ON SPACES OF ϕ–INTEGRABLE FUNCTIONS

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1. As is well known, numerous papers have been devoted to the study of generalizations of the classical spaces $L^{\alpha}(a,b)$ of which those, called $L_M^*(a,b)$ by some authors or $L_{\phi}^*(a,b)$ by others, should be quoted in the first place. In order to get some classes of spaces including those known extensions as a particular case let us introduce the following concepts. A continuous function non-decreasing for $u \geqq 0$ which vanishes only at 0 and tends to ∞ as $u \to \infty$ is said to be a *ϕ-function*. Let μ be a σ-additive and totally σ-finite measure on a σ-algebra $\mathfrak{F}$ of subsets of an abstract set E. Let $L^{\phi}(E,\mu)$ be the set of μ-measurable real functions for which the integral

$$\int_E \phi(|x(t)|)d\mu$$

is finite. A function $x \in L^{\phi}(E,\mu)$ will be said *ϕ-integrable*. $L^{\phi}(E,\mu)$ is a convex set in the space of μ-measurable functions, in general non-linear, so that for the sake of avoiding this inconvenience we shall introduce a wider class $L^{*\phi}(E,\mu)$ including x in it if and only if $\lambda x \in L^{\phi}(E,\mu)$ with a certain $\lambda > 0$ (in general depending on x). $L^{*\phi}(E,\mu)$ becomes then a linear space. If the ϕ-function ϕ is convex then $L^{*\phi}(E,\mu)$ becomes with a proper definition of norm a Banach space and for $\phi(u) = u^{\alpha}$, $\alpha \geqq 1$ we arrive at spaces of functions integrable with exponent α.

Many people occupied themselves, under the assumption of convexity of ϕ, with studying properties of $L^{*\phi}(E,\mu)$ and with extending this order of ideas replacing the spaces $L^{\alpha}(E,\mu)$ by this wider class of spaces. An exhaustive account on these questions can be found in the monograph of Zaanen [15] from 1953 and in the book published in Russian by Krasnosielsky and Ruticky [4]. The case of arbitrary ϕ-functions must necessarily be settled if we wish to include in the theory also such spaces as $L^{\alpha}(a,b)$ or more generally $L^{\alpha}(E,\mu)$ with $0 < \alpha < 1$ i.e. spaces which are not B-normed but only F-normed (linear metric spaces). It was first L. W. Kantorovitch [2] who has drawn attention to the spaces of functions which are ϕ-integrable in the sense of Lebesgue in a finite interval but with the additional assumption that the space is linear. He was not interested, though, in developing the theory itself but treated the spaces in question as certain examples of semi-ordered metric spaces. More systematic study of general spaces $L^{*\phi}(E,\mu)$ has become a subject of research only about 2 years ago in papers of S. Mazur and W. Orlicz [8] and S. Rolewicz [13]. In what follows I wish to present some results contained in the above-mentioned papers and also in the papers yet unpublished of W. Matuszewska [5], [6], [7] and myself [11], [12]. Since my aim here is to give only a preliminary study which would throw some light on the whole subject, it will be ample enough if we restrict ourselves to that typical case when E is a finite interval and μ the Lebesgue measure on the algebra of sets which are Lebesgue-measurable in (a,b). Under that hypothesis instead of writing $L^{\phi}(E,\mu)$ (resp. $L^{*\phi}(E,\mu)$) we shall use the notation $L^{\phi}(a,b)$ (resp. $L^{*\phi}(a,b)$) and put

$$\mathfrak{I}_{\phi}(x) = \int_a^b \phi(|x(t)|)dt.$$

These considerations are fit for generalization to $L^{*\phi}(E,\mu)$ with non-atomic measure μ. Also the case of E being union of enumerably many atoms e_n with measures uniformly bounded from above and from below (by a positive number) can similarly be dealt with. The latter case of $L^{\phi}(E,\mu)$ corresponds to spaces of sequences $x = \{t_n\}$ which are ϕ-convergent i.e. such that $\sum_{\nu=1}^{\infty} \phi(|t_\nu|) < \infty$.

2. The first observation is the following:

In $L^{*\phi}(a,b)$ a norm can be defined as follows:

$$\|x\|_{\phi} = \inf\{\varepsilon > 0 : \mathfrak{I}_{\phi}(\tfrac{x}{\varepsilon}) \leq \varepsilon\}. \tag{1}$$

The norm defined by (1) is a complete F-norm and if $\|x_n\|_{\phi} \to 0$ then $\mathfrak{I}_{\phi}(x_n) \to 0$. From the standpoint of the subject under consideration whatever norm $\|x\|$ might be introduced in $L^{*\phi}(a,b)$, it will be useful only in that case when the following condition holds:

$$\|x_n\| \to 0 \text{ implies } \mathfrak{I}_{\phi}(x_n) \to 0. \tag{M}$$

It is evident from the closed-graph theorem that if a complete F-norm in $L^{*\phi}(a,b)$ satisfies (M) then it is equivalent to $\| \ \|_{\phi}$. Further on when speaking of a norm in $L^{*\phi}(a,b)$ we always mean a *complete norm satisfying* (M), though this assumption does not explicitly occur in the following. The norm defined by (1) will be called the norm *generated by* ϕ. It can be trivially seen that this norm is not homogeneous for any ϕ and is s-homogeneous only for $\phi(u) = ku^{\alpha}$ $\alpha > 0$. In the latter case with $k = 1$ i.e. for $L^{\alpha}(a,b)$ we have

$$\|x\|_{\phi} = \left(\int_a^b |x(t)|^{\alpha} dt \right)^{\frac{1}{1+\alpha}},$$

therefore $\| \ \|_{\phi}$ is $\alpha/(1+\alpha)$-homogeneous, unlike the classical norm being homogeneous for $\alpha \geqq 1$ and α-homogeneous for $0 < \alpha < 1$.

A natural question is when one can manage to introduce a norm with better properties than that generated by ϕ, being e.g. defined by a simple formula, homogeneous or s-homogeneous etc. First, let us introduce some concepts of fundamental significance for the whole theory. We shall say that *a ϕ-function ϕ is not weaker than a ϕ-function ψ for large u*, in symbols $\psi \overset{l}{\prec} \phi$, if we have the inequality

$$\psi(u) \leqq b\phi(ku) \text{ for } u \geqq u_0 \geqq 0,$$

where b,k are positive constants. If $\phi \overset{l}{\prec} \psi$ and $\psi \overset{l}{\prec} \phi$ then ϕ and ψ are termed *equivalent for large u*, in symbols $\phi \overset{l}{\sim} \psi$. This sort of equivalence means that with suitable positive constants a,b,k,l we have

$$a\phi(lu) \leqq \psi(u) \leqq b\phi(ku) \text{ for } u \geqq u_0 \geqq 0.$$

The relation $\overset{l}{\prec}$ is transitive and the relation $\overset{l}{\sim}$ has the usual properties of relations of equivalence. The above definitions of $\overset{l}{\prec}$, $\overset{l}{\sim}$ are more general than those used e.g. in the quoted book of Krasnosielsky and Ruticky.

It may be noted that our definitions are specially fit to the considered case of a finite interval (a,b) of integrability. In the case of infinite (a,b) the relations $\psi \overset{a}{\prec} \phi, \psi \overset{a}{\sim} \phi$ (*ϕ not weaker than ψ for all u, ϕ equivalent to ψ for all u*) are to be defined as before with $u_0 = 0$.

For investigating the spaces of series the relations $\overset{s}{\prec}$, $\overset{s}{\sim}$ (for small u) are defined by means of the same inequalities as above but this time supposed to hold in a neighbourhood of 0.

The proofs of the following statements provide no difficulty:

(a) $L^{*\phi}(a,b) \subset L^{*\psi}(a,b)$ if and only if $\psi \overset{l}{\prec} \phi$, consequently $L^{*\phi}(a,b) = L^{*\psi}(a,b)$ if and only if $\phi \overset{l}{\sim} \psi$.

(b) if $\phi \overset{l}{\sim} \psi$ then the norm generated by ϕ is equivalent to the norm generated by ψ.

Owing to (a) and (b) it follows that it is possible to replace ϕ by some other ϕ-function which is equivalent to the former one and more handy for defining the norm in $L^{*\phi}(a,b)$.

Suppose now that

(2) $$\phi \overset{l}{\sim} \chi, \text{ with } \chi(u) = \psi(u^s),\ s > 0 \text{ and } \psi \text{ convex.}$$

Let $0 < s \leqq 1$. Then a norm can be defined by the formula

(3 $$\|x\|_s = \inf\left\{\varepsilon > 0 : \mathfrak{I}_\chi\left(\frac{x}{\varepsilon^{1/s}}\right) \leqq 1\right\}.$$

A simple piece of calculus work shows that $\| \ \|_s$ is an s-homogeneous norm. Hence we have defined by (3) in $L^{*\phi}(a,b)$ an s-homogeneous norm equivalent to $\| \ \|_\phi$. In the case of $s \geqq 1$, $\psi(u^s) = \bar{\psi}(u) = \bar{\chi}(u)$, $\bar{\psi}$ being convex, so that (3) applies to $\chi = \bar{\chi}$ and $s = 1$. In this case $L^{*\phi}(a,b)$ can be equipped with a homogeneous norm and, in fact, all the procedure of defining that B-norm is well-known from papers concerned with convex ϕ-functions.

It seems to be of interest that the above statement may be conversed:

If an s-homogeneous norm can be defined in $L^{*\phi}(a,b)$ then (2) holds. In particular, introducing a homogeneous norm is possible if and only if ϕ is equivalent in the previously established sense to a convex function.

There is another way of defining a norm if it is known that $\phi \overset{l}{\sim} \chi$, where $\chi(u) = \psi(u^s)$, $s > 0$, where ψ is concave. If $0 < s \leqq 1$ then χ is concave, whence subadditive, and it is evident that the formula

(4) $$\|x\| = \mathfrak{I}_\chi(x)$$

defines an F-norm in $L^{*\phi}(a,b)$. This method rests on the use of a definite integral formula which is applied in the classical case of $\chi(u) = u^\alpha$, $0 < \alpha \leqq 1$. Though it is trivial, I will note for the sake of completeness, that (4) defines an s-homogeneous norm only in that case when $\chi(u) = ku^s$.

Let us now state the following question:

When is a representation of the form $\phi \overset{l}{\sim} \chi$, $\chi(u) = \psi(u^s)$ with convex or concave ψ and possibly large or small exponent s, possible?

In order to answer this question we shall define some auxiliary functions

$$\underline{h}_\phi(\lambda) = \varliminf_{u\to\infty} \frac{\phi(u)}{\phi(\lambda u)}, \quad \bar{h}_\phi(\lambda) = \varlimsup_{u\to\infty} \frac{\phi(u)}{\phi(\lambda u)} \qquad \text{for } \lambda > 0.$$

Easy calculations show that the following limits exist (finite or not)

$$s_\phi = \lim_{\lambda\to 0^+} \frac{\lg \underline{h}_\phi(\lambda)}{-\lg \lambda}, \quad \sigma_\phi = \lim_{\lambda\to 0^+} \frac{\lg \bar{h}_\phi(\lambda)}{-\lg \lambda},$$

and that $\phi \overset{L}{\sim} \psi$ implies $s_\phi = s_\psi$, $\sigma_\phi = \sigma_\psi$. These constants are related to a certain class of ϕ-integrable functions which are equivalent to each other. The following theorem holds:

(a) if $s_\phi > 0$ then the representation

$$\phi \overset{L}{\sim} \chi_s \text{ with } \chi_s(u) = \psi_s(u^s), \tag{5}$$

ψ being convex, is possible for $s < s_\phi$ and impossible for $s > s_\phi$. When $s = s_\phi$ then (5) holds (again with convex ψ) if and only if $\phi(u) = u^s \rho(u)$, ρ being *pseudo-increasing for large* u i.e. having the following property:

$$\rho(u_2) \geqq m\,\rho(nu_1), \quad m,n > 0, \text{ for } u_2 \geqq u_1 \geqq u_0. \tag{6}$$

(b) If $\sigma_\phi < \infty$ then the representation (5) with a concave ψ is possible for $s > \sigma_\phi$ and impossible for $s < \sigma_\phi$. When $s = \sigma_\phi$ then (5) holds (again with concave ψ) if and only if $\phi(u) = u^s \rho(u)$, ρ being *pseudo-decreasing for large* u i.e. satisfying (6) with $\geqq$ changed to $\leqq$.

Summarizing the above results we arrive at the following

THEOREM 1. (a) *Introducing a homogeneous norm in $L^{*\phi}(a,b)$ is possible only if $s_\phi > 1$ or $s_\phi = 1$; in the latter case under the additional assumption that $\phi(u) = u\rho(u)$, where ρ is pseudo-increasing for large u.*

(b) *Introducing an s-homogeneous norm* $(0 < s < 1)$ *is possible only if $s_\phi > 0$. If $s_\phi \leqq 1$ then one may introduce a norm of order $< s_\phi$ and of order $s = s_\phi$ only if $\phi(u) = u^s \rho(u)$ with pseudo-increasing ρ.*

We shall conclude this section concerning introducing of norms in $L^{*\phi}(a,b)$ with the following remarks. ϕ is said to satisfy the *condition* (Λ_α) *for large* u if $\alpha > 1$ and if with a certain $c_\alpha > 1$ we have the inequality

$$\phi(\alpha u) \geqq c_\alpha \phi(u) \text{ for } u \geqq u_0(\alpha).$$

If we replace in this inequality $\geqq$ by $\leqq$, we get the *condition* (Δ_α) *for large* u.

It may be shown that the following properties are equivalent:

(α) $s_\phi > 0$,

(β) ϕ satisfies (Λ_α),

(γ) there exists a bounded neighbourhood of 0 in the space $[L^{*\phi}(a,b), \| \|_\phi]$.

On the other hand it is known that in an arbitrary F-normed space an s-homogeneous norm, equivalent to a given one, can be introduced if and only if there exists a bounded neighbourhood of 0, [14]. This theorem for $L^{*\phi}(a,b)$

may be established by a direct proof of the equivalence $(\beta)\leftrightarrows(\gamma)$, whence, owing to the equivalence $(\alpha)\leftrightarrows(\beta)$ and to the previous considerations, we can precisely find the degree s of homogeneity of the norm.

As regards the condition (Δ_α) we shall note that it is always possible to replace it by the equivalent condition (Δ_2), well-known from the theory of $L^{*\phi}(a,b)$ with convex ϕ. Similarly as in the latter case, this condition is of importance also for arbitrary ϕ's, e.g. when studying separability. To get back to questions considered so far, (Δ_2) is equivalent to $\sigma_\phi < \infty$, i.e. to representing ϕ as $\psi(u^s)$ with a concave ψ.

3. Now we shall briefly discuss the question of separability of $L^{*\phi}(a,b)$. It will be convenient, similarly as it is done for convex ϕ, to introduce first the concept of a finite element. Let $M^\phi(a,b)$ stand for the set of those x for which $\mathfrak{J}_\phi(\lambda x) < \infty$ for every $\lambda > 0$; $x \in M^\phi(a,b)$ will be called *finite element*. For instance bounded functions, in the considered case of finite (a,b), are finite elements. The following properties of $M^\phi(a,b)$ are easy to verify:

(α) $M^\phi(a,b)$ is the greatest linear subspace of $L^\phi(a,b)$,

(β) $M^\phi(a,b) = M^\psi(a,b)$ if and only if $\phi \overset{l}{\sim} \psi$,

(γ) $M^\phi(a,b)$ is separable with respect to the norm $\| \ \|_\phi$.

The question of separability of $L^{*\phi}(a,b)$ is satisfactorily settled by the following theorem:

THEOREM 2. *Separability of* $[L^{*\phi}(a,b), \| \ \|_\phi]$ *is equivalent to the following properties*:

(α) ϕ *satisfies* (Δ_2),

(β) $L^\phi(a,b) = L^{*\phi}(a,b)$,

(γ) $M^\phi(a,b) = L^{*\phi}(a,b)$.

4. The last section of my talk will be concerned with linear functions and operations over $L^{*\phi}(a,b)$. From the point of view of the subject under consideration it will be worthwhile to define in $L^{*\phi}(a,b)$, in addition to norm convergence, some other kind of convergence which will be referred to as ϕ-convergence or modular convergence. A sequence x_n of elements from $L^{*\phi}(a,b)$ will be called *ϕ-convergent* (or *modular convergent*) to x_0, in symbols $x_n \xrightarrow{\phi} x_0$, if with a certain $\lambda > 0$, $\mathfrak{J}_\phi(\lambda(x_n - x_0)) \to 0$ as $n \to \infty$. Since, as it has been already remarked, $\| x_n \|_\phi \to 0$ implies $\mathfrak{J}_\phi(x_n) \to 0$, norm-convergence implies ϕ-convergence.

The opposite implication holds, however, only when (Δ_2) holds. Hence distinguishing norm-convergence from ϕ-convergence is necessary only for rapidly increasing ϕ: that case when (Δ_2) fails to hold. In view of the above we distinguish in $L^{*\phi}(a,b)$ two kinds of distributive, continuous operations: linear operations in $[L^{*\phi}(a,b), \| \|_{\phi}]$ and *ϕ-linear* ones, i.e. those continuous in the sense of ϕ-convergence. Of course ϕ-linearity implies linearity with respect to $\| \|_{\phi}$. Some fundamental theorems concerning linear operations over normed spaces may be transferred without change to ϕ-linear operations. I should mention here e.g. the following theorem on sequences of operations: Let U_n be ϕ-linear operations over $L^{*\phi}(a,b)$ with values in an F-normed space. If

(7) $$U_n(x) \to U(x) \text{ for } x \in L^{*\phi}(a,b),$$

then U is ϕ-linear and U_n are equi-ϕ-continuous, in other words, $U_n(x_n) \to 0$ as $x_n \xrightarrow{\phi} 0$.

Nevertheless it should be noted that under the weaker assumption of boundedness of $U_n(x)$ in the whole space $L^{*\phi}(a,b)$ in place of (7), the equi-ϕ-continuity of U_n does not hold in general.

We now turn to the question of existence of non-trivial ϕ-linear functionals and their representation. According to the well-known theorem of M. Day [1], there exist over $L^{\alpha}(a,b)$, when $0 < \alpha < 1$, only trivial linear functionals. In this particular case ϕ-continuity must not be distinguished from norm-convergence, as $\phi(u) = u^{\alpha}$ satisfies (Δ_2). The general theorem states the following: In order that there exist non-trivial ϕ-linear functionals over $L^{*\phi}(a,b)$ it is necessary and sufficient that

(8) $$\underline{\lim}_{u \to \infty} \frac{\phi(u)}{u} > 0.$$

Now, let us replace (8) by the somewhat stronger

(9) $$\lim_{u \to \infty} \frac{\phi(u)}{u} = \infty.$$

This condition is not a so much imposing one, since it is, in particular, fulfilled by every convex ϕ, provided that the latter is not l-equivalent to the trivial ϕ-function $\phi(u) = u$. In order to obtain, under (9), a theorem on integral representability of ϕ-linear functionals, we replace ϕ by an equivalent function for which $\phi(u)/u \to 0$ when $u \to 0$ and we define the so-called *complementary function* ψ to ϕ

(10) $$\psi(v) = \sup_{u \geqq 0} (uv - \phi(u)) \quad \text{for } v \geqq 0.$$

It may be easily verified that ψ is a convex ϕ-function such that $\psi(v)/v \to \infty$ as $v \to \infty$. For $u,v \geqq 0$ the generalized Young inequality holds

$$uv \leqq \phi(u) + \psi(v) \tag{11}$$

(we get the Young inequality on supposing that ϕ is convex).

For every $v \geqq 0$ there exist u_v such that

$$u_v v = \phi(u_v) + \psi(v),$$

though, unlike the case of convex ϕ, the sign of equality in (11) must not occur for every u with a certain v_u. Nevertheless, arguments being usually applied, basing on Young inequality, for the sake of representing functionals, in the case of convex ϕ, may be extended also to the case of ϕ subject to (9) and the generalized Young inequality.

The following theorem may be deduced:

THEOREM 3. *The general form of a ϕ-linear functional over the space $L^{*\phi}(a,b)$ is as follows:*

$$\xi(x) = \int_a^b x(t)\,y(t)\,dt,$$

where $y(t)$ is a function from the space $L^{*\psi}(a,b)$, ψ being complementary to ϕ.

To conclude I will call your attention to the possibility of including the theory of the spaces $L^{*\phi}(a,b)$ into the more general theory of modular spaces. The starting point of this theory are semi-ordered linear spaces in which a functional is defined, called modular. H. Nakano [9] and his school developed the theory of modular spaces, embracing spaces $L^{*\phi}(a,b)$ with a convex ϕ as special cases. The axiomatic of modular space in the sense of Nakano may be generalized to include also spaces $L^{*\phi}(a,b)$ for arbitrary ϕ-functions.

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