THE SEMI-KADEC-KLEE CONDITION AND NEAREST-POINT PROPERTIES OF SETS IN NORMED LINEAR SPACES

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INTRODUCTION

In the Euclidean plane, a nonempty set is closed and convex if and only if it is *Chebyshev*; that is, for each point x in the plane there is a unique element of the set closest to x. This fact from approximation theory, usually called Motzkin's theorem, was actually first published by L. N. H. Bunt [9] in 1934, with T. S. Motzkin's articles [38] and [39] on the same subject appearing the next year. Since Bunt's theorem gives such a nice characterization of the Chebyshev sets in the Euclidean plane, efforts were soon underway to extend it to other normed linear spaces. The work of B. Jessen [30], H. Busemann [10], [11], and N. V. Efimov and S. B. Stechkin [20] in the 1940s and 1950s has led to a complete characterization of the finite-dimensional normed linear spaces in which the conclusion of Bunt's theorem holds. These are the finite-dimensional spaces that are *rotund* and *smooth*; that is, whose unit spheres contain no line segments and no points through which pass two distinct hyperplanes supporting the unit ball.

Much less is known in the infinite-dimensional case. There is no difficulty in deciding when nonempty closed convex sets are all Chebyshev. In 1941, M. M. Day [14] showed that this is true for any Banach space that is rotund and reflexive, while the converse is an easy application of the famous 1964 theorem of R. C. James [27]. Also, it is easy to see that

Chebyshev sets are always nonempty and closed. The problem is in deciding when all Chebyshev sets are convex. It was shown by Efimov and Stechkin [19] in 1958 that any space whose unit sphere contains an exposed point through which pass two distinct hyperplanes supporting the unit ball has a nonconvex Chebyshev set that is the union of two closed half-spaces. Beyond this, almost nothing is known about the convexity of arbitrary Chebyshev sets in infinite-dimensional spaces. In fact, there is no infinite-dimensional normed linear space in which the statement that every Chebyshev set is convex is known to be true, while there is no smooth space in which it is known to be false. In particular, *it is not known whether every Chebyshev set in classical Hilbert space is convex.* This is somewhat startling, since the geometry of classical Hilbert space is often treated as if it were completely understood.

Because of the difficulties involved with proving the convexity of an arbitrary Chebyshev set in a given infinite-dimensional space, the trend over the last two decades has been toward proving the convexity of Chebyshev sets subject to additional constraints. In 1961, V. L. Klee [35] showed that in a certain class of spaces that includes all Hilbert spaces, the nonempty closed convex sets are exactly the weakly closed Chebyshev sets. In 1970, I. Singer [46] improved Klee's result by extending it to a larger class of spaces. A further extension of this result will be given in this thesis. Other researchers have imposed different conditions on Chebyshev sets to force their convexity in particular spaces. Some of these results will be surveyed in the early parts of this thesis. More complete discussions of such results and of the connection between convexity and nearest-point properties of sets can be found in the survey articles of L. P. Vlasov [57] and T. D. Narang [40].

The results of Klee and Singer mentioned in the last paragraph have the following form:

> In a Banach space satisfying condition A, the nonempty closed convex sets are exactly the Chebyshev sets satisfying condition B.

One of the original goals of the research leading to this thesis was to discover theorems of this type with conditions A and B no stronger than necessary. By the result of Day and James mentioned above, condition A must include rotundity and reflexivity. If a Banach space is rotund and reflexive but not smooth, then it contains a nonconvex Chebyshev set that is the union of two closed half-spaces by one of the results of Efimov and Stechkin mentioned earlier. This union will share almost all of the attributes of a nonempty closed convex set other than convexity itself; for example, it will be weakly closed. In such a space, about the best we can expect is that the nonempty closed convex sets are exactly the Chebyshev sets that are convex, which is just the conclusion of the Day-James theorem in disguise. Thus, condition A should include rotundity, reflexivity, *and* smoothness if we are to obtain results besides the Day-James theorem. In fact, all of the choices for condition

B that have appeared in the literature have seemed to require even more than rotundity, smoothness, and reflexivity for condition A. For example, Vlasov [57] has shown that in a smooth space that is strongly rotund, a condition properly stronger than rotundity and reflexivity, the nonempty closed convex sets are exactly the Chebyshev sets such that the *metric projection*, the map from each point in the space to the nearest point in the set, is continuous. These additional geometric hypotheses inserted into condition A have always been reasonably strong, and in particular properly stronger than the *semi-Kadec-Klee* condition, a statement about the behavior of certain sequences in the dual unit sphere introduced by Vlasov in [56].

The first part of this thesis, consisting of Sections 1, 2, and 3, is concerned with the improvement of some known results of the type considered in the last paragraph by showing that they remain true when condition A includes only rotundity, reflexivity, smoothness, and the semi-Kadec-Klee condition. Along the way, we will obtain some other new results which will be fitted into the framework of a short survey of part of this branch of approximation theory. The second part of this thesis, consisting of Sections 4 through 7, is devoted to the study of the semi-Kadec-Klee condition and its connection with *supportive compactness*, a new property related to the well-known concept of approximative compactness. It will be shown that the existence of the semi-Kadec-Klee property in a space is closely related to the supportive compactness of the nonempty closed convex sets in that space.

We now describe more specifically the content of this thesis.

Section 1 is a preliminary section. Here we define many of the classes of normed linear spaces used in this thesis. We also translate many of these definitions into statements about the convergence properties of a certain type of sequence in the unit sphere, which immediately become statements about convergence properties of certain sequences, the *minimizing sequences*, in nonempty closed convex sets.

In Section 2, we obtain several new results characterizing certain classes of normed linear spaces in terms of the approximation-theoretic properties of their nonempty closed convex sets. A short survey of other known results of this type is also given. The proofs in this section tend to be very short, since most of the work is really done in Section 1.

The purpose of Section 3 is to prove some results of the following form:

> In a rotund, reflexive, smooth semi-Kadec-Klee space, the nonempty closed convex sets are exactly the Chebyshev sets satisfying some condition.

Seven different choices for this condition are offerred. In so doing, we extend some known results of Klee, Singer, and Vlasov. The results of Section 2 are used extensively here.

Beginning with Section 4, we embark on a study of semi-Kadec-Klee spaces. It is shown in Section 4 that the semi-Kadec-Klee condition is equivalent to a semicontinuity property of the norm-duality map J from a space into its dual. Some related results about the continuity properties of J are also proved.

Section 5 introduces the concept of supportive compactness of a set. The basic properties of supportive compactness are obtained.

In Section 6, we study the interaction between the supportive compactness of convex sets and the semi-Kadec-Klee condition. In particular, we characterize the rotund, reflexive, smooth semi-Kadec-Klee spaces as being the spaces where the nonempty closed convex sets are exactly the Chebyshev sets that are supportively weakly compact. This is in contrast to the results of Section 3, where none of the derived properties of rotund, reflexive, smooth semi-Kadec-Klee spaces are shown to characterize such spaces.

Section 7 contains some miscellaneous results about supportive compactness. Its relationship to approximative compactness is studied. A result of Vlasov about the convexity of certain Chebyshev sets is generalized to P-convex sets. In addition, we characterize the reflexive spaces in which every closed ball is a supportively weakly compact Chebyshev set.

There are also two appendices. The purpose of Appendix A is to show that many of the results of Sections 1, 2, and 3 that appear to depend on James's theorem can actually be obtained from more basic principles, though the proofs are somewhat less compact. This appendix also contains an elementary proof of James's theorem for a large class of spaces.

Appendix B contains some material about the approximative compactness of closed balls that is tangentially related to the results of Section 7.

The survey material in Section 2 could logically be omitted, since it is readily accessible in the literature. We have three reasons for inserting it. First, its inclusion makes this presentation reasonably self-contained. Second, many of these results are stated and proved in a slightly more general form than is usual. Third, it is shown in Appendix A that many of these results, previously believed to be quite deep because they are corollaries of James's theorem, can actually be obtained by our method of proof from the Bishop-Phelps theorem [4].

The notation we use is standard and follows that of Day [16]. The letter N always denotes a real normed linear space, and N* indicates its dual space. The letter B is used for a real Banach space. In spite of this standardization, we frequently preface our results with "Let N be a normed linear space" or "Let B be a Banach space". This is to emphasize the presence or absence of a completeness hypothesis. The unit sphere of N, $\{x \in N: \|x\| = 1\}$, is denoted by Σ, and the closed unit ball of N, $\{x \in N: \|x\| \leq 1\}$, by U. The symbols Σ' and U^{π} denote the corresponding objects for N*. Additional superscript primes and pi's extend this notation to higher duals in the obvious way. The map $Q: N \to N^{**}$ is the canonical embedding of N into its second dual. The Banach spaces c_0 and ℓ_p, $1 \leq p \leq \infty$, are defined as in Day [16].

The distance from a point x to a set M is the usual metric distance given by

$$d(x,M) = \inf \{\|x - y\|: y \in M\}.$$

The symbols "$\xrightarrow{\tau}$" and "τ-lim" denote, respectively, convergence and limit in the τ-topology. If τ is not specified, the norm topology is assumed. We frequently use "w" and "w*" as abbreviations for "weak" and "weak*" respectively, not to be confused with "ω", which usually designates a sequential property. The abbreviations "w. r. t." and "w. l. o. g." stand for "with respect to" and "without loss of generality" respectively. The symbol "∎" will mark the end of a proof or example.

Since we refer frequently to James's theorem, we should state here the version that we use.

<u>THEOREM</u> (James [27], [29]). *A Banach space is reflexive if and only if every continuous linear functional in the dual space achieves its supremum on the closed unit ball of the space.*

We also list two other results used frequently and sometimes implicitly.

<u>THEOREM</u> (Ascoli [2]). *Let f be a nonzero continuous linear functional on N, let c be a real number, and let x be an element of N. Then the distance from x to the hyperplane $H = \{y \in N: f(y) = c\}$ is given by $d(x,H) = \|f\|^{-1}|c - f(x)|$.*

<u>THEOREM</u>. *If* (x_α) *is a net in* N *and* $x_\alpha \overset{w}{\to} x$, *then* $\|x\| \leq \liminf \|x_\alpha\|$. *Also, if* (f_α) *is a net in* N^* *and* $f_\alpha \overset{w^*}{\to} f$, *then* $\|f\| \leq \liminf \|f_\alpha\|$. *That is, norms and conjugate norms are lower semicontinuous in the weak and weak* topologies respectively.*

This last result can be found in Day [16], as can all of the standard facts about normed spaces used in this thesis.

Our method of crediting results to others is as follows. If a result has appeared in the literature substantially as it is stated here, then a reference is given in our statement of that result. If a result is partly ours and partly another's, then an explanation of which part belongs to whom is given in the discussion preceding the result. Otherwise, the result is new. In some cases, a new result is based on a clever argument devised by someone else for a different purpose. In this case, appropriate credit and references are given.

We close this introduction with an explanation of our transliteration of Cyrillic names. With one exception, we use the English phonetic transliteration instead of the Czech diacritical; for example, we write "Chebyshev" instead of "Čebyšev". The lone exception is the last name of V. L. Šmulian. In several articles published in French and English, Šmulian transliterated his name as it appears in this sentence. Since one should be able to control the spelling of one's own name, we use Šmulian's transliteration.

SECTION 1

PRELIMINARIES

The purpose of this section is to give the definitions of some classes of normed linear spaces used throughout this thesis and then to obtain characterizations of some of these classes in terms of the behavior of certain sequences. These characterizations are used in Section 2 to simplify the proofs of the main results of that section.

1.1 <u>DEFINITION</u>. A normed linear space N is said to be

(R) *rotund* or *strictly convex* if Σ contains no intervals;

(UR) *uniformly rotund* if $\inf\{1 - \frac{1}{2}\|x + y\| : x, y \in \Sigma, \|x - y\| \geq \varepsilon\} > 0$ when $0 < \varepsilon \leq 2$ (Clarkson [12]);

(wUR) *weakly uniformly rotund* if $\inf\{1 - \frac{1}{2}\|x + y\| : x, y \in \Sigma, |f(x - y)| \geq \varepsilon\} > 0$ for each $f \in \Sigma'$ and each ε with $0 < \varepsilon \leq 2$ (Šmulian [51]);

(LUR) *locally uniformly rotund* if $\inf\{1 - \frac{1}{2}\|x + y\| : y \in \Sigma, \|x - y\| \geq \varepsilon\} > 0$ for each $x \in \Sigma$ and each ε with $0 < \varepsilon \leq 2$ (Lovaglia [36]);

(wLUR) *weakly locally uniformly rotund* if $\inf\{1 - \frac{1}{2}\|x + y\| : y \in \Sigma, |f(x - y)| \geq \varepsilon\} > 0$ for each $x \in \Sigma$, $f \in \Sigma'$, and ε with $0 < \varepsilon \leq 2$ (Lovaglia [36]);

(K) *strongly rotund* if, whenever K is a nonempty convex set in N, then the diameter of the intersection of K with tU tends to zero as t decreases toward the distance from 0 to K (Šmulian [51]);

(D) a *strongly rotund Banach space* if N is (K) and a Banach space;

(S) *smooth* if each point of Σ is a *point of smoothness* of U; that is, a point through which passes only one hyperplane supporting U;

(F) *Fréchet smooth* if the norm is Fréchet differentiable on Σ;

(UG) *uniformly Gateaux smooth* if the norm is uniformly Gateaux differentiable on Σ;

(US) *uniformly smooth* if the norm is uniformly Fréchet differentiable on Σ;

(H) a *Kadec-Klee* or *Radon-Riesz space* if, whenever (x_n) is a sequence in Σ, $x \in \Sigma$, and $x_n \xrightarrow{w} x$, then $x_n \to x$;

(Rf) a *reflexive space* if $Q(N) = N^{**}$.

The classes of spaces in the above definition are all well-known and have been extensively studied. For a summary of their most important properties, see Day [16].

Sequential Characterizations

While the definitions given on the last two pages are the usual ones for these classes of spaces, they are not always the most useful ones for our purposes. In many cases it is more convenient to have a characterization of a class of spaces in terms of the convergence properties of certain sequences in nonempty closed convex sets. We now prove a series of propositions doing exactly that.

In what follows, we use the usual convention that x_n denotes an element of a sequence, while x_α designates an element of a net. As always, N is a normed linear space.

1.2 DEFINITION. Let M be a nonempty subset of N, and let $x \in N$. A net (x_α) in M is a *minimizing net* for x if $\|x_\alpha - x\|$ tends to $d(x,M)$.

1.3 PROPOSITION. *The following are equivalent.*

(1) N is (R).

(2) Whenever $x_n \in \Sigma$, $f \in \Sigma'$, and $f(x_n) \to 1$, then all weakly convergent subsequences of (x_n) have the same limit.

(3) Whenever K is a nonempty closed convex set in N and (x_n) is a minimizing sequence in K for $x \in N$, then all weakly convergent subsequences of (x_n) have the same limit.

Proof. Suppose N is (R), and let (x_n) and f be as in the hypothesis of (2). If (x_{n_k}) is a subsequence of (x_n) converging weakly to some $x \in N$, then $f(x) = 1$ and $\|x\| = 1$. By the rotundity of N, there can be only one such $x \in \Sigma$ where f attains its supremum on U, and so all weakly convergent subsequences of (x_n) have the same limit. Conversely, if Σ contains a line segment, then it is easy to construct a sequence satisfying the hypothesis but not the conclusion of (2). Thus, (1) and (2) are equivalent.

Suppose (2) holds. Let K, (x_n), and x be as in the hypothesis of (3). W. l. o. g. $x \notin K$, and we can in fact assume that $x = 0$ and $d(0,K) = 1$. Let $f \in \Sigma'$ be such that the hyperplane $\{y \in N: f(y) = 1\}$ separates U from K. For each n, $1 \leq f(x_n) \leq \|x_n\| \to 1$. Thus, $y_n = \|x_n\|^{-1}x_n$ gives a sequence in Σ such that $f(y_n) \to 1$. By (2), all weakly convergent subsequences of (y_n) have the same limit, and so the same will be true of (x_n). Thus, (2) implies (3).

Finally, suppose that (3) holds. Let $y_n \in \Sigma$ and $f \in \Sigma'$ be such that $f(y_n) \to 1$. Let $K = \{y \in N: f(y) = 1\}$. W. l. o. g. $f(y_n) > 0$ for all n, so $x_n = (f(y_n))^{-1}y_n \in K$ for all n. Since $\|x_n\| = (f(y_n))^{-1} \to 1$, (x_n) is a minimizing sequence in K for 0, and so all weakly convergent subsequences of (x_n) have the same limit. Since the same will be true of (y_n), we see that (2) holds. Thus, (3) implies (2). ∎

In Propositions 1.4 through 1.7, we only prove that (1) and (2) are equivalent. The proof that (2) and (3) are equivalent then follows by an argument similar to that used in the proof of Proposition 1.3.

1.4 PROPOSITION. *The following are equivalent.*

(1) N is (Rf).

(2) Whenever $x_n \in \Sigma$, $f \in \Sigma'$, *and* $f(x_n) \to 1$, *then* (x_n) *has a weakly convergent subsequence.*

(3) Whenever K is a nonempty closed convex set in N and (x_n) *is a minimizing sequence in K for* $x \in N$, *then* (x_n) *has a weakly convergent subsequence.*

Proof. It is obvious that (1) implies (2). To get the reverse implication, we first suppose that N is not complete. Let (x_n) be a nonconvergent Cauchy sequence in Σ. Now (x_n) has a limit x in the completion N^c of N. Since N* can be identified with N^{c*} in the usual way, there is some $f \in \Sigma'$ such that $f(x_n)$ tends to $f(x) = 1$. Since (x_n) cannot have a subsequence weakly convergent in N, we see that (2) cannot hold.

Thus, (2) implies completeness. Now suppose that (2) holds. Let $f \in \Sigma'$, and let (x_n) be a sequence in Σ with $f(x_n) \to 1$. W. l. o. g. $x_n \overset{w}{\to} x$. Then $\|x\| \leq 1$ and $f(x) = 1$. Thus, each $f \in \Sigma'$ attains its supremum on U. By James's theorem, N is reflexive. ■

The use of James's theorem in the last proof should be noted. James's theorem is a very deep result, which would in turn seem to make Proposition 1.4 very deep. Actually, we have a proof of Proposition 1.4 that avoids the use of James's theorem altogether. The proof does use the Bishop-Phelps theorem [4], which is itself a nontrivial result. However, the proof of the Bishop-Phelps theorem is arguably more elementary than any known proof of James's theorem. It would be distracting to present this alternate proof of Proposition 1.4 here, especially since it depends on a lemma which we would like to postpone until Section 5. This alternate proof is presented in Appendix A, along with some observations about James's theorem and an elementary proof of that theorem for a large class of spaces.

The importance of having an elementary proof of Proposition 1.4 is that much of what follows depends on this result. In particular, we use it in the next section to prove Theorem 2.5. The equivalence of (1) and (3) in that theorem is a well-known result that has always been treated as a corollary of James's theorem, when it is in fact more elementary.

1.5 PROPOSITION. *The following are equivalent.*

(1) N is (R) & (Rf).

(2) Whenever $x_n \in \Sigma$, $f \in \Sigma'$, and $f(x_n) \to 1$, then (x_n) is weakly convergent.

(3) Whenever K is a nonempty closed convex set in N and (x_n) is a minimizing sequence in K for $x \in N$, then (x_n) is weakly convergent.

Proof. We just apply Propositions 1.3 and 1.4. ■

The next three propositions along these lines are well-known results.

1.6 PROPOSITION (Fan and Glicksberg [22]). *The following are equivalent.*

(1) N is (K).

(2) Whenever $x_n \in \Sigma$, $f \in \Sigma'$, *and* $f(x_n) \to 1$, *then* (x_n) *is Cauchy.*

(3) Whenever K is a nonempty closed convex set in N and (x_n) *is a minimizing sequence in K for* $x \in N$, *then* (x_n) *is Cauchy.*

Proof. We need the fact that (K) is equivalent to the following condition:

(ν) For every $f \in \Sigma'$, the diameter of the slice $\nu(f,\delta) = \{x \in U: f(x) \geqslant 1 - \delta\}$ tends to zero as δ decreases to zero.

This is easy to prove; see also Day [16]. Given this, it is easy to see that (1) implies (2).

Suppose N is not (ν). Then there is some $f \in \Sigma'$ such that $\nu(f,\delta)$ does not have diameter tending to zero as δ decreases to zero. It is not difficult to use this to find a sequence (x_n) in Σ satisfying the hypothesis but not the conclusion of (2). ∎

1.7 PROPOSITION (Fan and Glicksberg [22]). *The following are equivalent.*

(1) N *is (D).*

(2) *Whenever* $x_n \in \Sigma$, $f \in \Sigma'$, *and* $f(x_n) \to 1$, *then* (x_n) *converges.*

(3) *Whenever* K *is a nonempty closed convex set in* N *and* (x_n) *is a minimizing sequence in* K *for* $x \in N$, *then* (x_n) *converges.*

Proof. After applying Proposition 1.6, all that is left is to prove that (2) implies completeness, which is easy either directly or via Proposition 1.5. ∎

It is natural to ask for an analog of Proposition 1.7 for the case where we only require the sequences to have convergent subsequences. For this we need a definition.

1.8 DEFINITION (Singer [45]; Vlasov [57]). A normed linear space is called an *Efimov-Stechkin space* if it possesses the following property:

(CD) Whenever $x_n \in \Sigma$, $f \in \Sigma'$, and $f(x_n) \to 1$, then (x_n) has a convergent subsequence.

The proof of the following proposition is similar to that of the equivalence of (2) and (3) in Proposition 1.3.

1.9 PROPOSITION (Singer [45]). *The following are equivalent.*

(1) N is (CD).

(2) Whenever K is a nonempty closed convex set in N and (x_n) *is a minimizing sequence in K for* $x \in N$*, then* (x_n) *has a convergent subsequence.*

The following characterization of smooth spaces by Šmulian is analogous to the above propositions, but the resulting convergence property applies to sequences in N* rather than N. We offer a proof different from the one in Šmulian's paper.

1.10 PROPOSITION (Šmulian [49]). *The following are equivalent.*

(1) N is (S).

(2) Whenever $f_n \in \Sigma'$*,* $x \in \Sigma$*, and* $f_n(x) \to 1$*, then* (f_n) *is weak* convergent.*

The weak limit in (2) is the unique* $f \in \Sigma'$ *such that* $f(x) = 1$*.*

Proof. Suppose N is smooth, and let (f_n) and x be as in the hypothesis of (2). Let f be the unique element of

Σ' such that $f(x) = 1$. Suppose (f_n) does not converge weak* to f. Then there is a weak* neighborhood W of f and a subsequence of (f_n) that avoids W; w. l. o. g. (f_n) lies outside W. By the Banach-Alaoglu theorem, (f_n) contains a weak* convergent subnet (f_α). Let g be the limit of this subnet. Then

$$1 = \lim f_n(x) = \lim f_\alpha(x) = g(x) \leqslant \|g\| \leqslant 1,$$

and so $\|g\| = g(x) = 1$. Thus, $g = f$. Since (f_α) does not enter W, this is a contradiction.

Conversely, suppose (2) holds. If $x \in \Sigma$ and $\|f\| = \|g\| = f(x) = g(x) = 1$, then applying (2) to the sequence $(f, g, f, g, \ldots)$ shows that $f = g$. Thus, N is smooth. ■

We now prove several classical results, both for their later usefulness and because of their easy derivation from the above results.

1.11 <u>THEOREM</u> (Fan and Glicksberg [22]). *N is (D) if and only if N is (R) & (Rf) & (H).*

<u>Proof</u>. This is an easy application of the definition of condition (H) and Propositions 1.5 and 1.7. ■

The following theorem follows easily from the definitions and from Proposition 1.4. The proof of the corollary uses Theorem 1.11.

1.12 THEOREM (Vlasov [57]). *N is (CD) if and only if N is (Rf) & (H).*

1.13 COROLLARY (Singer [45]). *N is (D) if and only if N is (R) & (CD).*

Weak Collapse and (wK) Spaces

Proposition 1.6 characterizes the normed linear spaces in which minimizing sequences in nonempty closed convex sets are Cauchy, with the resulting class being the strongly rotund spaces. It is useful to have a similar characterization for spaces in which such minimizing sequences are weakly Cauchy. We do this by replacing the shrinking diameter in the definition of strongly rotund spaces with a weaker property.

1.14 DEFINITION. Let (A_α) be a net of nonempty subsets of a normed linear space N. Then (A_α) *weakly collapses* if $x_\alpha - y_\alpha \overset{w}{\to} 0$ whenever $x_\alpha, y_\alpha \in A_\alpha$ for each α.

The proof of the next lemma is elementary and will be omitted.

1.15 LEMMA. *The following are equivalent.*

(1) (A_α) weakly collapses.

(2) *For each* $f \in N^*$, $\lim_\alpha (\sup \{f(x): x \in A_\alpha\} - \inf \{f(x): x \in A_\alpha\}) = 0$.

(3) *For each* $f \in N^*$, $\lim_\alpha (\sup \{f(x - y): x, y \in A_\alpha\}) = 0$.

1.16 EXAMPLE. It is obvious that if the diameter of (A_α) tends to 0, then (A_α) weakly collapses. The converse is not true, even for a nested sequence of sets in a Banach space. Let (e_n) be the usual sequence of unit vectors in c_0, and let $A_n = \{e_m: m \geqslant n\}$. Then the diameter of A_n is 1 for all n, even though (A_n) weakly collapses. ■

Since the strongly rotund spaces are those for which certain sets shrink in diameter to zero, it makes sense to define the weakly rotund spaces to be those in which the same sets weakly collapse.

1.17 DEFINITION. A normed linear space N is said to have property

(wK) if, whenever K is a nonempty convex set in N, then the intersection of K with tU weakly collapses as t decreases to d(0,K); these spaces are called *weakly rotund*;

(wK_ω) if, whenever K is a nonempty convex set in N and (x_n) is a sequence of elements of K with norm tending to d(0,K), then (x_n) is weakly Cauchy;

(wν) if, for every $f \in \Sigma'$, the slice $\nu(f,\delta) = \{x \in U: f(x) \geqslant 1 - \delta\}$ weakly collapses as δ decreases to 0;

(wLν) if, for every $f \in \Sigma'$ that achieves its supremum on U, the slice $\nu(f,\delta)$ weakly collapses as δ decreases to 0.

We really need only consider closed half-spaces in the definition of (wK), rather than all nonempty convex sets. Suppose that the stated condition holds for closed half-spaces. Let K be any nonempty convex set. W. l. o. g. $t_0 = d(0,K) > 0$. Let H be a closed half-space containing K determined by a hyperplane separating t_0U and K. Since the intersection of H with tU weakly collapses as t decreases to t_0, so does the intersection of K with tU.

In the next proposition, we assign conditions (wK) and (wLν) their proper places among the common rotundity conditions and show that the first three classes of spaces in Definition 1.17 are really the same.

1.18 PROPOSITION.

(a) *(wK)* $\Longleftrightarrow$ *(wK$_\omega$)* $\Longleftrightarrow$ *(wν)*.

(b) *(K)* $\Longrightarrow$ *(wK)* $\Longrightarrow$ *(wLν)* $\Longrightarrow$ *(R)*.

(c) *(wUR)* $\Longrightarrow$ *(wK)*.

(d) *(wLUR)* $\Longrightarrow$ *(wLν)*.

<u>Proof</u>. Suppose N is (wK). Fix $f \in \Sigma'$, and let $M(\delta)$ be the intersection of $(1-\delta)^{-1}U$ with $\{x \in N: f(x) \geqslant 1\}$. Then $M(\delta)$ weakly collapses as $\delta \downarrow 0$, and hence so does $\nu(f,\delta) = (1-\delta)M(\delta)$. Thus, N is (w$\nu$).

Now suppose that N is (wν), and let K be a nonempty convex set in N and (x_n) a sequence of elements of K with $\|x_n\| \to d(0,K)$. W. l. o. g. $d(0,K) = 1$. Let $f \in \Sigma'$ determine a hyperplane $\{x \in N: f(x) = 1\}$ separating K from U. Since $\|x_n\|^{-1} \leqslant f(\|x_n\|^{-1}x_n)$, it follows that $\|x_n\|^{-1}x_n$ lies in $\nu(f, 1-\|x_n\|^{-1})$ for each n. Since $\nu(f, 1-\|x_n\|^{-1})$ weakly collapses as $n \to \infty$, $(\|x_n\|^{-1}x_n)$ is weakly Cauchy, and hence (x_n) is also. Thus, N is (wK_ω).

Next, suppose that N is (wK_ω). An easy application of the equivalence of (1) and (3) in Lemma 1.15 shows that N is (wK). This completes the proof of (a).

For (b), the first two implications are trivial consequences of the definitions and of (a). For the last, suppose that N is not rotund, and let L be a line segment in Σ. By the Hahn-Banach theorem, we can find a hyperplane H containing L and supporting U. Let $f \in \Sigma'$ be such that $H = \{x \in N: f(x) = 1\}$. Then f achieves its supremum on U, but $\nu(f,\delta)$ does not weakly collapse as $\delta \downarrow 0$, since L lies in $\nu(f,\delta)$ for all $\delta > 0$. Thus, N is not (wLν).

For (c), let N be (wUR). We will show that N is (wν). Let $f \in \Sigma'$. For any $\delta \in (0,1)$ and $x, y \in \nu(f,\delta)$, we note that $x' = \|x\|^{-1}x$, $y' = \|y\|^{-1}y$, and $\frac{1}{2}(x' + y')$ are all in $\nu(f,\delta)$. Thus, $1 - \delta \leqslant \frac{1}{2}\|x' + y'\|$ and $1 - \frac{1}{2}\|x' + y'\| \leqslant \delta$. Let

$g \in \Sigma'$ and $0 < \varepsilon \leqslant 2$, and restrict δ so that

$$0 < \delta < \inf \{1 - \tfrac{1}{2}\|u + v\| : u, v \in \Sigma, |g(u - v)| \geqslant \varepsilon\}.$$

Then $g(x' - y') < \varepsilon$. Since

$$\begin{aligned} g(x - y) &= g(x' - y') + g(x - x' - y + y') \\ &< \varepsilon + \|x' - x\| + \|y' - y\| \\ &\leqslant \varepsilon + 2\delta, \end{aligned}$$

it follows that $\sup \{g(u - v) : u, v \in \nu(f,\delta)\} \to 0$ as $\delta \downarrow 0$. By Lemma 1.15, $\nu(f,\delta)$ weakly collapses as $\delta \downarrow 0$, and so N is (wν).

To prove (d), let N be (wLUR). Suppose $f \in \Sigma'$ achieves its supremum on U at $x \in \Sigma$. By a proof analogous to that of the last paragraph, it can be shown that $\sup \{|g(u - x)| : u \in \nu(f,\delta)\}$ tends to 0 as $\delta \downarrow 0$. It follows immediately that $\sup \{g(u - v) : u, v \in \nu(f,\delta)\} \to 0$ as $\delta \downarrow 0$. Applying Lemma 1.15, $\nu(f,\delta)$ weakly collapses as $\delta \downarrow 0$, and so N is (wLν). ■

The class of weakly rotund spaces has appeared in the literature before, as Cudia's spaces "weakly uniformly rotund in each direction" in [13] and Yorke's spaces "weakly rotund at S(E*) in the S(E*) directions" from [61], where the definitions given were essentially in the form (wν). In view of the definition of strong rotundity, the term "weakly rotund" seems sufficient and will be used here.

This next proposition is really just a restatement of Proposition 1.18 (a), but we include it here because it complements Propositions 1.3 through 1.7 and Proposition 1.9. In particular, notice that it is a weak analog of Proposition 1.6.

1.19 PROPOSITION. *The following are equivalent.*

(1) N is (wK).

(2) Whenever $x_n \in \Sigma$, $f \in \Sigma'$, *and* $f(x_n) \to 1$, *then* (x_n) *is weakly Cauchy.*

(3) Whenever K is a nonempty closed convex set in N and (x_n) *is a minimizing sequence in K for* $x \in N$, *then* (x_n) *is weakly Cauchy.*

Proof. It is obvious that (2) holds whenever N is (wν), and so (1) implies (2). Conversely, if N is not (wν), then there is some $f \in \Sigma'$ such that $\nu(f,\delta)$ does not weakly collapse as $\delta \downarrow 0$. It is not difficult to use this to find a sequence (x_n) in Σ satisfying the hypothesis but not the conclusion of (2). Thus, (2) implies (1).

The equivalence of (2) and (3) is established as in Proposition 1.3. ∎

We can require more of the sequences in spaces of type (wLν), but we have to be a bit more restrictive about which sequences we consider.

1.20 <u>PROPOSITION</u>. *The following are equivalent.*

(1) *N is (wLν).*

(2) *Whenever* $x_n \in \Sigma$, $f \in \Sigma'$, *f achieves its supremum on U, and* $f(x_n) \to 1$, *then* (x_n) *is weakly Cauchy.*

(2') *Whenever* $x_n \in \Sigma$, $f \in \Sigma'$, *f achieves its supremum on U, and* $f(x_n) \to 1$, *then* (x_n) *is weakly convergent.*

(3) *Whenever K is a nonempty closed convex set in N,* (x_n) *is a minimizing sequence in K for* $x \in N$, *and there is a* $y \in K$ *with* $\|x - y\| = d(x,K)$, *then* (x_n) *is weakly Cauchy.*

(3') *With K,* (x_n), *x, and y as in (3),* (x_n) *is weakly convergent.*

In (2'), the weak limit is the unique $z \in \Sigma$ *where f achieves its supremum on U. In (3'), the weak limit is y.*

<u>Proof</u>. The equivalence of (1) and (2) can be established as in Proposition 1.19. The equivalence of (2) and (3) can be established as in Proposition 1.3 once we notice that the functional $f \in \Sigma'$ in the first half of that proof attains its supremum on Σ when we require K to have a point nearest x.

Suppose (2) holds. Let x_n and f be as in the hypothesis of (2'). Now N is (wLν) and hence rotund, so there is a unique z in Σ such that $f(z) = 1$. Let (y_n) be the sequence $(x_1, z, x_2, z, x_3, z, \dots)$. Then (y_n) lies in Σ and $f(y_n)$ tends to 1, so (y_n) is weakly Cauchy by (2). For any g in N*, it is immediate that the convergence of $(g(y_n))$ implies that $g(x_n) \to g(z)$. Thus, $x_n \overset{w}{\to} z$. This gives (2'). Since (2') obviously implies (2), (2) and (2') are equivalent. Notice that we have also proved the first remark following the list of equivalent conditions.

The proof that (3) and (3') are equivalent and the proof of the second remark following the list of equivalent conditions use the method of the previous paragraph. Note that $(x_1, y, x_2, y, \dots)$ is a minimizing sequence in K for x. ∎

Yorke and Cudia were interested in the weakly rotund spaces because they are exactly the spaces with smooth duals. The proofs of this given in [13] and [61] use the fact that a space is smooth if and only if its norm is Gateaux differentiable on Σ. The following proof, based on Propositions 1.10 and 1.19, is somewhat simpler and does not use differentiability properties of the norm.

1.21 THEOREM (Cudia [13]). *N is weakly rotund if and only if N* is smooth.*

Proof. Suppose N* is smooth. Let (x_n) be a sequence in Σ such that there is an $f \in \Sigma'$ with $f(x_n) \to 1$. Now $(Qx_n)(f)$ tends to 1, so (Qx_n) must be weak* convergent by Proposition 1.10 and the smoothness of N*. This implies that (x_n) is weakly Cauchy. By Proposition 1.19, N is weakly rotund.

Now suppose that N is weakly rotund. Since we need to treat N, N*, and N** simultaneously, we use here the common convention that the number of superscript asterisks on an element indicates where the element is located. Let (x_n^{**}) be a sequence in Σ'' such that there is an $x^* \in \Sigma'$ with $x_n^{**}(x^*) \to 1$. To show that N* is smooth, it is enough to show that (x_n^{**}) is weak* convergent and then apply Proposition 1.10. By the Banach-Alaoglu theorem, (x_n^{**}) has a weak* convergent subnet, so to show that (x_n^{**}) is weak* convergent, it is enough to show that any two such subnets have the same limit. To this end, let (x_α^{**}) be a weak* convergent subnet of (x_n^{**}) with limit x^{**}. Then

$$1 = \lim x_n^{**}(x^*) = \lim x_\alpha^{**}(x^*) = x^{**}(x^*) \leqslant \|x^{**}\| \leqslant \liminf \|x_\alpha^{**}\| = 1,$$

so $x^{**} \in \Sigma''$ and $x^{**}(x^*) = 1$. Now suppose that two such subnet limits y^{**} and z^{**} were unequal, and let $y^* \in \Sigma'$ be such that $y^{**}(y^*) \neq z^{**}(y^*)$. By an easy application of Goldstine's theorem that Q(U) is weak* dense in $U^{\pi\pi}$, we can find a sequence (y_n) in Σ such that $y^*(y_n) \to y^{**}(y^*)$ and $x^*(y_n) \to y^{**}(x^*) = 1$. Let (z_n) be a corresponding sequence for z^{**}. Let the sequence (x_n) be given by $(x_n) = (y_1, z_1, y_2, z_2, \ldots)$. Then (x_n) lies in Σ and $x^*(x_n) \to 1$, so (x_n) is weakly Cauchy by Proposition 1.19. Since $(y^*(x_n))$ must converge, $(y^*(y_n))$ and $(y^*(z_n))$ must have the same limit; that is, $y^{**}(y^*) = z^{**}(y^*)$. This contradiction establishes the theorem. ■

It is easy to prove that for reflexive spaces, rotundity of the space and smoothness of the dual are equivalent; see [16]. This, along with Proposition 1.18 (b) and the previous theorem, give the following result.

1.22 COROLLARY. *For reflexive spaces, rotundity, weak rotundity, and condition (wLν) are all equivalent.*

We can prove a result similar to Theorem 1.21 for (wLν) spaces, in which we cannot insist that all the points of Σ' be points of smoothness of U^{π}, but rather only the points representing functionals achieving their suprema on U. To do this, we need a lemma, whose proof is essentially the same as that of Proposition 1.10.

1.23 LEMMA (Šmulian [49]). *Let $x \in \Sigma$. Then the following are equivalent.*

(1) The point x is a point of smoothness of U.

(2) Whenever $f_n \in \Sigma'$ and $f_n(x) \to 1$, then (f_n) is weak convergent.*

The weak limit in (2) is the unique $f \in \Sigma'$ such that $f(x) = 1$.*

1.24 THEOREM. *N is (wLν) if and only if every f in Σ' that attains its supremum on U is a point of smoothness of U^{π}.*

Proof. The proof just follows that of Theorem 1.21, with Lemma 1.23 and Proposition 1.20 being used in place of Proposition 1.10 and Proposition 1.19 respectively. ■

In [53], Sullivan defined the *very rotund* spaces to be the spaces satisfying condition (2) of Theorem 1.24. Thus, our (wLν) spaces are exactly Sullivan's very rotund spaces. The following result of Sullivan's is immediate from Proposition 1.18 and the last theorem. We mention it because our proof avoids Sullivan's use of local reflexivity, though we do use Goldstine's theorem to prove Theorem 1.24.

1.25 COROLLARY (Sullivan [53]). *If N is (wLUR), then N is very rotund.*

Summary

Propositions 1.3 through 1.7, 1.9, and 1.19 all have basically the following form:

PROPOSITION. *The following are equivalent.*

(1) N is (class).

(2) Whenever $x_n \in \Sigma$, $f \in \Sigma'$, *and* $f(x_n) \to 1$, *then (condition on* (x_n)*).*

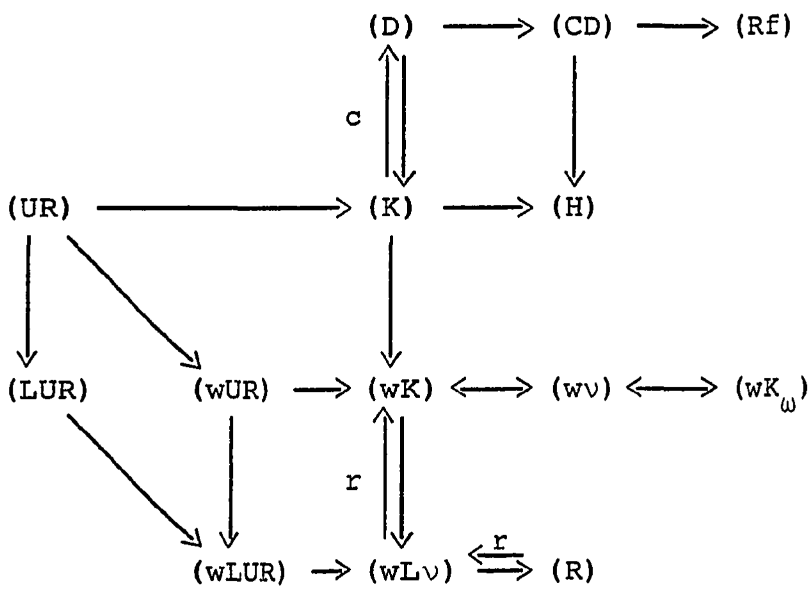
(3) Whenever K is a nonempty closed convex set in N and (x_n) is a minimizing sequence in K for $x \in N$, then (condition on (x_n)).

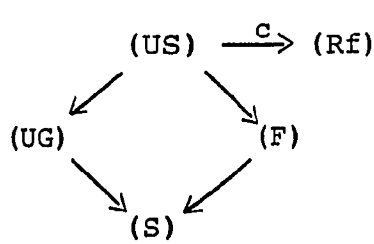
The following table summarizes the conditions corresponding to the classes.

1.26 TABLE. A summary of Propositions 1.3 through 1.7, 1.9, and 1.19.

Class	Condition on (x_n)
(R)	all weakly convergent subsequences of (x_n) have the same limit
(Rf)	(x_n) has a weakly convergent subsequence
(wK)	(x_n) is weakly Cauchy
(R) & (Rf)	(x_n) is weakly convergent
(CD)	(x_n) has a convergent subsequence
(K)	(x_n) is Cauchy
(D)	(x_n) converges

The following charts give the relationships between the classes of normed linear spaces defined above. All of these relationships are proved above, are easily deduced from the above, or can be found in Day [16]. An "r" above or beside an arrow indicates a relationship that exists in reflexive spaces, while a "c" above or beside an arrow indicates a relationship that holds for Banach spaces.





SECTION 2

APPROXIMATIVE COMPACTNESS AND CONTINUITY OF THE METRIC PROJECTION

This section contains several new results characterizing certain classes of normed linear spaces in terms of the approximation-theoretic properties of their closed convex sets, as well as a short survey of some well-known results of the same type. Although the classical results have always been stated and proved for Banach spaces, a minimal amount of extra effort yields these same results for arbitrary normed linear spaces, and so we offer them in this more general setting. The proofs tend to be very short anyway, since most of the work is done in Section 1.

The results of this section are used in the next section to prove the principal results of the first half of this thesis, Theorems 3.11 and 3.13.

Approximative Compactness

The following definition is due to Efimov and Stechkin [21] for the case of norm convergence, while the generalization to the weak topology is due to Breckner [6]. In this definition, as in the rest of this section, M is a nonempty subset of normed linear space N.

2.1 DEFINITION. M is *approximatively compact* (resp. *approximatively weakly compact*) if for each $x \in N$ every minimizing sequence in M has a subsequence converging in norm (resp. converging weakly) to an element of M.

2.2 DEFINITION. The *metric projection* onto M is the set-valued map P_M from N onto M that maps each $x \in N$ to the collection of all points of M closest to x. That is, y is in $P_M x$ if and only if y is in M and $\|x - y\| = d(x,M)$. As usual, the *domain of definition* of P_M is $\{x \in N: P_M x \neq \emptyset\}$.

When there is no possibility of confusion, we denote P_M by just P. When P is single-valued, we frequently treat it as if it were point-valued instead of set-valued; for example, we write $Px = y$ instead of $Px = \{y\}$.

2.3 DEFINITION. M is a *set of existence* or *proximinal* if Px is nonempty for each $x \in N$; that is, closest points always exist. M is a *set of uniqueness* if Px is either empty or a singleton for each $x \in N$. M is a *Chebyshev set* if Px is a singleton for each $x \in N$.

We collect here some standard properties of the above objects that follow from the definitions and from well-known properties of the norm; see Vlasov [57].

2.4 PROPOSITION.

(a) If (y_n) *is a minimizing sequence in* M *for* $x \in N$ *and converges weakly to* $y \in M$, *then* $y \in Px$.

(b) An approximatively compact set is approximatively weakly compact.

(c) An approximatively weakly compact set is norm closed and proximinal.

(d) If M *is approximatively (resp. approximatively weakly) compact and* Px *is a singleton for some* $x \in N$, *then every minimizing sequence in* M *for* x *converges (resp. converges weakly) to* Px.

(e) A proximinal set is nonempty and closed.

Approximative Compactness of Closed Convex Sets

In the following theorem, the equivalence of (1) and (3), and thus implicitly the equivalence of (1) and (2), was established by Breckner [6] for Banach spaces. Vlasov mentioned the equivalence of (1) and (2) for Banach spaces in [59]. In [57], Vlasov claimed the equivalence of (1) and (3) for arbitrary normed linear spaces, but his proof used James's theorem to show that (3) implies (1), and James has shown that his theorem does not always hold without the assumption that the space in question is complete; see [28]. Our proof is based on Proposition 1.4, which can be proved without James's theorem by the argument given in Appendix A.

2.5 THEOREM. *The following conditions on normed linear space N are equivalent.*

(1) N is (Rf).

(2) Every nonempty closed convex set in N is approximatively weakly compact.

(3) Every weakly sequentially closed set in N is approximatively weakly compact.

Proof. The equivalence of (1) and (2) is just a restatement of Proposition 1.4. To see that (1) implies (3), suppose that N is a reflexive Banach space and (x_n) is a minimizing sequence in weakly sequentially closed set M for $x \in N$. Since (x_n) is bounded, it has a weakly convergent subsequence, which must converge to an element of M. Thus, M is approximatively weakly compact, giving (3). It is obvious that (3) implies (2). ■

In light of Theorem 2.5, it is natural to ask for a characterization of the normed linear spaces in which every nonempty closed convex set is approximatively compact. The next theorem gives the answer and shows that the Efimov-Stechkin property is, in a certain sense, a strong analog of reflexivity. The proof is similar to that of Theorem 2.5 and uses Proposition 1.9. We leave the details to the reader.

2.6 THEOREM (Singer [45]). *The following conditions on normed linear space N are equivalent.*

(1) N is (CD).

(2) Every nonempty closed convex set in N is approximatively compact.

(3) Every weakly sequentially closed set in N is approximatively compact.

We might now ask what happens if we require the closed convex sets to be Chebyshev sets as well as approximatively compact. A moment's thought shows that the resulting spaces in which this happens would seem to be the rotund spaces that are otherwise like those of the previous theorem. This is in fact the case.

2.7 THEOREM (Fan and Glicksberg [22]). *The following conditions on a normed linear space N are equivalent.*

(1) N is (D).

(2) Every nonempty closed convex set in N is an approximatively compact Chebyshev set.

Proof. This follows immediately from Proposition 1.7. Note that condition (3) of that proposition does imply that the metric projection is single-valued. ∎

It is probably true that the most well-known theorem of this branch of approximation theory is the Day-James theorem: A Banach space is rotund and reflexive if and only if each of its closed convex subsets is a Chebyshev set. The forward implication in this theorem was first proved in 1941 by Day [14], but the reverse implication, long suspected, had to await the proof of James's theorem. Vlasov claimed the theorem for arbitrary normed linear spaces in [57], but his proof implicitly assumed completeness. We give a proof here for arbitrary normed linear spaces which depends on the following result of Jörg Blatter.

2.8 LEMMA (Blatter [5]). *Let N be a normed linear space such that every nonempty closed convex subset of N has a point of minimum norm. Then N is complete.*

Condition (2) in the following result is not part of the classical Day-James theorem. Note that the equivalence of (1) and (2) ultimately depends on Proposition 1.4, and can be obtained without the use of James's theorem by the argument of Appendix A.

2.9 THEOREM (Day and James). *The following conditions on a normed linear space N are equivalent.*

(1) N is (R) & (Rf).

(2) Every nonempty closed convex set in N is an approximatively weakly compact Chebyshev set.

(3) Every nonempty closed convex set in N is a Chebyshev set.

Proof. The proof of the equivalence of (1) and (2) follows immediately from Proposition 1.5. Since (2) obviously implies (3), we need only show that (3) implies (1).

Suppose N satisfies (3). By the lemma, N is complete. If N were not reflexive, then by James's theorem there would be some $f \in \Sigma'$ not attaining its supremum on U. Since the closed convex set $\{y \in N: f(y) = 1\}$ would have no point nearest the origin, we see that (3) implies (Rf). Finally, if N were not rotund, then there would be a line segment in Σ which would have many points nearest the origin. Thus, (3) implies (R), which finishes the proof. ∎

If we were to write out a detailed proof that (1) implies (2) in the above theorem without using Proposition 1.5, we would discover that reflexivity has two functions in that proof. First, it forces closed convex sets to be proximinal, since hyperplanes are so. Second, it forces minimizing sequences in such sets to have weakly convergent subsequences.

If we were to weaken the hypothesis in (2) so that only proximinal convex sets need be approximatively weakly compact Chebyshev sets, it is clear that the full strength of reflexivity would no longer be required, though rotundity would still be needed to force proximinal hyperplanes to have unique points nearest the origin. It is equally clear that some mild form of reflexivity would need to be retained to force minimizing sequences in proximinal convex sets to have weakly convergent subsequences. It turns out that the proper requirement on the normed linear space is that it be very rotund.

2.10 THEOREM. *A normed linear space is (wLν) if and only if each of its proximinal convex sets is an approximatively weakly compact Chebyshev set.*

Proof. Let N be (wLν), and let K be a proximinal convex set in N. By Proposition 1.20 (3'), minimizing sequences in K converge weakly to elements of K, so K is approximatively weakly compact. If there were two points in K closest to $x \in N$, then a minimizing sequence in K for x could be constructed by alternating these points, but it would not converge weakly, a contradiction. Thus, K is Chebyshev.

Conversely, suppose that every proximinal convex set in N is an approximatively weakly compact Chebyshev set. Let (x_n) and f be as in the hypothesis of Proposition 1.20 (2'). Then $H = \{x \in N: f(x) = 1\}$ is a proximinal convex set. It is easy to see that $(f(x_n)^{-1}x_n)$ is a minimizing sequence in H for the origin, and hence converges weakly by Proposition 2.4 (d). By Proposition 1.20, N is (wLν). ∎

The following problem remains open.

2.11 PROBLEM. Find a sensible characterization of the normed linear spaces in which each convex Chebyshev set is approximatively weakly compact.

This problem seems fundamentally more difficult than obtaining the characterization of Theorem 2.10. For that characterization, as for several of our previous ones, the fundamental technique used in the proof involves separating a certain closed convex set from the unit ball by a hyperplane and thereby reducing the problem to considering hyperplanes; see the proof of Proposition 1.20, upon which Theorem 2.10 is based. This does not seem to help in Problem 2.11, because even if the closed convex set involved is Chebyshev, the separating hyperplane may not be. The reader will have no trouble constructing examples of this in $\mathbb{R}^2$ with any nonrotund norm.

Continuity of P for Closed Convex Sets

A question of great importance in this branch of approximation theory is the characterization of the normed linear spaces in which the metric projection onto each nonempty closed convex set has certain continuity properties. Oshman [42] gave a quite technical characterization of the Banach spaces for which such a metric projection is always norm-to-norm continuous. It was conjectured at that time that Oshman's spaces were actually just the strongly rotund Banach spaces (D). Vlasov has recently confirmed this conjecture for Banach spaces. By an application of Lemma 2.8, we can state his theorem in a slightly strengthened form.

2.12 THEOREM (Vlasov [60]). *A normed linear space N is (D) if and only if the metric projection onto every nonempty closed convex set in N is single-valued and norm-to-norm continuous.*

We are going to see that the rotund reflexive Banach spaces play precisely the same role for norm-to-weak continuity. In addition, if we do not insist that the domain of definition of P be all of N, then we can prove its norm-to-weak continuity for arbitrary convex sets in a far larger class of spaces.

2.13 THEOREM. *Let N be a normed linear space of class (wLv). Then the metric projection onto each convex set is single-valued and norm-to-weak continuous on its domain of definition.*

Proof. Let K be a convex set in N, w. l. o. g. nonempty, and let (x_n) and x be in the domain of definition of P with $x_n \to x$. Suppose $z \in N$ and $y_1, y_2 \in Pz$ are such that $y_1 \neq y_2$. Then $[y_1, y_2]$ lies in Pz by a short argument involving the convexity of K and of the ball centered at z with radius d(z,K). Since this ball cannot have a line segment on its surface by the rotundity of N, this contradiction shows that P is single-valued on its domain of definition.

Since the distance function $d(\cdot,K)$ is continuous and $x_n \to x$,

$$\begin{aligned}\|Px_n - x\| &\leq \|Px_n - x_n\| + \|x_n - x\| \\ &= d(x_n,K) + \|x_n - x\| \\ &\to d(x,K),\end{aligned}$$

implying that (Px_n) is a minimizing sequence in K for x. By Proposition 1.20 applied to the norm closure of K, $Px_n \xrightarrow{w} Px$. Thus, P is norm-to-weak continuous on its domain of definition, as claimed. ■

2.14 <u>COROLLARY</u>. *In a normed linear space of type (wLν), the metric projection onto each proximinal convex set is single-valued and norm-to-weak continuous.*

2.15 <u>QUESTION</u>. Does the conclusion of either Theorem 2.13 or Corollary 2.14 characterize spaces of type (wLν)?

If we add reflexivity to the hypothesis, then all the nonempty closed convex sets become proximinal by Theorem 2.9, and we do obtain the following characterization.

2.16 THEOREM. *The following conditions on a normed linear space N are equivalent.*

(1) N is (R) & (Rf).

(2) The metric projection onto every nonempty closed convex set in N is single-valued and norm-to-weak continuous.

Proof. If N is (R) & (Rf), then N is (wLν) by Corollary 1.22. Since every nonempty closed convex set in N is proximinal by Theorem 2.9, an application of Corollary 2.14 yields (2).

Conversely, if N satisfies (2), then every nonempty closed convex set in N is Chebyshev, so N is (R) & (Rf) by Theorem 2.9. ∎

Thus, in a rotund reflexive space the nonempty closed convex sets are all Chebyshev sets with norm-to-weak continuous metric projections. The next reasonable question to ask is for a characterization of the spaces where the nonempty closed convex sets are *exactly* the Chebyshev sets with this continuity property. In the next section, we obtain a partial solution to this problem.

SECTION 3

SEMI-KADEC-KLEE SPACES AND A PROBLEM OF KLEE'S

In [35], Klee took up the problem of the convexity of Chebyshev sets in Hilbert space subject to additional conditions. In particular, he proved that in Hilbert space every weakly closed Chebyshev set is convex, and in fact obtained the following stronger result.

3.1 <u>THEOREM</u> (Klee [35]). *In a Banach space that is (UR) & (UG), the nonempty closed convex sets are exactly the weakly closed Chebyshev sets.*

Singer later observed that the smoothness hypothesis can be weakened substantially.

3.2 <u>THEOREM</u> (Singer [46]). *In a Banach space that is (UR) & (S), the nonempty closed convex sets are exactly the weakly closed Chebyshev sets.*

The purpose of this section is to prove a result stronger than Theorem 3.2 and to obtain some similar results for some other classes of normed spaces and with other conditions on the Chebyshev sets besides being weakly closed. In order to do this, we need to examine a condition on the norm introduced by Vlasov.

Semi-Kadec-Klee Spaces

3.3 DEFINITION (Vlasov [56]). A normed linear space is called a *semi-Kadec-Klee space* if it possesses the following property:

(SH) Whenever x, $x_n \in \Sigma$, $f_n \in \Sigma'$, $f_n(x_n) = 1$, and $x_n \overset{w}{\to} x$, then $f_n(x) \to 1$.

Condition (SH) says that whenever (x_n) converges weakly to x on the unit sphere and (H_n) is a sequence of hyperplanes such that H_n supports the unit ball at x_n, then the distance from H_n to x tends to zero. Vlasov originally called this condition property (SA), consistent with his use of (A) for the Kadec-Klee property.

The following result gives some common conditions on normed spaces that imply condition (SH).

3.4 PROPOSITION (Vlasov [56], [57]).

(a) (H) $\Longrightarrow$ *(SH).*

(b) (UG) $\Longrightarrow$ *(SH).*

Vlasov gave an example in [56] showing that the implication in (a) is not reversible. The implication in (b) is also not reversible, since any finite-dimensional nonsmooth space has property (H) and hence (SH). There are, however, infinite-dimensional spaces that do not have property (SH). For example, in c_0 let $x_n = e_1 + e_n$ for $n \geq 2$, where e_i is the i'th unit vector. In $\ell_1 = c_0{}^*$ let $f_n = e_n$ for $n \geq 2$. then $x_n \overset{w}{\to} x = e_1$, $x_n \in \Sigma$, $f_n \in \Sigma'$, and $f_n(x_n) = 1$ for all n, but $f_n(x) = 0 \not\to 1$. Thus, c_0 is not a semi-Kadec-Klee space.

The importance of semi-Kadec-Klee spaces resides in Theorem 3.6 below, presented in [57] as Theorem 4.28 (k). To state this theorem, we need a definition giving a weak analog of the concept of ORL continuity from [7] and [8]; see also [18].

3.5 DEFINITION. The metric projection onto Chebyshev set M is said to be *outer radially norm-to-weak continuous* (*ORNW continuous*) if, whenever $x \notin M$, the restriction of P to the ray $\{x + \lambda(x - Px): \lambda \geq 0\}$ is norm-to-weak continuous.

3.6 THEOREM (Vlasov [57]). *In a semi-Kadec-Klee space with a rotund dual, every Chebyshev set with an ORNW continuous metric projection is convex.*

Vlasov actually stated this theorem under the hypothesis that the metric projection is norm-to-weak continuous, but an examination of his proof shows that ORNW continuity suffices.

We use this result along with some of our previous results to prove a strengthened version of Theorem 3.2. We can also replace weak closure by other properties of the Chebyshev set in the conclusion of that theorem, as we now show.

Convexity of Special Chebyshev Sets

As mentioned in the introduction to this thesis, the difficulties involved in proving that arbitrary Chebyshev sets are convex in a given normed linear space have led workers in this field to resort to proving the convexity of Chebyshev sets subject to additional conditions, such as approximative compactness or some type of continuity of the metric projection. Theorem 3.6 above is a typical example. Here are some other such special conditions sometimes used.

3.7 DEFINITION (Vlasov [58]). A nonempty set M is a *Δ-set* if its intersection with each closed half-space is either empty or proximinal.

3.8 DEFINITION (Klee [34]). A nonempty set M is *boundedly (weakly) compact* if its intersection with each closed ball is (weakly) compact.

In [3], Asplund proved that a Chebyshev Δ-set in a Hilbert space is convex. Vlasov [58] extended this result to uniformly smooth Banach spaces, and his proof actually goes through whenever the Banach space is (SH) with a rotund dual, a weaker requirement.

We are going to see that in certain spaces the convexity of some classes of Chebyshev sets, such as the Chebyshev Δ-sets, implies that the space is smooth. To do this, we need to look at a certain construction of Efimov and Stechkin that shows what can happen when the space is not smooth.

3.9 EXAMPLE (Efimov and Stechkin [19]). Recall that $f \in \Sigma'$ *exposes* U at $x \in \Sigma$ if $f(x) = 1$ and $f(y) < 1$ everywhere else on U. We then say that x is an *exposed point* of U.

Let N be a normed linear space with a point $x \in \Sigma$ that is both an exposed point and a point of nonsmoothness of U. Let $f \in \Sigma'$ be an exposing functional for x. Let f_1 and f_2 be distinct elements of Σ' such that $f_1(x) = f_2(x) = 1$. By replacing f_1 and f_2 by $\frac{1}{2}(f_1 + f)$ and $\frac{1}{2}(f_2 + f)$ if necessary, we can assume that f_1 and f_2 both expose U at x. For i = 1, 2, let $M_i = \{y \in N: f_i(y) \geqslant 1\}$. Since each f_i exposes U at x, it is not difficult to see that each M_i is a Chebyshev set. Let M be the union of M_1 and M_2. We claim that M is also a Chebyshev set. To see this, suppose that $y \notin M$. The only case we need worry about is when $d(y,M_1) = d(y,M_2)$, and w. l. o. g. we can assume that this common distance is 1 and that $y = 0$. Since the only point in M at distance 1 from 0 is x, we see that M is Chebyshev.

Hence, N contains a nonconvex Chebyshev set. Note that any rotund nonsmooth normed linear space thus contains a nonconvex Chebyshev set. ∎

It will also be useful to have the following lemma. The proof is not difficult; see [57].

3.10 LEMMA (Vlasov [57]). *An approximatively weakly compact Chebyshev set has a norm-to-weak continuous metric projection.*

The following is the first of the two principal results of this section. Recall that condition (wLν) can be stated as a smoothness condition on the dual space by Theorem 1.24, and note that the following result also involves a rotundity condition on the dual.

3.11 THEOREM. *Let M be a subset of normed linear space N, and consider the following statements about M.*

(1) M is a proximinal convex set.

(2) M is Chebyshev and P_M is ORNW continuous.

(3) M is Chebyshev and P_M is norm-to-weak continuous.

(4) M is Chebyshev and approximatively weakly compact.

If N is (wLν) & (SH) and has a rotund dual, then all four statements are equivalent. Conversely, if (1) through (4) are equivalent for all M, then N is (wLν) and smooth.

Proof. Suppose first that N is (wLν) & (SH) and N* is rotund. By Lemma 3.10, (4) implies (3). It is obvious that (3) implies (2). By Theorem 3.6, (2) implies (1). Finally, (1) implies (4) by Theorem 2.10.

Now suppose that (1) through (4) are equivalent for each set M in N. By Theorem 2.10, N is (wLν) and hence (R). Suppose N were not smooth. Consider the nonconvex Chebyshev set M of Example 3.9. It is reasonably obvious that whenever $x \notin M$, then Px is constant on the ray $[Px, x, \infty)$, but the proof is somewhat tedious. For a rigorous argument, see the proof in Theorem 3.9 of [57] that a space is smooth if each of its suns is convex. In particular, P_M is ORNW continuous. This contradiction shows that N must be smooth. ∎

We are going to show that for reflexive spaces that are (wLν) & (SH) and have rotund duals, several equivalent conditions can be added to Theorem 3.11, including that M be a Chebyshev Δ-set and that M be a boundedly weakly compact Chebyshev set. We first give an example to show that this is false in the absence of reflexivity.

3.12 EXAMPLE. Every separable Banach space has an equivalent norm that is (LUR) with the corresponding dual norm being rotund; see [16], Theorem VII.4.1 (a). Let $\|\cdot\|_a$ be such a norm on ℓ_1, and let B be the resulting Banach space. Then B is (wLUR) and hence (wLν) by Proposition 1.18 (d). Also, any (LUR) space is (H) (see [16]) and so (SH). Thus, B is (wLν) & (SH) and has a rotund dual, so conditions (1) through (4) of Theorem 3.11 are equivalent for subsets of B.

However, not every proximinal convex subset of B is boundedly weakly compact. For example, the closed unit ball is convex and is easily seen to be proximinal. However, if U were boundedly weakly compact, then it would be weakly compact and B would be reflexive. Since reflexivity is isomorphism-invariant, ℓ_1 would be reflexive, a contradiction.

Also, not every proximinal convex set in B is a Δ-set. For example, B itself is certainly a proximinal convex set. However, since B is not reflexive, there is some $f \in \Sigma'$ that does not achieve its supremum on Σ_B. Then $A = \{x \in B: f(x) \geqslant 1\}$ is a closed half-space, and the intersection of B with A, which is just A, is not proximinal. ∎

The next theorem is the strengthened version of Theorem 3.2 that we have been promising. In order to understand its relationship to Theorem 3.11, consider a reflexive space that is (wLν) & (SH) and has a rotund dual. Then the space is obviously (R) & (Rf) & (S) & (SH). Conversely, if N is a space that is (R) & (Rf) & (S) & (SH), then N* is rotund because rotundity and smoothness are dual concepts in reflexive spaces; see [16]. N is also (wLν) by Corollary 1.22. Thus, adding reflexivity to the requirement that a space be (wLν) & (SH) and have a rotund dual yields exactly (R) & (Rf) & (S) & (SH). Thus, the next theorem and its corollary are analogs of Theorem 3.11 for reflexive spaces.

3.13 THEOREM. *Consider the following statements about normed linear space N.*

(1) N is (R) & (Rf) & (S) & (SH).

(2) The nonempty closed convex subsets of N are exactly the Chebyshev sets with ORNW continuous metric projections.

(3) The nonempty closed convex subsets of N are exactly the Chebyshev sets with norm-to-weak continuous metric projections.

(4) The nonempty closed convex subsets of N are exactly the approximatively weakly compact Chebyshev sets.

(5) The nonempty closed convex subsets of N are exactly the boundedly weakly compact Chebyshev sets.

(6) The nonempty closed convex subsets of N are exactly the weakly sequentially closed Chebyshev sets.

(7) The nonempty closed convex subsets of N are exactly the weakly closed Chebyshev sets.

(8) The nonempty closed convex subsets of N are exactly the Chebyshev Δ-sets.

(9) N is (R) & (Rf) & (S).

Then:

(a) (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6) $\Longrightarrow$ (7) $\Longrightarrow$ (9).

(b) (4) $\Longrightarrow$ (8) $\Longrightarrow$ (9).

(c) For spaces of type (SH), (2) through (9) are equivalent.

Proof. (1) $\Longrightarrow$ (2). This is an easy consequence of Theorems 2.16 and 3.6.

(2) $\Longrightarrow$ (3). Suppose (2) holds. By Theorem 2.9, N is (R) & (Rf), so by Theorem 2.16, every nonempty closed convex set is Chebyshev with a norm-to-weak continuous metric projection. The reverse inclusion follows easily from (2).

(3) $\Longrightarrow$ (4). Suppose (3) holds. By Theorem 2.9, every nonempty closed convex set is an approximatively weakly compact Chebyshev set. For the reverse inclusion, just apply Lemma 3.10 to condition (3).

(4) $\Longrightarrow$ (5). Suppose (4) holds. Since boundedly weakly compact sets are easily seen to be approximatively weakly compact, all we need to show is that nonempty closed convex sets are boundedly weakly compact. Since (4) implies the reflexivity of N by Theorem 2.5, this is immediate.

(5) $\Longrightarrow$ (6). Suppose (5) holds. By Theorem 2.9, N is reflexive. In reflexive spaces, weakly sequentially closed sets are easily seen to be boundedly weakly compact; see [57], Proposition 2.3. Thus, we need only show that nonempty closed convex subsets of N are weakly sequentially closed, which is immediate.

(6) $\Longrightarrow$ (7). This follows easily, since every closed convex set is weakly closed and a weakly closed set is weakly sequentially closed.

(7) $\Longrightarrow$ (9). Suppose (7) holds. By Theorem 2.9, N is (R) & (Rf). Suppose N were not smooth. The nonconvex Chebyshev set of Example 3.9 is the union of two weakly closed

half-spaces and hence is a weakly closed Chebyshev set. This contradiction establishes that N is (S), which gives (9). This finishes the proof of (a).

(4) $\Longrightarrow$ (8). Suppose (4) holds. By Theorem 2.9, N is (R) & (Rf), and in particular is complete. In [58], Vlasov proved that every Chebyshev Δ-set in a Banach space is approximatively weakly compact, and so every Chebyshev Δ-set in N is convex. Conversely, suppose M is a nonempty closed convex set and H is a closed half-space. If M intersects H, then the intersection is a nonempty closed convex set and hence is Chebyshev by Theorem 2.9; in particular, the intersection is proximinal. Thus, M is a Δ-set. By Theorem 2.9 again, M is itself Chebyshev, which gives (8).

(8) $\Longrightarrow$ (9). Suppose (8) holds. By Theorem 2.9, N is (R) & (Rf). Suppose N were not smooth, and consider the nonconvex Chebyshev set M of Example 3.8. Now M is the union of two closed half-spaces, each of which is a Δ-set by Theorem 2.9 (3). From this, it is easy to deduce that M is itself a Chebyshev Δ-set. This contradiction establishes that N is smooth, and hence (9) holds. This proves (b).

A comparison of (1) and (8), along with (a) and (b), establishes (c). ∎

3.14 <u>COROLLARY</u>. *Let N be (R) & (Rf) & (S) & (SH), and let M be a subset of N. Then the following are equivalent.*

(1) M is a nonempty closed convex set.

(2) M is Chebyshev and P_M is ORNW continuous.

(3) M is Chebyshev and P_M is norm-to-weak continuous.

(4) M is Chebyshev and approximatively weakly compact.

(5) M is Chebyshev and boundedly weakly compact.

(6) M is Chebyshev and weakly sequentially closed.

(7) M is Chebyshev and weakly closed.

(8) M is a Chebyshev Δ-set.

Any (UR) space is (LUR) and hence (H); see [16]. Also, (UR) Banach spaces are reflexive by a result of Milman [37] and Pettis [43]. Thus, any Banach space that is (UR) & (S) is also (R) & (Rf) & (S) & (SH). It follows that Theorem 3.13 is at least as strong a result as Theorem 3.2. To show that it is stronger, we display a space that is (R) & (Rf) & (S) & (SH) but not (UR).

3.15 <u>EXAMPLE</u>. In [14], Day constructed a Banach space B that is separable and reflexive but not isomorphic to any uniformly rotund space. As in Example 3.12, B can be given an equivalent norm $\|\cdot\|_a$ such that the resulting Banach space B_a is (LUR) & (S). By the comments preceding this example, B_a is (R) & (Rf) & (S) & (SH), even though it cannot be (UR). ■

It is instructive to compare Theorem 3.14 with the following result.

3.16 THEOREM (Vlasov [57]). *Let B be a Banach space. Then the following are equivalent.*

(1) B is (R) & (Rf) & (S) & (H); i. e. (D) & (S).

(2) The nonempty closed convex subsets of B are exactly the approximatively compact Chebyshev sets.

(3) The nonempty closed convex subsets of B are exactly the Chebyshev sets with norm-to-norm continuous metric projections.

Incidentally, the completeness hypothesis in this last theorem can be removed by an application of Lemma 2.8.

In Theorem 3.16, conditions (2) and (3) imply (H), so it is reasonable to ask if any of conditions (2) through (8) in Theorem 3.13 imply (SH). If any one did, then that condition would be equivalent to N being (R) & (Rf) & (S) & (SH), which would bring to an end one line of investigation. It would be particularly interesting to discover if condition (7) implies that N is (SH), for in that case conditions (1) through (7) would be equivalent in any normed linear space.

3.17 QUESTION. Suppose that the nonempty closed convex subsets of N are exactly the weakly closed Chebyshev sets. Must N be a semi-Kadec-Klee space?

We might also ask if all these special conditions on the Chebyshev sets are necessary. The following question along these lines is still open.

3.18 <u>QUESTION</u>. If N is (R) & (Rf) & (S) & (SH), are the nonempty closed convex sets exactly the Chebyshev sets?

For finite-dimensional spaces, the answer is yes, as can be deduced from the following result. We show that this result is contained in Theorem 3.13.

3.19 <u>THEOREM</u> (Busemann [10], [11]; Efimov and Stechkin [20]). *For a finite-dimensional normed linear space N, the following are equivalent.*

(1) N is (R) & (S).

(2) The nonempty closed convex sets are exactly the Chebyshev sets.

<u>Proof</u>. (1) $\Rightarrow$ (2). Any finite-dimensional normed space is (H) and hence (SH), as well as (Rf). Thus, (1) implies that N is (R) & (Rf) & (S) & (SH), so by Theorem 3.13, the nonempty closed convex sets are exactly the weakly closed Chebyshev sets. Since the norm and weak topologies agree on N and any Chebyshev set is norm closed, the words "weakly closed" are redundant.

(2) $\Longrightarrow$ (1). If (2) holds, then the nonempty closed convex sets are exactly the weakly closed Chebyshev sets, so N is (R) & (S) by Theorem 3.13. ∎

As mentioned in the introduction to this thesis, much less is known in the infinite-dimensional case. Of course, the problem does not lie in the "Chebyshevness" of nonempty closed convex sets; this was settled in full by Theorem 2.9. The difficulty lies in proving the convexity of Chebyshev sets. There is no infinite-dimensional space known in which every Chebyshev set is convex. There is also no smooth space of any kind known to contain a nonconvex Chebyshev set, so much remains to be done in this area. In fact, the convexity of Chebyshev sets in classical Hilbert space is considered to be the major open problem of this branch of approximation theory.

While Theorem 3.13 is an improvement on Theorem 3.2, it still falls short of being a characterization of spaces in which the nonempty closed convex sets are exactly the weakly closed Chebyshev sets. It does, however, point out the importance of learning more about spaces that are (R) & (Rf) & (S) & (SH), and about semi-Kadec-Klee spaces in general. The next few sections are devoted to doing this, with one result being an approximation-theoretic characterization of the spaces that are (R) & (Rf) & (S) & (SH). The tool we use for this study is the new concept of supportive compactness. However, before we can study supportive compactness, we need to take a look at the norm-duality map and its continuity properties. It turns out that the semi-Kadec-Klee condition is just a statement about the continuity of this map.

SECTION 4
SOME CONTINUITY PROPERTIES OF THE NORM-DUALITY MAP

The semi-Kadec-Klee condition is a statement about the interaction between certain sequences in Σ and certain corresponding sequences in Σ'. To study this relationship, it proves to be important to have some simple map between N and N* whose properties are directly related to the geometry of these two spaces. There are several reasonable choices for this map that have in the past been useful in this branch of approximation theory.

In 1969, Asplund [3] gave an elementary proof that in any Hilbert space, every Chebyshev set with a norm-to-norm continuous metric projection is convex. To do this he applied the map $T: H \setminus \{0\} \to H \setminus \{0\}$ given by $T(x) = \|x\|^{-2}x$ to a hypothetical nonconvex Chebyshev set as the first step in a certain construction. This transformation, called *inversion in the unit sphere,* had earlier been used by Klee [35] to prove certain theorems about Chebyshev sets in Hilbert space. Klee in turn attributed the method to F. A. Ficken, who used it in some of his unpublished work.

Another map frequently encountered is the *spherical image map* ν studied by Cudia in [13]; see also the discussion of the subdifferential of the norm by Giles in [24]. This is the set-valued map sending each x in Σ to the collection of all f in Σ' with $f(x) = 1$. Note that for a Hilbert space H, the restriction of the inversion map T to Σ just gives ν, provided that H* is identified with H in the usual way.

Ficken's inversion map can be extended to arbitrary normed linear spaces in a natural way, but it seems to be somewhat difficult to analyze in this general setting. The spherical image map has friendlier properties, but does not contain enough information for our purposes. There is another well-known natural map, the *norm-duality map*, that will prove more suitable for our purposes. We actually need a slightly generalized form of this map.

The Norm-Duality Map

4.1 DEFINITION. Let N be a normed linear space, and let $z \in N$. Then J_z is the set-valued map from N into $2^{N^*} \setminus \{\emptyset\}$ given by:

$$J_z x = \|x - z\|\{f \in \Sigma': f(x - z) = \|x - z\|\}.$$

The map $J = J_0$ is called the *norm-duality map*.

Note that $J_z z = \{0\}$ and that whenever f is in $J_z x$, then $f(x - z) = \|x - z\|^2$ and $\|f\| = \|x - z\|$; conversely, whenever f satisfies these last two equalities, then f is in $J_z x$. In particular, f is in Jx if and only if $f(x) = \|x\|^2$ and $\|f\| = \|x\|$, and so the restriction of J to Σ just yields the spherical image map ν. See Holmes [26] for more about the map J.

The map J_z possesses a certain useful continuity property. Recall the following definition.

4.2 <u>DEFINITION</u>. Let (A, τ_1) and (B, τ_2) be topological spaces, and let $\Phi: A \to 2^B \setminus \{\emptyset\}$ be a set-valued map. Then Φ is *τ_1-to-τ_2 upper semicontinuous* (τ_1-τ_2 *u. s. c.*) if, for every τ_2-open set G in B, the set $\{x \in A: \Phi x \text{ lies in } G\}$ is τ_1-open in A.

The following fact is well-known for J, and the extension to J_z causes no problem. A proof can be constructed along the lines of the proof in [24] that subgradient mappings are norm-to-weak* upper semicontinuous.

4.3 <u>PROPOSITION</u>. *Let $z \in N$. Then J_z is norm-to-weak* upper semicontinuous.*

4.4 <u>COROLLARY</u>. *Let $z \in N$. If J_z is single-valued on N, then J_z is norm-to-weak* continuous.*

Many of the geometric properties of the normed linear spaces that we have been studying can be expressed as corresponding properties of the norm-duality map. We list here some of the more basic and well-known of these relationships. The proofs are elementary and will be omitted, though note that the proof of (c) uses James's theorem.

4.5 PROPOSITION. *Let N be a normed linear space and B a Banach space.*

(a) *N is smooth if and only if J is single-valued.*

(b) *N is rotund if and only if Jx and Jy are disjoint whenever x and y are distinct elements of N.*

(c) *B is reflexive if and only if $J(B) = B^*$.*

(d) *If $J^*: N^* \to N^{**}$ is the norm-duality map for N^*, then for any $x \in N$, $Qx \in J^*(Jx)$. In fact, $Qx \in J^*(f)$ for each f in Jx.*

(e) *B is (R) & (Rf) & (S) if and only if J is a bijection from B onto B^*, in which case J^{-1} can be identified with J^* in the obvious way.*

Part (c) of the theorem does not hold for arbitrary normed linear spaces. James [28] has constructed a normed linear space that is not complete but in which every continuous linear functional attains its supremum on Σ.

It is well-known that a smooth space is Fréchet smooth if and only if the spherical image map ν is norm-to-norm continuous; see [24], [13], and [51]. If we require norm-to-weak continuity instead, we obtain a smoothness condition between (F) and (S). The following is not the original definition of such spaces, but is an equivalent formulation due to Giles [23].

4.6 <u>DEFINITION</u>. A normed linear space is *very smooth* and is said to satisfy condition (VS) if the spherical image map is norm-to-weak continuous.

The following characterizations are easy to derive from the above results and comments. As is the usual convention, continuity, as opposed to semicontinuity, implies that the map is single-valued.

4.7 <u>PROPOSITION</u>. *Let N be a normed linear space.*

(a) N is (S) if and only if J is norm-to-weak continuous.*

(b) N is (VS) if and only if J is norm-to-weak continuous.

(c) N is (F) if and only if J is norm-to-norm continuous.

It turns out that the semi-Kadec-Klee spaces are exactly the spaces where J has another semicontinuity property, but the relevant topology on N is not one of the usual ones.

The Lambda Topology

4.8 <u>DEFINITION</u>. The λ *topology* on a normed linear space N is the topology with closed sets defined as follows. A set A is λ-closed if, whenever (x_n) is a sequence in A such that $x_n \xrightarrow{w} x \in N$ and $\|x_n\| \to \|x\|$, then $x \in A$. The λ^* *topology* on N* is defined similarly, with weak convergence replaced by weak* convergence.

We are not claiming that N and N* are topological vector spaces under these topologies, and in fact they are frequently not, as we show in Proposition 4.10 below. However, the λ and λ^* topologies are at least topologies.

4.9 PROPOSITION.

(a) *The λ and λ^* topologies are Hausdorff topologies on N and N^* respectively.*

(b) *The λ (resp. λ^*) topology is at least as strong as the weak (resp. weak*) topology, but is no stronger than the norm topology.*

(c) *If (x_n) is a sequence in N, then $x_n \overset{\lambda}{\to} x$ if and only if $x_n \overset{w}{\to} x$ and $\|x_n\| \to \|x\|$. A similar result holds for λ^*-sequential convergence.*

(d) *If (x_α) is a net in N and $x_\alpha \overset{\lambda}{\to} x$, then $x_\alpha \overset{w}{\to} x$ and $\|x_\alpha\| \to \|x\|$. A similar result holds for λ^*-convergence.*

Proof. We give proofs for the λ topology. The proofs for the λ^* topology are analogous.

For (a), note that the empty set and N are λ-closed and that arbitrary intersections of λ-closed sets are λ-closed. If $A_1, \dots, A_n$ is a finite collection of λ-closed sets with union A and (x_n) is a sequence with $x_n \overset{w}{\to} x \in N$ and $\|x_n\| \to \|x\|$, then some infinite subsequence of (x_n) belongs to some A_i. It easily follows that $x \in A$, and hence A is λ-closed. Thus, the λ-closed sets do define a topology on N. We will show it to be Hausdorff in a moment.

For (b), it is obvious that a weakly closed set is λ-closed. This finishes the proof of (a), since a topology as strong as a Hausdorff topology is itself Hausdorff. Also, if A is λ-closed, (x_n) is a sequence in A, and $\|x_n - x\| \to 0$, then $x \in A$, so A is norm closed. This proves (b).

For (d), note that for any $\varepsilon > 0$, the set $D(\varepsilon) = \{y \in N: |\|y\| - \|x\|| < \varepsilon\}$ is λ-open and is thus a λ-neighborhood of x. It immediately follows that $\|x_\alpha\| \to \|x\|$. By (b), $x_\alpha \overset{w}{\to} x$. This gives (d) and one part of (c).

For the rest of (c), suppose that $x_n \overset{w}{\to} x$ and $\|x_n\| \to \|x\|$. If A is a λ-closed set not containing x, then there is an n_A such that $x_n \notin A$ whenever $n \geq n_A$. Thus, $x_n \overset{\lambda}{\to} x$. ∎

It is not difficult to see that a set is approximatively weakly compact if and only if it is approximatively sequentially λ-compact; that is, minimizing sequences have λ-convergent subsequences with limits in the set. Thus, for many of our purposes we could deal with the λ topology rather than the weak. Unfortunately, the λ topology displays a certain bit of unpleasant pathology in spaces that do not have Kadec-Klee norms.

4.10 PROPOSITION. *In a normed linear space N, the following are equivalent.*

(1) N is (H).

(2) The norm and λ topologies agree on N.

(3) Vector addition is λ-continuous.

Proof. The proof of the equivalence of (1) and (2) is just an exercise in definitions. Also, (2) obviously implies (3), so we need only show that (3) implies (2). Suppose that the λ topology is properly weaker than the norm topology. Then there is some norm open ball B centered at the origin and an $x \in N$ such that $B + x$ is not λ-open. Since $N \setminus B$ is easily seen to be λ-closed, B is λ-open. Thus, vector addition cannot be λ-continuous. ■

We can now prove the major results of this section, which say that condition (SH) is just a continuity statement about the norm-duality map.

4.11 THEOREM. *A normed linear space is a semi-Kadec-Klee space if and only if its norm-duality map is λ-to-weak* upper semicontinuous.*

Proof. Suppose N is (SH). Let G be a weak* open set in N*. We need to show that $\{y \in N: Jy \text{ lies in } G\}$ is

λ-open; that is, that $A = \{y \in N\colon Jy \text{ intersects } N^* \setminus G\}$ is λ-closed. Let (x_n) be a sequence in A with $x_n \overset{w}{\to} x \in N$ and $\|x_n\| \to \|x\|$. We need to show that $x \in A$.

Suppose $x = 0$. Since $J0 = \{0\}$, we need to show that $0 \notin G$. Suppose to the contrary that $0 \in G$. Since G is norm open, there is a $\delta > 0$ such that if $f \in N^*$ and $\|f\| < \delta$, then $f \in G$. Since $x_n \to 0$, some x_{n_0} has norm less than δ, and so $\|f\| < \delta$ for all $f \in Jx_{n_0}$. Thus, Jx_{n_0} lies in G, which contradicts the assumption that $x_{n_0} \in A$.

Now suppose that $x \neq 0$. For each n let f_n be an element of Jx_n lying outside of G. Since $\|f_n\| = \|x_n\| \to \|x\|$, we can assume w. l. o. g. that $f_n \neq 0$ and $x_n \neq 0$ for each n. Then $\|x_n\|^{-1}x_n \overset{w}{\to} \|x\|^{-1}x$. For each n,

$$\|f_n\|^{-1}f_n(\|x_n\|^{-1}x_n) = \|f_n\|^{-1}\|x_n\|^{-1}f_n(x_n) = \|x_n\|^{-2}\|x_n\|^2 = 1.$$

Since N is (SH), $\|f_n\|^{-1}f_n(\|x\|^{-1}x) \to 1$. Multiplying by $\|f_n\|\|x\|$, we see that $f_n(x) \to \|x\|^2$. By the Banach-Alaoglu theorem, (f_n) has a subnet (f_α) with a weak* limit f. Then

$$\|x\|^2 = \lim f_\alpha(x) = f(x) \leqslant \|f\|\|x\| \leqslant \lim \|f_\alpha\|\|x\| = \|x\|^2,$$

and so $\|f\| = \|x\|$ and $f(x) = \|x\|^2$. Thus, $f \in Jx$. Also, since each $f_\alpha \in N^* \setminus G$, a weak* closed set, $f \in N^* \setminus G$. Thus, Jx intersects $N^* \setminus G$, implying that $x \in A$. This proves that J is λ-w* u. s. c. whenever N is (SH).

Now suppose that J is λ-w* u. s. c. Let $x_n, x \in \Sigma$ with $x_n \overset{w}{\to} x$, and let $f_n \in \Sigma'$ be such that $f_n(x_n) = 1$ for all n. To show that N is (SH), we need to show that $f_n(x) \to 1$.

For each $\varepsilon > 0$, let $G_\varepsilon = \{g \in N^*: |g(x) - 1| < \varepsilon\}$, a weak* open set. Then $\{y \in N: Jy \text{ lies in } G_\varepsilon\}$ is λ-open and contains x. Since $x_n \overset{\lambda}{\to} x$, there is an n_ε such that:

$$n \geq n_\varepsilon \Rightarrow Jx_n \text{ lies in } G_\varepsilon \Rightarrow f_n \in G_\varepsilon \Rightarrow |f_n(x) - 1| < \varepsilon.$$

Thus, $f_n(x) \to 1$, as required. ■

4.12 COROLLARY. *The following are equivalent.*

(1) N is (S) & (SH).

(2) J is λ-to-weak continuous.*

(3) J is λ-to-λ^* continuous.

Proof. The equivalence of (1) and (2) is immediate from the theorem and Proposition 4.5 (a). Since (3) obviously implies (2), we need only show that (2) implies (3). Suppose (2) holds. Let K be a λ^*-closed set in N^*, and let $A = J^{-1}(K)$. It is enough to show that A is λ-closed. Let (x_n) be a sequence in A with $x_n \overset{w}{\to} x$ and $\|x_n\| \to \|x\|$. Then $Jx_n \overset{w^*}{\to} Jx$ and $\|Jx_n\| = \|x_n\| \to \|x\| = \|Jx\|$, so $Jx_n \overset{\lambda^*}{\to} Jx$. Since K is λ^*-closed, $Jx \in K$. Thus, $x \in A$. ■

For a large class of spaces, the λ-to-weak* upper semicontinuity of Theorem 4.11 can be replaced by λ-to-λ^* upper semicontinuity.

4.13 THEOREM. *Suppose N is a normed linear space whose dual space has a weak* sequentially compact unit ball. Then N is a semi-Kadec-Klee space if and only if its norm-duality map is λ-to-λ* upper semicontinuous.*

Proof. If J is λ-λ* u. s. c., then N is (SH) by Theorem 4.11. Conversely, suppose that N is (SH). Let G be λ*-open in N* and let A, (x_n), and x be as in the first paragraph of the proof of Theorem 4.11. The proof now proceeds word-for-word like that proof until the application of the Banach-Alaoglu theorem. We pick it up there.

By the weak* sequential compactness of U^{π}, (f_n) has a subsequence (f_{n_j}) with weak* limit f. As in the proof of Theorem 4.11, $f \in Jx$ and $\|f\| = \|x\|$. Since $f_{n_j} \xrightarrow{w^*} f$ and $\|f_{n_j}\| = \|x_{n_j}\| \to \|x\| = \|f\|$, $f_{n_j} \xrightarrow{\lambda^*} f$. Since each f_{n_j} lies in $N^* \setminus G$, a λ*-closed set, f also lies in $N^* \setminus G$. Thus, Jx intersects $N^* \setminus G$, and so $x \in A$. As before, this is enough to prove that J is λ-λ* u. s. c. ∎

4.14 COROLLARY. *Suppose N satisfies one of the following conditions.*

(a) N has an equivalent smooth norm and is complete.

(b) N is weakly compactly generated (see [16]) and complete.

(c) N is reflexive.

(d) N is separable.

Then N is a semi-Kadec-Klee space if and only if its norm-duality map is λ-to-λ upper semicontinuous.*

<u>Proof</u>. Hagler and Sullivan [25] have shown that if N is a Banach space with an equivalent smooth norm, then U^{π} is weak* sequentially compact, so in case (a) we just apply the previous theorem. Case (b) follows from case (a), because weakly compactly generated Banach spaces have equivalent smooth norms; see [16]. Case (c) is immediate from the Eberlein-Šmulian theorem, while case (d) follows from the fact that the weak* topology on U^{π} is metrizable whenever N is separable. ∎

Case (c) of the corollary also follows from case (b), since reflexive spaces are weakly compactly generated. Though separable Banach spaces are also weakly compactly generated, invoking case (b) to prove case (d) would require that N be complete.

With a small amount of extra proof, Proposition 4.7 and Corollary 4.12 can be combined to yield the following.

4.15 <u>PROPOSITION</u>. *Let N be a normed linear space.*

(a) N is (S) & (SH) if and only if J is λ-to-λ continuous.*

(b) N is (S) if and only if J is norm-to-λ continuous.*

(c) N is (VS) if and only if J is norm-to-λ continuous.

(d) N is (F) if and only if J is norm-to-norm continuous.

We finish this section by noting two cases in which J is bicontinuous. Klee [34] asked for a characterization of the spaces for which J is a norm-to-norm homeomorphism. Cudia obtained the answer in [13] by an argument analogous to the one we offer.

4.16 THEOREM (Cudia [13]). *Let N be a normed linear space. Then the following are equivalent.*

(1) N is (R) & (Rf) & (F) & (H); that is, (D) & (F).

(2) J is a norm-to-norm homeomorphism of N onto N.*

Proof. If N is (R) & (Rf) & (F) & (H), then J is a norm-to-norm continuous bijection onto N* by Propositions 4.5 (e) and 4.15 (d). Also, by a result of Šmulian [52], N being D is equivalent to N* being (F), provided N is complete. Thus, T^{-1} is also norm-to-norm continuous.

Now suppose that (2) holds. Since an incomplete normed linear space cannot be homeomorphic to a Banach space (Klee [33]), N is complete. Then Propositions 4.5 (e) and 4.15 (d) imply that N is (R) & (Rf) & (F) and that N* is (F). By Šmulian's result, N is (D) and hence (H). ∎

Note that condition (1) in the previous theorem implies that both N and N* are (H), because N being reflexive and Fréchet smooth implies that N* is (D) by Šmulian's result mentioned above. It then becomes a trivial corollary of

Proposition 4.10 that J is a λ-to-λ homeomorphism whenever it is a norm-to-norm homeomorphism. The following more substantial result can be proved by the same type of argument as in the last proof. We leave the details to the reader.

4.17 THEOREM. *Let N be a normed linear space. Then the following are equivalent.*

(1) N is (R) & (Rf) & (S) & (SH) and N^* is (SH).

(2) J is a λ-to-λ homeomorphism of N onto N^*.

SECTION 5
BASIC PROPERTIES OF SUPPORTIVE COMPACTNESS

In this section, we define a new analog of approximative compactness, called supportive compactness, and obtain some of its basic properties. The eventual goal, to be accomplished in Section 6, is to obtain an approximation-theoretic characterization of the spaces that are (R) & (Rf) & (S) & (SH) to compare with Theorem 3.13, which falls just short of giving such a characterization. Along the way, we will discover that supportively compact sets have certain interesting properties of their own. For instance, if a Banach space has a rotund dual but no other assumed geometric properties, then its supportively compact Chebyshev sets are convex.

Supportive Compactness

5.1 DEFINITION. Let M be a nonempty subset of normed linear space N. A net (f_α) in N* is a *supportive net* for M with respect to $x \in N$ if there is a minimizing net (x_α) in M for x such that $f_\alpha \in J_x x_\alpha$ for each α.

5.2 DEFINITION. Let τ be a topology on N*. Then nonempty set M in N is *supportively τ-compact* if, for every $x \in N$ and every supportive net (f_α) for M with respect to x, (f_α) has a subnet that is τ-convergent to some $f \in J_x(M)$. If this condition holds with nets replaced by sequences, then we say that M is *supportively* $\tau\omega$*-compact*. If τ is not specified, the norm topology is assumed.

The three topologies we use for τ in the above definitions are the norm, weak, and weak*. We list here some basic properties of sets that are supportively τ-compact in these topologies.

5.3 PROPOSITION. *Let M be supportively* τ*-compact, where* τ *is the norm, weak, or weak* topology. Then the following hold.*

(a) M is proximinal and hence closed.

(b) Whenever $z \in N$ *and* $y \in P_M z$*, then* $J_z y$ *is* τ*-compact.*

(c) Whenever $z \in N$*, then* $J_z(P_M z)$ *is* τ*-compact.*

(d) If N is smooth, then whenever $P_M z$ *is a singleton, every supportive net for M w. r. t. z is* τ*-convergent to* $J_z(P_M z)$*.*

Proof. For (a), let $x \in N$, and let (f_α) be any supportive net for M w. r. t. x that is τ-convergent to some $f \in J_x(M)$. Let $y \in M$ be such that $f \in J_x y$, and let (x_α) be a minimizing net in N that corresponds to (f_α). Then

$$d(x,M) \leqslant \|x - y\| = \|f\| \leqslant \liminf \|f_\alpha\|$$
$$= \liminf \|x - x_\alpha\| = d(x,M).$$

It follows that $\|x - y\| = d(x,M)$ and $y \in P_M x$. This finishes the proof of (a).

For (c), suppose that (f_α) is a net in $J_z(P_M z)$. It is easy to see that (f_α) is a supportive net for M w. r. t. z and so has a subnet (f_β) τ-convergent to some $f \in J_z(M)$. By the proof of (a), $f \in J_z(P_M z)$.

For (b), let (f_α) be a net in $J_z y$, where $y \in P_M z$. By (c), (f_α) has a subnet (f_β) that is τ-convergent to some $f \in J_z(P_M z)$. Then $\|f\| = d(z,M) = \|z - y\|$ and $f(y - z) = \lim f_\beta(y - z) = \|y - z\|^2$. Thus, $f \in J_z y$, and so $J_z y$ is τ-compact.

After recalling that J_z is single-valued whenever N is smooth, the reader will have no difficulty supplying the proof of (d). ∎

The following lemma is proved in the same way as parts (a) and (c) of the last proposition.

5.4 <u>LEMMA</u>. *Let M be supportively $\tau\omega$-compact, where τ is the norm, weak, or weak* topology. Then M is proximinal. Also, for any $z \in N$, $J_z(P_M z)$ is sequentially τ-compact.*

Since sequences are frequently easier to manipulate than nets, it would be nice to know when the definition of supportive τ-compactness is equivalent to that of supportive $\tau\omega$-compactness. The next theorem answers this question for the most interesting cases. To prove the theorem, we need the following lemma, whose proof follows the proof given in Vlasov [58] that approximative sequential weak compactness implies approximative weak compactness, which is in turn an adaptation of an argument due to Day from the second edition of his book [15]; see his Theorem III.2.4.

5.5 LEMMA. *Suppose N is a normed linear space and A is a subset of N having the property that any minimizing sequence in A for 0 has a weakly convergent subsequence with limit in A. Then any minimizing net in A for 0 has a weakly convergent subnet with limit in A.*

Proof. *Claim:* Whenever L is a finite-dimensional subspace of N^{**}, then there exist $f_i \in \Sigma'$ ($i = 1, 2, \ldots$) such that $\sup_i z''(f_i) = \|z''\|$ for each $z'' \in L$. To see this, let (z_i'') be a sequence dense in L. For each i, let $f_i \in \Sigma'$ be such that $z_i''(f_i) \geqslant \|z_i''\| - i^{-1}$. This sequence works.

Since a minimizing net (y_α) has a bounded tail, (Qy_α) contains a weak* convergent subnet in N^{**}. Therefore, it is sufficient to establish the following fact:

If $(Qy_\alpha)_{\alpha \in \Phi} \overset{w^*}{\to} y'' \in N^{**}$, $\|y_\alpha\| \to d(0,A)$, and (y_α) lies in A, then $y'' \in Q(A)$.

We first inductively construct a sequence (α_n) in Φ and a system of sequences (f_{1j}), (f_{2j}), ... in Σ'. To this end, we select sequence (f_{1j}) in Σ' arbitrarily and an α_1 such that both $|(Qy_{\alpha_1} - y'')(f_{11})| \leqslant 1$ and $\|y_{\alpha_1}\| - d(0,A) \leqslant 1$. Suppose now that $\alpha_1, \ldots, \alpha_{n-1}$ and sequences $(f_{1j}), \ldots, (f_{n-1,j})$ have been chosen so that

$$|(Qy_{\alpha_k} - y'')(f_{ij})| \leq k^{-1} \text{ and } \|y_{\alpha_k}\| - d(0,A) \leq k^{-1} \qquad (1)$$

whenever $i, j \leq k \leq n - 1$. Let L_n be the linear hull of $\{Qy_{\alpha_1}, \ldots, Qy_{\alpha_{n-1}}\}$ and let L_n' be the linear hull of L_n and y''. By applying our first claim to $L = L_n'$, we can find a sequence $(f_{nj})_{j=1}^{\infty}$ in Σ' such that

$$\sup_j |(Qx - y'')(f_{nj})| = \|Qx - y''\| \text{ for all } Qx \in L_n. \qquad (2)$$

Now choose α_n so that (1) is satisfied when $k = n$ and $i, j \leq n$. We have now inductively constructed $(\alpha_n)_{n=1}^{\infty}$ and $(f_{ij})_{i,j=1}^{\infty}$ so that (1) is satisfied whenever $i, j \leq k$ and (2) is satisfied for all $n \geq 2$.

Notice that $(y_{\alpha_n})_{n=1}^{\infty}$ has the following properties:

$$\|y_{\alpha_n}\| \to d(0,A) \text{ and } f_{ij}(y_{\alpha_n}) \to y''(f_{ij}) \text{ as } n \to \infty \text{ for all } i, j.$$

By hypothesis, there is a subsequence $(y_{\alpha_{n_k}})$ of (y_{α_n}) such that $y_{\alpha_{n_k}} \xrightarrow{w} z \in A$. Then

$$f_{ij}(z) = y''(f_{ij}) \text{ for all } i, j. \qquad (3)$$

Claim: If S is the union of $L_1, L_2, \ldots$ and B is the norm closure of S, then $Qz \in B$. Suppose not. Note that S is a subspace of $Q(N)$ as the increasing union of a collection of subspaces, and hence B is a closed subspace of $Q(N)$. By the Hahn-Banach theorem, there is an $f \in \Sigma'$ such that $f(Q^{-1}(B)) = 0$ and $f(z) > 0$. Since $y_{\alpha_{n_k}} \in Q^{-1}(B)$ for each k, it follows that

$0 = \lim_k f(y_{\alpha_{n_k}}) = f(z) > 0$. This contradiction proves the claim.

Let $\varepsilon > 0$ and let $Qy \in S$ be such that $\|Qz - Qy\| \leqslant \varepsilon$. By virtue of (3),

$$|(Qy - y'')(f_{ij})| = |(Qy - Qz)(f_{ij})| \leqslant \varepsilon \text{ for all } i, j.$$

But $\|Qy - y''\| \leqslant \sup_{i,j} |(Qy - y'')(f_{ij})| \leqslant \varepsilon$ by (2). Since $\|Qz - y''\| \leqslant 2\varepsilon$ for all $\varepsilon > 0$, we see that $y'' = Qz \in Q(A)$, which completes the proof. ■

5.6 THEOREM. *Let M be a nonempty subset of a normed linear space N.*

(a) M is supportively compact if and only if it is supportively ω-compact.

(b) M is supportively weakly compact if and only if it is supportively weakly ω-compact.

(c) If N is separable, then M is supportively weak compact if and only if it is supportively weak* ω-compact.*

(d) If U^{π} is weak sequentially compact and M is supportively weak* compact, then M is supportively weak* ω-compact.*

Proof. For (a) and (c), the relevant topologies are metric topologies on norm-bounded subsets, and so the forward implications present no difficulties, as the reader can easily verify.

For the reverse implication for (c), suppose the sequential condition holds, and let (f_α) be a supportive net for M w. r. t. x with corresponding minimizing net (x_α). By the Banach-Alaoglu theorem, (f_α) contains a weak* convergent subnet, which w. l. o. g. is (f_α). Let f be its limit. It is enough to show that $f \in J_x(P_M x)$. If not, then since $J_x(P_M x)$ is weak* compact by Lemma 5.4 and the metrizability of the weak* topology on norm-bounded subsets, there are disjoint weak* open sets W_1 and W_2 such that $J_x(P_M x)$ lies in W_1 and $f \in W_2$. W. l. o. g. (f_α) lies entirely in W_2. Since $\|x_\alpha\| \to d(x,M)$, it follows that the unordered set $\{f_\alpha\}$ contains a supportive sequence for M w. r. t. x, which might not be a subnet of (f_α), weak* convergent to some $g \in J_x(M)$. Since $g \in J_x(P_M x)$, we have a contradiction. This proves (c).

We now obtain the reverse implication for (a). Suppose M is supportively ω-compact. Let (f_α) be a supportive net for M w. r. t. x with corresponding minimizing net (x_α). Let $\varepsilon > 0$, and let W_ε be the union of all open balls of radius ε centered at an element of $J_x(P_M x)$. Thus, W_ε is an open set containing $J_x(P_M x)$. We claim that there is an α_0 such that $\alpha \geqslant \alpha_0$ implies $f_\alpha \in W_\varepsilon$. If not, then there is a subnet (f_β) of (f_α) that lies outside W_ε. Since a minimizing sequence can be extracted from the unordered set $\{x_\beta\}$, there must be a sequence (g_n) in $\{f_\beta\}$ that converges to an element of $J_x(P_M x)$, which is a contradiction.

Thus, $d(f_\alpha, J_x(P_M x)) \to 0$. Now $J_x(P_M x)$ is norm compact by Lemma 5.4, and so it is approximatively compact and hence proximinal. For each α, let g_α be an element of $J_x(P_M x)$ closest to f_α. Note that $f_\alpha - g_\alpha \to 0$. By compactness, (g_α) has a subnet (g_β) converging to some $g \in J_x(P_M x)$. Since (f_β) also converges to g, M is supportively compact. This proves (a).

The reverse implication for (b) is an immediate consequence of Lemma 5.5 applied to N* with $A = J_x(M)$. For the forward implication, suppose that (f_n) is a supportive sequence for supportively weakly compact set M w. r. t. $x \in N$. By hypothesis, every countably infinite subset of $\{f_n\}$ has a weak limit point, and so $\{f_n\}$ is relatively weakly sequentially compact by the Eberlein-Šmulian theorem. Let (f_{n_j}) be a weakly convergent subsequence of (f_n). Since (f_{n_j}) must itself have a subnet converging weakly to some $f \in J_x(M)$, it follows that $f_{n_j} \overset{w}{\to} f$, which completes the proof of (b).

Finally, suppose the hypothesis of (d) holds. Then any supportive sequence (f_n) for M w. r. t. $x \in N$ has a weak* convergent subsequence, which w. l. o. g. is (f_n). Since (f_n) also has a subnet weak* convergent to an element of $J_x(M)$, we are done. ∎

The following corollary is obtained from part (d) of the theorem in the same way that Corollary 4.14 was obtained from Theorem 4.13.

5.7 <u>COROLLARY</u>. *Whenever Banach space B has an equivalent smooth norm, in particular whenever it is weakly compactly* *generated, then every supportively weak* compact set in B is supportively weak* ω-compact.*

It should be noted that many common spaces to which the corollary applies are also either separable or reflexive, and so the conclusion of the corollary can be obtained more readily from parts (b) and (c) of the preceding theorem. However, if S is an uncountable index set, then $c_0(S)$ is neither reflexive nor separable, and yet is weakly compactly generated; see [16].

SECTION 6

SUPPORTIVE COMPACTNESS AND CONVEXITY

The purpose of this section is to study the interaction between convexity and the various forms of supportive compactness defined in the previous section. We first study the convexity of supportively compact Chebyshev sets. Following this, we take up the converse problem of deciding when convex sets have some form of supportive compactness. By combining our results from these two studies, we obtain the approximation-theoretic characterization of the normed linear spaces that are (R) & (Rf) & (S) & (SH) that was promised at the end of Section 3.

Convexity of Supportively Compact Chebyshev Sets

In order to prove that a Chebyshev set is convex, it is frequently necessary to prove that it has some "solar" property; see [57]. The one that we use is the following.

6.1 DEFINITION (Vlasov [55]). A nonempty closed set M is called a δ-*sun* if, for each $x \notin M$, there is a sequence (z_n) for which $z_n \neq x$, $z_n \to x$, and $\frac{d(z_n,M) - d(x,M)}{\|z_n - x\|} \to 1$.

The following two facts are going to prove useful for our study.

6.2 LEMMA (Vlasov [57]). *Let M be a proximinal set in normed linear space N with metric projection P, and suppose that x, x', v, $v' \in N$, $x \neq v$, $x \notin M$, and $f \in \Sigma'$ are such that $x' \in Px$, $x \in (x',v)$, $v' \in Pv$, and $f(v' - x) = \|v' - x\|$. Then*

$$0 \leqslant 1 - \frac{d(v,M) - d(x,M)}{\|v - x\|} \leqslant 1 - \frac{f(x' - x)}{\|x' - x\|}.$$

6.3 LEMMA (Vlasov [54]). *If B is a Banach space, then B^* is rotund if and only if each δ-sun in B is convex.*

The method of proof of the following lemma was inspired by Vlasov's proof in [57] that a Chebyshev set with a norm-to-weak continuous metric projection in a semi-Kadec-Klee space is a -sun.

6.4 LEMMA. *Let M be a supportively weak* ω-compact set. Suppose that whenever $x \in N$ and y, $z \in Px$, then $J_x y = J_x z$. Then M is a δ-sun. In particular, every supportively weak* ω-compact Chebyshev set is a δ-sun.*

Proof. Let $x \notin M$. W. l. o. g. $x = 0$ and $d(0,M) = 1$, because the property of being a δ-sun is easily seen to be invariant under translations and expansions. Let $y \in P0$, and let (v_n) be a sequence such that $v_n \to 0$ and 0 lies in (y,v_n); that is, $v_n \to 0$ down the "far side" of the ray $(y,0,\infty)$. Let $v_n' \in Pv_n$. By the continuity of $d(\cdot,M)$,

$$1 \leqslant \|v_n'\| \leqslant \|v_n' - v_n\| + \|v_n\| = d(v_n,M) + \|v_n\| \to d(0,M) = 1.$$

Thus, (v_n') is a minimizing sequence in M for 0. Let $f_n \in Jv_n'$ for each n, and let (f_{n_j}) be a weak* convergent subsequence, $f_{n_j} \xrightarrow{w^*} f \in J(P0) = Jy$. By thinning, we can assume that $(f_{n_j}) = (f_n)$. Thus, $g_n = \|v_n'\|^{-1} f_n \xrightarrow{w^*} f$, and so $g_n(y) \to f(y) = \|y\|^2 = 1$. By Lemma 6.2,

$$0 \leqslant 1 - \frac{d(v_n,M) - d(0,M)}{\|v_n\|} \leqslant 1 - g_n(y) \to 0.$$

Thus, (v_n) is the sequence required in the definition of a δ-sun. ∎

By combining Corollary 5.7 with Lemmas 6.3 and 6.4, we immediately obtain the following theorem. Note that there is only one geometric condition on the Banach space, though it is a reasonably strong one.

6.5 <u>THEOREM</u>. *In a Banach space with a rotund dual, every supportively weak* ω-compact Chebyshev set (a fortiori every supportively weak* compact Chebyshev set) is convex.*

6.6 <u>COROLLARY</u>. *In a Banach space with a rotund dual, every Chebyshev set that is supportively compact or supportively weakly compact is convex.*

Thus, if Banach space B has a rotund dual, a Chebyshev set with any of the types of supportive compactness we have treated is convex.

Supportive Compactness of Convex Sets

We now work on the problem converse to the one discussed above. We need the following technical lemma, which is used both here and in Section 7.

6.7 LEMMA. *Let N be a normed linear space such that U^{π} is weak* sequentially compact. Let τ be a topology on N such that the norm is τ-lower semicontinuous and J is τ-to-weak* upper semicontinuous. Let M be a subset of N with $d(0,M) = 1$ such that every minimizing sequence in M for 0 has a τ-convergent subsequence with limit in M. Then every supportive sequence for M w. r. t. 0 has a weak* convergent subsequence with limit in $J(P0)$.*

Proof. Let (f_n) be a supportive sequence for M w. r. t. 0, and let (x_n) be a corresponding minimizing sequence. W. l. o. g. $x_n \overset{\tau}{\to} y \in M$. By the τ-lower semicontinuity of $\|\cdot\|$, $y \in P0$. Since U^{π} is weak* sequentially compact, we can assume w. l. o. g. that $f_n \overset{w^*}{\to} f$. We will be done if we can show that $f \in Jy$.

Suppose $f \notin Jy$. It is not difficult to see that Jy is weak* closed. Since the weak* topology is completely regular (see [16], p. 12), there must be disjoint weak* open sets W_1 and W_2 with $f \in W_1$ and Jy lying in W_2. Since $x_n \overset{\tau}{\to} y$ and J is τ-to-weak* upper semicontinuous, there is an n_0 such that

$n \geq n_0$ implies $f_n \in W_2$ and hence $f_n \notin W_1$. Since $f_n \xrightarrow{w^*} f \in W_1$, this is a contradiction. ∎

6.8 THEOREM. *Let N be a normed linear space such that* U^{π} *is weak* sequentially compact. In the following collection of statements, (1)* $\Rightarrow$ *(2)* $\Rightarrow$ *(3).*

(1) N is (wLν) & (SH).

(2) Every proximinal convex set in N is a supportively weak ω-compact Chebyshev set.*

(3) N is (R) & (SH).

Proof. (1) $\Rightarrow$ (2). Suppose N is (wLν) & (SH). Let M be a proximinal convex set in N, and let $x \in N$. Let (f_n) be a supportive sequence for M w. r. t. x. W. l. o. g. $x = 0$ and $d(0,M) = 1$. By Theorem 2.10, M is an approximatively weakly compact Chebyshev set, so if (x_n) is a minimizing sequence in M for 0, then $x_n \xrightarrow{w} P0$ by Proposition 2.4 (d). Also, $\|x_n\| \to \|P0\|$, so $x_n \xrightarrow{\lambda} P0$. Now J is λ-w* u. s. c. by Theorem 4.11. Since the norm is λ-lower semicontinuous, Lemma 6.7 shows that (f_n) has a weak* convergent subsequence with limit in $J(P0)$. Thus, M is supportively weak* ω-compact.

(2) $\Rightarrow$ (3). Suppose that (2) holds. Since every proximinal hyperplane is Chebyshev, it is not hard to see that N is rotund. Now suppose that $x \in \Sigma$, $x_n \in \Sigma$, $x_n \xrightarrow{w} x$, $f_n \in \Sigma'$, and $f_n(x_n) = 1$ for each n. We need to show that $f_n(x) \to 1$ to

establish that N is (SH). Let $f \in \Sigma'$ be such that $f(x) = 1$. W. l. o. g. $f(x_n) > 0$ for all n, and so $y_n = f(x_n)^{-1}x_n \in K = \{y \in N: f(y) = 1\}$. Note that K is a proximinal convex set. Now $\|y_n\| = f(x_n)^{-1} \to 1$, so (y_n) is a minimizing sequence in K for 0. It follows that $(g_n) = (f(x_n)^{-1}f_n)$ is a supportive sequence for K w. r. t. 0 corresponding to (y_n). If (g_{n_j}) is a subsequence of (g_n), then (g_{n_j}) must have a subsequence $(g_{n_{j_k}})$ converging weak* to some $g \in J(P_K 0) = Jx$, by (2). Thus, $g_{n_{j_k}}(x) \to g(x) = 1$. It follows that $g_n(x) \to 1$, and so $f_n(x)$ is also convergent to 1. ∎

As we saw in the proof of Corollary 4.14, U^π is weak* sequentially compact whenever N is separable or is a Banach space with an equivalent smooth norm, such as a weakly compactly generated Banach space. More obviously, the last theorem holds for reflexive spaces, which gives the following result.

6.9 <u>THEOREM</u>. *Let N be a normed linear space. Then the following are equivalent.*

(1) N is (R) & (Rf) & (SH).

(2) Every nonempty closed convex set in N is a supportively weakly compact Chebyshev set.

(3) Every nonempty closed convex set in N is a supportively weak compact Chebyshev set.*

Proof. (1) $\Rightarrow$ (2). Suppose (1) holds. By Corollary 1.22, N is (wLν). By Theorem 2.9, every nonempty closed convex set in N is proximinal, and hence a supportively weak* ω-compact Chebyshev set by Theorem 6.8. An easy application of reflexivity and Theorem 5.6 (b) yields (2).

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (1). Suppose (3) holds. By Theorem 2.9, N is (R) & (Rf). By Theorem 5.6 (b), every nonempty closed convex set in N is a supportively weak* ω-compact Chebyshev set. By Theorem 6.8, N is (SH). ∎

A Characterization of (R) & (Rf) & (S) & (SH) Spaces

We can now combine the results of this section to give the long-promised approximation-theoretic characterization of the normed linear spaces that are (R) & (Rf) & (S) & (SH). Compare this next theorem to Theorem 3.13, which falls short of being such a characterization.

6.10 THEOREM. *Let N be a normed linear space. Then the following are equivalent.*

(1) N is (R) & (Rf) & (S) & (SH).

(2) The nonempty closed convex sets in N are exactly the supportively weakly compact Chebyshev sets.

(3) The nonempty closed convex sets in N are exactly the supportively weak compact Chebyshev sets.*

<u>Proof</u>. (1) $\Longrightarrow$ (2). This is an immediate consequence of Corollary 6.6 and Theorem 6.9.

(2) $\Longrightarrow$ (1). If (2) holds, then N is (R) & (Rf) & (SH) by Theorem 6.9. If N were not smooth, then by Example 3.9 N would contain a nonconvex Chebyshev set that would be the union of two supportively weakly compact half-spaces. Such a set would itself be supportively weakly compact, a contradiction.

The equivalence of (2) and (3) is immediate from the fact that either condition implies reflexivity. ∎

Theorem 6.10 naturally raises the problem of a characterization of the spaces in which the nonempty closed convex sets are exactly the supportively norm compact Chebyshev sets. It turns out that the resulting spaces are obtained by replacing smoothness by Fréchet smoothness in Theorem 6.10 (1).

6.11 <u>THEOREM</u>. *Let N be a normed linear space. Then the following are equivalent.*

(1) N is (R) & (Rf) & (F) & (SH).

(2) The nonempty closed convex sets in N are exactly the supportively compact Chebyshev sets.

Proof. Suppose N is (R) & (Rf) & (F) & (SH). Šmulian [51] showed that whenever a Banach space B has a dual that is (F), then B is (D) and hence (H); see also our Theorem 1.11. In our case, N* is (H), and so supportive compactness and supportive weak compactness agree for sets in N. An application of Theorem 6.10 yields (2).

Now suppose that (2) holds. By Theorem 6.9, N is (R) & (Rf) & (SH). If N were not smooth, then by Example 3.9 N would contain a nonconvex Chebyshev set that would be the union of two closed half-spaces, each supportively compact. As in the proof of Theorem 6.10, this would yield a contradiction. Thus, N is smooth and J is single-valued. By Proposition 4.7 (c), we will be done if we can show that J is norm-to-norm continuous. In fact, it suffices to show that J is norm-to-norm continuous on Σ. Let x_n, $x \in \Sigma$, $x_n \to x$, $f_n = Jx_n$, and $f = Jx$. Now $f(x_n) = 1$, so $(f(x_n)^{-1}x_n)$ is a minimizing sequence in $H = \{y \in N: f(y) = 1\}$ for 0 with corresponding supportive sequence $(f(x_n)^{-1}f_n)$. By (2) and an easy argument, $f(x_n)^{-1}f_n \to Jx = f$. Thus, $f_n \to f$, and so J is norm-to-norm continuous. ∎

It might seem interesting to find approximation-theoretic characterizations of spaces that are (R) & (Rf) & (σ) & (SH), where (σ) is some form of smoothness besides (S) or (F). Theorem 6.10 does this trivially for (σ) = (VS), since there is no difference between smoothness and very smoothness for reflexive spaces; see Proposition 4.7. It is not so easy to find approximation-theoretic characterizations of the type we have been considering when the smoothness property in question is a uniform property, such as (UG) or (US). In fact, this entire thesis to this point does not contain a single characterization of a class of normed linear spaces involving some uniform rotundity or smoothness property, or even a localization of one, such as local uniform rotundity. The closest we ever come is in Appendix B, where we obtain an approximation-theoretic characterization of the midpoint locally uniformly rotund spaces by studying the behavior of closed balls.

SECTION 7

CLASSES OF SUPPORTIVELY COMPACT SETS

In the previous section, we explored the connection between supportive compactness and convexity, with the goal of obtaining the characterization of Theorem 6.10. The purpose of this final section is to study supportive compactness in some other classes of sets useful in approximation theory besides the closed convex sets.

Approximatively Compact Sets

It might seem that the norm-to-weak* upper semicontinuity of the norm-duality map would force approximatively compact sets to be supportively weak* ω-compact. The following example shows that this is not so. In fact, not even singletons need be supportively weak* ω-compact.

7.1 EXAMPLE. In ℓ_∞, let $x = (1, 1, 1, \dots)$, and let $M = \{x\}$. In ℓ_1, let $x_n = \frac{1}{2}(e_1 + e_n)$ for $n \geqslant 2$, where e_i is the i'th unit vector. Let $f_n = Q_0(x_n)$, where Q_0 is the canonical map from ℓ_1 into $\ell_1^{**} = \ell_\infty^*$. It is easy to see that (f_n) is a supportive sequence for M with respect to 0, but that (f_n) has no weak* convergent subsequence. Thus, M is not supportively weak* ω-compact. ■

The problem in the above example is that the closed unit ball of ℓ_∞* is not weak* sequentially compact.

7.2 PROPOSITION. *Let N be a normed linear space such that* U^π *is weak* sequentially compact. Then every approximatively compact set in N is supportively weak** ω*-compact.*

Proof. Let M be an approximatively compact set in N, and let (f_n) be a supportive sequence for M w. r. t. $x \in N$ with corresponding minimizing sequence (x_n). W. l. o. g. $x = 0$ and $d(0,M) = 1$. An application of Lemma 6.7 now finishes the proof. ■

7.3 COROLLARY. *If* U^π *is weak* sequentially compact, then every boundedly compact set, a fortiori every compact set, in N is supportively weak** ω*-compact.*

As noted in the proof of Corollary 4.14, U^π is weak* sequentially compact whenever N is a weakly compactly generated Banach space. Also note that U^π is weak* sequentially compact whenever N is a separable normed space, since the weak* topology on U^π is metrizable.

7.4 COROLLARY. *If N is reflexive, then every approximatively compact set in N is supportively weakly compact.*

By imposing various smoothness conditions on N, we can force the norm-duality map to have certain continuity properties. This in turn causes a supportive sequence to be convergent in some sense when the corresponding minimizing sequence is norm convergent. In particular, the following result follows easily from Proposition 4.7.

7.5 PROPOSITION. *Let N be a normed linear space.*

(a) *If N is smooth, then every approximatively compact set in N is supportively weak* ω-compact.*

(b) *If N is very smooth, then every approximatively compact set in N is supportively weakly compact.*

(c) *If N is Fréchet smooth, then every approximatively compact set in N is supportively compact.*

If we require that N be a semi-Kadec-Klee space, then we can obtain the result for approximatively weakly compact sets corresponding to Proposition 7.2.

7.6 PROPOSITION. *Let N be a normed linear space of type (SH) such that* U^{π} *is weak* sequentially compact. Then every approximatively weakly compact set in N is supportively weak* ω-compact.*

Proof. Note that any weakly convergent minimizing sequence is λ-convergent. Also, J is λ-w* u. s. c. by Theorem 4.11. With these observations, the proof now continues like that of Proposition 7.2. ∎

Corollaries to Proposition 7.6 corresponding to the corollaries of Proposition 7.2 can now be obtained. We leave this to the reader.

P-Convex Sets

The following definition is a well-known generalization of the Chebyshev property; see [57].

7.7 DEFINITION. Set M in normed space N is called *P-convex* if, for every $x \in N$, the set Px is nonempty and convex.

The following lemma says that in a smooth space, an important property of Chebyshev sets is actually true of all P-convex sets.

7.8 LEMMA. *Let M be a P-convex set in smooth space N. Then for every $x \in N$, $J_x(P_M x)$ is a singleton.*

Proof. W. l. o. g. $x = 0$ and $d(0,M) = 1$. Let $f \in \Sigma'$ be such that $H = \{y \in N\colon f(y) = 1\}$ separates $P_M 0$ and U. It is easy to see that P_M0 lies in the intersection of H with Σ. Thus, for each $y \in P_M0$, f is the unique element of Σ' such that $f(y) = 1$; that is, $Jy = \{f\}$. Thus, $J(P_M0) = \{f\}$. ∎

Vlasov has shown that in a Banach space of type (SH) with a rotund dual, an approximatively weakly compact Chebyshev set is convex; see [57], Theorem 4.28 (k). We now use Proposition 7.6 to show that Vlasov's result remains true if the Chebyshev property is weakened to P-convexity.

7.9 THEOREM. *Let B be a Banach space of type (SH) with a rotund dual. Then every approximatively weakly compact P-convex set in B is convex.*

Proof. Let M be P-convex and approximatively weakly compact. B is smooth, and so U^{π} is weak* sequentially compact by a theorem of Hagler and Sullivan [25]. By Proposition 7.6, M is supportively weak* ω-compact. By Lemma 7.8, $J_x(P_Mx)$ is a singleton for every $x \in B$. By Lemma 6.4, M is a δ-sun. By Lemma 6.3, M is convex. ∎

Closed Balls

It can be shown that every closed ball in a normed linear space is an approximatively compact Chebyshev set if and only if the space has the midpoint local uniform rotundity property introduced by Anderson in [1]. The purpose of this subsection is to obtain some similar results with approximative compactness replaced by supportive weak compactness.

The following condition is a weakening of the semi-Kadec-Klee property.

7.10 DEFINITION. A normed linear space is said to have property (SH') if, whenever $x, x_n \in \Sigma$, $f_n \in \Sigma'$, $f_n(x_n) = 1$, $x_n \overset{w}{\to} x$, and $\lim_n \min \{t: \|tx_n - x\| = \alpha\} = 1 - \alpha$ for some $\alpha \in (0,1)$, then $f_n(x) \to 1$.

7.11 THEOREM. *In the following collection of assertions about normed linear space N, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4).*

(1) N is (R) & (Rf) & (SH').

(2) Every closed ball in N is a supportively weakly compact Chebyshev set.

(3) Every closed ball in N is a supportively weak ω-compact Chebyshev set.*

(4) N is (R) & (SH').

Proof. Suppose (1) holds. Let V be a closed ball in N and let (f_n) be a supportive sequence for V w. r. t. $x \in N$. W. l. o. g. $x = 0$ and $d(0,V) = 1$. Let (x_n) be a corresponding minimizing sequence for (f_n), and let $y_n = \|x_n\|^{-1}x_n$ for each n. By Theorem 2.9, V is an approximatively weakly compact Chebyshev set, so $x_n \overset{w}{\to} y_0 = P0$. Thus, $y_n \overset{w}{\to} y_0$ also. Letting β be the radius of V, we note that $(1 + \beta)y_0$ is the center of V. Since $x_n \in V$ for each n,

$$1 \leq \min \{t: \|ty_n - (1 + \beta)y_0\| = \beta\} \leq \|x_n\|.$$

Since $\|x_n\| \to 1$, $\lim_n \min \{t: \|ty_n - (1 + \beta)y_0\| = \beta\} = 1$, and so $\lim_n \min \{t: \|ty_n - y_0\| = \frac{\beta}{1 + \beta}\} = \frac{1}{1 + \beta}$. Letting $g_n = \|x_n\|^{-1} f_n$, we see that $g_n \in \Sigma'$ and $g_n(y_n) = 1$ for each n. Since N is (SH'), $g_n(y_0) \to 1$. Since N^* is reflexive, (g_n) has a weakly convergent subsequence, which w. l. o. g. is (g_n). Let f be its limit. Then

$$1 = \lim g_n(y_0) = f(y_0) \leq \|f\| \leq 1,$$

so $\|f\| = f(y_0) = 1$ and $f \in Jy_0 = J(P0)$. Since $f_n \overset{w}{\to} f$, V is supportively weakly compact. Thus, (2) holds.

It is obvious that (2) implies (3).

Now suppose that (3) holds. Since closed balls are Chebyshev, it is easy to see that N is rotund. Now let x, x_n, f_n, and α be as in the hypothesis of the definition of condition (SH'). Let $\beta = \frac{\alpha}{1 - \alpha}$ and let V be the closed ball of radius β and center $(1 + \beta)x$. Let $m_n = \min \{t: \|tx_n - x\| = \alpha\}$; w. l. o. g. m_n is finite for each n. Then

$$\min \{t: \|tx_n - (1 + \beta)x\| = \beta\} = (1 + \beta)m_n \to (1 + \beta)(1 - \alpha) = 1.$$

Thus, $((1 + \beta)m_n x_n)$ is a minimizing sequence in V w. r. t. 0. Let (f_{n_j}) be a subsequence of (f_n). By supportive weak* ω-compactness, there is a subsequence $((1 + \beta)m_{n_{j_k}} f_{n_{j_k}})$ of the sequence $((1+\beta)m_{n_j}f_{n_j})$ converging weak* to some $f \in Jx$. Since $(1+\beta)m_{n_{j_k}}f_{n_{j_k}}(x) \to f(x) = 1$, it follows that

$$f_{n_{j_k}}(x) \to 1.$$

Since every subsequence of $(f_n(x))$ has a subsequence tending to 1, $f_n(x) \to 1$. Thus, N is (SH'). ■

7.12 <u>COROLLARY</u>. *A reflexive space is (R) & (SH') if and only if each of its closed balls is a supportively weakly compact Chebyshev set.*

Incidentally, the fact about midpoint locally uniformly rotund spaces mentioned in the first sentence of this subsection is a new result. It can be found in Appendix B, along with some related results about the approximative properties of closed balls.

APPENDIX A

PROPOSITION 1.4 AND JAMES'S THEOREM

In this appendix, we give a proof of Proposition 1.4 that uses only elementary methods and the Bishop-Phelps theorem and does not rely on James's theorem. We also give an extension of James's theorem to certain classes of normed linear spaces without a completeness hypothesis. This theorem also has an elementary proof not using the classical version of James's theorem. Finally, we indicate one possible direction of search for anyone seeking an elementary proof of James's theorem itself.

1.4 PROPOSITION. *The following are equivalent.*

(1) N is (Rf).

(2) Whenever $x_n \in \Sigma$, $f \in \Sigma'$, and $f(x_n) \to 1$, then (x_n) has a weakly convergent subsequence.

(3) Whenever K is a nonempty closed convex set in N and (x_n) is a minimizing sequence in K for $x \in N$, then (x_n) has a weakly convergent subsequence.

Proof (without James's theorem): It is obvious that (1) implies (2). The equivalence of (2) and (3) follows by elementary methods as in Proposition 1.3. Thus, it suffices to prove that (3) implies (1).

Suppose (3) holds. Then N is complete by the elementary argument given in Chapter 1. Let F be any *support functional* in N**; that is, an element of N** taking on its supremum on U^{π}. We wish to prove that $F \in Q(N)$, so w. l. o. g. $\|F\| = 1$. By Goldstine's theorem, there is a net (x_α) in U with $Qx_\alpha \overset{w^*}{\to} F$. Let $f \in \Sigma'$ be such that $F(f) = 1$. Since $Qx_\alpha(f) \to F(f)$, $1 \geqslant \|x_\alpha\| \geqslant f(x_\alpha) \to F(f) = 1$, and so $\|x_\alpha\| \to 1$ and $f(x_\alpha) \to 1$. Let $K = \{x \in N: f(x) = 1\}$, a nonempty closed convex set. Since (3) holds, any minimizing sequence in K for 0 has a weakly convergent subsequence, whose limit is in K because K is weakly closed. By Lemma 5.5, whose proof was elementary, any minimizing net in K for 0 has a weakly convergent subnet with limit in K. In particular, the minimizing net $(f(x_\alpha)^{-1}x_\alpha)$ has a weakly convergent subnet $(f(x_\beta)^{-1}x_\beta)$ converging to some $x \in K$. Thus, $x_\beta \overset{w}{\to} x$ and $Qx_\beta \overset{w^*}{\to} Qx$. Since $Qx_\beta \overset{w^*}{\to} F$, we see that $F = Qx$.

Thus, every support functional in N** lies in Q(N). The Bishop-Phelps theorem [4] says that whenever B is a Banach space, the support functionals in B* are dense in B*. It follows immediately that Q(N) = N**, and so N is reflexive. ∎

We can now check to see how much of what we did in the first two sections really depended on James's theorem. Nothing in Section 1 required it, while the only results in Section 2 that needed it were the implications (3) $\Longrightarrow$ (1) in Theorem 2.9 and (2) $\Longrightarrow$ (1) in Theorem 2.16, as well as Vlasov's Theorem 2.12 that we mentioned but did not prove or later use.

Vlasov needed a result of Oshman from [41] that used James's theorem in a crucial way. It is interesting that none of our other results need James's theorem, because Theorem 1.11, Theorem 1.12, the equivalence of (1) and (3) in Theorem 2.5, and Theorem 2.7 are all well-known results whose known proofs have all relied on James's theorem.

It would be nice, though of course not crucial, to remove completely our dependence on James's theorem. This could be done by solving the following problem, also mentioned by Blatter in [5].

A.1 PROBLEM. Find an elementary proof, not using James's theorem, of the following fact:

> A normed linear space is reflexive whenever it is rotund and each of its nonempty closed convex sets has a point nearest the origin.

While we have no solution to this problem, we can give such a proof if we replace rotundity with the somewhat stronger property (wLν). In fact, we can get by with the following weakening of (wLν).

A.2 DEFINITION (Vlasov [59]). A normed linear space has property (wCDL) if, whenever $x_n \in \Sigma$, $f \in \Sigma'$, f achieves its supremum on U, and $f(x_n) \to 1$, then (x_n) has a weakly convergent subsequence.

It is clear from Proposition 1.20 that condition (wCDL) is condition (wLν) with the rotundity requirement eliminated. By an easy argument, it can be seen that (wCDL) just says that hyperplanes generated by support functionals in Σ' are approximatively weakly compact; see [59].

A.3 <u>THEOREM</u>. *Let N be a normed linear space with property (wCDL). Then N is reflexive if and only if every $f \in \Sigma'$ achieves its supremum on U.*

<u>Proof</u> (without James's theorem): The forward implication is elementary. For the reverse, suppose that every $f \in \Sigma'$ is a support functional. We now just compare Definition A.2 with Proposition 1.4. ■

As we mentioned before the proof of Theorem 2.5, James has shown that his theorem does not hold, in general, for normed linear spaces not assumed to be complete. However, Theorem A.3 shows that James's theorem does hold for all normed spaces of class (wCDL), and hence for all spaces that have stronger properties, sich as (wK), (wLUR), (wUR), or (LUR). Also, Theorem A.3 is in a sense the strongest theorem possible in this direction, since a space that is not (wCDL) obviously has no hope of being reflexive.

Since reflexivity is isomorphism-invariant, a simple corollary of Theorem A.3 is that a normed space N with an equivalent (wCDL) norm $/\cdot/$ such that every $f \in N^*$ achieves its supremum on the closed unit ball of (N, /•/) is reflexive. It is interesting to compare this result with Klee's result in [32], also obtained by elementary methods, that a Banach space B such that every $f \in B^*$ achieves its supremum on every isomorph of U is reflexive.

Several comments about property (wCDL) are in order. First, property (wCDL) by itself does not imply completeness. Any dense subspace of ℓ_2 is uniformly rotund and hence (wCDL). We make this comment to point out that Theorem A.3 does have some content beyond James's theorem itself. Second, property (wCDL) does not imply reflexivity by itself, even for complete spaces. Any separable Banach space can be given an equivalent (LUR) and hence (wCDL) norm; see [16]. Third, the following example shows that not all rotund Banach spaces are (wCDL). Thus, one cannot solve Problem A.1 by proving that all rotund spaces are (wCDL) and then applying Theorem A.3.

A.4 <u>EXAMPLE</u>: $(\ell_1, \|\cdot\|_H)$. This space was constructed by Mark Smith in [47] to show that not all (URWC) spaces are (MLUR); see Smith's paper for the definitions.

For $y = (y^1, y^2, \ldots)$ in ℓ_2 let $y' = (0, y^2, y^3, \ldots)$ and let $\|y\|_S = \max\{|y^1|, \|y'\|_2\}$. Let (α_n) be a sequence of positive reals with $\alpha_n \to 0$, and define $T\colon \ell_2 \to \ell_2$ by $T(y^1, y^2, \ldots) = (y^1, \alpha_2 y^2, \alpha_3 y^3, \ldots)$. Define $\|\cdot\|_W\colon \ell_2 \to \mathbb{R}$ by

$$\|y\|_W = (\|y\|_S^2 + \|Ty\|_2^2)^{\frac{1}{2}}.$$

It can be shown that $\|\cdot\|_W$ is a norm on ℓ_2 equivalent to the usual norm; see Smith's paper.

For $x = (x^1, x^2, \ldots)$ in ℓ_1, let $x' = (0, x^2, x^3, \ldots)$, and let $\|x\|_M = \max\{|x^1|, \|x'\|_1\}$. Let $I: \ell_1 \to \ell_2$ be the inclusion mapping. Define $\|\cdot\|_H: \ell_1 \to \mathbb{R}$ by

$$\|x\|_H = (\|x\|_M^2 + \|Ix\|_W^2)^{\frac{1}{2}}.$$

Smith showed that $\|\cdot\|_H$ is an equivalent rotund norm on ℓ_1. We now show that $(\ell_1, \|\cdot\|_H)$ is not (wCDL).

Let (e_n) be the usual sequence of unit vectors in ℓ_1 and ℓ_∞. Let $f \in \ell_1{}^*$ be given in the usual ℓ_∞ representation by $f = 3^{\frac{1}{2}}e_1$. In ℓ_1, let $x_0 = 3^{-\frac{1}{2}}e_1$. It is easy to check that $\|x_0\|_H = 1$ and that if $x = (x^1, x^2, \ldots)$ and $\|x\|_H \leqslant 1$, then $|x^1| \leqslant 3^{-\frac{1}{2}}$. It follows immediately that $\|f\|_H = 1$ and that f attains its supremum on the closed unit ball of $(\ell_1, \|\cdot\|_H)$ at x_0. Now let $x_n = 3^{-\frac{1}{2}}(e_1 + e_n)$ for $n \geqslant 2$. It is not difficult to check that $\|x_n\|_H \to 1$. If we let $w_n = \|x_n\|_H^{-1}x_n$, then $\|w_n\|_H = 1$ and $f(w_n) \to 1$. However, (w_n) cannot have a weakly convergent subsequence, because the sequence (e_n) in ℓ_1 does not. Thus, $(\ell_1, \|\cdot\|_H)$ is not (wCDL). ■

We close this appendix by noting that a solution to the following problem, when combined with Theorem A.3, would give an elementary proof of James's theorem.

A.5 <u>PROBLEM</u>. Find an elementary proof, not using James's theorem, of the following fact:

If B is a Banach space such that every $f \in \Sigma'$ achieves its supremum on U, then every hyperplane supporting U is approximatively weakly compact.

Any solution to Problem A.5 will use the completeness of B in some crucial way. James gave an example in [28] of an incomplete space with every $f \in \Sigma'$ achieving its supremum on U. By Theorem A.3, such a space cannot be (wCDL).

APPENDIX B

APPROXIMATIVE PROPERTIES OF CLOSED BALLS

Many of the theorems proved previously in this thesis characterize the normed linear spaces in which the nonempty closed convex sets have certain approximative properties. We might also ask for similar characterizations relative to certain interesting subclasses of the closed convex sets. For instance, an important problem in approximation theory is to find the approximative properties of closed subspaces of certain normed linear spaces and to characterize the spaces in which the closed subspaces all have certain of these properties.

The purpose of this appendix is to characterize the normed linear spaces in which the closed balls all have some approximative property. This effort will disclose a surprising connection between the approximative compactness of closed balls and the property of midpoint local uniform rotundity of a normed linear space originally studied by Anderson. We now give Anderson's definition and some useful extensions of our own.

B.1 <u>DEFINITION</u>. A normed linear space N is said to be

(MLUR) *midpoint locally uniformly rotund* if $x_n \to x_0$ whenever $x_n, y_n, x_0 \in \Sigma$ and $\frac{1}{2}(x_n + y_n) \to x_0$ (see Anderson [1]);

(wMLR) *weakly midpoint locally uniformly rotund* if $x_n \xrightarrow{w} x_0$ whenever x_n, y_n, $x_0 \in \Sigma$ and $\frac{1}{2}(x_n + y_n) \to x_0$;

(MSC) *midpoint sequentially compact* if (x_n) has a convergent subsequence whenever x_n, y_n, $x_0 \in \Sigma$ and $\frac{1}{2}(x_n + y_n) \to x_0$;

(wMSC) *weakly midpoint sequentially compact* if (x_n) has a weakly convergent subsequence whenever x_n, y_n, $x_0 \in \Sigma$ and $\frac{1}{2}(x_n + y_n) \to x_0$.

It turns out that midpoint sequential compactness is just midpoint local uniform rotundity with the rotundity removed.

B.2 PROPOSITION.

(a) (MLUR) $\Longleftrightarrow$ (R) & (MSC).

(b) (wMLR) $\Longleftrightarrow$ (R) & (wMSC).

Proof. Part (a) is easy, as is the forward implication in (b). For the reverse implication in (b), suppose that N is (R) & (wMSC). Let x_n, y_n, $x_0 \in \Sigma$ with $m_n = \frac{1}{2}(x_n + y_n) \to x_0$. By thinning, we can assume that $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ for some $x, y \in U$. In fact, $x, y \in \Sigma$ since $x_0 = \frac{1}{2}(x + y) \in \Sigma$. Since N is (R), $x = y = x_0$. It follows that every subsequence of (x_n) has a subsequence converging weakly to x_0, and so $x_n \xrightarrow{w} x_0$. ∎

The following characterization of midpoint sequentially compact spaces is going to be quite useful.

B.3 <u>LEMMA</u>. *Let N be a normed linear space. Then the following are equivalent.*

(1) N is (MSC) (resp. (wMSC)).

(2) Whenever $x_0 \in \Sigma$, $\|x_n\| \to 1$, $\|y_n\| \to 1$, *and* $\frac{1}{2}(x_n + y_n) \to x_0$, *then* (x_n) *has a convergent (resp. weakly convergent) subsequence.*

<u>Proof</u>. Since (2) obviously implies (1), all we need to show is that (1) implies (2). Suppose that N is (MSC) (resp. (wMSC)). Let x_0, x_n, and y_n be as in the hypothesis of (2). Then:

$$\begin{aligned}&\|\tfrac{1}{2}(\|x_n\|^{-1}x_n + \|y_n\|^{-1}y_n) - x_0\| \\ &\qquad \leq \tfrac{1}{2}\|x_n + y_n - \|x_n\|^{-1}x_n - \|y_n\|^{-1}y_n\| + \|\tfrac{1}{2}(x_n + y_n) - x_0\| \\ &\qquad \to 0.\end{aligned}$$

Thus, $(\|x_n\|^{-1}x_n)$ has a convergent (resp. weakly convergent) subsequence, and hence (x_n) does also. ■

Recall from Theorem 2.6 that the Efimov-Stechkin spaces are exactly the spaces in which all nonempty closed convex sets are approximatively compact. Our next result says that the midpoint sequentially compact spaces play the same role for closed balls.

B.4 THEOREM. *If N is a normed linear space, then N is (MSC) if and only if every closed ball in N is approximatively compact.*

Proof. Suppose that N is (MSC). Let V be a closed ball in N, and let (x_n) be a minimizing sequence in V for some $x \in N$; w. l. o. g. $x = 0$ and $d(0,V) = 1$. Let x_0 be the point on the line connecting 0 with the center of V that is on the surface of both V and U.

Case 1: Radius (V) = 1. Let $y_n = 2x_0 - x_n$. Then $\frac{1}{2}(x_n + y_n) = x_0$ and $\|x_n\| \to 1$. Since $2x_0$ is the center of V,

$$2 = \|2x_0\| \leqslant \|2x_0 - x_n\| + \|x_n\| \leqslant 1 + \|x_n\| \to 2,$$

and so $\|y_n\| = \|2x_0 - x_n\| \to 1$. By Lemma B.3, (x_n) has a convergent subsequence, whose limit lies in the closed set V.

Case 2: Radius (V) < 1. Since V is contained in the closed ball of radius 1 centered at $2x_0$, (x_n) is a minimizing sequence for 0 in this larger ball, and we need only appeal to Case 1.

Case 3: Radius (V) = r > 1. Let V' be the ball of radius 1 centered at $2x_0$. Let $x_n' = x_0 + r^{-1}(x_n - x_0)$; that is, x_n' is obtained by drawing x_n back toward x_0 enough that $x_n' \in V'$. Then $1 \leqslant \|x_n'\| \leqslant (1 - r^{-1}) + r^{-1}\|x_n\| \to 1$, so (x_n') has a convergent subsequence by Case 1, and hence (x_n) does also. As in Case 1, the limit is in V.

Thus, V is approximatively compact, no matter what its radius is. This gives the forward implication.

Now suppose that every closed ball in N is approximatively compact. Let $x_n, y_n, x_0 \in \Sigma$ with $\frac{1}{2}(x_n + y_n) \to x_0$. Let V be the closed ball with center $2x_0$ and radius 1. Let $x_n' = 2x_0 - x_n$, so that (x_n') is a sequence on the surface of V. Then

$$1 \leq \|x_n'\| \leq \|x_n' - y_n\| + \|y_n\| = \|2x_0 - x_n - y_n\| + \|y_n\| \to 1,$$

so (x_n') is a minimizing sequence in V for 0. Since (x_n') has a convergent subsequence, so does (x_n). ∎

If we examine the statements of the previous theorem and Theorem 2.6, the following result is immediate.

B.5 <u>COROLLARY</u>. *An Efimov-Stechkin space is midpoint sequentially compact.*

By Theorem 2.5, the reflexive spaces are exactly the spaces in which all nonempty closed convex sets are approximatively weakly compact. The weakly midpoint sequentially compact spaces play the same role for closed balls. The proof of the following theorem is essentially the same as that of Theorem B.4.

B.6 <u>THEOREM</u>. *If N is a normed linear space, then N is (wMSC) if and only if every closed ball in N is approximatively weakly compact.*

B.7 <u>COROLLARY</u>. *A reflexive space is weakly midpoint sequentially compact.*

Of course, the corollary is also a trivial consequence of the definition of condition (wMSC).

It is easy to see that a normed linear space N is rotund if and only if each closed ball in N is a Chebyshev set. By combining this fact with Theorem B.4 and Proposition B.2 (a), the following result is immediate.

B.8 <u>THEOREM</u>. *A normed linear space N is (MLUR) if and only if every closed ball in N is an approximatively compact Chebyshev set.*

It is somewhat surprising that the well-known class of midpoint locally uniformly rotund spaces should have such a simple approximation-theoretic characterization, since these spaces are rarely discussed in approximation theory. It is interesting to note that the (MLUR) spaces play the same role for closed balls as do the strongly rotund Banach spaces for arbitrary nonempty closed convex sets; see Theorem 2.7.

If we combine Theorem B.6 with Proposition B.2 (b), we obtain the following weak analog of Theorem B.8.

B.9 <u>THEOREM</u>. *A normed linear space N is (wMLR) if and only if every closed ball in N is an approximatively weakly compact Chebyshev set.*

Recall that the rotund reflexive spaces are exactly the spaces in which all nonempty closed convex sets are approximatively weakly compact Chebyshev sets; see Theorem 2.9. In one sense, the last theorem says that the (wMLR) spaces play exactly the same role for closed balls as do the rotund reflexive spaces for arbitrary nonempty closed convex sets. In another important sense, they do not. By Theorem 2.9 again, the rotund reflexive spaces are exactly the spaces in which the nonempty closed convex sets are all Chebyshev, while the rotund spaces are the spaces in which closed balls are always Chebyshev sets. At the end of this appendix, we show that the rotund spaces do form a class of spaces properly larger than the (wMLR) class.

The following result can be viewed either as an easy corollary of Theorems B.8 and B.9 proved with the use of Theorems 2.7 and 2.9, or as a reasonably straightforward consequence of the appropriate definitions.

B.10 COROLLARY.

(a) (Anderson [1]) *A strongly rotund Banach space is midpoint locally uniformly rotund;*

(b) A rotund reflexive space is weakly midpoint locally uniformly rotund.

Recall for a moment Theorems 2.6 and 1.12, which together show that every nonempty closed convex set in a reflexive Kadec-Klee space is approximatively compact. The essential idea of the proof is that reflexivity forces minimizing sequences to have weakly convergent subsequences, which the Kadec-Klee property converts into norm convergent subsequences. It might seem that if we give up reflexivity and just settle for bounded sequences having weakly Cauchy subsequences, then we would lose everything essential to the proof. It is therefore somewhat surprising that we still retain approximative compactness for closed balls. To see this, we now obtain new characterizations for the spaces studied earlier in this appendix.

B.11 <u>DEFINITION</u>. A normed linear space N is said to have property

($n\ell_1$) if it contains no subspace isomorphic to ℓ_1;

(G_1) if, whenever (x_n) is a minimizing sequence in a closed ball for some point in N, then (x_n) has a weakly Cauchy subsequence;

(G_2) if, whenever (x_n) is a weakly Cauchy minimizing sequence in a closed ball for some point in N, then (x_n) has a norm convergent subsequence;

(wG_2) if the definition of (G_2) holds with "norm convergent" replaced by "weakly convergent".

B.12 <u>PROPOSITION</u>. *For Banach spaces,*

(a) $(n\ell_1) \Longrightarrow (G_1)$;

(b) $(H) \Longrightarrow (G_2)$.

<u>Proof</u>. Rosenthal [44] has proved that a Banach space is $(n\ell_1)$ if and only if each of its bounded sequences contains a weakly Cauchy subsequence. This gives (a) immediately.

For (b), we use an argument inspired by Kadec's proof in [31] that, for Banach spaces, $(n\ell_1)$ & (H) & (R) together imply (MLUR). Let V be a closed ball in Banach space B of type (H). Let (x_n) be a weakly Cauchy minimizing sequence in V for $x \in B$. W. l. o. g. $x = 0$ and $d(0,V) = 1$. We can assume that the radius of V is 1; the other cases follow from this as in the proof of Theorem B.4.

Let $2x_0$ be the center of V. Then x_0 lies on the surfaces of both V and U. Let $y_n = x_n - x_0$. Notice that (y_n) is weakly Cauchy, and that $\|x_0 + y_n\| \to 1$.

Claim: $\|x_0 - y_n\| \to 1$. Just notice that $\|2x_0 - x_n\| \to 1$ because $\|x_n\| \to 1$; that is, (x_n) tends toward the surface of V. This proves the claim.

Suppose that (x_n) were not norm convergent. Since (y_n) is thus not Cauchy, there is an $\varepsilon > 0$ and subsequences $(y^{(1)}_{n_k})$ and $(y^{(2)}_{n_k})$ of (y_n) with $\|y^{(1)}_{n_k} - y^{(2)}_{n_k}\| \geqslant \varepsilon$ for all k. We have:

$$\|x_0 + \tfrac{1}{2}(y^{(1)}_{n_k} - y^{(2)}_{n_k})\| \leqslant \tfrac{1}{2}(\|x_0 + y^{(1)}_{n_k}\| + \|x_0 - y^{(2)}_{n_k}\|) \to 1,$$

so $\limsup \|x_0 + \frac{1}{2}(y_{n_k}^{(1)} - y_{n_k}^{(2)})\| \leq 1$. Since $x_0 + \frac{1}{2}(y_{n_k}^{(1)} - y_{n_k}^{(2)})$ converges weakly to x_0,

$$1 = \|x_0\| \leq \liminf \|x_0 + \tfrac{1}{2}(y_{n_k}^{(1)} - y_{n_k}^{(2)})\|.$$

Thus, $\|x_0 + \frac{1}{2}(y_{n_k}^{(1)} - y_{n_k}^{(2)})\| \to 1$ and $x_0 + \frac{1}{2}(y_{n_k}^{(1)} - y_{n_k}^{(2)}) \overset{w}{\to} x_0$. Since B is (H), it follows that $x_0 + \frac{1}{2}(y_{n_k}^{(1)} - y_{n_k}^{(2)}) \to x_0$, and hence that $\|y_{n_k}^{(1)} - y_{n_k}^{(2)}\| \to 0$, a contradiction. ∎

B.13 <u>PROPOSITION</u>. *For Banach spaces,*

(a) (G_1) & $(G_2) \iff$ *(MSC);*

(b) (G_1) & (G_2) & *(R)* $\iff$ *(MLUR);*

(c) (G_1) & $(wG_2) \iff$ *(wMSC);*

(d) (G_1) & (wG_2) & *(R)* $\iff$ *(wMLR).*

<u>Proof</u>. See Proposition B.2 and Theorems B.4 and B.6. ∎

Propositions B.12 and B.13, together with Theorems B.4 and B.8, give the following two results. The implication $(n\ell_1)$ & (H) & (R) $\Longrightarrow$ (MLUR) in the second result was obtained by Kadec in [31], but the rest of the result is new.

B.14 <u>THEOREM</u>. *If Banach space B is* $(n\ell_1)$ *& (H), then B is (MSC), and so every closed ball in B is approximatively compact.*

B.15 THEOREM. *If Banach space B is* $(n\ell_1)$ *& (H) & (R), then B is (MLUR), and so every closed ball in B is an approximatively compact Chebyshev set.*

It is interesting to compare these results with Theorem 1.12, Corollary 1.13, and Theorems 2.6 and 2.7, which show that whenever a normed linear space is (Rf) & (H) (resp. (Rf) & (H) & (R)), every nonempty closed convex set in N is approximatively compact (resp. approximatively compact and Chebyshev).

Let us return for a moment to Corollary B.10 (a). While it is true that (D) implies (MLUR), there are hordes of spaces that are (MLUR) but not (D). For example, Anderson [1] has shown that any (LUR) space is (MLUR). Since any nonreflexive separable Banach space can be given an equivalent (LUR) norm (see [16]), while spaces of type (D) are reflexive, it is not difficult to construct (MLUR) spaces that are not (D). The following question does remain open, however.

B.16 QUESTION. Are the reflexive midpoint locally uniformly rotund spaces exactly the strongly rotund Banach spaces?

It is not difficult to show that this question is equivalent to the following one.

B.17 QUESTION. For reflexive spaces, does condition (MLUR) imply condition (H)?

It was shown by Smith in [48] that not all (MLUR) spaces have property (H), thus settling in the negative a question asked by Anderson in [1]. However, Smith's counter-example is not reflexive, so Question B.17 remains open. A positive answer to this question would certainly be interesting. However, if we examine Propositions B.12 (b) and B.13 (b), we see that Question B.17 is equivalent to the following.

B.18 <u>QUESTION</u>. For reflexive spaces that are (G_1) & (R), are conditions (H) and (G_2) equivalent?

Put this way, an affirmative answer does not seem likely.

To end this appendix, we examine the relationships between the main classes of spaces studied above. As we mentioned above, Anderson has shown that every (LUR) space is (MLUR). Thus, the following result is not surprising.

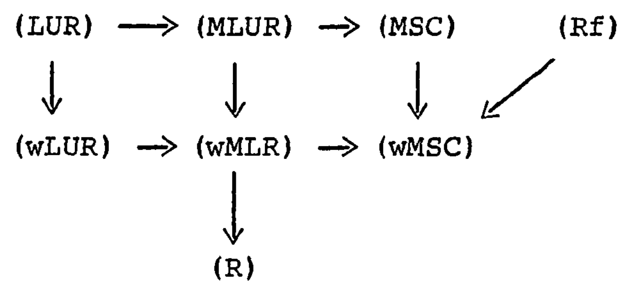
B.19 <u>PROPOSITION</u>. *A weakly locally uniformly rotund normed linear space is weakly midpoint locally uniformly rotund.*

<u>Proof</u>. Suppose N is (wLUR). Let $x_n, y_n, x_0 \in \Sigma$ be such that $m_n = \frac{1}{2}(x_n + y_n) \to x_0$. We need to show that $x_n \xrightarrow{w} x_0$. Let $f \in \Sigma'$ be such that $f(x_0) = 1$. Now $f(y_n) \leqslant 1$, $f(x_n) \leqslant 1$, and $f(m_n) \to 1$, so $f(x_n) \to 1$ also. Thus, $f(\frac{1}{2}(x_n + m_n)) \to 1$, and so

$$1 \geqslant \tfrac{1}{2}\|x_n + x_0\| \geqslant \tfrac{1}{2}\|x_n + m_n\| - \tfrac{1}{2}\|m_n - x_0\|$$
$$\geqslant f(\tfrac{1}{2}(x_n + m_n)) - \tfrac{1}{2}\|m_n - x_0\| \to 1.$$

Since N is (wLUR) and $\|\tfrac{1}{2}(x_n + x_0)\| \to 1$, $x_n \overset{w}{\to} x_0$. ∎

The following implication diagram can now be derived easily from the above results.



We now give some examples, most of which are adapted from the work of Mark Smith, to show that no more implication arrows can be added to the above diagram.

B.20 <u>EXAMPLE</u>: $\ell_1^{(2)}$. Let $\ell_1^{(2)}$ be the Banach space $(\mathbb{R}^2, \|\cdot\|_1)$, where $\|(x_1,x_2)\|_1 = |x_1| + |x_2|$. Then $\ell_1^{(2)}$ is (Rf) & (MSC) & (wMSC), as is any finite-dimensional space, but $\ell_1^{(2)}$ has none of the other properties in the diagram, because it is not (R). ∎

B.21 <u>EXAMPLE</u>: $(\ell_1, \|\cdot\|_E)$. This space was constructed in [17] to show that ℓ_1 has an equivalent (UCED) norm; see the reference for the definition of condition (UCED). Let I be the inclusion map from ℓ_1 into ℓ_2. Define $\|\cdot\|_E: \ell_1 \to \mathbb{R}$ by

$$\|x\|_E = (\|x\|_1^2 + \|Ix\|_2^2)^{\frac{1}{2}}.$$

In [47], Smith showed that $\|\cdot\|_E$ is a norm on ℓ_1 that is (LUR) and equivalent to the usual ℓ_1 norm. Since $(\ell_1, \|\cdot\|_E)$ is not reflexive, we see that none of the other properties in our diagram implies reflexivity. ■

B.22 <u>EXAMPLE</u>: $(\ell_2, \|\cdot\|_W)$. This equivalent norm on ℓ_2 is defined in our Example A.4. Smith [47] showed that this norm is (wLUR) but not (MLUR). Since $\|\cdot\|_W$ is a rotund norm, $(\ell_2, \|\cdot\|_W)$ is not (MSC). Thus, (wLUR) does not imply (MSC). Since $(\ell_2, \|\cdot\|_W)$ is a reflexive space, neither does (Rf) imply (MSC). ■

B.23 <u>EXAMPLE</u>: $(\ell_1, \|\cdot\|_H)$. This space, constructed by equivalently renorming ℓ_1, is defined in our Example A.4. Smith [47] used this norm to show that (R) does not imply (MLUR). We now use it to show that (R) does not even imply (wMSC).

Let (e_n) be the usual sequence of unit vectors in ℓ_1. Let $x_0 = 3^{-\frac{1}{2}}e_1$, $x_n = 3^{-\frac{1}{2}}(e_1 + e_n)$, and $y_n = 3^{-\frac{1}{2}}(e_1 - e_n)$. Then $m_n = \frac{1}{2}(x_n + y_n) = x_0$. It is not too difficult to check that $\|x_0\|_H = 1$, $\|x_n\|_H \to 1$, and $\|y_n\|_H \to 1$. However, $x_n - y_n = 2\cdot 3^{-\frac{1}{2}}e_n$ has no weakly convergent subsequence, so $(\ell_1, \|\cdot\|_H)$ is not (wMSC).

Thus, (R) does not imply any of the other properties in the diagram. ■

Smith [47] also constructed a space $(\ell_2, \|\cdot\|_A)$ that is (MLUR) but not (wLUR). Thus, our implication diagram is complete.

INDEX OF CLASSES OF SPACES

This alphabetical index of all the classes of normed linear spaces used in this thesis includes the abbreviation for each class, the name of the class where applicable, and the page containing its definition. Alternative names are given in parentheses following the name we use.

CLASS	NAME OF CLASS	PAGE
(CD)	Efimov-Stechkin	17
(D)	strongly rotund Banach	11
(F)	Fréchet smooth	11
(G_1)	-	116
(G_2)	-	116
(H)	Kadec-Klee (Radon-Riesz)	11
(K)	strongly rotund	11
(LUR)	locally uniformly rotund	10
(MLUR)	midpoint locally uniformly rotund	109
(MSC)	midpoint sequentially compact	110
$(n\ell_1)$	-	116
(R)	rotund (strictly convex)	10
(Rf)	reflexive	11
(S)	smooth	11
(SH)	semi-Kadec-Klee	46
(SH')	-	99

CLASS	NAME OF CLASS	PAGE
(UG)	uniformly Gateaux smooth	11
(UR)	uniformly rotund	10
(US)	uniformly smooth	11
(VS)	very smooth	65
(wCDL)	-	104
(wG_2)	-	116
(wK)	weakly rotund	21
(wK_ω)	-	21
(wLUR)	weakly locally uniformly rotund	10
($wL\nu$)	very rotund	22
(wMLR)	weakly midpoint locally uniformly rotund	110
(wMSC)	weakly midpoint sequentially compact	110
(wUR)	weakly uniformly rotund	10
($w\nu$)	-	22

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