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FUNCTIONAL ANALYSIS

The Property (H) in Orlicz Spaces

by

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Summary. In the paper some geometrical and order properties of Orlicz spaces equipped with Luxemburg norm and with Orlicz norm are investigated. In particular, the necessary and sufficient conditions for an Orlicz space with Luxemburg norm to have the Radon-Riesz property (H) are obtained.

1. Preliminaries. In the sequel it is supposed that the interval $(0,1)$ is supplied with the usual structure of measure space with Lebesgue measure μ. Let $x(t)$ be a real measurable function on $(0,1)$. The rearrangement of $x(t)$ is a non-negative, non-increasing and continuous from the left function defined by the equality

$$x^*(t) = \inf\{\lambda > 0 : n_x(\lambda) < t\}, \quad 0 < t < 1,$$

where $n_x(\lambda) = \mu(\{t \in (0,1) : |x(t)| > \lambda\})$ is the distribution function for $|x(t)|$. The following preorder relation on the set of all measurable functions is connected with the notion of rearrangement:

$$x \prec y \text{ iff } \int_0^s x^*(t)\,dt \leqslant \int_0^s y^*(t)\,dt \quad \text{for all } s \in (0,1).$$

A Banach space $(E, \|\cdot\|)$ of real measurable functions on $(0,1)$ is called a symmetric space if for any $x \in E$ and for any measurable function $y(t)$ it follows from $y^* \leqslant x^*$ that $y \in E$ and $\|y\| \leqslant \|x\|$. For any symmetric space E the space $L_\infty(0,1)$ is identified in a natural manner with a subspace of the conjugate space E^* which makes possible to consider the topology $\sigma(E, L_\infty)$ on E.

Let E be a symmetric space; E is said to have the property (H) if for each $x \in E$ and a sequence $(x_n) \subset E$ such that $\|x_n\| \to \|x\|$ and $x_n \to x$ in the weak topology $\sigma(E, E^*)$ we have $x_n \to x$ in the norm. If we replace the topology $\sigma(E, E^*)$ in this definition by the topology $\sigma(E, L_\infty)$ (respectively, by the measure topology) we get definition of the property (H_∞) (respectively, the property (Hm)).

We say that the norm of a symmetric space E is

— strictly monotone if for each positive $x, y \in E$ the equality $\|x + y\| = \|x\|$ implies $y = 0$;

— K-monotone if for each $x, y \in E$ such that $x \prec y$ we have $\|x\| \leqslant \|y\|$; if, in addition, $x^* \neq y^*$ implies $\|x\| \neq \|y\|$, then the norm of E is said to be strictly K-monotone;

— locally uniformly monotone if for each $x \in E$ and $(x_n) \subset E$ such that $0 \leqslant x \leqslant x_n$ for all $n \in \mathbb{N}$ and $\|x_n\| \to \|x\|$ we have $x_n \to x$ in the norm;

— order semi-continuous if for each $x \in E$ and $(x_n) \subset E$ such that $0 \leqslant x_n \uparrow x$ we have $\|x_n\| \to \|x\|$.

Note that the property of K-monotones of the norm takes place in a wide class of symmetric spaces, in particular, in spaces with the Fatou property and in separable spaces.

There are the following correlations between the notions defined above [1]:

THEOREM 1. *Let E be a separable symmetric space.*

(a) *The following conditions are equivalent:*

(i) *the norm of E is strictly monotone and E has the property (Hm),*

(ii) *the norm of E is locally uniformly monotone,*

(b) *the following conditions are equivalent:*

(i) *E has the property (H_∞),*

(ii) *the norm of E is strictly K-monotone and E has the property (Hm),*

(c) *if E has the property (H), then the norm of E is strictly K-monotone.*

For definitions of geometrical properties of Banach spaces we refer to [2]. We will also use the terminology and notations of the theory of Orlicz spaces from [3, 4].

Let $M(x)$ and $N(y)$ be complementary N-functions. Orlicz space is a linear space of all real measurable functions $x(t)$ on $(0,1)$ such that $I_M(\lambda x) = \int_0^1 M[\lambda x(t)]dt < \infty$ for some $\lambda > 0$ dependent on $x(t)$. Two equivalent norms are considered on Orlicz space: the Orlicz norm

$$\|x\|_M = \sup\left\{ \int_0^1 |x(t)y(t)|\, dt : I_N(y) \leqslant 1 \right\} = \inf_{k>0} (1 + I_M(kx))/k$$

and the Luxemburg norm

$$\|x\|_{(M)} = \inf\{\lambda > 0 : I_M(x/\lambda) \leqslant 1\}.$$

Denote by L_M, $L_{(M)}$ the Orlicz spaces generated by $M(x)$ and equipped with Orlicz norm and Luxemburg norm, respectively; E_M, $E_{(M)}$ the closure of $L_\infty(0,1)$ in them, respectively. All these four spaces are symmetric spaces with K-monotone and order semi-continuous norms.

2. Luxemburg norm. Everywhere below $M(x)$ is an N-function.

THEOREM 2. *The following conditions are equivalent:*
(i) *the function $M(x)$ satisfies the Δ_2-condition,*
(ii) *the norm of $L_{(M)}$ is strictly monotone.*
(iii) *the norm of $L_{(M)}$ is locally uniformly monotone,*
(iv) *$L_{(M)}$ has the property (Hm).*

Proof. The equivalence of (i) and (ii) has been established in [5, 6].

(i) ⇒ (iii). Let $x \in L_{(M)}$, $(x_n) \subset L_{(M)}$, $0 \leqslant x \leqslant x_n$ for all $n \in \mathbb{N}$ and $\|x_n\|_{(M)} \to \|x\|_{(M)}$. Obviously, we may assume that $x \neq 0$. Put for the sake of brevity $\lambda_n = 1/\|x_n\|_{(M)}$. Since $M(t-s) \leqslant M(t) - M(s)$ for $t \geqslant s \geqslant 0$, we get $I_M(\lambda_n(x_n - x)) \leqslant I_M(\lambda_n x_n) - I_M(\lambda_n x)$ for all $n \in \mathbb{N}$. We have $I_M(\lambda_n - x_n) = 1$, and, by the Lebesgue theorem, $I_M(\lambda_n x) \to 1$. Therefore $I_M(\lambda_n(x_n - x)) \to 0$. Then $\lambda_n\|x_n - x\|_{(M)} \to 0$, and since $\lim_{n\to\infty} \lambda_n = 1/\|x\|_{(M)} > 0$, we have $\|x_n - x\|_{(M)} \to 0$.

(iii) ⇒ (ii). This implication is obvious.

We have established the equivalence of (i), (ii) and (iii). Thus by Theorem 1, each of these conditions implies (iv).

(iv) ⇒ (i). Let x be an arbitrary function from $L_{(M)}$. Let us consider its rearrangement $x^* \in L_{(M)}$ and the sequence $x_n = x^*\chi_{(1/n,1)}$. Then $x_n \to x^*$ in the measure topology. Besides, $x_n \uparrow x^*$, so, by the order semi-continuity of the norm of $L_{(M)}$, we get $\|x_n\|_{(M)} \to \|x^*\|_{(M)}$. Therefore, $\|x_n - x^*\|_{(M)} \to 0$. Since $x_n \in L_\infty(0,1)$ for $n \in \mathbb{N}$, we have $x^* \in E_{(M)}$ and so $x \in E_{(M)}$. Thus $L_{(M)} = E_{(M)}$ which is equivalent to the Δ_2-condition for $M(x)$.

THEOREM 3. *The following conditions are equivalent:*
(i) *$M(x)$ is strictly convex and satisfies the Δ_2-condition,*
(ii) *$L_{(M)}$ is rotund,*
(iii) *$L_{(M)}$ is locally uniformly convex,*
(iv) *$L_{(M)}$ has the property (H),*
(v) *$L_{(M)}$ has the property (H_∞).*
(vi) *$L_{(M)}$ is separable and its norm is strictly K-monotone.*

Proof. The equivalence of (i), (ii) and (iii) has been established in [7, 8] (see also [9]).

(iii) ⇒ (iv). This implication holds true for arbitrary Banach spaces [2].

(iv) ⇒ (v). If $L_{(M)}$ has the property (H), then it is separable [10]. Thus, by Theorem 2, $L_{(M)}$ has the property (Hm). In addition, the norm of $L_{(M)}$

is strictly K-monotone, due to Theorem 1(c). Therefore, by Theorem 1(b), $L_{(M)}$ has the property (H_∞).

(v) $\Rightarrow$ (vi). If $L_{(M)}$ has the property (H_∞), then it obviously has the property (H), too. Hence, $L_{(M)}$ is separable. By Theorem 1(b), the norm of $L_{(M)}$ is strictly K-monotone.

(vi) $\Rightarrow$ (i). If $L_{(M)}$ is separable, then $M(x)$ satisfies the Δ_2-condition. Let us suppose that $M(x)$ fails to be strictly convex. Then it is linear on some interval $[a,b]$, $a>0$, i.e.

$$M(x) = \frac{B-A}{b-a}(x-a)+A, \quad x \in [a,b]$$

where $A = M(a)$, $B = M(b)$. Let us show that in this case the norm of $L_{(M)}$ is not strictly K-monotone. To this end we construct two functions $x, y \in L_{(M)}$ such that $y \prec x$, $y^* \neq x^*$ and $\|x\|_{(M)} = \|y\|_{(M)}$. Let us consider separately two cases.

1. If $A+B<2$, then we put $x_1 = b\chi_{(0,1/4)} + a\chi_{(1/4,1/2)}$, $y_1 = (a+b)\chi_{(0,1/2)}/2$ and $c = \max\{b, M^{-1}(2)\}$. Then the number $t_0 = (1-(A+B)/4)/M(c)+1/2$ belongs to the interval $(1/2,1)$, and the functions $x = x_1 + c\chi_{(1/2,t_0]}$, $y = y_1 + c\chi_{(1/2,t_0]}$ are desired.

2. If $A+B \geqslant 2$, we denote $r = 2/(A+B)$ and put $x = b\chi_{(0,r/2]} + a\chi(r/2,r]$, $y = (a+b)\chi_{(0,r]}/2$.

3. Orlicz norm. Turn now to the investigation of geometrical properties of Orlicz space equipped with Orlicz norm.

THEOREM 4. *If $M(x)$ satisfies the Δ_2-condition, then the norm of $L_{(M)}$ is locally uniformly monotone.*

Proof. Let $x \in L_M$, $(x_n) \subset L_M$, $0 \leqslant x \leqslant x_n$ for all $n \in \mathbb{N}$ and $\|x_n\|_M \to \|x\|_M$. We may assume that $x \neq 0$. We have

$$\|x_n\|_M = \inf_{k>0}(1+I_M(kx_n))/k \geqslant \inf_{k>0}(1+I_M(kx))/k = \|x\|_M.$$

For each n, we choose a number $k_n > 0$ such that

$$\|x_n\|_M + 1/n \geqslant (1+I_M(k_n x_n))/k_n, \quad n \in \mathbb{N}. \tag{1}$$

We have then for all $n \in \mathbb{N}$

$$\|x_n\|_M + 1/n \geqslant (1+I_M(k_n x_n))/k_n \geqslant (1+I_M(k_n x))/k_n \geqslant \|x\|_M,$$

from which it follows that

$$\lim_{n\to\infty}(I_M(k_n x_n) - I_M(k_n x))/k_n = 0$$

and

$$\lim_{n\to\infty} I_M(k_n(x_n - x))/k_n = 0. \tag{2}$$

It follows from (1) that

$$\liminf k_n > 0. \tag{3}$$

Show that $\limsup k_n < \infty$. If this is not the case, then passing, if necessary, to a subsequence we may consider that $k_n \to \infty$. Choose a measurable set $\Omega \subset [0,1]$ of positive measure such that $x(t) \geqslant \lambda\chi_\Omega(t)$ for some $\lambda > 0$. Then we have

$$I_M(k_n x) \geqslant \int_\Omega M(k_n\lambda)\,d\mu = M(k_n\lambda)\mu(\Omega)$$

and

$$(1 + I_M(k_n x))/k_n \geqslant \lambda\mu(\Omega)M(k_n\lambda)/(k_n\lambda).$$

The right side of the above inequality tends to infinity as $n \to \infty$ by the definition of N-function which contradicts the inequality (1). Hence,

$$\limsup k_n < \infty. \tag{4}$$

We get from (2), (3) and (4) that $I_M(x_n - x) \to 0$. This means that $\|x_n - x\|_M \to 0$.

COROLLARY 1. *The following conditions are equivalent:*
(i) *$M(x)$ satisfies the Δ_2 condition.*
(ii) *$L_{(M)}$ has the property (Hm).*

COROLLARY 2. *Let the function $M(x)$ satisfy the Δ_2-condition. If the space $L_{(M)}$ is rotund, then it has the property (H).*

In order to proof the next theorem we will need two auxiliary lemmas.

Denote by B_E the unit ball of a symmetric space E and by S_E its unit sphere.

LEMMA 1. *Let E be a separable symmetric space. If for any $x \in S_E$ and a sequence $(x_n) \subset S_E$ it follows from $\|x_n^* + x^*\| \to 2$ that $\|x_n^* - x^*\| \to 0$, then the space E is locally uniformly convex.*

Proof. Let $x \in S_E$, $(x_n) \subset S_E$ and $\|x_n + x\| \to 2$. It follows from the properties of rearrangements [11] that $(x_n + x) \prec (x_n^* + x^*)$, so $\|x_n + x\| \leqslant \|x_n^* + x^*\| \leqslant \|x_n^*\| + \|x^*\| = 2$. Therefore $\|x_n^* + x^*\| \to 2$ and hence $\|x_n^* - x^*\| \to 0$.

There exists an equivalent norm $\|\cdot\|_0$ on E such that $(E, \|\cdot\|_0)$ is a symmetric locally uniformly convex space [12]. Put $\alpha = 1/\|x\|_0$. We have then $\|\alpha x_n^* - \alpha x^*\|_0 \to 0$, hence $\|\alpha x_n\|_0 \to 1$. We have

$$\|(x_n + x)^*/2 + x^*\| \leqslant \|x_n + x\|/2 + \|x\| \leqslant 2,$$

$$\lim_{n\to\infty} \|(x_n + x)^*/2 + (x_n^* + x^*)/2\| = \lim_{n\to\infty} \|(x_n + x)^*/2 + x^*\|$$

and

$$(x_n + x) \prec (x_n + x)^*/2 + (x_n^* + x^*)/2.$$

Therefore $\|(x_n + x)^*/2 + x^*\| \to 2$ from which it follows that $\|(x_n + x)^*/2 - x^*\| \to 0$. Hence $\|\alpha(x_n + x)^*/2 - \alpha x^*\|_0 \to 0$, so $\|\alpha x_n + \alpha x\|_0 = \|\alpha(x_n + x)^*\|_0 \to 2$. Thus we have $\|\alpha x_n - \alpha x\|_0 \to 0$, therefore $\|x_n - x\| \to 0$.

LEMMA 2. *Let E be a symmetric space, $x \in E$, $(x_n) \subset E$, $x_n = x_n^*$ for all $n \in \mathbb{N}$ and $x_n \to x$ in the topology $\sigma(E, L_\infty)$. Then $x_n \to x$ in the measure topology.*

Proof. It is sufficient to show that for any subsequence (y_k) of the sequence (x_n) there exists a subsequence (z_m) converging to x in the measure topology. Since $x_n \to x$ in the topology $\sigma(E, L_\infty)$, then there exists a constant $C > 0$ such that $\int_0^1 x_n(t)dt \leqslant C$. Fix a number $s > 0$. Then we have

$$sx_n^*(s) \leqslant \int_0^1 x_n^*(s)\chi_{(0,s)}(t)\, dt \leqslant \int_0^1 x_n^*(t)\, dt \leqslant C,$$

or $x_n^*(s) \leqslant C/s$. By this estimate and by Helley theorem, there exists subsequence (z_m) of a sequence (y_k) converging to a non-increasing function z everywhere on $(0,1)$. Then for any $0 < a < b < 1$ we have $\int_a^b z_m(t)dt \to \int_a^b z(t)dt$. On the other hand, by the condition of the Lemma, $\int_a^b z_m(t)dt \to \int_a^b x(t)dt$, from which the equality $\int_a^b z(t)dt = \int_a^b x(t)dt$ follows. Since a and b are arbitrary, we get $x = z$. Thus we have $z_m \to x$ everywhere on (0,1) and therefore $z_m \to x$ in the measure topology.

THEOREM 5. *Let E be a rotund reflexive symmetric space such that E and E^* have the property (Hm). Then E is locally uniformly convex.*

Proof. Let $x \in S_E$, $(x_n) \subset S_E$, $x = x^*$, $x_n = x_n^*$ for all $n \in \mathbb{N}$ and $\|x_n + x\|_E \to 2$. Choose a sequence $(f_n) \subset S_{E^*}$ such that the equalities $f_n = f_n^*$ and

$$\lim_{n\to\infty} [x_n + x, f_n] = 2 \tag{5}$$

hold, where $[x, f]$ denotes the bilinear form connected with the duality between E and E^*.

Passing, if necessary, to subsequences we may consider that (x_n) and (f_n) weakly converge to some elements $y \in B_E$ and $f \in B_{E^*}$, respectively. By Lemma 2, $f_n \to f$ in the measure topology. It follows from (5) and from the inequality $|[x_n, f_n]| \leqslant 1$ that $|[x, f]| = \lim_{n\to\infty} |[x, f_n]| = 1$, i.e. $\|f\|_{E^*} = 1$. Since E^* has the property (Hm), we have

$$\|f_n - f\|_{E^*} \to 0 \tag{6}$$

Since $x_n \to y$ weakly, then using (5) and (6) we get

$$[x+y,f] = \lim_{n\to\infty}[x+x_n,f] = \lim_{n\to\infty}([x+x_n,f-f_n]+[x+x_n,f_n]) = 2.$$

Therefore, $\|x+y\|_E = 2$ and so $\|y\|_E = 1$. Since E is rotund, we have $x = y$, i.e. x_n converges weakly to x. By Theorem 1, E has the property (H). Thus, we have $\|x_n - x\|_E \to 0$. Applying Lemma 1 concludes the proof of the theorem.

In particular, it follows from this theorem that a reflexive Orlicz space L_M is locally uniformly convex.

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