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MATHEMATICS

A Note on Modular Spaces. I

by

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1. Let X denote a real linear space, ϱ — a functional defined in X, $-\infty < \varrho(x) \leqslant \infty$. The functional ϱ will be called a *modular*, if the conditions

A. 1. $\varrho(0) = 0$; A. 2. $\varrho(\lambda x) = 0$ for all λ implies $x = 0$; A. 3. $\varrho(-x) = \varrho(x)$;

A. 4. $\varrho(\alpha x + \beta y) \leqslant \varrho(x) + \varrho(y)$ for $x, y \in X$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.

1.1. A functional ϱ will be called *s-convex*, $0 < s \leqslant 1$, if

A_s. 4. $\varrho(\alpha x + \beta y) \leqslant \alpha^s \varrho(x) + \beta^s \varrho(y)$ for $x, y \in X$, $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$.

A 1-convex functional will be called briefly *convex*.

Evidently, A_1. 4. implies A. 4. If $\varrho(0) = 0$ for an s-convex functional ϱ, then $\varrho(\alpha x) \leqslant \alpha^s \varrho(x)$ for $0 \leqslant \alpha \leqslant 1$ and consequently, $\varrho(tx)\, t^{-s}$ is a nondecreasing function of t, $t > 0$.

1.2. The following properties of modulars are of importance in our considerations:

B. 1. $\alpha_n \to 0$ implies $\varrho(\alpha_n x) \to 0$;

B. 2. $\varrho(x_n) \to 0$ implies $\varrho(\alpha x_n) \to 0$ for every $\alpha > 0$.

1.21. We shall apply the following notation: $X_b^*(\varrho) = \{x : \varrho(x) < \infty,\ x \in X\}$, $X^*(\varrho) = \{x : \varrho(\lambda x) < \infty$ for a certain $\lambda(x) > 0,\ x \in X\}$. Of course, every x satisfying B. 1. belongs to $X^*(\varrho)$. Denote by $X(\varrho)$ the set of x satisfying B. 1. $X^*(\varrho)$ is a linear space, $X(\varrho)$ — its subspace.

1.3. It is shown in [2] that in $X(\varrho)$ a norm may be defined as follows *):

$$(*) \qquad \|x\|_\rho = \inf\{\varepsilon > 0 : \varrho(x/\varepsilon) \leqslant \varepsilon\},$$

possessing the following properties: a) $\| \ \|_\rho$ is an F-norm, b) $\varrho(x) \leqslant \|x\|_\rho$ for $\|x\|_\rho < 1$, c) $\|\lambda x\|_\rho$ is nondecreasing for $\lambda \geqslant 0$. In the sequel, we call this norm

*) In [2] the norm (*) is defined assuming a condition stronger that A. 2. However, the proof that (*) is a norm possessing the properties in 1.3 can be performed by our assumption A.2 without any changes. In [3] modulars, defined as in 1, were termed semimodulars.

the norm generated by the modular ϱ. Under some additional assumptions on the modular ϱ, other definitions of the norm may be introduced in $X(\varrho)$ by means of ϱ. There holds the theorem:

1.4. *If* ϱ *is an s-convex modular, then* $X(\varrho) = X^*(\varrho)$ *and an s-homogeneous norm can be defined in* $X^*(\varrho)$ *as follows*:

$$(+) \qquad \|x\|_{0s} = \inf\left\{\varepsilon > 0 : \varrho\left(\frac{x}{\varepsilon^{1/s}}\right) \leqslant 1\right\}.$$

We prove, e.g., the triangle inequality. Let $\|x\|_{0s} < \varepsilon_1$, $\|y\|_{0s} < \varepsilon_2$, $\varepsilon = \varepsilon_1 + \varepsilon_2$. Then

$$\varrho((x+y)\varepsilon^{-1/s}) = \varrho(\varepsilon_1^{1/s}\varepsilon^{-1/s}x\varepsilon_1^{-1/s} + \varepsilon_2^{1/s}\varepsilon^{-1/s}y\varepsilon_2^{-1/s}) \leqslant$$

$$\leqslant (\varepsilon_1^{1/s}\varepsilon^{-1/s})^s\varrho(x\varepsilon_1^{-1/s}) + (\varepsilon_2^{1/s}\varepsilon^{-1/s})^s\varrho(y\varepsilon_2^{-1/s}) \leqslant 1,$$

for $(\varepsilon_1^{1/s}\varepsilon^{-1/s})^s + (\varepsilon_2^{1/s}\varepsilon^{-1/s})^s = 1$. Thus, $\|x+y\|_{0s} \leqslant \varepsilon_1 + \varepsilon_2$.

1.41. The norms $\|\ \|_{0s}$ and $\|\ \|_\varrho$ are equivalent, more exactly, there hold the inequalities: (a) $\|x\|_{0s} \leqslant (\|x\|_\rho)^s$, if $\|x\|_\rho < 1$, (b) $(\|x\|_\rho)^{s+1} \leqslant \|x\|_{0s}$, if $\|x\|_{0s} < 1$.

If $\|x\|_\rho < 1$ and $\|x\|_\rho < \varepsilon < 1$, then $\varrho(x(\varepsilon^s)^{-1/s}) \leqslant \varepsilon < 1$, whence $\|x\|_{0s} \leqslant \varepsilon^s$. If $\|x\|_{0s} < 1$ and $\|x\|_{0s} < \varepsilon < 1$, then $\varrho(x\varepsilon^{-1/1+s}) \leqslant (\varepsilon^{1/s}\ \ \varrho(x\varepsilon^{-1/1+s}) \leqslant \varepsilon^{1/1+s}$, whence $\|x\|_\rho \leqslant \varepsilon^{1/1+s}$.

1.5. If ϱ is an s-convex modular, then

$$\|x\|_{0s} = \inf_{t>0}[\sup(t^{-s}, \varrho(tx)t^{-s})].$$

It follows from (+) that $\varrho(x) \leqslant 1$ implies $\|x\|_{0s} \leqslant 1$. If $\varrho(x) > 1$, then $\varrho(x\varrho(x)^{-1/s}) \leqslant (\varrho(x)^{-1/s})^s\varrho(x) = 1$ and $\|x\varrho(x)^{-1/s}\|_{0s} = \|x\|_{0s}\varrho(x)^{-1} \leqslant 1$. Hence there follow the inequalities

$$\|x\|_{0s} \leqslant \sup(1, \varrho(x)),\ \ \|x\|_{0s} \leqslant \sup(t^{-s}, \varrho(tx)t^{-s}) \text{ for } t > 0,$$

i.e. $\|x\|_{0s} \leqslant \inf_{t>0}[\sup(t^{-s}, \varrho(tx)t^{-s}]$. The definition of the norm (+) implies that if $0 < t^0 < (\|x\|_{0s})^{-1/s}$, then $\varrho(t_0 x) \leqslant 1$, i.e.

$$\inf_{t>0}[\sup(t^{-s}, \varrho(tx)t^{-s}] \leqslant \sup(t_0^{-s}, \varrho(t_0x)t_0^{-s}) \leqslant t_0^{-s},$$

where t_0^{-s} is arbitrarily near to $\|x\|_{0s}$.

1.51. If ϱ is an s-convex modular, then the functional

$$\|x\|_s = \inf_{t>0}(t^{-s} + \varrho(tx)t^{-s})$$

is an s-homogeneous norm in $X^*(\varrho)$ and there hold the inequalities

$$(\circ) \qquad \tfrac{1}{2}\|x\|_s \leqslant \|x\|_{0s} \leqslant \|x\|_s.$$

Denoting $\gamma(x) = 1 + \varrho(x)$, we have $\|x\|_s = \inf_{t>0} \gamma(tx)\, t^{-s}$. Moreover, $0 \leqslant \|x\|_s < \infty$; in fact, $\varrho(x) \geqslant 0$ for $x \in X$ and $\varrho(tx) < \infty$ for $x \in X^*(\varrho)$, if $t > 0$ is sufficiently small. The s-homogeneity being obvious, we prove only the triangle inequality. Let $t', t'' > 0$; $\gamma(x)$ is an s-homogeneous functional, whence

$$\gamma\left(\frac{t'^{1/s}\, t''^{1/s}}{(t'+t'')^{1/s}}(x+y)\right) = \gamma(\alpha t'^{1/s} x + \beta t''^{1/s} y) \leqslant \alpha^s \gamma(t'^{1/s} x) + \beta^s \gamma(t''^{1/s} y),$$

where

$$\alpha = t''^{1/s}(t'+t'')^{-1/s}, \qquad \beta = t'^{1/s}(t'+t'')^{-1/s}, \qquad \alpha^s + \beta^s = 1.$$

Choosing $t_0 = t'^{1/s}\, t''^{1/s}(t'+t'')^{-1/s}$ we obtain

$$\|x+y\|_s \leqslant \gamma(t_0(x+y))\, t_0^{-s} \leqslant \gamma(t'^{1/s}x)\, t'^{-1} + \gamma(t''^{1/s} y)\, t''^{-1}.$$

The inequalities (o) follow from **1.5**, $\frac{1}{2}\gamma(tx)\, t^{-s} \leqslant \sup(t^{-s}, \varrho(tx)\, t^{-s}) \leqslant \gamma(tx)\, t^{-s}$. If $\|x\|_s = 0$, then (o) implies $\|x\|_{0s} = 0$, i.e. $x = 0$. Hence, $\| \;\|_s$ is a norm and not only a quasinorm.

1.6. If ϱ is an s-convex modular and if $n(x)$ is a quasinorm in $X^*(\varrho)$ continuous with respect to the norm $\| \;\|_\varrho$, then

(a) $$\sup_{\varrho(x)\leqslant 1} n(x) = k$$

is finite; moreover, if $n(x)$ is s-homogeneous, then

(b) $$n(x) \leqslant k(1+\varrho(x)).$$

We prove (a). Suppose $\|x\|_\varrho \leqslant \eta$ implies $n(x) \leqslant 1$. If x satisfies the inequality $\varrho(x) \leqslant 1$, we choose a positive integer l such that $l\eta > 1$, $l > \eta^{-(s+1)/s}$. We have $\varrho(l^{-1}\eta^{-1}x) \leqslant l^{-s}\eta^{-s} < \eta$, i.e. $\|l^{-1}x\|_\varrho \leqslant \eta$, thence $n(l^{-1}x) \leqslant 1$, $n(x) \leqslant l$. Now, we prove (b). If $\varrho(x) > 1$, then $\varrho(x\,\varrho(x)^{-1/s}) \leqslant 1$, whence $n(x\,\varrho(x)^{-1/s}) \leqslant k$, $n(x)\,\varrho(x)^{-1} \leqslant k$; thus $n(x) \leqslant k \sup(1, \varrho(x))$ for an arbitrary $x \in X^*(\varrho)$.

The last inequality implies easily that, by the assumptions of **1.6** (b), $n(x) \leqslant k\|x\|_{0s}$.

1.61. If ϱ is an arbitrary modular and if $n(x)$ is an s-homogeneous quasinorm in $X(\varrho)$ satisfying the inequality

$$n(x) \leqslant k(1+\varrho(x)) \quad \text{for} \quad x \in X(\varrho),$$

then $\|x_n\|_\varrho \to 0$ implies $n(x_n) \to 0$.

Evidently, $n(x) \leqslant k(t^{-s} + \varrho(tx)\, t^{-s})$, $t > 0$; it suffices to note that $\|x_n\|_\varrho \to 0$ implies $\varrho(tx_n) \to 0$ for an arbitrary $t > 0$.

1.7. If ϱ is a convex (i.e. a 1-convex) modular, then the norm $\| \;\|_1$ is the Amemiya's norm, well-known in the theory of modular spaces (cf. [4]). It can be also defined as follows. By the definition of the norm $\| \;\|_1$, a distributive functional ξ over $X^*(\varrho)$ is continuous with respect to the norm $\| \;\|_1$ and has a norm $\|\xi\| \leqslant 1$ if and only if

(++) $$\xi(x) \leqslant 1 + \varrho(x) \text{ for } x \in X^*(\varrho).$$

Let $n(x) = \sup \xi(x)$, where the supremum is taken over all distributive functionals over $X^*(\varrho)$ satisfying $(++)$. Since for every ξ satisfying $(++)$ we have $\xi(x) \leqslant \leqslant t^{-1} + \varrho(tx)t^{-1}$ for $t > 0$, there holds $n(x) \leqslant \|x\|_1$. On the other hand, there exists a distributive functional ξ continuous in $X^*(\varrho)$ with respect to $\| \ \|_1$ satisfying the conditions $\xi(x_0) = \|x_0\|_1$, $\xi(x) \leqslant \|x\|_1 \leqslant 1+\varrho(x)$; hence, $n(x_0) \geqslant \|x_0\|_1$. Thus, $n(x) = \|x\|_1$.

1.8. A modular ϱ satisfies the *conditions* (D), if it has the following property: to every $0 < \alpha < 1$ there exists a decomposition $x = x_1 + x_2$ such that $\varrho(x_1) \leqslant \alpha\varrho(x)$, $\varrho(x_2) \leqslant (1-\alpha)\varrho(x)$.

Convex modulars are examples of modulars satisfying the condition (D). In **2** we give further examples of such modulars.

1.81. If ϱ is a modular satisfying the condition (D) and if $n(x)$ is a quasinorm in $X(\varrho)$ continuous with respect to $\| \ \|_\rho$, then **1.6** (a), (b) hold, where $k < \infty$.

Let $\varrho(x) \leqslant 1$. Given any positive integer k, there exists a decomposition $x = x_1 + x_2 + \ldots + x_{2k}$, $\varrho(x_i) \leqslant 2^{-k}\varrho(x) \leqslant 2^{-k}$. Assume $\|x\|_\rho < l^{-1}$ (l—an integer) implies $n(x) \leqslant 1$; choosing k sufficiently large (k is independent of x) we have $\varrho(lx_i l^{-1}) < l^{-1}$, whence $\|l^{-1}x_i\|_\rho < l^{-1}$, $n(l^{-1}x_i) \leqslant 1$, $n(x) \leqslant ln(xl^{-1}) \leqslant \leqslant l[n(x_1 l^{-1}) + n(x_2 l^{-1}) + \ldots + n(x_{2k} l^{-1})] \leqslant l\, 2^k$. Let $\varrho(x) = r + l$, where $r \geqslant 1$, $0 < l \leqslant 1$. There exist x_1, x_2, $x = x_1 + x_2$ such that $\varrho(x_1) \leqslant r\varrho(x)^{-1}\varrho(x) = r$, $\varrho(x_2 \leqslant (1 - r\varrho(x)^{-1})\varrho(x) = l\varrho(x)^{-1}\varrho(x) = l$. It is shown by induction that a decomposition $x = x_1 + x_2 + \ldots + x_r + x_0$ exists such that $\varrho(x_i) \leqslant 1$, $\varrho(x_0) \leqslant \leqslant l \leqslant 1$. Hence it follows

$$n(x) \leqslant n(x_1) + n(x_2) + \ldots + n(x_r) + n(x_0) \leqslant rk + k \leqslant k(\varrho(x) + 1).$$

1.9. If ϱ is a modular in $X(\varrho)$ then a quasinorm $n(x)$ (a distributive functional ξ) in $X(\varrho)$ is called *ϱ-continuous*, if $\varrho(x_n) \to 0$ implies $n(x_n) \to 0$ ($\xi(x_n) \to 0$). A distributive and ϱ-continuous functional is called *ϱ-linear*. Evidently, a ϱ-continuous $n(x)$ is continuous with respect to the norm generated by ϱ (by **1.3**, b)), but not conversely. If the condition B.2 is satisfied, then $\varrho(x_n) \to 0$ implies $\|x_n\|_\rho \to 0$ and then the ϱ-continuity is equivalent to the continuity with respect to $\| \ \|_\rho$. A set $X^0 \subset X(\varrho)$ is called *modular-dense* in $X(\varrho)$, if to every $x \in X(\varrho)$ there exist $x_n \in X(\varrho)$ and $\lambda > 0$ such that $\varrho(\lambda(x_n - x)) \to 0$.

1.91. If ϱ is a modular satisfying the condition (D) and if there exists an s-homogeneous norm in $X(\varrho)$ continuous with respect to $\| \ \|_\rho$, then there is a constant $c > 0$ for which

$$\liminf_{t\to\infty} \varrho(tx)t^{-s} \geqslant cn(x) > 0 \quad \text{for} \quad x \in X(\varrho).$$

If n is the norm satisfying the assumptions, then $n(x) \leqslant k(t^{-s} + \varrho(tx)t^{-s})$, by **1.81** and **1.6**, (b).

1.92. If ϱ is a modular satisfying the condition (D) and if

$$\liminf_{t\to\infty} \varrho(tx)t^{-1} = 0$$

in a set modular-dense in $X(\varrho)$, then there exists only a trivial ϱ-linear functional in $X(\varrho)$ ([3]).

We put $n(x) = |\xi(x)|$; then, by **1.91** with $s = 1$, we get $\xi(x) = 0$ in a set modular-dense in $X(\varrho)$. Hence the ϱ-continuity implies $\xi(x) = 0$ always in $X(\varrho)$.

2. In this section the term *φ-function* will mean a function continuous and nondecreasing for $u \geqslant 0$, vanishing only at $u = 0$ and tending to ∞ as $u \to \infty$. If $\varphi(au + \beta v) \leqslant a^s \varphi(u) + \beta^s \varphi(v)$, $0 < s \leqslant 1$, $a, \beta \geqslant 0$, $a^s + \beta^s = 1$, then this φ-function will be called *s-convex* (it is an s-convex modular in the space of real numbers). E.g. $\varphi(u) = \psi(u^s)$, where ψ is a convex φ-function, is s-convex.

2.1. We shall consider the following examples of modular spaces:

A. Let μ denote a finite, σ-additive measure, defined on a σ-algebra F of subsets of a set E, X — the space of functions μ-measurable in E, $\varrho(x) = \int_E \varphi(|x(t)|)\, d\mu$, $L^\varphi(\mu) = X^*(\varrho) = X(\varrho)$.

B. X — the space of sequences $x = \{t_\nu\}$, $\varrho(x) = \Sigma\varphi(|t_\nu|)$, $l^\varphi = X^*(\varrho) = X(\varrho)$.

C. X — the space the elements of which are classes of bounded functions in $\langle a, b\rangle$; two functions belong to the same class if they differ by a constant. $\varrho(x) = \sup \Sigma\varphi(|x(t_\nu) - x(t_{\nu-1})|)$, where the supremum is taken over all partitions $\pi : a = t_0 < t_1 < \ldots < t_n = b$, $V^*_\varphi = X^*(\varrho)$ ([1]).

2.2 If φ is an s-convex φ-function, then the modulars defined in A—C are s-convex. By this assumption, the modular in V^*_φ satisfies B. 1, whence a norm generated by ϱ can be introduced in V^*_φ. If the measure μ is nonatomic then the modular defined in A satisfies the condition (D). In general, modulars defined in B by means of arbitrary φ do not satisfy the condition (D), e.g. if $\varphi(t)/t \to 0$ as $t \to \infty$ or if $\varphi(t' + t'') < \varphi(t') + \varphi(t'')$ for $t', t'' > 0$.

2.3. A function $x \in V^*_\varphi$ is called *absolutely continuous with respect to* φ ([1]), if to any $\varepsilon > 0$ there is a $\delta > 0$ such that $\Sigma\varphi(|x(\beta_i) - x(a_i)|) < \varepsilon$ for all systems of non-overlapping intervals $(a_i, \beta_i) \subset \langle a, b\rangle$ satisfying the inequality $\Sigma\varphi(\beta_i - a_i) < \delta$. We denote by AC^*_φ the set of all x for which λx is absolutely continuous with respect to φ for a $\lambda > 0$ (two functions which differ only by a constant are considered to be the same element of the space). AC^*_φ is a linear subspace of V^*_φ.

2.4. Let $\varphi(u)/u \to 0$ as $u \to 0$. By this assumption:

(a) the modular ϱ defined in C satisfies the condition B. 1 for $x \in AC^*_\varphi$,

(b) ϱ satisfies the condition (D).

Proof of (a). Denote $V_\varphi(x; a, \beta) = \sup \Sigma\varphi(|x(t_\nu) - x(t_{\nu-1})|)$, where the supremum is taken over all partitions of the interval $\langle a, \beta\rangle \subset \langle a, b\rangle$, $V_\varphi(x; a, a) = 0$. The definition of ϱ is $\varrho(x) = V_\varphi(x; a, b)$. The partition being given, denote $\mathcal{J}'(x) = \Sigma' \varphi(|x(t_\nu) - x(t_{\nu-1})|)$, $\mathcal{J}''(x) = \Sigma'' \varphi(|x(t_\nu) - x(t_{\nu-1})|)$, where the summation is extended over all intervals of the length $< \delta$ in the first sum, and over all intervals of the length $\geqslant \delta$ — in the second one. Let $\varrho(x) < \infty$, $x \in AC^*_\varphi$. Since $\varphi(u)/u < \varepsilon(b - a)^{-1}$ for $\delta > 0$ sufficiently small, we have $\mathcal{J}'(x) \leqslant \varepsilon(b - a)^{-1} \Sigma'(t_\nu - t_{\nu-1}) \leqslant \varepsilon$. Moreover, there holds the inequality $\mathcal{J}''(x) \leqslant (b - a)\,\delta^{-1} \varphi(2 \sup_{\langle a, b\rangle} |x(t)|)$, for the number of intervals in this sum is $\leqslant (b - a)\,\delta^{-1}$. Hence we obtain for $0 \leqslant a \leqslant 1$,

$$\Sigma \varphi (a |x (t_\nu) - x (t_{\nu-1})|) \leqslant \mathcal{J}' (x) + (b - a) \delta^{-1} \varphi (2 a \sup |x (t)|),$$

$$\varrho (ax) \leqslant \varepsilon + \varepsilon \text{ for } a \text{ sufficiently small.}$$

Proof of (b). Let $0 < \varrho (x) < \infty$, $x \in AC_\varphi^*$. It is easily shown that $V_\varphi (x; a, \tau)$ and $V_\varphi (x; \tau, b)$ are continuous functions of τ for $a \leqslant \tau \leqslant b$. We define the functions

$$x_\tau' (t) = \begin{cases} x (t) & \text{for } a \leqslant t < \tau \\ x (\tau) & \text{for } b \geqslant t \geqslant \tau, \end{cases} \qquad x_\tau'' (t) = \begin{cases} x (\tau) & \text{for } a \leqslant t < \tau \\ x (t) & \text{for } b \geqslant t \geqslant \tau. \end{cases}$$

Since $x(t) + x(\tau) = x_\tau'(t) + x_\tau''(t)$ for $t \in \langle a, b \rangle$, we have $x = x_\tau' + x_\tau''$ and moreover, $\varrho (x_\tau') = V_\varphi (x; a, \tau)$, $\varrho (x_\tau'') = V_\varphi (x; \tau, b)$. Let $0 < \alpha < 1$; there is a τ^*, $a < \tau^* < b$, such that $\varrho (x_{\tau^*}') = \alpha \varrho (x)$. Since $\varrho (x_{\tau^*}') + \varrho (x_{\tau^*}'') = V_\varphi (x; a, \tau^*) + V_\varphi (x; \tau^*, b) \leqslant \leqslant V_\varphi (x, a, b) = \varrho (x)$, we have $\varrho (x_{\tau^*}'') \leqslant (1 - \alpha) \varrho (x)$.

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