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MARKOV MAPS IN NONCOMMUTATIVE PROBABILITY THEORY AND MATHEMATICAL STATISTICS

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The general concept of statistical decision rule belongs to the foundations of the modern statistical theory, see [1] . Markov maps (of probability distribution collections Cap $(\Omega, \mathcal{A})$ which describe these rules form an algebraic cetegory [2] ,[3] , with an additional operation of (tensor) multiplication [4] . This fact allows to treat many aspects of mathematical statistics in terms of appeared categorical geometry [5], [6], [4]. In this paper we examine the possibility of such an approach to the statistical problems of noncommutative probability theory. A number of our results is new even in the frame of classical statistics.

1.Let $(\Omega, \mathcal{A})$ be a mesurable space of elementary events ω corresponding to an observable random phenomenon, and let $\{P_\theta, \theta \in \Theta\}$ be a family of probability distributions on $(\Omega, \mathcal{A})$ being a priori possible in the fram of observation. According to [1] any Wald's statistical decision rule can be written as a transient probability distribution $\Pi(\omega; d\delta)$ from Ω onto a measurable space $(\Delta, \mathcal{B})$ of decision $\delta \in \Delta$. Thus if we exploit the rule Π then our decisions will be distributed in accordance with the law

$$Q_\theta = P_\theta \Pi, \quad Q_\theta(\cdot) = \int_\Omega P_\theta(d\omega)\Pi(\omega; \cdot). \tag{1}$$

The observer does not know the true distribution P which completely describes the observed random phenomenon . He knows only that P belongs to the family $\{P_\theta, \theta \in \Theta\}$.

Consequently, all a priori conclusions about a quality of decision rule Π are based on the properties of the family $\{P_\theta \Pi\}$.

Definition I. Two families $\{P_\theta^{(i)}\}$ on $(\Omega^{(i)}, \mathcal{A}^{(i)})$, i=1,2, parametrized by the common parameter $\theta \in \Theta$ are equivalent providing that for any decision space $(\Delta, \mathcal{B})$ and any rule $\Pi^{(i)}(\omega^{(i)}, d\delta)$, i= 1,2, there exists a rule $\Pi^{(j)}(\omega^{(j)}, d\delta)$, j= 2 when i = I, and j=I when i =2, such that

$$P_\theta^{(i)} \Pi^{(i)} = Q_\theta = P_\theta^{(j)} \Pi^{(j)} \tag{1.2}$$

The statistical decision rules form a category so that a composition of two decision rules is againg such a rule. Hence it is easy to prove the result going back to D.Blackwell [7] :

Theorem I.I. Two families $\{P_\theta^{(i)}, \theta \in \Theta\}$, i =1,2, with a common parameter space Θ are equivalent in the statistical inference theory iff they are congruent in the categoric geometry, $\{P_\theta^{(1)}\} \sim \{P_\theta^{(2)}\}$, i.e. providing there exist such morphisms $Ш^{12}$ and $Ш^{21}$ that

$$P_\theta^{(1)} Ш^{12} = P_\theta^{(2)}, \quad P_\theta^{(2)} Ш^{21} = P_\theta^{(1)}, \quad \forall \theta \in \Theta \tag{1.3}$$

Thus we treat probability distribution families as oarametrized figures and study the geometry (i.e. the invariant properties) of those under the category of Markov maps. The geometry is transitive: one can markovly transfer any "point" P into any another one Q. Sets with at least two points have non-trivial invariants.

Definition 2. A real function f defined on all object squares will be called an invariant provided

$$\{(P_1, P_2) \sim (Q_1, Q_2)\} \Rightarrow \{f(P_1, P_2) = f(Q_1, Q_2)\} \tag{1.4}$$

for all congruent couples, and f will be called a monotone invariant if always

$$f(P_1, P_2) \geq f(P_1 Ш, P_2 Ш). \tag{1.5}$$

Every monotone invariant is evidently an ivariant in the sense of (1.4). Invariants of more rich families are defined in a similar way. The following results concerning invariant metrics on collections Cap $(\Omega, \mathcal{A})$ we have obtained in [5] .

<u>Theorem 1.2.</u> If a Markov monotonically invariant matric ϱ is positive homogeneous, i.e.

$$\{P_1 - Q_1 = \alpha(P_2 - Q_2)\} \Rightarrow \{\varrho(P_1, Q_1) = |\alpha| \varrho(P_2, Q_2)\}, \tag{1.6}$$

then

$$\varrho(P, Q) = \gamma(\varrho) |P - Q| \tag{1.7}$$

where $\gamma(\varrho)$ is a constant, $|P - Q|$ is the variation of difference.

Let us point out that in the discrete case, i.e. when algebra $\mathcal{A}$ is generated by atoms A_k (k=1,2,...),and $P(A_k) = p_k$, $Q(A_k) = q_k$, we have

$$|P - Q| = \sum |p_k - q_k|. \tag{1.8}$$

Theorem 1.3. If a Markov monotonically invariant metric ϱ is a Riemannian one, then

$$\varrho(P, Q) = \beta(\varrho)\, s(P, Q), \tag{1.9}$$

where $\beta(\varrho)$ is a constant, $s(P, Q)$ is the Bhattacharyya distance generated by Fisher quadratic form;

$$s(P, Q) = 2 \arccos \sum \sqrt{p_k q_k}, \quad ds^2 = \sum (dp_k)^2 / p_k. \tag{1.10}$$

In this paper we consider the possible generalizations of theorems 1.2 and 1.3 as well as the validity of generalizations of the following statements.

Theorem 1.4. If a metric ρ is monotone invariant under Markov map category, then

$$\rho(P,Q) \geq \frac{1}{8}\rho\left(R(\tfrac{1}{2}),R(\tfrac{1}{4})\right)\cdot|P-Q|, \tag{1.11}$$

where $R(\frac{1}{2})$ and $R(\frac{1}{4})$ are probability distributions on two-atom algebra; $R(\frac{1}{2}):\{r_1=r_2=\frac{1}{2}\}$, $R(\frac{1}{4}):\{r_1=\frac{1}{4}, r_2=\frac{3}{4}\}$.

Theorem 1.5. If $f(P,Q)$ is a Markov monotone invariant, then there exist a monotone real function $g(x)$ on $\mathbb{R}^+$, $g(c)=0$, and a constant C such that

$$f(P,Q) \geq c+g(|P-Q|) \tag{1.12}$$

If in addition $\{P\neq Q\} \Rightarrow \{f(P,Q)\neq f(P,P)\}$, then $g(x)>0 \quad \forall x>0$.

Two last theorems are new.

2. Random phenomena in mycrophysics cannot be described by the schemes of classical (or commutative as it is said now) probability theory because the logic of quantum events is non-Aristotelian. The following algebraic scheme has proved to be the most convenient method to assign an object of noncommutative theory. An injective von Neumann algebra $\mathcal{L}$ of bounded linear operators acting on a Hilbert space $\mathcal{H}$ is assumed to be given, in particular, the algebra $\mathcal{L}(\mathcal{H})$ of all such operators. Algebra $\mathcal{L}$ is (in general) a noncommutative generalization of classic (commutative) algebra of all bounded measurable functions on the space of elementary events. Hermitian elements of algebra $\mathcal{L}$ are called (bounded) observables. The probability state of the object is given by a nonnegative normed normal (i.e. ultra weak continuous, or, that is the same, monotonically

continuous) linear functional Φ on $\mathcal{L}$, $\Phi : \mathcal{L}^H \to \mathbb{R}$, being an analogue of the mathematical expectation (i.e. mean value) induced by probability measure, see [8], [9]. The collection $\mathfrak{S}(\mathcal{L})$ of all object states Φ is a convex closed set in the pre-dual space $\mathcal{L}_*$, $(\mathcal{L}_*)^* = \mathcal{L}$.

All the Hermitian idempotents e of algebra $\mathcal{L}$ are orthoprojectors on corresponding (closed) subspaces $\mathcal{E} = e\mathcal{H}$ of the space $\mathcal{H}$. By the analogy with the commutative case, where the idempotents are indicators of measurable sets of the elementary event space, these subspaces are called events (as well as "yes-no" - experiments). Evidently the classical scheme is a special case of noncommutative one, $\mathcal{L}$ being the commutative algebra of all bounded measurable functions on elementary event space.

<u>Definition 3.</u> Any affine map

$$\Pi : \mathfrak{S}(\mathcal{L}_1) \to \mathfrak{S}(\mathcal{L}_2)$$

will be called Markov according to the natural analogy with the classical case.

A map Π may be extended by linearity to a positive monotone continuous linear map of pre-dual spaces)of charges), $\Pi : \mathcal{L}_{1*} \to \mathcal{L}_{2*}$. The conjugate map $\Pi^* : \mathcal{L}_2^H \to \mathcal{L}_1^H$ is $\mathbb{R}$-linear, positive, normal (i.e. monotonically continuous) and normed. It is uniquely extended to a $\mathbb{C}$-linear map $\mathcal{L}_2 \to \mathcal{L}_1$. To simplify the notations we do not distinguish between Π and Π^* writing Π on the right side of a state and on the left side of a observable. It is easy to see that the system of all Markov maps of all collections $\mathfrak{S}(\mathcal{L})$ forms an algebraic category in the sense of Eilenberg and MacLane [10]. First of all, the identity map of any object is evidently a Markov one. Secondly, the composition (successive realization) of two Markov maps is again a Markov one. Thirdly, this operation is associative because of being a map composition. The mentioned system will be below called the broad Markov category.

It is possible to define a tensor multiplication of algebras $\mathcal{L}$ as well as of their pre-dual spaces. Stinespring [11] has shown that the induced tensor products of broadly Markovian maps need not to be positive and, in particular, Markov. He has extracted the completely positive map system being closed under operation of tensor multiplication.

Definition 4. A linear map $\Pi : \mathcal{L}_2 \to \mathcal{L}_1$ is called completely positive providing all the maps $\Pi \otimes id_r$ are positive,

$$\Pi \otimes id_r \geqslant 0, \ \forall r \in \mathbb{N}, \tag{2.1}$$

where id_r is the identity endomorphism of $\mathcal{L}(\mathcal{H}_r)$, $\dim \mathcal{H}_r = r$.

The completely positive Markov maps form also a category. The latter will be called the restricted Markov category. Specifically, the result of interaction between the quantum particle and particles of a random media is described by a completely positive Markov map. Further Stinespring [11] found that all Markov maps into a commutative von Neumann algebra as well as those from such algebra are completely positive. The latters describe acts of (classic) measurement of some numerical-valued physical characteristics [12]. This Stinespring's result is based on the well-known theorem of Naimark [13] stating that any Hermitian resolution of identity operator is extendable up to an orthogonal one.

Thus, by the scheme of definition 1 in noncommutative theory there appear the two geometrical equivalence of families being called the broad and the restricted ones. Moreover, the scheme of definition 1 allows us to introduce the family equivalence with respect to measurements.

By analogies with theorem 1.1 it is easy to show for these equivalence the following chain of implications:

$$\{\text{restricted}\} \Rightarrow \{\text{broad}\} \Rightarrow \{\text{measuring}\} . \qquad (2.2)$$

It is of interest to consider the maximal Markovly congruent families.

<u>Theorem 2.1.</u> If the congruence of families $\{\Phi_\Theta^{(i)}, \Theta \in \Theta\} \subset \mathfrak{S}(\mathcal{B}_i)$, i = 1,2, is realized by completely positive Markov maps $\sqcup\!\sqcup^{12}$ and $\sqcup\!\sqcup^{21}$ then

$$\mathfrak{S}(\mathcal{B}_i')\sqcup\!\sqcup^{ij} = \mathfrak{S}(\mathcal{B}_j') \supset \{\Phi_\Theta^{(j)}, \Theta \in \Theta\}, \; i \neq j = 1,2, \qquad (2.3)$$

where $\mathcal{B}_i' \subset \mathcal{B}_i$, i = 1, 2, are some injective von Neumann subalgebras.

A similar proposition holds for broad Markov congruence. The proof follows immediately from the ergodic theorem for Markov endomorphisms, [14], [15], see also [16], [17].

3. For convenience we consider in the paper separable Hilbert spaces only. Moreover, by injectivity of algebras $\mathcal{B}$ we may consider nothing but the total algebras $\mathcal{B}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{B}(\mathcal{H})) = \mathfrak{S}(\mathcal{H})$. Hence to describe the observables, the states, and the Markov maps, the matrix notations may be used, [18], [19]. For that purpose one must fixe some orthonormal basis in $\mathcal{H}$. Then the observables A and the states Φ will be described respectively by Hermitian matrices $({}^i_j a)$ and by positive matrices (φ_j^i) with the unit trace. In the case of observables the superscripts and subscripts are written on the left of the letter to distinguish then from states. To describe a Markow map $\Pi : \mathfrak{S}(\mathcal{H}^{(1)}) \to \mathfrak{S}(\mathcal{H}^{(2)})$ one must construct the bases of $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$. There is a possibility of the initial basis and the final one

being taken distinct even if $\mathcal{H}^{(1)} = \mathcal{H}^{(2)}$.

Lemma 3.1. The properties of four-valent (unitary) tensor $({}^i_j\mathfrak{m}^k_l)$ describing a Markov map $\mathfrak{M} : \mathfrak{S}(\mathcal{H}^{(1)}) \to \mathfrak{S}(\mathcal{H}^{(2)})$ are the following:

$$ {}^i_j\mathfrak{m}^k_l = {}^j_i\bar{\mathfrak{m}}^l_k , \quad \forall i,j,k,l, \tag{3.1}$$

$$ tr_{(2)}\mathfrak{M} = \sum_j {}^i_j\mathfrak{m}^k_k = {}^i_j\delta , \quad \forall i,j, \tag{3.2}$$

$$ \sum \xi^i \xi_j \, {}^i_j\mathfrak{m}^k_l \, {}^k\eta_l\eta \geq 0 \tag{3.3}$$

$$\forall(\xi^1,\ldots,\xi^m,\ldots) = (\bar{\xi}_1,\ldots,\bar{\xi}_m,\ldots); \forall({}^1\eta,\ldots,{}^n\eta,\ldots) = ({}_1\bar{\eta},\ldots,{}_n\bar{\eta},\ldots).$$

Here ${}^i_j\delta$ is the Kronecker symbol, $\bar{\xi}$ is the complex conjugate of ξ and the summation is taken over coinciding upper and lower indices (i.e. superscript and subscript) or over coinciding left and right ones. The correct sense of summation in the case of dim $\mathcal{H} = \infty$ is stated in [19]. The conditions (3.1) - (3.3) imply respectively the Hermitian property, the positivity, and the trace preservation. Now let us show how the Stinespring's construction is simplified in finitedimensional theory.

Lemma 3.2. A matrix $({}^i_j\mathfrak{m}^k_l)$ of any completely positive Markov map satisfies the conditions (3.1), (3.2) and

$$ \sum {}^k\xi^i_l \, {}_k\xi_j \, {}^i_j\mathfrak{m}^k_l \geq 0, \tag{3.4}$$

$$\forall({}^1\xi^1,\ldots,{}^m\xi^n) = ({}_1\bar{\xi}_1,\ldots,{}_m\bar{\xi}_n).$$

The condition (3.4) is stronger that (3.3).

<u>Lemma 3.3.</u> A matrix ${}^{i}_{j}\mathcal{W}^{k}_{e}$ of any completely positive Markov map permits [20] the decompositions of form

$$\left({}^{i}_{j}\mathcal{W}^{k}_{e}\right) = \sum\left({}^{i}\mathcal{U}^{k}(s)\,{}_{j}\mathcal{U}_{e}(s)\right), \qquad (3.5)$$
$${}^{i}\mathcal{U}^{k}(s) = {}_{i}\overline{\mathcal{U}}_{k}(s), \quad \forall i, k, s.$$

<u>Corollary.</u> The composition of completely positive Markov maps is again such a map.

<u>Proof.</u> The condition (3.4) allows to regard the tensor $\left({}^{i}_{j}\mathcal{W}^{k}_{e}\right)$ as a matrix of positive Hermitian operator on a space $\mathcal{H}_m \otimes \mathcal{H}_n$. The positivity of this matrix implies (3.5). Further, any positive trace-class operator matrix is also decomposable in the sum of the form $(b^{ip}\, b_{jq})$. The $\mathcal{W} \otimes id_z$ image of such a tensor is positive in virtue of

$$\sum\sum\left(b^{ip}\, b_{jq}\, {}^{i}\mathcal{U}^{k}\, {}_{j}\mathcal{U}_{e}\, {}^{p}\delta^{u}{}_{q}\,\delta_{v}\right)^{ku} a_{ev}\, a =$$
$$= \left|\sum b^{ip}\, {}^{i}\mathcal{U}^{k}\, {}^{kp}a\right|^{2} \geq 0.$$

Finally, the matrix $(\delta^{ip}\,\delta_{jq})$ of some special positive Hermitian operator on $\mathcal{H}_m \otimes \mathcal{H}_m$ is mapped by a matrix of $\mathcal{W} \otimes id_z$ into a matrix $\left(f^{ku}_{ev}\right)$ of operator on $\mathcal{H}_m \otimes \mathcal{H}_n$, where $f^{ku}_{ev} = {}^{u}_{v}\mathcal{W}^{k}_{e}, \forall k, l, u, v$. The positivity of $\left(f^{ku}_{ev}\right)$ implies (3.4). In addition, it should be pointed out that the Kraus decomposition (3.5) is correct in a denumerable case too.

4. Any $\mathbb{R}$-linear combination of two states of an object belongs to a Hermitian part of its pre-dual space, $\mathbb{R}-Lin\,\mathfrak{S}(\mathcal{L}) = \mathcal{L}_*^H$, and is described by the trace-class operator.

<u>Definition 5.</u> The positive part $\Gamma^{(+)}$ and the negative part $\Gamma^{(-)}$ of an operator $\Gamma \in \mathcal{L}_*^H$ are described by spectral decompositions:

$$\Gamma = \sum \gamma_k E_k, \quad \Gamma^{(+)} = \sum_{\gamma_k > 0} \gamma_k E_k, \qquad (4.1)$$

$$\Gamma^{(-)} = -\sum_{\gamma_k<0} \gamma_k E_k .$$

The values

$$Tr^{(+)}\Gamma = Tr\,\Gamma^{(+)} = \sum_{\gamma_k>0} \gamma_k , \; Tr^{(-)}\Gamma = Tr\,\Gamma^{(-)}, \; |\Gamma| = \sum |\gamma_k| , \qquad (4.2)$$

where $|\Gamma| = Tr^{(+)}\Gamma + Tr^{(-)}\Gamma$ will be respectively called the positive variation, the negative and the total ones. Decompositions $\Gamma = \Gamma^{(+)} - \Gamma^{(-)}$ and $\mathcal{H} = \mathcal{E}^{(+)} \oplus \mathcal{E}^{(-)}$, where $E^{(+)} = \sum_{\gamma_k>0} E_k$, $E^{(-)} = \sum_{\gamma_k<0} E_k$, $\mathcal{E}^{(+)} = E^{(+)}\mathcal{H}$, $\mathcal{E}^{(-)} = E^{(-)}\mathcal{H}$, we shall call the forks.

<u>Lemma 4.1.</u> Positive and negative variations of $\Gamma \in \mathcal{L}_*^H$ are monotone invariant under the broad Markov category. Any $\Gamma \in \mathcal{L}_*^H$ is congruent to the cahrge Γ_{reg} on two-atom measured space $\Omega = \{\omega_1, \omega_2\}$,

$$\Gamma_{reg}(\omega_1) = Tr^{(+)}\Gamma , \quad \Gamma_{reg}(\omega_2) = -Tr^{(-)}\Gamma . \qquad (4.3)$$

Corolary. $Tr^{(+)}\Gamma$ and $Tr^{(-)}\Gamma$ form a complete system of invariants of $\Gamma \in \mathcal{L}_*^H$ under both both the Markov categories.

Proof. There are bases of $\mathcal{H}^{(1)}$ and of $\mathcal{H}^{(2)}$ formed by eigenvectors of Γ, respectively, of $\Gamma Ш$. Then in appeared matrix representation the connection between matrix elements of Γ and of $\Gamma Ш$ is reduced to that between the diagonal elements only. Hence, we are led to a classical, see [5], lemma 5.9.

An application of this results to the linear combination of state operators $\Psi - z\Phi$ gives us two families

$$\mu^{(\pm)}(z) = tr(\Psi - z\Phi)^{(\pm)}, \; \mu^{(+)}(z) - \mu^{(-)}(z) = 1 - z, \; z \in \mathbb{R}, \qquad (4.4)$$

of monotone invariants of the couple (Φ, Ψ). Each them forms a complete system of invariants when Φ and Ψ are (operatorly) commuting. For Φ and Ψ being commuting it is necessary and, may be, sufficient that

$$[(\Psi - z_1\Phi)^{(+)} - z_2\Phi]^{(+)} = [\Psi - (z_1+z_2)\Phi]^{(+)}, \quad \forall z_1, z_2 \geq 0$$

When Φ and Ψ are commuting

$$tr[(\sqrt{\Psi} - \sqrt{\Phi})^2] = \int_0^1 \mu^{(-)}(z) z^{-3/2} dz + \int_1^\infty \mu^{(+)}(z) z^{-3/2} dz, \quad (4.5)$$

$$tr(\Psi(\ln\Psi - \ln\Phi)] = \int_0^1 \mu^{(-)}(z) z^{-1} dz + \int_1^\infty \mu^{(+)}(z) z^{-1} dz \quad (4.6)$$

In the general case the transparent system of invariants and the generalizations of the formulas (4.5) and (4.6) are not known yet.

5. The axioms of a metric space are the following: 1° $\rho(x,y) \geq 0$, 2° $\rho(x,y) = 0 \Rightarrow y = x$, 3° $\rho(x,y) = \rho(y,x)$, 4° $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. If the function $\rho(x,y)$ doesn't satisfy the axiom 2° it is called a pseudometric. We shall consider the more exotic case of asymmetrical distance ρ, when axiom 3° is not fulfilled, and the inequality 4° takes place with just given order of "points".

Theorem 5.1. If the metric $\rho(\Phi, \Psi)$ given on all the objects $\mathfrak{S}(\mathcal{L})$ is a monotone invariant under the restricted Markov category, the

$$\rho(\Phi, \Psi) \geq \tfrac{1}{8}\left(R(\tfrac{1}{2}), R(\tfrac{1}{4})\right) |\Phi - \Psi|, \quad (5.1)$$

where $|\Phi - \Psi|$ is the total variation (trace norm) of the difference $\Phi - \Psi$, $R(\theta)$ is a probability distribution on the twoatom measurable space:

$$R(\theta):\{R(\omega_1;\theta)=r_1(\theta)=\theta,\ R(\omega_2;\theta)=r_2(\theta)=1-\theta\} \quad (5.2)$$

Remark. If a distance ρ is asymmetric, the

$$\rho(\Phi,\Psi)\geq \frac{1}{8}|\Phi-\Psi|\min\{\rho(R(\tfrac{1}{2}),R(\tfrac{1}{4})),\rho(R(\tfrac{1}{4}),R(\tfrac{1}{2}))\}.$$

Proff. Let $\mathcal{H}=\mathcal{E}^{(+)}\oplus\mathcal{E}^{(-)}$ be the fork of $\mathcal{H}$ for the trace-class operator $\Phi-\Psi$. The restriction of states on the measurable space $\Omega=\{\mathcal{E}^{(+)},\mathcal{E}^{(-)}\}$ is a completely positive Markov map. Hence, $\rho(\Phi,\Psi)\geq\rho(\Phi_{red},\Psi_{red})$ and $|\Phi-\Psi|=|\Phi_{red}-\Psi_{red}|$ according to lemma 4.1. Consequently, it is sufficient to prove the theorem in the case when Φ and Ψ are the probability distributions on a two-atom measured space. Any probability distribution on the latter is of the form (5.2), $0\leq\theta\leq 1$, where θ is a probability of the first atom, and

$$|R(\theta)-R(\theta')|=2|\theta-\theta'|. \quad (5.3)$$

Any Markov endomorphism of the collection $\{R(\theta),\ 0\leq\theta\leq 1\}$ is described by an affine endomorphism of the segment $\{\theta: 0\leq\theta\leq 1\}$

Lemma 5.2. If $0\leq\theta_1\leq\theta_1'\leq\theta_2'\leq\theta_2\leq 1$, then the Markov endomorphism Π of collection $\{R(\theta)\}$, which maps the distributions $R(\theta_i)$ into $R(\theta_i')$, i = 1,2, is given by the matrix

$$\frac{1}{\theta_2-\theta_1}\left\|\begin{matrix}(1-\theta_1)(1-\theta_2')-(1-\theta_2)(1-\theta_2') & (1-\theta_1)\theta_2'-(1-\theta_2)\theta_1' \\ \theta_2(1-\theta_1')-\theta_1(1-\theta_2') & \theta_2\theta_1'-\theta_1\theta_2'\end{matrix}\right\|$$

<u>Corollary</u>. If $f(\Phi, \Psi)$ is a monotone invariant under the restricted Markov category then $f(R(\Theta_1), R(\Theta_2)) \geq$
$\geq f(R(\Theta_1'), R(\Theta_2'))$ under the lemma's conditions.

<u>Lemma 5.3</u>. Let us construct the points $\Theta_k = \Theta_0 q^k$ and the powers of the matrix

$$\begin{pmatrix} q & 1-q \\ 0 & 1 \end{pmatrix} \sim Ш, \quad \begin{pmatrix} q^n & 1-q^n \\ 0 & 1 \end{pmatrix} \sim Ш^n, \qquad (5.5)$$

where $0 \leq \Theta_0 \leq 1$, $0 \leq q \leq 1$. The powers $Ш^n$ of the Markov endomorphism $Ш$ associated with matrix (5.5) maps the collection $\{R(\Theta)\}$ by the rule $R(\Theta_k)Ш^n = R(\Theta_{k+n})$. In addition

$$|R(\Theta_{k-1}) - R(\Theta_k)| = q^{k-1} |R(\Theta_0) - R(\Theta_1)|. \qquad (5.6)$$

This lemmas are proved by easy calculations. Let us return to the proof of the theorem and denote $\Phi_{red} = R(P_1)$, $\Psi_{red} = R(P_2)$, $\Theta_0 = \max\{P_1, P_2, 1-P_1, 1-P_2\}$. Without loss of generality, it is possible to assume $\Theta_0 = P_1$. Otherwise, we shall renumber the events (renumeration generates the invertible Markov map of distributions) or renumerate the indices. Moreover, if $P_1 = P_2$ the inequality (5.1) is trivially filfilled. This, we can suggest $\Theta_0 = P_1 > \frac{1}{2} > P_2$. Let us denote $q = P_2 P_1^{-1} < 1$ and construct the points $\Theta_k = \Theta_0 q^k$, $k = 1, \ldots, \nu$, where $\Theta_{\nu-1} > \frac{1}{4}$, $\Theta_\nu < \frac{1}{4}$, so that $q^k = \Theta_k \Theta_0^{-1} \geq \frac{1}{4}$, $k = 0, \ldots, \nu-1$. By the monotone invariance of ρ, the triangle inequality, corollary from lemma 5.2, and the equalities (5.6) we have

$$\nu\rho(\Phi_{red}, \Psi_{red}) \geq \rho(R(\Theta_0), R(\Theta_1)) + \cdots + \rho(R(\Theta_{\nu-1}), R(\Theta_\nu)) \geq$$
$$\geq \rho(R(\Theta_0), R(\Theta_\nu)) \geq \rho(R(\tfrac{1}{2}), R(\tfrac{1}{4}));$$

$$\nu \left| \Phi_{red} - \Psi_{red} \right| = 2 \left| \Theta_0 - \Theta_1 \right| + \dots + 2q^{-\nu+1} \left| \Theta_{\nu-1} - \Theta_\nu \right| \leq$$
$$\leq 8 \left| \Theta_0 - \Theta_\nu \right| \leq 8.$$

We are to divide the first inequality by the second one for (5.1) to be established in the case under consideration, Q.E.D.

For an asymmetric distance it is possible that $\Theta_0 = p_2 > p_1$ Then one must construct the points $\Theta_k = \Theta_0 q^k$, $q = p_1 p_2^{-1}$

6. Let us study the structure of natural loss function in the problem of state estimating. It is assumed that the function L is defined on all the squares of objects $\mathfrak{S}(\mathcal{L})$, is positive

$$L(\Phi, \Phi) = 0, \quad \Phi \neq \Psi \Rightarrow 0 < L(\Phi, \Psi) \leq +\infty, \tag{6.1}$$

and is monotonic under the restricted Markov category:

$$L(\Phi \Pi, \Psi \Pi) \leq L(\Phi, \Psi). \tag{6.2}$$

First of all let us consider the values of L on the probability distributions on two-atom measurable space, see (5.2)

$$\ell(x, y) = L(R(x), R(y)). \tag{6.3}$$

<u>Lemma 6.1.</u> The function $\ell(x,y)$ defined on the square $0 \leq x, y \leq 1$ vanishes on the diagonal $x = y$ and is strongly positive outside. The function

$$f(z) = \inf_{|x-y| \geq z} \ell(x, y) \tag{6.4}$$

is monotonic and strongly positive for all $z > 0$, $f(0) = 0$

Proof. The first statements follow from (6.1). The monotonicity of f is implied by the definition (6.4).

Let us consider the subset $Q(z) = \{(x,y): |x-y| \geq z, 0 \leq x, y \leq 1\}$. It is compact. Hence, for any z there exists a convergent sequence, $(x_n, y_n) \to (x^*, y^*) \in Q(z)$, $\lim \ell(x_n, y_n) = f(z)$. In the case $x^* < y^*$ let us take a segment $[x_0, y_0]$ such that $x^* < x_0 < y_0 < y^*$. Then $[x_0, y_0] \subset [x_n, y_n]$ for all $n \geq n_0$. Hence, by the corollary from lemma 5.2, $\ell(x_0, y_0) \leq \ell(x_n, y_n)$ for all $n \geq n_0$, and $0 < \ell(x_0, y_0) \leq f(z)$. In the case $x^* > y^*$ one must consider the segments of the form $[y, x]$.

<u>Theorem 6.2</u>. If the function L defined on all the squares of objects $\mathfrak{S}(\mathcal{L})$ satisfies the conditions (6.1) and (6.2), then

$$L(\Phi, \Psi) \geq f(|\Phi - \Psi|), \tag{6.5}$$

where the function $f(z)$ defined by (6.3) is monotonic and strongly positive for all $z > 0$, $f(0) = 0$.

<u>Proof</u>. By the monotony of L and lemma 4.1 applied to the difference $(\Phi - \Psi) \in \mathcal{L}_*^H$ we have $L(\Phi, \Psi) \geq$

$$\geq L(\Phi_{red}, \Psi_{red}) \geq f(|\Phi_{red} - \Psi_{red}|) = f(|\Phi - \Psi|),$$

where the middle inequality follows from (6.4). Q.E.D.

If the monotone invariant $F(\Phi, \Psi)$ satisfies the condition: $\Phi \neq \Psi \Rightarrow F(\Phi, \Psi) \neq F(\Phi, \Phi)$ instead of (6.1), then $F(\Phi, \Psi) \geq c + f(|\Phi - \Psi|)$, where $c = F(\Phi, \Phi) = F(\Psi, \Psi)$, and the function f is constructed accordingly to (6.4) for $L(\Phi, \Psi) = F(\Phi, \Psi) - c$.

7. Every invariant Riemmanian metric and its equivariant differential quadratic form are of great importance because the latter allows the corresponding information inequality to be written out, see [12].

Let us fixe some eigenbasis of state trace-class operator Φ, $\mathcal{H} = \oplus \mathcal{L}^k$ In this basis Φ will be described by diagonal matrix diag $(p_1, p_2, \dots)$ where $p_k = \varphi_k^k$ are the corresponding eigenvalues. A differential $d\Phi$

will be described in the basis by the trace zero Hermitian matrix

$$d\varphi_e^k = \xi_e^k + \sqrt{-1}\,\eta_e^k, \tag{7.1}$$

where ξ (resp. η) is the real (resp.the imaginary) part of differential, the symbol d being omitted for the convenience of writing. There are relations

$$\xi_e^k = \xi_k^\ell,\ \eta_e^k = -\eta_k^\ell,\ \eta_k^k = 0,\ \forall_{k,\ell};\ \sum \xi_k^k = 0. \tag{7.2}$$

Thus, a complete system of real linear coordinates in a tangent space of $\mathfrak{S}$ is formed by the variables ξ_e^k and η_e^k with $k<\ell$, and the variables ξ_k^k without one of the latters. But a desired quadratic form is preferred to be written out by means of all the variables ξ_k^k , comp. [5], and with the coefficients $u(\Phi), v(\Phi), w(\Phi)$ of polar bilinear form:

$$Q_\Phi(d\Phi) = \tag{7.3}$$

$$= \sum u_{je}^{ik}\xi_j^i\xi_e^k + 2\sum v_{je}^{ik}\xi_j^i\eta_e^k + \sum w_{je}^{ik}\eta_j^i\eta_e^k.$$

Any unitary transformation S of $\mathcal{H}$ generates the completely positive Markov map $Ш_S: \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{H})$ acting by formula

$$\Phi Ш_S = S\Phi S^*,\ ({}^i_j u_e^k) = ({}^i s^k{}_j\, s_e). \tag{7.4}$$

These simplest maps $Ш_S$ are evidently invertible, and $\Phi \sim \Phi Ш_S$ for all $\Phi \in \mathfrak{S}$.

<u>Lemma 7.1</u>. If the differential quadratic form (7.3) is unitary equivariant, $Q_\Phi(d\Phi) = Q_{S\Phi S^*}(dS\Phi S^*) =$

$= Q_\Phi^{(S)}(d\Phi)$, then it must be identically equal to the sum

$$\sum_j u^{jj}_{jj}(\xi^j_j)^2 + \sum_{j<k}\sum (u^{jk}_{jk}+u^{kj}_{kj})\xi^j_j\xi^k_k +$$
$$+ 2\sum_{j<k}\sum u^{jj}_{kk}[(\xi^j_k)^2+(\eta^j_k)^2]. \tag{7.5}$$

Proof. Let us consider the symmetry S of $\mathcal{H}$ being the reflection in the first axis $\mathcal{L}^{(1)}$, $k=1$. A map changes the sign of all elements of the first row and of the first column without the diagonal elements ξ^1_1 and η^1_1 . As

$$(\Phi, d\Phi) \sim (\Phi Ш_S, d\Phi Ш_S),$$
$$Q_\Phi(d\Phi) = Q^{(S)}_\Phi(d\Phi) = \frac{1}{2}[Q_\Phi(d\Phi) + Q^{(S)}_\Phi(d\Phi)].$$

The form (7.3) differs from that of $Q^{(S)}_\Phi(d\Phi)$ in signs of terms being indexed by 1 once or three times. In the halfsum of Q and $Q^{(S)}$ this terms annihilate while the other ones are preserved . Beiterating such an operation at = 2,3, ... leads us to equivalent form:

$$Q_\Phi(d\Phi) = \sum u^{jj}_{jj}(\xi^j_j)^2 + \sum\sum (u^{jk}_{jk}+u^{kj}_{kj})\xi^j_j\xi^k_k + \tag{7.6}$$
$$+ \sum\sum u^{jj}_{kk}(\xi^j_k)^2 + 2\sum\sum v^{jj}_{kk}\xi^k_j\eta^j_k +$$
$$+ \sum\sum w^{jj}_{kk}(\eta^j_k)^2$$

where the summation is taken as in (7.5).

Now let us consider the rotation of the first axis $\mathcal{L}^{(1)}$, $k=1$, onto itself induced by multiplying of $\mathcal{L}^{(1)}$ by $(-1)^{1/2}$. It can be prolonged up to the unitary transformation S leaving the orthocompement $\mathcal{H} \ominus \mathcal{L}^{(1)}$ to be fixed. Then $\Phi = S\Phi S^*$, $S\xi^1_e S^* = \eta^1_e$, $S\eta^1_e S^* = -\xi^1_e$ at $e \neq 1$ while the other variables will remain fixed. Producing the half- sum of Q and $Q^{(S)}$ we annihilate the terms with $\xi_\ell^1 \eta_\ell^1$ and join the summands $(\xi_\ell^1)^2+(\eta_\ell^1)^2$. Further, if we assume all $\eta_j^i=\xi_j^i=0$ except ξ_ℓ^1, then the invariance of Q will imply

$u_{\ell\ell}^{11}=w_{\ell\ell}^{11}$, and so for all ℓ. Reiterating such an operation at $k=2,3,\ldots$ will lead us to the form (7.6).

<u>Theorem 7.2.</u> If a differential quadratic form $Q_{\bar\Phi}(d\bar\Phi)$ is equivariant under the restricted Markov category, $Q_{\bar\Phi}(d\bar\Phi)=Q_{\bar\Phi Ш}(d\bar\Phi Ш)$, then it must be identically equal to the form

$$Q_{\bar\Phi}(d\bar\Phi)=c\sum_k (p_k)^{-1}(d\varphi_k^k)^2+2\sum_{j<k} c(p_j,p_k)|d\varphi_k^j|, \qquad (7.7)$$

where $c(x,y)$ is a real function in the domain $\{(x,y): 0\le x,y\le x+y\le 1\}$ and $c(x,y)=c(y,x)$, $c(x,x)=cx^{-1}$.

Proof. In the commutative theory a Markov equivariant field of differential quadratic forms is proved to be unique up to a scalar factor. Hence, by (1.10) $u_{jk}^{jk}=0$ when $j\neq k$, and $u_{kk}^{kk}=c(p_k)^{-1}$. Thus, the first two sumsof (7.5) reduce th the first sum in (7.7). Now, let us construct two Markov maps connected with some state $\bar\Phi$ and with von Neumann algebra $\mathcal{B}_{2,1}=\mathcal{B}(\mathcal{L}^{(j)}\oplus \oplus\mathcal{L}^{(k)})\oplus\mathbb{C}$, $\mathcal{L}^{(j)}$ and $\mathcal{L}^{(k)}$ being eigenaxes of $\bar\Phi$. $Ш$ maps the collection $\mathfrak{S}(\mathcal{H})$ onto $\mathfrak{S}_{2,1}=\mathfrak{S}(\mathcal{B}_{2,1})$ by the rule $\Psi Ш=\Gamma\oplus\gamma$

$$\Gamma=(E_j+E_k)\Psi(E_j+E_k),\ \gamma=1-\Psi_j^j-\Psi_k^k. \qquad (7.8)$$

$\mathcal{K}$ maps $\mathfrak{S}_{2,1}$ into $\mathfrak{S}(\mathcal{H})$ by the rule

$$(\Gamma \oplus \gamma)\mathcal{K} = \Gamma \oplus \gamma(H - E_j - E_k)\Phi. \tag{7.9}$$

By the criterion (3.4) $Ш$ and $\mathcal{K}$ are easily proved to be completely positive. Further, the maps $Ш$ and $\mathcal{K}$ are reciprocal on the operator Φ and the differential $d\Phi = \xi_k^j$ Hence,

$$(\Phi, \xi_k^j) \sim (\Phi Ш, \xi_k^j Ш), \gamma(\Phi) = 1 - p_j - p_k, u_{kk}^{jj}(\Phi) =$$

$$= u_{22}^{11}(\Phi Ш) = u_{22}^{11}(p_1, p_2) = c(p_1, p_2).$$

In the case $p_j = p_k$ any rotation G of the plane $\mathcal{L}^{(j)} \oplus \mathcal{L}^{(k)}$ leaving fixed its orthocomplement does not change the operator Φ, $G\Phi G^* = \Phi$ When a rotation G reduce the tensor (ξ_k^j, η_k^j) to the principial axes (their angles of inclination being $\pi/4$), $d\Phi$ is mapped into $dG\Phi G^* = (\zeta_j^j, \zeta_k^k)$, where $\zeta_j^j = -\zeta_k^k, (\zeta_j^j)^2 = (\zeta_k^k)^2 = (\xi_k^j)^2 + (\eta_k^j)^2$. Then $c(p,p) = c \cdot p^{-1}$ by invariance of Φ .Q.E.D.

<u>Lemma 7.3</u>. When a Markovly equivariant differential quadratic form Q_Φ is weakly continuous in Φ , the function $c(x,y)$ of its decomposition (7.7) is positively homegeneous in degree - 1:

$$c(x,y) = \lambda \cdot c(\lambda x, \lambda y), \ \forall \lambda : 0 < \lambda \leq (x+y)^{-1}. \tag{7.10}$$

<u>Proof</u>. Let us consider the Markov maps of multiplying $\Pi_n : \mathfrak{S}(\mathcal{L}) \to \mathfrak{S}(\mathcal{L} \otimes \mathcal{L}(\mathcal{H}_n))$ acting by the rule

$$\Psi \to \Psi \otimes n^{-1} \mathbb{1}_n \tag{7.11}$$

where $\mathbb{1}_n$ is an identity operator on $\mathcal{H}_n$, dim $\mathcal{H}_n = n$ On the image $\mathfrak{S}(\mathcal{L})\Pi_n$, it may be invered by partial trace operation Tr_n .Hence, $(\Phi, d\Phi) \sim (\Phi\Pi, d\Phi\Pi)$.

The congruency implies $c(x',y') = nc(x'n^{-1}, y'n^{-1}) = \nu c(x'\nu^{-1}, y'\nu^{-1})$. If λ is rational, $\lambda = \nu \cdot n^{-1}$, (7.10) follows from the second equality, $x = x' \cdot n^{-1}$, $y = y' \cdot n^{-1}$. By the presupposed continuity (7.10) is valid for all λ.

<u>Theorem 7.4.</u> Any equivariant weakly continuous differential quadratic form satisfies the information additivity principle:

$$Q_{\Phi \otimes \Psi}(d[\Phi \otimes \Psi]) = Q_{\Phi}(d\Phi) + Q_{\Psi}(d\Psi) \tag{7.12}$$

Proof. Let us fix the eigenbases of Φ and Ψ. Then there are matrix representations $\Phi = \mathrm{diag}(p_1, p_2, \dots)$, $\Psi = \mathrm{diag}(q_1, q_2, \dots)$. Their tensor (Kronecker) product is diagonal, and the matrix of differential $d(\Phi \otimes \Psi) = \Phi \otimes d\Psi + d\Phi \otimes \Psi$ is sparce. So.

$$Q(d[\Phi \otimes \Psi]) = \sum (p_i q_k)^{-1} [p_i \, d\psi_k^k + q_k \, d\varphi_i^i]^2 +$$

$$+ 2 \sum_{i, k<l} c(p_i q_k, p_i q_l) |p_i \, d\psi_l^k|^2 + 2 \sum_{k, i<j} c(p_i q_k, p_j q_k) |q_k \, d\varphi_j^i|^2 =$$

$$= \sum (q_k)^{-1} (d\psi_k^k)^2 + \sum (p_i)^{-1} (d\varphi_i^i)^2 +$$

$$+ 2 \sum c(q_k, q_l) |d\psi_l^k|^2 + 2 \sum c(p_i, p_j) |d\varphi_j^i|^2,$$

where the rule (7.10) and the equalities

$$\sum p_i = \sum q_k = 1, \quad \sum d\varphi_i^i = \sum d\psi_k^k = 0$$

have been used.

8. When a Riemann metric defined by form Q is not only invariant but also monotone under Markov category,

the bounds of the coefficient $c(x,y)$ in (7.7) can be given.

Lemma 8.1. If a Riemann metric ρ is monotone, then

$$c(x,y) \geq \theta^2 c(\theta x, \theta y); \ \forall \theta: 0 \leq \theta \leq 1. \tag{8.1}$$

Corollary. Under condition of the lemma the homogenity property (7.10) is valid without the weak continuity assumption.

To prove the statements we'll construct the completely positive Markov endomorphism $Ш$ of collection $\mathfrak{S}_{2,1} = \mathfrak{S}(\mathcal{L}(\mathcal{H}_2) \oplus \mathbb{C})$ by the rule

$$\Phi Ш = \Theta(E_1+E_2)\Phi(E_1+E_2) \oplus (1-\theta)E_3 \Phi E_3, \tag{8.2}$$

where Φ diag $(x, y, 1-x-y)$, $\Phi Ш =$ diag$(\theta x, \theta y, 1-\theta x-\theta y)$. The detail calculation is omitted.

In view of homogenity (7.1), let us study the boundary behaviour of $c(x,y)$ i.e. when $x+y=1$, $y=1-x$.

Theorem 8.2. If a Riemannian metric ρ is monotone under restricted Markov category, then

$$c(\tfrac{1}{2}, \tfrac{1}{2}) \leq c(x, 1-x) \leq [4x(1-x)]^{-1} c(\tfrac{1}{2}, \tfrac{1}{2}); \tag{8.3}$$

$$c(x, 1-x) \geq c(x', 1-x'), \quad x > x' \geq \tfrac{1}{2}; \tag{8.4}$$

$$c(x, 1-x)\,dx \geq (1-x)\,dc(x, 1-x); \tag{8.5}$$

where in (8.5) the right differential is taken.

To prove these inequalities some irreversible Markov endomorphisms of $\mathfrak{S}(\mathcal{H}_2)$ have been constructed. The dimension of their tensors makes it impossible to write them out here.

As the collections $\mathfrak{S}(\mathcal{L})$ are convex, it is possible to consider the quadratic form values of finite differences. For the classic Fisher information, ~~and~~, consequently for the couples of commuting operator Φ and difference $\Delta\Phi$ we have

$$Q_{\Phi\otimes\Psi}(\Delta[\Phi\otimes\Psi]) = Q_{\Phi}(\Delta\Phi) + Q_{\Psi}(\Delta\Psi) + + Q_{\Phi}(\Delta\Phi)\cdot Q_{\Psi}(\Delta\Psi). \tag{8.6}$$

A form Q satisfies (8.6) in a general case of non-commuting couples provided $c(pu,qv)+c(pv,qu) = = 2c(p,q)c(u,v)$ everywhere in the domain of admissible values of the arguments. This functional equation is satisfied by the functions $c(u,v) = c[u^{\alpha}v^{-\alpha}+u^{-\alpha}v^{\alpha}]\times \times(uv)^{-1/2}$, $0\le\alpha\le\frac{1}{2}$. When $\alpha = 1/2$, resp. $\alpha=0$, then $Q_{\Phi}(d\Phi) = c\cdot Tr[\Phi^{-1}(d\Phi)^2]$,resp.

$Q_{\Phi}(d\Phi) = c\,Tr[(d\Phi)\Phi^{-1/2}(d\Phi)\Phi^{-1/2}]$, see[22].

9. The statistical problem of estimation of unknown probability distribution P on $E=\{x: 0\le x\le 1\}$ by means of independent P-distributed observations without additional a priori information about P is proved by us 23 to be incorrect if the error of estimate P^* is measured by the variance $|P^*-P|$ .Being the special cases of theorems 5.1 and 6.2, and, in their turn, the far generalizations of results [24], [25], the theorems 1.4 and 1.5 implies this statistical point estimation problem to be incorrect whether a Markovly invariant measuring of estimation error is made by a metric or by a loss function.

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