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MATHEMATICS
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Representation Theorem for Semi-Boolean Algebras. I

by

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Summary. In the presented paper a class of lattices — which are here called semi-Boolean algebras — is introduced and investigated. An abstract algebra $(A, \cup, \cap, \Rightarrow, \dot{-}, \urcorner, \ulcorner)$ will be called semi-Boolean algebra provided that $(A, \cup, \cap, \Rightarrow, \urcorner)$ is a pseudo-Boolean algebra and $(A, \cup, \cap, \dot{-}, \ulcorner)$ is a Brouwerian algebra. The main result of this note is a representation theorem for semi-Boolean algebras and an example of these algebras. Bi-topological Boolean algebras play an important role in the general theory of semi-Boolean algebras. For these algebras a representation theorem is formulated and proved.

In this paper a class of lattices-which are here called semi-Boolean algebras — is introduced and investigated. Semi-Boolean algebras can be applied to algebraic treatment of intuitionistic logic with two additional connectives $\dot{-}, \ulcorner$, which are dual to intuitionistic implication and intuitionistic negation, respectively. These algebras play the analogous part for the just mentioned logic to that played by Boolean algebras for classical logic. The logic mentioned above will be considered in a separate paper. The main purpose of this note is to give certain representation theorems for semi-Boolean algebras.

We will say that an abstract algebra $\mathfrak{A}=(A, \cup, \cap, \Rightarrow, \dot{-})$ is a semi-Boolean algebra provided that

i) $(A, \cup, \cap, \Rightarrow)$ is a relatively pseudo-complemented lattice

ii) $\dot{-}$ is a binary operation which satisfies the following condition: $a \dot{-} b \leqslant x$ if and only if $a \leqslant b \cup x$ for any $a, b, x \in A$.

The operation $\dot{-}$ will be called the pseudo-difference.

Every semi-Boolean algebra $\mathfrak{A}=(A, \cup, \cap, \Rightarrow, \dot{-})$ has the zero element $\wedge$ and the unit element $\vee$. Every element $a \in A$ has $\cap$—complement and $\cup$—complement [1], namely

$$\urcorner a = a \Rightarrow \wedge$$

is $\cap$—complement of an element a in A, and

$$\ulcorner a = \vee \dot{-} a$$

is $\cup$—complement of an element a in A. Hence the above definition of semi-Boolean algebra is equivalent to the following one: An abstract algebra $(A, \cup, \cap,$

$\Rightarrow, \dot{-}, \urcorner, \ulcorner$) will be called a semi-Boolean algebra provided that $(A, \cup, \cap, \Rightarrow, \urcorner)$ is a pseudo-Boolean algebra [1] and $(A, \cup, \cap, \dot{-}, \ulcorner)$ is a Brouwerian algebra [2]. We will say that an abstract algebra $\mathfrak{B}=(B, \cup, \cap, \rightarrow, -, I, C)$ is a bi-topological Boolean algebra if $(B, \cup, \cap, \rightarrow, -)$ is a Boolean algebra, I and C are interior and closure operations respectively, such that the following condition is satisfied

$$(*)\qquad \begin{aligned} Ia &= CIa \\ Ca &= ICa \end{aligned}$$

for every $a \in B$.
The operations I and C will be called conjugate operations over B when they satisfy (*). An element $a \in B$ is said to be I—open (C—closed) when $a=Ia$ ($a=Ca$). For any bi-topological Boolean algebra $\mathfrak{B}$ we will denote by $G_I(B)$ the set of all I—open elements in B. On account of (*) the elements of $G_I(B)$ are simultaneously I—open and C—closed. It is easy to verify that the following statement is true

THEOREM 1. *The algebra $(G_I(B), \cup, \cap, \Rightarrow, \dot{-})$ where $G_I(B)$ is the set of all I—open elements in a bi-topological Boolean algebra $\mathfrak{B}=(B, \cup, \cap, \rightarrow, -, I C)$, is a semi-Boolean algebra. For all $a, b \in G_I(B)$ we have:*

$$(1)\qquad a \Rightarrow b = I(-a \cup b)$$

$$(2)\qquad a \dot{-} b = C(a \cap -b).$$

The following theorem explains the connection between semi-Boolean algebras and bi-topological Boolean algebras.

THEOREM 2. *For every semi-Boolean algebra $\mathfrak{A}=(A, \cup, \cap, \Rightarrow, \dot{-})$ there exists a bi-topological Boolean algebra $\mathfrak{B}=(B, \cup, \cap, \rightarrow, -, I, C)$ such that $A=G_I(B)$.*

By a topological space we will understand a system $\langle X, I\rangle$ ($\langle X, C\rangle$) where the set X is non-empty and I is an interior operation (C is a closure operation).

If the systems $\langle X, I\rangle$ and $\langle X, C\rangle$ are any topological spaces then the system $\langle X, I, C\rangle$ will be called a bi-topological space. Let $\mathcal{B}_0(X)$ denote the class of all subsets of the space X. If for every $Y \in \mathcal{B}(X) \subset \mathcal{B}_0(X)$

$$(**)\qquad \begin{aligned} IY &= CIY \\ CY &= ICY \end{aligned}$$

then we will say that the operations I and C are conjugate over $\mathcal{B}(X)$.

If $\langle X, I, C\rangle$ is a bi-topological space, $\mathcal{B}(X)$ is a field of subset of X and the operations I and C are conjugate over $\mathcal{B}(X)$, then the algebra $\mathfrak{P}=(\mathcal{B}(X), \cup, \cap, -$ $I, C)$ as well as every its subalgebra will be called a bi-topological field of sets (more exactly: a bi-topological field of subsets of X).

THEOREM 3. *For every bi-topological Boolean algebra $\mathfrak{B}$ there exists a bi-topological field of sets $\mathfrak{P}$ and an isomorphism of $\mathfrak{B}$ onto $\mathfrak{P}$.*

Let $\mathfrak{B}=(B, \cup, \cap, \rightarrow, -, I^*, C^*)$ be a bi-topological Boolean algebra. Let us denote by X the set of all prime filters ∇ of a Boolean algebra $\mathfrak{B}_0=(B, \cup, \cap, -)$

and for every $a \in B$ let $h(a)$ denote the set of all $\nabla \in X$ such that $a \in \nabla$. It follows from [3] that the Stone space $\langle X, \mathcal{I} \rangle$ where the interior operation $\mathcal{I}$ is determined by the class $\{h(a)\}_{a \in B}$ assumed as a subbasis, is a compact totally disconnected Hausdorff space. Moreover, the class $\{h(a)\}_{a \in B}$ is the field of all both open and closed subsets of the topological space $\langle X, \mathcal{I} \rangle$ and h is an isomorphism of $\mathfrak{B}_0$ onto $\mathcal{B}(X)$, where $\mathcal{B}(X) = \{h(a)\}_{a \in B}$.

Now, a new interior operation and a new closure operation in X will be defined in the following way:

$$IY = \bigcup_{\substack{h(a) \subset Y \\ a = I^* a}} h(a) \tag{3}$$

$$CY = \bigcup_{\substack{Y \subset h(b) \\ b = C^* b}} h(b) \tag{4}$$

for every $Y \subset X$.

It will be shown that the operations defined above are conjugate over $\mathcal{B}(X)$ i.e. the condition (**) is satisfied. Let $Y \in \mathcal{B}(X)$ i.e. $Y = h(x)$ for some $x \in B$. Thus from (3) it follows that

$$IY = Ih(x) = \bigcup_{\substack{h(a) \subset h(x) \\ a = I^* a}} h(a).$$

Since h is an isomorphism of the Boolean algebra $\mathfrak{B}_0$ onto $\mathcal{B}(X)$, the condition $h(a) \subset h(x)$ is equivalent to $a \leqslant x$, for $a, x \in B$. Since we have $a = I^* a$, the last inequality is equivalent to the following one: $a \leqslant I^* x$ i.e. to $h(a) \subset h(I^* x)$. Thus

$$Ih(x) = \bigcup_{\substack{h(a) \subset h(x) \\ a = I^* a}} h(a) = \bigcup_{\substack{h(a) \subset h(I^* x) \\ a = I^* a}} h(a) = h(I^* x).$$

In the same way it could be shown, that

$$Ch(x) = h(C^* x).$$

Since the operations I^* and C^* are conjugate over B it is true that

$$IY = Ih(x) = h(I^* x) = h(C^* I^* x) = CIh(x) = CIY$$

and

$$CY = Ch(x) = h(C^* x) = h(I^* C^* x) = ICh(x) = ICY.$$

This proves that the condition (**) is satisfied. The algebra $\mathfrak{P} = (\mathcal{B}(X), \cup, \cap, -, I, C)$ is the required bi-topological field of sets and h is an isomorphism of the bi-topological Boolean algebra $\mathfrak{B}$ onto $\mathfrak{P}$.

Let $\langle X, \mathcal{I} \rangle$ be an arbitrary compact topological space i.e. for every indexed set $\{A_t\}_{t \in T}$ of open subsets, the equation $X = \bigcup_{t \in T} A_t$ implies the existence of a finite set $T_0 \subset T$ such that $X = \bigcup_{t \in T_0} A_t$. Let $\mathcal{B}(X)$ be a field of all both open and closed subset of the topological space $\langle X, \mathcal{I} \rangle$. Let $\mathcal{R}$ be an arbitrary ring of sets such that the field $\mathcal{B}(X)$ is generated by $\mathcal{R}$ and the following conditions are satisfied:

i) $\emptyset \in \mathcal{R}$

ii) $X \in \mathcal{R}$

iii) if $Z \in \mathcal{B}(X)$, then $\bigcup_{\substack{A \in \mathcal{R} \\ A \subset Z}} A \in \mathcal{B}(X)$ and $\bigcap_{\substack{B \in \mathcal{R} \\ Z \subset B}} B \in \mathcal{B}(X)$.

Let I be an interior operation in X defined as follows

(5) $$IY = \bigcup_{\substack{A \in \mathcal{R} \\ A \subset Y}} A$$

for every $Y \subset X$.
The system $\langle X, I \rangle$ is a topological space. Let C be a closure operation in the set X defined as follows

(6) $$CY = \bigcap_{\substack{B \in \mathcal{R} \\ Y \subset B}} B.$$

The system $\langle X, C \rangle$ is a topological space. Thus the system $\langle X, I, C \rangle$ is a bi-topological space. We observe that if $A \in \mathcal{R}$ then $IA = A$ and $CA = A$ i.e. the elements of the ring $\mathcal{R}$ are both I-open and C-closed.

THEOREM 4. *If* $A \in \mathcal{B}(X)$ *then* $IA \in \mathcal{R}$ *and* $CA \in \mathcal{R}$.

In fact, if $A \in \mathcal{B}(X)$ then by iii) and the definition of the interior operation I it is true that IA is simultaneously an open and a closed subset of a compact space $\langle X, \mathcal{T} \rangle$. Hence, IA is a finite union of the elements of the ring $\mathcal{R}$ i.e. $IA \in \mathcal{R}$. In the same way it could be shown that $CA \in \mathcal{R}$. Thus the following statement holds:

THEOREM 5. *The field* $\mathcal{B}(X)$ *is a bi-topological field of sets.*

From Theorems 3 and 2 we obtain

THEOREM 6. *The algebra* $\mathfrak{R} = (\mathcal{R}, \cup, \cap, \Rightarrow, \dot{-})$ — *where* $\mathcal{R}$ *is the ring defined above, the operations* $\cup$, $\cap$ *are set-theoretical union and intersection respectively, and operations* $\Rightarrow$, $\dot{-}$ *are defined as follows*:

(7) $$Y \Rightarrow Z = I((X - Y) \cup Z)$$

(8) $$Y \dot{-} Z = C(Y \cap (X - Z))$$

for every $Y, Z \in \mathcal{R}$ — *is a semi-Boolean algebra.*

Every semi-Boolean algebra of this kind is said to be $(X, \mathcal{R})$-topological semi-Boolean algebra.

THEOREM 7. *For every semi-Boolean algebra* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ *there exists* $(X, \mathcal{R})$-*topological semi-Boolean algebra* $\mathfrak{R} = (\mathcal{R}, \cup, \cap, \Rightarrow, \dot{-})$ *and an isomorphism* h *of* $\mathfrak{A}$ *onto* $\mathfrak{R}$.

By Theorem 2 we can assume that $A = G_{I*}(B)$ where B is the set of all elements of a bi-topological Boolean algebra $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, I^*, C^*)$

Let $\langle X, \mathcal{T} \rangle$ be the Stone space of the Boolean algebra $\mathfrak{B}_0 = (B, \cup, \cap, -)$. Let I and C be the interior and closure operations respectively, in the set X which are defined by (3) and (4). It follows from Theorem 3 that these operations are conjugate over the Stone field $\mathcal{B}(X) = \{h(a)\}_{a \in B}$. Let $\mathcal{R}$ be the class of all $h(a)$ such that $a \in \nabla$ for $a \in G_{I*}(B)$. Suppose that $Y \in \mathcal{R}$, then $Y = h(x)$ for some $x = I^* x = C^* x \in G_{I*}(B)$. Thus $IY = Y$ and $CY = Y$ i.e. the elements of $\mathcal{R}$ are simultaneously I-open and C-closed. It is easy to see that if $Y \in \mathcal{B}(X)$ then $IY \in \mathcal{R}$ and $CY \in \mathcal{R}$.

From the Theorem 6 it follows that the algebra $\mathfrak{R}=(\mathcal{R}, \cup, \cap, \Rightarrow, \dot{-})$, where $\cup, \cap$ are set-theoretical union and intersection respectively, and $\Rightarrow, \dot{-}$ are defined by (7) and (8), is a $(X, \mathcal{R})$-topological semi-Boolean algebra.

We will now prove that the mapping h is the required isomorphism of $\mathfrak{A}$ onto $\mathfrak{R}$. Obviously the mapping h is one-to-one and

$$h(a \cup b)=h(a) \cup h(b)$$
$$h(a \cap b)=h(a) \cap h(b)$$

for $a, b \in A$.
Let us prove that

$$h(a \Rightarrow b)=h(a) \Rightarrow h(b)$$
$$h(a \dot{-} b)=h(a) \dot{-} h(b)$$

for $a, b \in A$.
By the definition of the operation $\Rightarrow$ in $\mathcal{R}$ we have $h(a) \Rightarrow h(b)=I\big((X-h(a)) \cup h(b)\big)$. On the other hand, $h(a \Rightarrow b)=h\big(I^*(-a \cup b)\big)$, where the signs $-, \cup, I^*$ denote the complement, the join and the interior operation in the bi-topological Boolean algebra $\mathfrak{B}$. Thus

$$h(a \Rightarrow b)=h\big(I^*(-a \cup b)\big)=Ih(-a \cup b)=I\big((X-h(a)) \cup h(b)\big)=h(a) \Rightarrow h(b).$$

The proof of the equation $h(a \dot{-} b)=h(a) \dot{-} h(b)$ is similar. This completes the proof of Theorem 7.

To illustrate the notation of $(X, \mathcal{R})$-topological semi-Boolean algebra, let us consider the case in which X is the Cantor discontinuum [4] i.e. X is the Cartesian product U^E, where U is the set consisting of the integers 0 and 1 only, and E is a non-empty set. By definition, X is the set of all mapping $u=\{u_\alpha\}_{\alpha \in E}$ such that $u_\alpha=0$ or $u_\alpha=1$, $a \in E$. Let A^α $(a \in E)$ be the set of all $u \in X$ such that $u_\alpha=1$. Denote by $\mathcal{D}$ the class of all sets A^α and their complements. Let $\mathcal{B}(X)$ be the field of subsets of X generated by $\mathcal{D}$. It is known that $\mathcal{B}(X)$ is the field of all both open and closed subsets of the topological space $\langle X, \mathcal{I}\rangle$, where $\mathcal{I}$ is the interior operation in X determined by the class $\mathcal{D}$ assumed as a subbasis. Now, let $\mathcal{R}$ be the ring of the sets which belong to the class $\{A^\alpha\}_{\alpha \in E}$ such that

i′) $\emptyset \in \mathcal{R}$

ii′) $X \in \mathcal{R}$.

It is easy to see that the field $\mathcal{B}(X)$ is generated by the ring $\mathcal{R}$ i.e. if $Y \in B(X)$ then

$$Y=\bigcap_{i=1}^{k}(A^{\alpha_{i1}} \cup \ldots \cup A^{\alpha_{in}} \cup B^{\beta_{i1}} \cup \ldots \cup B^{\beta_{im}}), \tag{9}$$

where for every i, j: $A^{\alpha_{ij}} \in R$, $B^{\beta_{ij}}$ is the complement of some $A^\alpha \in \mathcal{R}$ $((X-B^{\beta_{ij}}) \in \mathcal{B})$ and $a_{ij} \neq \beta_{ij}$.

Let I be the interior operation defined by (5), and C be a closure operation defined by (6). It will be shown that if $Y \in \mathcal{B}(X)$ i.e. if Y is of the form (9), then

$$IY=\bigcap_{i=1}^{k}(A^{\alpha_{i1}} \cup \ldots \cup A^{\alpha_{in}}) \tag{10}$$

$$CY=\bigcup_{i=1}^{k}\big((X-B^{\beta_{i1}}) \cap \ldots \cap (X-B^{\beta_{im}})\big). \tag{11}$$

We prove the condition (10). The proof of (11) is similar. Obviously, if $Y=\emptyset$ or $Y=X$ the condition (10) is satisfied. Let $Y\neq\emptyset$ and $Y\neq X$. On account of the definition of the interior operation I it is sufficient to show that if Y is of the form (9) then the following equation is fulfilled:

$$(12)\qquad \bigcup_{\substack{A\in\mathcal{R}\\ A\subset Y}} = \bigcap_{i=1}^{k} (A^{\alpha_{i1}}\cup\ldots\cup A^{\alpha_{im}}).$$

It is easy to see that

$$\bigcap_{i=1}^{k} (A^{\alpha_{i1}}\cup\ldots\cup A^{\alpha_{in}})\subset Y$$

and

$$\bigcap_{i=1}^{k} (A^{\alpha_{i1}}\cup\ldots\cup A^{\alpha_{im}})\in\mathcal{R}.$$

Thus, it is sufficient to shown that

$$(13)\qquad \text{if } Z\in\mathcal{R} \text{ and } Z\subset Y \text{ then } Z\subset \bigcap_{i=1}^{k} (A^{\alpha_{i1}}\cup\ldots\cup A^{\alpha_{in}}).$$

Let us suppose that $Z\in\mathcal{R}$ and $Z\subset Y$. Hence $Z=\bigcup_{p=1}^{l} (A^{\gamma_{p1}}\cap\ldots\cap A^{\gamma_{ps}})$. Obviously, for every i and p we have the inclusion

$$(14)\qquad A^{\gamma_{p1}}\cap\ldots\cap A^{\gamma_{ps}}\subset A^{\alpha_{i1}}\cup\ldots\cup A^{\alpha_{in}}\cup B^{\beta_{i1}}\cup\ldots\cup B^{\beta_{im}}.$$

We observe, that for every i and p there exists an integer $j (1\leqslant j\leqslant s)$ such that $\gamma_{pj}\in\in\{a_{i1}\ldots a_{in}\}$ i.e. there exist j $(1\leqslant j\leqslant s)$ and t $(1\leqslant t\leqslant n)$ such that $A^{\kappa_{pj}}=A^{\alpha_{it}}$. Suppose the contrary i.e. for all j $\gamma_{pj}\notin\{a_{i1}\ldots a_{in}\}$. Let $u=\{u_\alpha\}_{\alpha\in E}$ be the mapping such that for fixed i and p

$$u_{\gamma_{p1}}=\ldots=u_{\gamma_{ps}}=u_{\beta_{i1}}=\ldots=u_{\beta_{im}}=1$$

and

$$u_{\alpha_{i1}}=\ldots=u_{\alpha_{in}}=0.$$

Thus u belongs to $A^{\gamma_{i1}}\cap\ldots\cap A^{\gamma_{is}}$ but $u\notin A^{\alpha_{i1}}\cup\ldots\cup A^{\alpha_{in}}\cup B^{\beta_{i1}}\cup\ldots\cup B^{\beta_{im}}$. This is impossible on account of (14).

Hence for every i and p

$$A^{\gamma_{p1}}\cap\ldots\cap A^{\gamma_{ps}}\subset A^{\gamma_{pj}}=A^{\alpha_{it}}\subset A^{\alpha_{i1}}\cup\ldots\cup A^{\alpha_{in}}.$$

Consequently

$$Z=\bigcup_{p=1}^{l} (A^{\gamma_{p1}}\cap\ldots\cap A^{\gamma_{ps}})\subset\bigcap_{i=1}^{k} (A^{\alpha_{z1}}\cup\ldots\cup A^{\alpha_{in}}).$$

Thus the condition (13) is fulfilled.

It follows immediately from i'), ii'), (10) and (11) that $\mathcal{R}$ is a ring satisfying the conditions i)—iii). By Theorem 6 we infer that $(\mathcal{R},\cup,\cap,\Rightarrow,\dot{-})$ — where $\mathcal{R}$ is the ring defined above, the operations $\cup$, $\cap$ are set-theoretical union and intersection respectively, and operations $\Rightarrow$, $\dot{-}$ are defined by (7) and (8) — is an $(X,\mathcal{R})$-topological semi-Boolean algebra.

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[4] R. Sikorski, *Boolean algebras*, Berlin, Heidelberg New, York, 1969.

Ц. Рацшэр, **Теорема о представлении для полу-Ьулевых алгебр. I часть**

Содержание. В настоящей работе рассматривается некоторый тип абстрактных алгебр, называемых полу-Булевыми алгебрами. Абстрактная алгебра $(A, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$ является полу-Булевой алгеброй, если $(A, \cup, \cap, \Rightarrow, \neg)$ псевдо-Булева алгебра, тогда как алгебра $(A, \cup, \cap, \dot{-}, \ulcorner)$ это алгебра Брауэра. Главной целью работы является построение нетривиального примера таких алгебр, а также доказательства теоремы о представлении. Важную роль в теории полу-Булевых алгебр играют так называемые би-топологические алгебры Буля, а в особенности теорема о представлении для этих алгебр, формулировка и доказательство которой дается в начале работы.