varphi(\lambda u)}, \qquad (\circ\circ') \quad \overline{h}^a_\varphi(\lambda) = \sup_{u>0} \frac{\varphi(u)}{\varphi(\lambda u)}.$$

1.1. *There exist limits*

$$(*) \quad s_\varphi = \lim_{\lambda\to 0+} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg\lambda}, \qquad (**) \quad \sigma_\varphi = \lim_{\lambda\to 0+} \frac{\lg \overline{h}_\varphi(\lambda)}{-\lg\lambda},$$

and there exist limits (let s^a_φ resp. σ^a_φ stand for these) of analogous expression which appear at (*), (**) on replacing $\underline{h}_\varphi(\lambda)$, resp. $\overline{h}_\varphi(\lambda)$, by $\underline{h}^a_\varphi(\lambda)$, resp. $\overline{h}^a_\varphi(\lambda)$.

To prove this let us notice that $\underline{h}_\varphi(\lambda) \geqslant 1$ for $0 < \lambda \leqslant 1$,

$$\underline{h}_\varphi(\lambda\mu) = \varliminf_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)} \cdot \frac{\varphi(\lambda u)}{\varphi(\lambda\mu u)} \geqslant \underline{h}_\varphi(\lambda)\, \underline{h}_\varphi(\mu), \; 0 < \lambda, \mu \leqslant 1,$$

and $\overline{h}_\varphi(\lambda\mu) \leqslant \overline{h}_\varphi(\lambda)\, \overline{h}_\varphi(\mu)$. By suitable substitution we reduce the supermultiplicative function $\underline{h}_\varphi(\lambda)$, resp. the submultiplicative function $\overline{h}_\varphi(\lambda)$ to a superadditive, resp. subadditive one and apply the well-known theorem on limits of such functions. For the reader's convenience we shall supply a direct proof in the case (*). Let $0 < \lambda_0 < 1$; given $0 < \lambda < 1$, we determine a positive integer n so that $\lambda = \lambda_0^n \mu$, where $\lambda_0 < \mu \leqslant 1$. $\underline{h}_\varphi(\lambda)$ being supermultiplicative, we obtain $\underline{h}_\varphi(\lambda) = \underline{h}_\varphi(\lambda_0^n \mu) \geqslant (\underline{h}_\varphi(\lambda_0))^n\, \underline{h}_\varphi(\mu)$. Hence,

$$\frac{\lg \underline{h}_\varphi(\lambda)}{-\lg\lambda} \geqslant -\frac{n}{n\lg\lambda_0 + \lg\mu} \lg \underline{h}_\varphi(\lambda_0) + \frac{\lg \underline{h}_\varphi(\mu)}{-\lg\lambda}.$$

If $\underline{h}_\varphi(\lambda_0) < \infty$, we get with $\lambda \to 0+$

$$\lim_{\lambda\to 0+} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg \lambda} \geqslant \frac{\lg \underline{h}_\varphi(\lambda_0)}{-\lg \lambda_0}.$$

The latter inequality holds also in the case of $\underline{h}_\varphi(\lambda_0) = \infty$ (in that case $\underline{h}_\varphi(\lambda) = \infty$ for every $\lambda, 0 < \lambda < x_0$). Thus we have

$$\underline{\lim}_{\lambda\to 0+} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg \lambda} \geqslant \sup_{0<\lambda<1} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg \lambda} \geqslant \overline{\lim}_{\lambda\to 0+} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg \lambda},$$

whence

$$(+) \qquad \lim_{\lambda\to 0+} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg \lambda} = \sup_{0<\lambda<1} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg \lambda}.$$

Similarly we find

$$(++) \qquad \lim_{\lambda\to 0+} \frac{\lg \overline{h}_\varphi(\lambda)}{-\lg \lambda} = \inf_{0<\lambda<1} \frac{\lg \overline{h}_\varphi(\lambda)}{-\lg \lambda}.$$

Relations analogous to $(+)$ and $(++)$ hold for $\underline{h}^a_\varphi(\lambda)$, $\overline{h}^a_\varphi(\lambda)$.

1.2. *If* $\varphi \overset{l}{\sim} \psi$ *then* $s_\varphi = s_\psi$, $\sigma_\varphi = \sigma_\psi$; *if* $\varphi \overset{a}{\sim} \psi$ *then* $s^a_\varphi = s^a_\psi$, $\sigma^a_\varphi = \sigma^a_\psi$.

If $\varphi \overset{l}{\sim} \psi$, we have with certain positive constants a, b, k_1, k_2

$$a\varphi(k_1 u) \leqslant \psi(u) \leqslant b\varphi(k_2 u)$$

for sufficiently large u. Hence, we obtain

$$\frac{a}{b}\,\frac{\varphi(k_1 u)}{\varphi\left(\frac{k_2}{k_1}\lambda k_1 u\right)} = \frac{a}{b}\,\frac{\varphi(k_1 u)}{\varphi(k_2 \lambda u)} \leqslant \frac{\psi(u)}{\psi(\lambda u)} \leqslant \frac{b}{a}\,\frac{\varphi(k_2 u)}{\varphi(\lambda k_1 u)} = \frac{b}{a}\,\frac{\varphi(k_2 u)}{\varphi\left(\frac{k_1}{k_2}\lambda k_2 u\right)},$$

$$\frac{a}{b}\,\underline{h}_\varphi\left(\frac{k_2}{k_1}\lambda\right) \leqslant \underline{h}_\psi(\lambda) \leqslant \frac{b}{a}\,\underline{h}_\varphi\left(\frac{k_1}{k_2}\lambda\right),$$

$$\frac{\lg \underline{h}_\psi(\lambda)}{-\lg \lambda} \leqslant \frac{\lg \frac{b}{a}}{-\lg \lambda} + \frac{\lg \underline{h}_\varphi\left(\frac{k_1}{k_2}\lambda\right)}{-\lg \frac{k_1}{k_2}\lambda}\,\frac{-\lg \frac{k_1}{k_2}\lambda}{-\lg \lambda}, \quad \text{for } 0 < \lambda < 1,$$

whence $s_\psi \leqslant s_\varphi$; we show similarly that $s_\varphi \leqslant s_\psi$. Our argument is untouched in the case $\overset{a}{\sim}$. One can prove similarly $\sigma_\varphi = \sigma_\psi$, resp. $\sigma^a_\varphi = \sigma^a_\psi$.

1.3. (a) *Let* $0 < s < \infty$; *in order that* $s_\varphi = s$, *it is necessary and sufficient that*

$$(\bigcirc) \qquad \varphi(u) = u^s \varrho(u),$$

where $\varrho(u)$ *is continuous, positive for* $u > 0$ *and subject to*

$$(+) \qquad \lim_{\lambda\to 0+} \frac{\lg \underline{\lim}_{u\to\infty} \frac{\varrho(u)}{\varrho(\lambda u)}}{-\lg \lambda} = 0.$$

(b) *Let* $0 < \sigma < \infty$; *in order that* $\sigma_\varphi = \sigma$ *it is necessary and sufficient that*

$$(\circ\circ) \qquad \varphi(u) = u^\sigma \varrho_1(u),$$

where $\varrho_1(u)$ *is continuous, positive for* $u > 0$ *and subject to*

$$(++) \qquad \lim_{\lambda\to 0+} \frac{\lg \overline{\lim}_{u\to\infty} \dfrac{\varrho_1(u)}{\varrho_1(\lambda u)}}{-\lg\lambda} = 0.$$

We shall get analogous theorems for indices s_φ^a, σ_φ^a *if we replace in the above formulae* $\underline{\lim}_{u\to\infty} \dfrac{\varrho(u)}{\varrho(\lambda u)}$ *by* $\inf_{u>0} \dfrac{\varrho(u)}{\varrho(\lambda u)}$ resp. $\overline{\lim}_{u\to\infty} \dfrac{\varrho_1(u)}{\varrho_1(\lambda u)}$ *by* $\sup_{u>0} \dfrac{\varrho_1(u)}{\varrho_1(\lambda u)}$, (on the whole ϱ, ϱ_1 are not φ-functions).

1.4. φ is said to satisfy the *condition* (Δ_α), $a > 1$, *for large* u, if $\varphi(au) \leqslant d_\alpha \varphi(u)$ for $u \geqslant u(a)$; φ is said to satisfy the *condition* (Λ_α), $a > 1$, *for large* u, if with a $c_\alpha > 1$ we have $\varphi(u) c_\alpha \leqslant \varphi(au)$ for $u \geqslant u_1(a)$. If $u(a)$ resp. $u_1(a)$ are zero, then we say that the condition (Δ_α) resp. (Λ_α) is satisfied *for all* u (cf. [3]).

1.4.1. (a) *If* $s_\varphi > 0$, *then* (Λ_α) *is satisfied for large* u *and one may pick up, given an arbitrary* $\varepsilon > 0$, *some* $a > 1$ *and* c_α *so that* $(\lg c_\alpha)(\lg a)^{-1} > s_\varphi - \varepsilon$. *If* (Λ_α) *holds for large* u, *we have* $s_\varphi \geqslant (\lg c_\alpha)(\lg a)^{-1} > 0$.

(b) *If* $\sigma_\varphi < \infty$ *then* (Δ_α) *is satisfied for large* u *and one may pick up, given an arbitrary* $\varepsilon > 0$, *some* $a > 1$, d_α *so that* $(\lg d_\alpha)(\lg a)^{-1} < \sigma_\varphi + \varepsilon$. *If* (Δ_α) *holds for large* u, *we have* $\sigma_\varphi \leqslant (\lg d_\alpha)(\lg a)^{-1} < \infty$.

Analogous theorems hold for indices s_φ^a *resp.* σ_φ^a *and conditions* (Λ_α) *resp.* (Δ_α) *for all* u.

Let $s_\varphi < \infty$, $s_\varphi - \varepsilon/2 = s'$, $s_\varphi - \varepsilon = s$ and let with a certain $0 < \lambda_0 < 1$ be

$$\frac{\lg \underline{h}_\varphi(\lambda_0)}{-\lg\lambda_0} > s',$$

whence $\underline{h}_\varphi(\lambda_0) \geqslant \lambda_0^{-s'}$ and for $0 < c < 1$ we have, provided $u \geqslant u_0(c)$, $\varphi(u)/\varphi(\lambda_0 u) \geqslant c^{s'} \lambda_0^{-s'}$. Let $a = 1/\lambda_0$; for $u \geqslant u_0(c)$ we have $\varphi(au) \geqslant c^{s'} a^{s'} \varphi(u)$. It follows that choosing c sufficiently near to 1, $ca > 1$, we get for sufficiently large u the inequality $\varphi(au) \geqslant c_\alpha \varphi(u)$, where $c_\alpha = (ca)^{s'}$,

$$\frac{\lg c_\alpha}{\lg a} = s' + s' \frac{\lg c}{\lg a} > s.$$

For $s_\varphi = \infty$ we choose $s < s' < \infty$ arbitrarily. Suppose φ satisfying (Λ_α) for large u with a certain $a > 1$ and with constant c_α. Hence, for large u,

$$\frac{\varphi(u)}{\varphi(\lambda_0 u)} \geqslant c_\alpha, \text{ with } \lambda_0 = \frac{1}{a},$$

$\underline{h}_\varphi(\lambda_0) \geqslant c_\alpha$, $\lg \underline{h}_\varphi(\lambda_0)/-\lg\lambda_0 \geqslant \lg c_\alpha/\lg a$ and it remains to appeal to 1.1 (+).

(b) can be proved similarly.

We arrive at the following corollary: $\sigma_\varphi < \infty$ *is equivalent to* (Δ_α), $s_\varphi > 0$ *is equivalent* to (Λ_α).

2. Let ϱ be an arbitrary function continuous and non-vanishing for $u > 0$. ϱ will be said *pseudo-increasing for large u* if there exist positive constants m, n such that

$$\varrho(u_2) \geqslant m\varrho(nu_1) \text{ for } u_2 \geqslant u_1 \geqslant u_0;$$

changing in the above inequality sign $\geqslant$ to $\leqslant$ we come to the definition of *pseudo-decreasing (for large u)* function; if $u_0 = 0$ we get the definitions of pseudo-increasing (-decreasing) function *for all u*.

φ-function φ is said to be (Λ, s) resp. (Δ, σ)-representable $(s > 0, \sigma > 0)$ if $\varphi(u) = \psi(u^s)$, where ψ is a convex φ-function, resp. $\varphi(u) = \psi(u^\sigma)$, where ψ is a concave φ-function. We have for (Λ, s)-representable functions $s_\varphi = s s_\psi$, $\sigma_\varphi = s\sigma_\psi$ and analogous relations hold for (Δ, σ)-representable functions.

2.1. (a) *In order that* $\varphi \overset{l}{\sim} \chi$, *where* χ *is* (Λ, s)*-representable, it is necessary and sufficient that* φ *be of form* 1.3, (o), *where* ϱ *is pseudo--increasing for large u.*

(b) *In order that* $\varphi \overset{l}{\sim} \chi$, *where* χ *is* (Δ, σ)*-representable, it is necessary and sufficient that* φ *be of form* 1.3, (oo), *where* ϱ_1 *is pseudo--decreasing for large* u.

The theorem does not cease to hold if we replace $\overset{l}{\sim}$ by $\overset{a}{\sim}$ and the words "for large u" by "for all u".

For $s = 1$, resp. $\sigma = 1$, the above is equivalent to theorem 2.3 from [3] and the case of arbitrary s, σ may be switched to the latter one. For the details of proof see [4].

2.2. *If* (+) $\varphi(au) \leqslant c a^s \varphi(u/\varepsilon)$ $(s > 0)$, *where* $c > 0$, *for* $0 \leqslant a \leqslant 1$ *and for* (++) $a^s \varphi(u/\varepsilon) \geqslant a > 0$, *then* $\psi \overset{l}{\sim} \chi$, *where* χ *is* (Λ, s)*-representable.*

Let $a^s = (\varphi(u/\varepsilon))^{-1} a$; we have $\varphi(a^{1/s} u \varphi(u/\varepsilon)^{-1/s}) \leqslant ca$, whence $\varliminf_{u\to\infty} u^{-s}\varphi(u/\varepsilon) > d > 0$. Let $u_2 \geqslant u_1 \geqslant u_0$, where $u^{-s}\varphi(u/\varepsilon) \geqslant d/2$ for $u \geqslant u_0$, $a = u_1/u_2$. Since for sufficiently large u_0 we have $(u_1/u_2)^s \varphi(u_2/\varepsilon) \geqslant u_0^s d/2 \geqslant a$ it follows that $\varphi(au_2) = \varphi(u_1) \leqslant c(u_1/u_2)^s \varphi(u_2/\varepsilon)$, i.e. writing $\varrho(u) = \varphi(u) u^{-s}$ we get $\varrho(u_1) \leqslant c\varrho(u_2/\varepsilon)$ for large u.

2.2.1. *If* (+) *holds for those* a, u *for which* (++) *is satisfied and, moreover, for* $0 \leqslant a \leqslant 1$ *and* u *sufficiently small, then* $\varphi \overset{a}{\sim} \chi$, *where* χ *is* (Λ, s)*-representable.*

2.3. (a) *If* $s_\varphi > 0$ *then with every* $s < s_\varphi$ *we have* $\varphi \overset{l}{\sim} \chi_s$, *where* χ_s *is* (Λ, s)*-representable, we have* $\varphi \overset{l}{\sim} \chi$, *where* χ *is* (Λ, s_φ)*-representable, if and only if* φ *is of form* 1.3, (o) *with* $s = s_\varphi$ *and* ϱ *pseudo-increasing for large u and satisfying* 1.3, (+). *With no* $s > s_\varphi$ φ *is l-equivalent to a* (Λ, s)*-representable* χ_s.

(b) *If* $\sigma_\varphi < \infty$, *then with every* $\sigma_\varphi < \sigma$ *we have* $\varphi \overset{l}{\sim} \chi_\sigma$, *where* χ_σ *is* (Δ, σ)*-representable; we have* $\varphi \overset{l}{\sim} \chi$, *where* χ *is* (Δ, σ_φ)*-representable, if and only if* ψ *is of form* 1.3, (oo) *with* $\sigma = \sigma_\varphi$ *and* ϱ_1 *pseudo-decreasing for large* u *and satisfying* 1.3, (++). *With no* $\sigma < \sigma_\varphi$ φ *is l-equivalent to a* (Δ, σ)*-representable* χ_σ.

The theorem is still valid if we replace in its formulation s_φ, σ_φ *by* s_φ^a, σ_φ^a, $\overset{l}{\sim}$ *by* $\overset{a}{\sim}$, *the words "for large u"* by *"for all u"*.

In the proof of (a) we make use of Theorem 2.2 [3] (for details see [4]) and 1.4.1, (a), 2.1, (a), in the proof of (b) of Theorem 2.1 [3] and 1.4.1, (b), 2.1, (b).

2.4. Now we shall give few examples of φ-functions and the corresponding indices s_φ, σ_φ. If φ is convex, then from the inequality $\varphi(\lambda u) < \lambda\varphi(u)$ for $0 < \lambda < 1$ it follows $s_\varphi \geqslant s_\varphi^a \geqslant 1$; if φ is concave, then $\sigma_\varphi \leqslant \sigma_\varphi^a \leqslant 1$. If there exist

$$\lim_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)} = \underline{h}_\varphi(\lambda) = \overline{h}_\varphi(\lambda) < \infty \text{ for every } \lambda > 0,$$

then $\underline{h}_\varphi(\lambda\mu) = \underline{h}_\varphi(\lambda)\underline{h}_\varphi(\mu)$ for $\lambda, \mu > 0$, whence $\underline{h}_\varphi(\lambda) = 1$ or λ^{-s} $(s > 0)$ for $\lambda > 0$; therefore two cases are possible: $s_\varphi = \sigma_\varphi = 0$ or $s_\varphi = \sigma_\varphi = s$. If $\underline{h}_\varphi(\lambda) = \overline{h}_\varphi(\lambda) = 1$ for $\lambda > 0$, then we have the so-called slowly oscillating function in the sense of Karamata [1]. If $\varphi(u) \sim \psi(u)\varphi(u)$, where $s_\psi > 0$, then, owing to 1.2, we get $s_\varphi = \sigma_\varphi = \infty$, since we have always $s_{\varphi\psi} \geqslant s_\varphi + s_\psi$, $\sigma_{\varphi\psi} \leqslant \sigma_\varphi + \sigma_\psi$. The case $\psi(u) = u$ means the functions satisfying the condition (Δ_3), to adopt the notation of [2]. Let $\varphi(u) = u^s/\varrho(u)$, $\varphi_1(u) = u^s\varrho(u)$, where $\varrho(u)$ is a slowly oscillating function (e.g. $\varrho(u) = \lg(1+u)$. We obtain $s_\varphi = \sigma_\varphi = s$, $s_{\varphi_1} = \sigma_{\varphi_1}$, but φ is not l-equivalent to a (Δ, s)-representable φ-function and φ_1 is not l-equivalent to a (Δ, s)-representable φ-function. For it may be easily seen that $1/\varrho(u)$ is not pseudo-increasing for large u and $\varrho(u)$ is not pseudo-decreasing for large u.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) **4** (1930), 38—53.

[2] M. A. Krasnosielskiï, J. B. Rutickiï, *Convex functions and Orlicz spaces*, (in Russian) Moscow 1958.

[3] W. Matuszewska, *On generalized Orlicz spaces*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **8** (1960), 349.

[4] — , *Przestrzenie funkcji φ-całkowalnych*, (doctoral-thesis), A. Mickiewicz University, Poznań, 1960, Prace Matem. **5**, (in press).