The theory of W*-algebras

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Preface

There are essentially three different ways of studying the operator *-algebras on hilbert spaces. The first alternative is to assume that the algebra is weakly closed (called the W^*-algebra). These algebras are also called Rings of operators and, more recently, von Neumann algebras.

The earliest attack to the study along such lines is due to von Neumann in 1929. In a series of five memoirs beginning with [16], Murray and von Neumann made important strides to the theory of W^*-algebras. Call a W^*-algebra a factor if its center is just the complex numbers. To a large extent the study of W^*-algebras may be reduced to the case of a factor by a reduction theory devised by von Neumann [26]. At the same time a number of authors, notably Dixmier, have pushed through the major portions of the theory for general W^*-algebras.

The second alternative is to assume that the algebra is uniformly closed (called the C^*-algebra or the B^*-algebra).

The earliest attack to the study along such lines is due to Gelfand and Naimark in 1943.

A notable advantage of the C^*-algebra is the existence of an elegant system of intrinsic postulates defined by Gelfand and Naimark; so one can and does study the C^*-algebra in abstract fashion that pays no attention to any particular representation.

Because of this reason, the theory of C^*-algebras has naturally been placed into the theory of general Banach algebras, and in certain respects they are among the best behaved examples of infinite dimensional Banach algebras. These situations have been extremely different from the case of W^*-algebras.

The theory of W^*-algebras has been always developed in association with underlying hilbert spaces, because we could not have a thorough non-spatial characterization of W^*-algebras like one of C^*-algebras.

However, nowadays, we can push the non-spatial theory of W^*-algebras, since we have much information concerning the removal of this pathology.

Early attempts along these lines are due to von Neumann [22] and Steen [34]. Rickart [29] made a start on such treatment which was picked up by Kaplansky who carried the study more or less to its completion in his series of papers on AW^*-algebras.

This AW^*-algebra is the third alternative. Although much of the non-spatial theory of W^*-algebras can be extended to AW^*-algebras, additionals on an AW^*-algebra are needed for it to be representable as a W^*-algebra. That this is already the case for commutative algebras is proved by Dixmier [2] who gave a characterization of commutative W^*-algebras among algebras $C(\Omega)$, Ω a compact Hausdorff space. Finally, characterizations of general W^*-algebras have been obtained by Kadison [9] and the present author [32]. The characterization of Kadison is a non-commutative extension of Dixmier's one; which uses the order properties and normal positive functionals.

On the other hand, Dixmier [3] showed that a W^*-algebra is the dual space of some Banach space. Then the author has shown that a necessary and sufficient condition for a B^*-algebra to admit a faithful *-representation as a W^*-algebra is that it be a dual space as a Banach space.

Using this characterization, in this note, we shall mainly develop the non-spatial theory of W^*-algebras and place the theory of W^*-algebras properly into the general theory of Banach algebras.

On those portions of the theory of W^*-algebras concerned with the representation of the algebras on hilbert spaces, there is a comprehensive book by Dixmier [4], in which there is an extensive literature.

The purely algebraic treatment of those portions which are positive even in AW^*-algebras is given by Kaplansky [14] in his Chicago notes on Rings of operators, and a substantial number of additional papers on AW^*-algebras and related matters are found in the Bibliography. Therefore, we have no intention of giving a complete coverage of the subject. Also, it is, indeed, impossible, as there are many topics concerning W^*-algebras. In this note, a Banach space-like point of view dominates strongly the selection and organization of material. The reader is referred for further information on the subject to the book of Dixmier and the note of Kaplansky.

Moreover we shall suppose some results concerning B^*-algebras, the reader is referred for their information to the books of Rickart [31] and Naimark [19]. Also, we shall deal with locally convex topological linear spaces. For this, we shall refer to the books of Bourbaki.

The main body of this note is divided into three chapters: I Banach space-like considerations. II Algebraic considerations, III The theory of representation. In addition to the main text, there is an Appendix devoted primarily to questions.

Contents

Chapter I Banach space-like considerations

§1. Definition of W^*-algebras.

We suppose that the reader has some knowledge concerning B^*-algebras. Our subjects for the research are a special class of B^*-algebras, called W^*-algebras.

Definition 1.1. A B^*-algebra is called a W^*-algebra if it is a dual space as a Banach space.

Let $\{M_\alpha \mid \alpha \in J\}$ be a family of W^*-algebras, we define the direct sum $\sum_{\alpha \in J} \oplus M_\alpha$ as follows: elements of $\sum_{\alpha \in J} \oplus M_\alpha$ are composed of all family $(a_\alpha)_{\alpha \in J}$ such that $a_\alpha \in M_\alpha$ and $\sup_\alpha \| a_\alpha \| < +\infty$ and define : $(a_\alpha) + (b_\alpha) = (a_\alpha + b_\alpha)$, $(a_\alpha)(b_\alpha) = (a_\alpha b_\alpha)$, $(a_\alpha)^* = (a_\alpha^*)$ and $\|(a_\alpha)\| = \sup_\alpha \| a_\alpha \|$. Then it is also a B^*-algebra and a dual space; hence it is also a W^*-algebra.

Let M be a W^*-algebra, then there is a Banach space F such that M is the dual of F. According to the general theory of Banach spaces, it is not necessarily assured that such F is unique -- in fact, $\ell^1 \times \ell^1$ is isometrically isomorphic to ℓ^1, but $c_0 \times c_0$ is not so to c_0.

However, afterward we shall show that such F is unique.

Since M is a dual space, its unit sphere has sufficiently many extreme points by the theorem of Krein-Milman. Therefore, firstly we shall study the properties of extreme points in the unit sphere of B^*-algebras.

§2. Extreme points in the unit spheres of B^*-algebras.

Let B be a B^*-algebra, S its unit sphere and x be an extreme point of S. Let A be the commutative B^*-subalgebra generated by x^*x and $C_0(\Omega)$ the function-representation of A, where Ω is a locally compact space. Then one can easily take a sequence $\{y_n\}$ of positive elements of $C_0(\Omega)$ such that $\|y_n\| \leq 1$ for all n, $\|(x^*x)y_n - (x^*x)\| \to 0$ $(n \to \infty)$ and $\|(x^*x)y_n^2 - (x^*x)\| \to 0$ $(n \to \infty)$.

Suppose that at some point p of Ω, x^*x takes a non-zero value less than one. Then we can take a positive element c of A, non-zero at p such that if $r_n = y_n + c$, $s_n = y_n - c$, then $\|(x^*x)r_n^2\| \leq 1$ and $\|(x^*x)s_n^2\| \leq 1$. Hence xr_n and xs_n are in S. On the other hand, $\|(xy_n - x)^*(xy_n - x)\| =$ $\|x^*xy_n^2 - x^*xy_n - x^*xy_n + x^*x\| \to 0$ $(n \to \infty)$: hence $xy_n \to x$, so that $xr_n \to x + xc$ and $xs_n \to x - xc$. Since $x + xc$, $x - xc$ belong to S and $x = \frac{(x+xc) + (x-xc)}{2}$, $x = x+xc =$ $x-xc$; hence $xc = 0$ and so $\|cx^*xc\| = \|x^*xc^2\| = 0$, this contradicts that $x^*x(p)c^2(p) \neq 0$. Therefore x^*x has not any non-zero value less than one on Ω.

In other words, x^*x is a projection, we shall call such x a partially isometry, x^*x the initial projection of x and xx^* the final projection of x (since $(xx^*)(xx^*) = x(x^*x)x^* =$ xx^*, xx^* is also a projection). Put $x^*x + xx^* = h$, if h is not invertible, there is a sequence $\{z_n\}$ of positive elements commuting with h as follows: $\|z_n^2\| = 1$, $\|hz_n^2\| \to 0$ $(n \to \infty)$, so that $\|xz_n\| = \|z_nx^*\| = \|z_nx^*xz_n\|^{1/2} \leq \|z_nhz_n\|^{1/2} \to 0$ $(n \to \infty)$ and analogously $\|z_nx\| = \|x^*z_n\| \to 0$ $(n \to \infty)$.

So,

$$\|z_n - xx^*z_n - z_nx^*x + xx^*z_nx^*x\| \to 1 \quad (n \to \infty).$$

Now we shall use a notation $y(1-x) = y-yx$, $(1-x)y = y-xy$, then we show that $(1-x^*x)B(1-x^*x) = (0)$.

Suppose that a is an element such that $\|a\| \leq 1$, $a \in (1-xx^*)B(1-x^*x)$. Then,

$$\|x \pm a\| = \|(x^* \pm a^*)(x \pm a)\|^{1/2} = \\ = \|x^*x \pm (x^*a + a^*x) + a^*a\|^{1/2} .$$

Since $a^*xx^*a = a^*(xx^* - xx^*)a = 0$, $x^*a = a^*x = 0$ and moreover $x^*xa^*a = x^*x(1 - x^*x)a = (x^*x - x^*x)a = 0$; hence $\|x \pm a\| = \max(\|x^*x\|^{1/2}, \|a^*a\|^{1/2}) \leq 1$, so that by the extremity of x, $a = 0$.

On the other hand, since $z_n - xx^*z_n - z_nx^*x + xx^*z_nx^*x$ belong to $(1 - xx^*)B(1 - x^*x)$, $z_n - xx^*z_n - z_nx^*x + xx^*z_nx^*x = 0$; hence we have a contradiction, so that $x^*x + xx^*$ is invertible in B. Therefore B has a unit I.

Next, we shall show that unit I is an extreme point. In fact, if $I = \frac{1}{2}(a+b)$ $(a,b \in S)$, put $c = \frac{1}{2}(a^*+a)$, $d = \frac{1}{2}(b^*+b)$, then $I = \frac{1}{2}(c+d)$ $(c,d \in S)$.

Since $d = 2I - c$, d and c commute. Representing the B^*-algebra generated by I, c, d we can easily conclude that $d = c = I$; hence $a^* = 2I-a$, so that a is normal, and, passing to the function space, this time shows us that $a = a^* = I$; hence I is an extreme point.

Therefore we obtain

Theorem 2.1. The unit sphere of a B^*-algebra has an extreme point if and only if it has the unit.

Next we shall show a characterization of extreme points in the unit sphere.

We have already shown that if x is an extreme point, $(I - x^*x)B(I - x^*x) = (0)$. Now conversely we shall show that if $\|x\| \leq 1$ and $(I-xx^*)B(1-x^*x) = (0)$, then x is an extreme point.

Lemma 2.1. Let P be the set of all positive elements of B, then extreme points of $P \cap S$ are all projections of B.

Proof. Let e be a projection, and put $e = \frac{1}{2}(a+b)$ $(a,b \in P \cap S)$, then $a = 2e-b$, and so a and b commute; hence we obtain easily $a = b = e$. Conversely suppose h is an extreme point in $P \cap S$ and $C(\Omega)$ be the function space generated by h, then we can easily conclude that h is a projection.

Now let x be an element such that $(1-xx^*)B(1-x^*x) = (0)$ and $\|x\| \leq 1$, then $x^*(1-xx^*)x(1-x^*x) = (0)$; hence $x^*x = e$ and $xx^* = f$ are projections.

Suppose that $x = \frac{1}{2}(a+b)$ with a, b in S. Then

$$e = x^*x = \frac{1}{2}(x^*a + x^*b) \quad \text{and} \quad e = \frac{1}{2}(x^*ae + x^*be).$$

Since e is the unit of the B^*-algebra eBe, and x^*ae and x^*bc belong to eSe, $e = x^*ae = x^*be$; hence $x = xx^*x = xe = xx^*ae = fae = fbe$. On the other hand, $ae = fae + (1-f)ae$, so that

$$1 \geq \|ea^*ae\| = \|(fae + (1-f)ae)^*(fae + (1-f)ae)\|$$
$$= \|ea^*fae + ea^*(1-f)ae\| = \|e + ea^*(1-f)ae\|;$$

hence $(1-f)ae = 0$ and so $ae = fae$, and analogously $be = fbe$; hence $x = ae = be$.

Now $x^* = \frac{1}{2}(a^*+b^*)$ and, by symmetry, $a^*f = b^*f = x^*$ or $fa = fb = x$. Our hypothesis, $(1-e)B(1-f) = (0)$, tells us that $a = fa(1-e) + ae = fb(1-e) + be = b$.

Hence we obtain

Theorem 2.2. An element in the unit sphere of a B^*-algebra B is extreme if and only if it satisfies $(1-xx^*)B(1-x^*x) = (0)$.

It is very interesting to extend Theorem 2.1 to general Banach algebras. Kakutani has shown the following

Theorem 2.3. Let $\mathfrak{A}$ be a Banach algebra with unit 1, then 1 is an extreme point in its unit sphere.

Proof. Since $\mathfrak{A}$ is isometrically representable as a subalgebra of the algebra $B(E)$ of all bounded operators on a Banach space E, it is enough to show that the identity operator on E is an extreme point in the unit sphere of $B(E)$.

Let E^* be the dual of E, then $\|1 \pm a\| \leq 1$ implies $\|1^* \pm a^*\| \leq 1$. For any $f \in E^*$, put $f_1 = (1^* + a^*)f$ and $f_2 = (1^* - a^*)f$, then $2f = f_1 + f_2$ and $\|f_1\| \leq \|f\|$, $\|f_2\| \leq \|f\|$; therefore if f is an extreme point of the unit sphere of E^*, $f = f_1 = f_2$; hence $a^*f = 0$, so that $a^* = 0$ and so $a = 0$. Now let $1 = \frac{b+c}{2}$ ($\|b\|$, $\|c\| \leq 1$) and put $a = 1 - b$, then $1 - a = b$ and $1 + a = 21 - b = c$; hence by the above considerations, $b = 1$ and so $c = 1$. This completes the proof.

Remark. The converse question of Theorem 2.3 is negative

Notices of § 2

Von Neumann [22], using the strong operator topology, had proved the existence of unit in W^*-algebras, and Kaplansky [13], using the lattice property of projections, did it.

Our proof of using extreme points is another one. The extremity of unit in Theorem 2.1 and Theorem 2.2 are due to Kadison [8]; he proved Theorem 2.2 in B^*-algebras with unit; however from our considerations, it is easily seen that the assumption of unit is unnecessary.

§3. Topologies on W^*-algebras.

Let M be a B^*-algebra which is the dual of a Banach space F, S the unit sphere of M, A the self-adjoint portion of M, P the positive portion of A. Henceforward we shall always use the topology $\sigma(M,F)$ on M; we shall call this topology $\sigma(M,F)$ the weak topology of M; it is well known that S is $\sigma(M,F)$-compact.

Lemma 3.1. A and P are $\sigma(M,F)$-closed.

Proof. First, we shall show that $A \cap S$ is closed. If it is not closed, there is a directed set $\{x_\alpha\}$ in $A \cap S$ such that it converges to an element $a + ib$ $(b \neq 0)$, where a and b are self-adjoint.

Suppose that there exists a positive number $\lambda > 0$ in the spectrum of b (otherwise consider $\{-x_\alpha\}$). Then,

$$\| x_\alpha + inI \| \leq (1+n^2)^{1/2} < \lambda + n \leq \| b + nI \| \leq \| a + ib + inI \|$$

for a large number n.

Since $\{x_\alpha + inI\}$ converges to $a + ib + inI$ and belongs to $(1 + n^2)^{1/2}S$, the compactness of $(1 + n^2)^{1/2}S$ means that $a + ib + inI$ belongs also to $(1 + n^2)^{1/2}S$. This contradicts the above inequality; hence $A \cap S$ is closed, so that A is closed by the theorem of Banach.

Moreover, since $P \cap S \subset (A \cap S) + 1 \subset P$, we have $P \cap S = (A \cap S) \cap \{(A \cap S) + I\}$; hence $P \cap S$ is closed, so that P is closed.

Lemma 3.2. Let T be the totality of $\sigma(M,F)$-continuous positive linear functionals on M. Then for any self-adjoint element $a \bar{\varepsilon} P$, there is an element φ of T such that $\varphi(a) < 0$; in particular, $\psi(b) = 0$ for all $\psi \varepsilon T$ implies $b = 0$.

This follows immediately from Lemma 3.1 and the theorem in the theory of locally convex vector space.

Definition 3.1. We call a directed set $\{x_\alpha\}$ in A increasing, if $x_\alpha \geq x_\beta$ whenever $\alpha \geq \beta$.

Lemma 3.3. Every uniformly bounded, increasing directed set converges to its least upper bound. If $x = \text{l.u.b}\, \{x_\alpha\}$, then $a^* x a = \text{l.u b}\, \{a^* x_\alpha a\}$.

Proof. Let E be the set of all finite linear combinations of elements of T. It is clear that the topology $\sigma(M,E)$ is weaker than the topology $\sigma(M,F)$. Moreover, $\sigma(M,E)$ is a Hausdorff topology by Lemma 3.2; since S is $\sigma(M,F)$-compact, $\sigma(M,E)$ is equivalent to $\sigma(M,F)$ on λS $(\lambda > 0)$.

Therefore, to show that a uniformly bounded directed set $\{x_\alpha\}$ is a Cauchy directed set in $\sigma(M,F)$-topology, it is enough to show that for any $\varphi \varepsilon T$ and positive number ϵ there is an index α_0 such that $|\varphi(x_\alpha - x_\beta)| \leq \epsilon$ for $\alpha, \beta \geq \alpha_0$.

Let $\{x_\alpha\}$ be uniformly bounded and increasing. Then $\{\varphi(x_\alpha)\}$ is so for every $\varphi \varepsilon T$; hence $\{x_\alpha\}$ is $\sigma(M,F)$-Cauchy, so that by the compactness of S, it converges to some element x.

Moreover, it is clear by Lemma 3.2 that $x = \text{l.u. b}\{x_\alpha\}$. If u is an invertible element, then clearly

$$\underset{\alpha}{\text{l.u.b}}\, \{u^* x_\alpha u\} = u^* \{\underset{\alpha}{\text{l.u.b}}\, x_\alpha\} u = u^* x u \ .$$

Finally, if a is an arbitrary element of M, then there is a suitable number $\lambda > 0$ such that $\lambda 1 + a$ is invertible.

Then,

$$\varphi((\lambda 1+a)^* x_\alpha (\lambda 1+a)) = \lambda^2 \varphi(x_\alpha) + \lambda \varphi(u^* x_\alpha) + \lambda \varphi(x_\alpha a)$$
$$+ \varphi(a^* x_\alpha a) \longrightarrow \varphi((\lambda 1+a)^* x(\lambda 1+a)) \quad \text{for any} \quad \varphi \in T.$$

On the other hand,

$$|\varphi(a^*(x_\alpha - x_\beta))| = |\varphi(a^*(x_\alpha - x_\beta)^{1/2}(x_\alpha - x_\beta)^{1/2})|$$
$$\leq \varphi(a^*(x_\alpha - x_\beta)a)^{1/2} \varphi(x_\alpha - x_\beta)^{1/2} \quad \text{for} \quad \alpha \geq \beta,$$

and analogously,

$$|\varphi((x_\alpha - x_\beta)a)| \leq \varphi(x_\alpha - x_\beta)^{1/2} \varphi(a^*(x_\alpha - x_\beta)a)^{1/2} \quad \text{for} \quad \alpha \geq \beta;$$

hence

$$\lambda^2 \varphi(x_\alpha) + \lambda \varphi(a^* x_\alpha) + \lambda \varphi(x_\alpha a) \longrightarrow \lambda^2 \varphi(x) + \lambda\varphi(a^* x) + \lambda\varphi(xa),$$

so that $\underset{\alpha}{\text{l.u.b}} \{a^* x_\alpha a\} = a^* xa$.

Lemma 3.4. Let C be any maximal commutative B^*-subalgebra of M and Ω be its spectrum space, then Ω is a Stonean space.

Proof. Let $\{f_\alpha\}$ be a uniformly bounded, increasing directed set and $x_0 = \underset{\alpha}{\text{l.u.b}} \{f_\alpha\}$. For any unitary element u of C, $u^* f_\alpha u = u^{-1} f_\alpha u = f_\alpha$ converges to $u^{-1} x_0 u = x_0$; as C is maximal, x_0 belongs to C and so Ω is Stonean.

Lemma 3.5. Let e be any projection of M. Then the subalgebra eMe is $\sigma(M,F)$-closed, and moreover the mapping $x \longrightarrow exe$ is $\sigma(M,F)$-continuous.

Proof. $e(P\cap S)e$ consists clearly of those elements of $P\cap S$ which are $\leq e$. If $\{x_\alpha\}$ is a directed set in $e(P\cap S)e$ converging to an element $x_0 \geq 0$, then $e - x_\alpha \geq 0$, so that $e - x_0 \geq 0$; hence $e(P\cap S)e$ is closed. Since $e(A\cap S)e = e(P\cap S)e - e(P\cap S)e$, the compactness of $e(P\cap S)e$ implies that $e(A\cap S)e$ is closed; hence eMe is closed.

Next, we shall show the continuity of the mapping. For this, it is enough to show that the kernel $(1-e)M + M(1-e)$ of the mapping is closed, because M is an algebraic direct sum of eMe and $(1-e)M + M(1-e)$.

Now, we shall show that $\{ea_\alpha(1-e)\}$ $(a_\alpha \in A\cap S)$ converges to a, then $eae = (1-e)a(1-e) = 0$. For any integer n and complex number $c(|c| = 1)$,

$$\| ea_\alpha(1-e) + cne \| = \| \{ea_\alpha(1-e) + cne\} \{(1-e)a_\alpha e + \bar{c}ne\} \|^{1/2}$$
$$= \| ea_\alpha(1-e) + n^2 e \|^{1/2} \leq (1+n^2)^{1/2}.$$

Now suppose that $eae \neq 0$ and there is a positive number $\lambda > 0$ in the spectrum of $\dfrac{eae + ea^*e}{2}$ (otherwise consider $(-a_\alpha)$), then

$$\| eae + ne + ea(1-e) + (1-e)ae + (1-e)a(1-e) \| \geq \| e(a+n1)e \|$$
$$\geq \| \frac{eae + ea^*e}{2} + ne \| \geq \lambda + n.$$

Therefore,

$$\| a + ne \| > (1+n^2)^{1/2} \quad \text{for a large number } n.$$

This is a contradiction; hence

$$\frac{eae + ea^*e}{2} = 0 ,$$

and analogously

$$\frac{iea^*e - ia^*e}{2} = 0 ,$$

so that $eae = 0$.

Similarly, suppose that $(1-e)a(1-e) \neq 0$. Then

$$\| ea_\alpha(1-e) + cn(1-e)\| = \| \{(1-e)a_\alpha e + \bar{c}n(1-e)\} \{ea_\alpha(1-e) + cn(1-e)\}\|^{1/2}$$
$$= \| (1-e)a_\alpha ea_\alpha(1-e) + n^2(1-e)\|^{1/2} \leq (1+n^2)^{1/2};$$

hence we shall obtain an analogous contradiction, so that $a = ea(1-e) + (1-e)ae$; hence the closure of $(1-e)Se$ is contained in $eM(1-e) + (1-e)Me$. By symmetry, the closure of $eS(1-e)$ is contained in $eM(1-e) + (1-e)Me$. From the above discussion and the compactness of S, we easily conclude that $eS(1-e) + (1-e)Se$ is closed, so that $eM(1-e) + (1-e)Me$ is closed; hence

$$(1-e)M + M(1-e) = (1-e)Me + eM(1-e) + (1-e)M(1-e)$$

is closed.

Lemma 3.6. Let e be any projection of M, then the mapping $x \to ex$ and $x \to xe$ are $\sigma(M,F)$-continuous.

Proof. Suppose that $\{ea_\alpha(1-e)\}$ $(a_\alpha \in S)$ converges to a and $(1-e)ae \neq 0$. Since by Lemma 3.5,

$$a = ea(1-e) + (1-e)ae$$

$$\| a+n(1-e)ae\| = \| ea(1-e) + (n+1)(1-e)ae\|$$
$$= \max \{\| ea(1-e)\|, (n+1)\| (1-e)ae\| \};$$

hence

$$\| a+n(1-e)ae\| = (n+1)\| (1-e)ae\|$$

for a large number n. On the other hand,

$$\| ea_\alpha(1-e) + n(1-e)ae\| \leq \max\{1, n\|(1-e)ae\|\} = n\|(1-e)ae\|$$

for a large number n, and this contradicts the above inequality; hence $eM(1-e)$ is closed. Therefore, the mappings $x \to ex(1-e)$ and $(1-e)xe$, and so ex and xe are $\sigma(M,F)$-continuous.

Theorem 3.1. The mapping $x \to x^*$, and ax, xa and so a^*xa are $\sigma(M,F)$-continuous for any $a \in M$.

Proof. By Lemma 3.1, A is $\sigma(M,F)$-closed, so that we can easily conclude that the mapping $x \to x^*$ is $\sigma(M,F)$-continuous. Next, let C be a maximal commutative B^*-subalgebra containing a self-adjoint element h, then by Lemma 3.4 the spectrum space Ω of C is stonean and so for any arbitrary positive number $\epsilon\ (> 0)$ there is a finite family $\{e_i\}$ of orthogonal projections belonging to C as follows:

$$\| h - \sum_{i=1}^{n} \lambda_i e_i \| < \epsilon,$$

where $\{\lambda_i\}$ is a family of complex numbers.

Let $\{x_\alpha\}$ $(\|x_\alpha\| \leq 1)$ be a directed set converging to 0, then for any $\sigma(M,F)$-continuous linear functional f,

$$|f(hx_\alpha)| = |f((h - \sum_{i=1}^{n} \lambda_i e_i)x_\alpha) + f((\sum_{i=1}^{n} \lambda_i e_i)x_\alpha)|$$
$$\leq \|f\|\epsilon + \sum_{i=1}^{n} |\lambda_i|\ |f(e_i x_\alpha)|$$

By Lemma 3.6,

$$\overline{\lim_\alpha}\ |f(hx_\alpha)| \leq \|f\|\epsilon .$$

Since ϵ is arbitrary, $\lim_\alpha |f(hx_\alpha)| = 0$; hence a linear functional $g(x) = f(hx)$ is continuous on S, so that g is continuous on M by the theorem of Banach and so the mapping $x \longrightarrow hx$ and so $x \longrightarrow ax$ is $\sigma(M,F)$-continuous; finally the mapping $x \longrightarrow (a^*x)^* = xa$ and a^*xa are continuous. This completes the proof.

From the theory of locally convex spaces, we can identify the Banach space F with the Banach space of all $\sigma(M,F)$-continuous linear functionals. Now let $\tau(M,F)$ be the Mackey topology on M, that is, the topology of uniform convergences on all relatively $\sigma(F,M)$-compact convex sets in F. Then,

Theorem 3.2. The mapping $x \Longrightarrow x^*$, ax, xa and so a^*xa for any $a \in M$ are $\tau(M,F)$-continuous.

Notation. Denote $f(x)$ by $\langle x,f\rangle$ and define $\langle x,f^*\rangle = \overline{\langle x^*,f\rangle}$, $\langle x,Laf\rangle = \langle ax,f\rangle$ and $\langle x,Raf\rangle = \langle xa,f\rangle$ for $x,a \in M$ and $f \in F$, then f^*, Laf and Raf belong to F by Theorem 3.1.

Lemma 3.7. The mapping $f \Longrightarrow f^*$, Laf and Raf in F are $\sigma(F,M)$-continuous.

Proof. Let $\{f_\alpha\}$ be a directed set of F converging to 0 in the $\sigma(F,M)$-topology, then

$$\langle x,f_\alpha^*\rangle = \overline{\langle x^*,f_\alpha\rangle} \Longrightarrow 0 \quad \text{for all} \quad x \in M \ ;$$

hence the mapping $f \longrightarrow f^*$ is $\sigma(M,F)$-continuous.

Next

$$\langle x,Laf_\alpha\rangle = \langle ax,f_\alpha\rangle \Longrightarrow 0 \quad \text{for all} \quad x \in M \ ;$$

hence the mapping $f \longrightarrow Laf$ and analogously $f \longrightarrow Raf$ are $\sigma(F,M)$-continuous.

The proof of Theorem 3.2. Let $\{x_\alpha\}$ be a directed set of

M converging to 0 in the $\tau(M,F)$, G any relatively $\sigma(F,M)$-compact set in F, then by Lemma 3.7, G^*, LaG and RaG are relatively $\sigma(F,M)$-compact, and moreover

$$\langle x_\alpha^*, G\rangle = \overline{\langle x_\alpha, G^*\rangle} \Longrightarrow 0 \quad \text{(uniformly)} \quad ;$$

hence the mapping $x \longrightarrow x^*$ is $\tau(M,F)$-continuous.

Next

$$\langle ax_\alpha, G\rangle = \langle x_\alpha, LaG\rangle \Longrightarrow 0 \quad \text{(uniformly)} \quad ;$$

hence $x \longrightarrow ax$ and analogously xa are $\tau(M,F)$-continuous. This completes the proof of Theorem 3.2.

Finally we shall introduce another topology. Put $\alpha_\varphi(x) = \varphi(x^*x)^{1/2}$ for any $\varphi \in T$, then α_φ is a semi-norm on M. The family of semi-norm $\{\alpha_\varphi \mid \text{all } \varphi \in T\}$ defines a locally convex Hausdorff topology on M. This topology is called the strong topology and denoted by $S(M,F)$.

Theorem 3.3. $\tau(M,F) \precsim S(M,F) \precsim \sigma(M,F)$

Proof. Let f be a $\tau(M,F)$-continuous linear functional on S, V_f the null space of f, then $V_f \cap S$ is $\tau(M,F)$-closed. Since $V_f \cap S$ is convex, by the theorem of Mackey, $V_f \cap S$ is also $\sigma(M,F)$-closed; hence V_f is $\sigma(M,F)$-closed, so that f is $\sigma(M,F)$-continuous and so $\tau(M,F)$-continuous.

Now let $\{x_\alpha\}$ $(\subset S)$ be a directed set converging to zero in $\tau(M,F)$, then

$$\alpha_\varphi(x_\alpha)^2 = \varphi(x_\alpha^* x_\alpha) = \langle x_\alpha^* x_\alpha, \varphi\rangle = \langle x_\alpha, L_{x_\alpha^*}\varphi\rangle \quad ,$$

where $\{L_{x_\alpha^*}\varphi\} \subset L_S\varphi$. Moreover, the mapping $x \longrightarrow L_x\varphi$ of M in $\sigma(M,F)$ into F in $\sigma(F,M)$ is continuous, for if $\{y_\alpha\}$

converges to zero in $\sigma(M,F)$, then by Theorem 3.1, $\langle x, Ly_\alpha\varphi\rangle = \langle y_\alpha x, \varphi\rangle \Longrightarrow 0$ for any $x \varepsilon M$. Therefore $L_S\varphi$ is $\sigma(F,M)$-compact; hence

$$\alpha_\varphi(x_\alpha)^2 = \langle x_\alpha, L_{x_\alpha^*}\varphi\rangle \Longrightarrow 0 ;$$

hence $\{x_\alpha\}$ converges to 0 in $S(M,F)$. Therefore any $S(M,F)$ continuous linear functional on S is also $\tau(M,F)$-continuous on S; hence it is $\sigma(M,F)$-continuous. So we obtain that the dual of M with the topology $S(M,F)$ is also F; hence by the theorem of Mackey $\tau(M,F) \precsim S(M,F) \precsim \sigma(M,F)$. This completes the proof.

Corollary 1. Let f be a linear functional on M, then the following conditions are equivalent.

(i) f is $\sigma(M,F)$-continuous (iv) f is $\sigma(M,F)$-continuous on S

(ii) f is $S(M,F)$-continuous (v) f is $S(M,F)$-continuous on S

(iii) f is $\tau(M,F)$-continuous (vi) f is $\tau(M,F)$-continuous on S.

Corollary 2. Let R be a convex set in M, then the following conditions are equivalent

(i) R is $\sigma(M,F)$-closed (iv) $R \cap \lambda S$ is $\sigma(M,F)$-closed for $\lambda > 0$

(ii) R is $S(M,F)$-closed (v) $R \cap \lambda S$ is $S(M,F)$-closed for $\lambda > 0$

(iii) R is $\tau(M,F)$-closed (vi) $R \cap \lambda S$ is $\tau(M,F)$-closed for $\lambda > 0$.

Theorem 3.4. The mapping $x \Longrightarrow ax, xa$ is $S(M,F)$-continuous and moreover the mapping $(x,y) \Longrightarrow xy$ of two variables is $S(M,F)$-continuous on $\lambda S \times M$. $(\lambda > 0)$.

Proof. Suppose that $\{x_\alpha\}$ converges to zero in $S(M,F)$, then

$$\alpha_\varphi(ax_\alpha)^2 = \langle x_\alpha^* a^* a x_\alpha, \varphi\rangle \leq \|a^* a\| \langle x_\alpha^* x_\alpha, \varphi\rangle$$
$$= \|a^* a\| \alpha_\varphi(x_\alpha)^2 \longrightarrow 0 \text{ for all } \varphi \in T ;$$

hence $\{ax_\alpha\}$ converges to zero in $S(M,F)$.

Moreover,

$$\alpha_\varphi(x_\alpha a)^2 = \langle a^* x_\alpha^* x_\alpha a, \varphi\rangle = \langle x_\alpha^* x_\alpha, L_{a^*} R_a \varphi\rangle \text{ for all } \varphi \in T.$$

Since $L_{a^*} R_a \varphi \in T$, $\{x_\alpha a\}$ converges strongly to zero.

Finally, suppose that $\{x_\alpha\}$ ($\subset S$) converges to zero and $\{y_\alpha\}$ converges to zero, then

$$\alpha_\varphi(x_\alpha y_\alpha)^2 \leq \|x_\alpha^* x_\alpha\| \alpha_\varphi(y_\alpha)^2 \leq \alpha_\varphi(y_\alpha)^2 ;$$

hence $x_\alpha y_\alpha$ converges strongly to zero.

This completes the proof.

Remark 3.1. In general, the *-operation is not $S(M,F)$-continuous. Concerning this, we shall state the details in chapter II.

Remark 3.2. We can consider other locally convex topologies on W^*-algebras. For instance, put $\alpha^*_\varphi(x) = \varphi(xx^*)^{1/2}$, then the family of semi-norms $\{\alpha_\varphi, \alpha^*_\varphi \mid \varphi \in T\}$ defines a locally convex topology $S^*(M,F)$ and $\tau(M,F) \precsim S^*(M,F) \precsim S(M,F) \precsim \sigma(M,F)$; clearly the *-operation is continuous under the $S^*(M,F)$. It is meaningful that the reader shall study more other topologies which are weaker than the $\tau(M,F)$.

The discontinuity of the *-operation does not necessarily mean the weak point of the $S(M,F)$; in fact, studying this property more deeply, we can obtain a criterion concerning the types of W^*-algebras [cf. Chapter II].

Notices of §3.

In the theory of W^*-algebras, the Mackey topology had hardly been used; but the author believes that it must be a useful tool in the theory; indeed, using the Mackey topology, we shall prove the density theorem of Kaplansky in the next section.

An inportant theorem of Banach spaces which is used in this section is the following theorem of Banach:

Theorem. Let E be a Banach space, E^* the dual of E and C be a convex set of E^*. Then C is $\sigma(E^*,E)$-closed if and only if $C \cap B$ for any $\sigma(E^*,E)$-compact set B is $\sigma(E^*,E)$-closed.

This theorem is very often used in the field of functional analysis. In the theory of W^*-algebras, the first one who used it seems to be Dixmier [15].

There are many topics concerning this Banach's theorem. It is true in Frechet spaces. Grothendieck showed that it is true for complete locally convex space E and subspaces C of E^* with deficiency one. Ptàk studied a necessary and sufficient condition in order that it be true. The reader can find related matters in the references of Ptàk [27].

§4. Density theorem.

We show the following

Theorem 4.1 (Kaplansky). Let B be a $*$-subalgebra of a W^*-algebra M which is $\sigma(M,F)$-dense in M, then $B \cap S$ is $\sigma(M,F)$-dense in S.

Proof. We can assume that B is uniformly closed. Let $a \in M$, then there is a directed set $\{a_\alpha\}$ in B such that $\tau(M,F)\text{-}\lim_\alpha a_\alpha = a$. Since $\|(1+a_\alpha^* a_\alpha)^{-1}\| \leq 1$, $\{R_{(1+a_\alpha^* a_\alpha)^{-1}} f\}$ for any $f \in F$ is relatively $\sigma(F,M)$-compact; hence

$$|\langle a, R_{(1+a_\alpha^* a_\alpha)^{-1}} f\rangle - \langle a_\alpha, R_{(1+a_\alpha^* a_\alpha)^{-1}} f\rangle| < \epsilon \text{ for all } \alpha \geq \alpha_0.$$

Therefore,

$$|\langle a(1+a_\alpha^* a_\alpha)^{-1}, f\rangle - \langle a_\alpha(1+a_\alpha^* a_\alpha)^{-1}, f\rangle| < \epsilon \text{ for all } \alpha \geq \alpha_0 \ldots (1) .$$

Moreover,

$$a(1+a^*a)^{-1} - a(1+a_\alpha^* a_\alpha)^{-1} = a[(1+a^*a)^{-1}\{(1+a_\alpha^* a_\alpha) - (1+a^*a)\}\cdot \cdot(1+a_\alpha^* a_\alpha)^{-1}]$$

$$= a(1+a^*a)^{-1}(a_\alpha^* a_\alpha - a^*a)(1+a_\alpha^* a_\alpha)^{-1}$$

$$= a(1+a^*a)^{-1}a_\alpha^* a_\alpha(1+a_\alpha^* a_\alpha)^{-1} - a(1+a^*a)^{-1}a^*a(1+a_\alpha^* a_\alpha)^{-1}.$$

Since $\|a_\alpha(1+a_\alpha^* a_\alpha)^{-1}\| \leq \frac{1}{2}$ and $a_\alpha^* \Longrightarrow a^*(\tau(M,F))$,

$$|\langle a(1+a^*a)^{-1}a_\alpha^* a_\alpha(1+a_\alpha^* a_\alpha)^{-1}, f\rangle - \langle a(1+a^*a)^{-1}a^* a_\alpha(1+a_\alpha^* a_\alpha)^{-1}, f\rangle| < \epsilon \text{ for all } \alpha \geq \alpha_1 \ldots (2).$$

On the other hand,

$$a(1+a^*a)^{-1}a^* a_\alpha(1+a_\alpha^* a_\alpha)^{-1} - a(1+a^*a)^{-1}a^*a(1+a_\alpha^* a_\alpha)^{-1}$$
$$= a(1+a^*a)^{-1}a^*(a_\alpha - a)(1+a_\alpha^* a_\alpha)^{-1} ;$$

hence

$$| \langle a(1+a^*a)^{-1}a^*(a_\alpha - a)(1+a_\alpha^* a_\alpha)^{-1}, f\rangle | < \epsilon \text{ for all } \alpha \geq \alpha_2 \ldots (3)$$

Therefore,

$$| \langle a(1+a^*a)^{-1} - a_\alpha(1+a_\alpha^* a_\alpha)^{-1}, f\rangle |$$

$$\leq | \langle a(1+a^*a)^{-1} - a(1+a_\alpha^* a_\alpha)^{-1}, f\rangle | + | \langle a(1+a_\alpha^* a_\alpha)^{-1} - a_\alpha(1+a_\alpha^* a_\alpha)^{-1}, f\rangle |$$

$$\leq | \langle a(1+a^*a)^{-1}a_\alpha^* a_\alpha(1+a_\alpha^* a_\alpha)^{-1} - a(1+a^*a)^{-1}a^*a_\alpha(1+a_\alpha^* a_\alpha)^{-1}, f\rangle |$$

$$+ | \langle a(1+a^*a)^{-1}a^*a_\alpha(1+a_\alpha^* a_\alpha)^{-1} - a(1+a^*a)^{-1}a^*a(1+a_\alpha^* a_\alpha)^{-1}, f\rangle |$$

$$+ | \langle a(1+a_\alpha^* a_\alpha)^{-1} - a_\alpha(1+a_\alpha^* a_\alpha)^{-1}, f \rangle |$$

$$< 3\epsilon \text{ for all } \alpha \geq \alpha_3 ;$$

hence $\sigma(M,F)\text{-}\lim_\alpha 2(a_\alpha(1+a_\alpha^* a_\alpha)^{-1}) = 2a(1+a^*a)^{-1}$.

Since $\| 2a_\alpha(1+a_\alpha^* a_\alpha)^{-1} \| \leq 1$, the $\sigma(M,F)$-closure of $B \cap S$ contains all elements of S such that $\{2a(1+a^*a)^{-1} \mid a \in M\}$.

Let V be an extreme point of S, then

$$2V(1+V^*V)^{-1} = 2V(\tfrac{1}{2}p + (1-p)), \quad \text{where} \quad p = V^*V;$$

hence $2V(1+V^*V)^{-1} = V$, so that the $\sigma(M,F)$-closure of $B \cap S$ contains all extreme points of S and so it coincides with S. This completes the proof.

Notices of §4.

In the proof of the density theorem, we use the following fact: if $a_\alpha \in B$, then $2a_\alpha(1+a_\alpha^* a_\alpha)^{-1} \in B$. This is trivial in the case of B having unit. If B has no unit, $B + \lambda 1$ is a C^*-algebra; hence $(1+a_\alpha^* a_\alpha)^{-1} \in B + \lambda 1$, so that $2a_\alpha(1+a_\alpha^* a)^{-1} \in B$.

Although, using the $\sigma(M,F)$-topology, we stated the density theorem, it is, of course, clear that $B \cap S$ is also $\tau(M,F)$-dense in S by a corollary 2 of Theorem 3.3.

The density theorem of Kaplansky is one of the most useful theorems in the theory of operator algebras. To emphasize the depth of this theorem, we shall state a counter-example in Banach spaces: Consider a $L^\infty(\Omega)$ on some measure space Ω, then it is a commutative W^*-algebra. Let $\mathfrak{M}$ be a maximal ideal of $L^\infty(\Omega)$ and V be a $\sigma(L^\infty, L^1)$-closed subspace of $L^\infty(\Omega)$ with deficiency one, then there is a linear isomorphism of $\mathfrak{M}$ onto V. Let V be the dual of E, V^* and $\mathfrak{M}^*$ the duals of V and $\mathfrak{M}$ respectively; since E is $\sigma(V^*,V)$-dense in V^*, $\phi^*(E)$ is $\sigma(\mathfrak{M}^*,\mathfrak{M})$-dense in $\mathfrak{M}^*$. If the unit sphere of $\phi^*(E)$ is $\sigma(\mathfrak{M}^*,\mathfrak{M})$-dense in the unit sphere of $\mathfrak{M}^*$, we can easily conclude that the unit sphere of $\mathfrak{M}$ is $\sigma(\mathfrak{M},\phi^*(E))$-compact, so that $\mathfrak{M}$ is the dual of $\phi^*(E)$; hence $\mathfrak{M}$ is a W^*-algebra.

On the other hand, if Ω has no atomic part, $\mathfrak{M}$ cannot have the unit; hence in this case, $\phi^*(E) \cap S$ is not $\sigma(\mathfrak{M}^*,\mathfrak{M})$-dense in S, where S is the unit sphere of $\mathfrak{M}^*$.

§5. Linear functionals on W^*-algebras.

Definition 5.1. A positive linear functional φ on M is said normal if it satisfies $\varphi(\text{l.u.b}_\alpha\, x_\alpha) = \text{l.u.b}_\alpha\, \varphi(x_\alpha)$ for any uniformly bounded increasing directed set (x_α).

Then,

Theorem 5.1. Let φ be a positive linear functional on M, then the following conditions are equivalent

(i') φ is normal

(ii') φ is $\sigma(M,F)$-continuous.

Proof. The implication ii) $\Longrightarrow$ i) was already proved. Prove the implication i) $\Longrightarrow$ ii). Let $\{p_\alpha\}$ be an increasing directed set of projections such that $x \longrightarrow \varphi(xp_\alpha)$ is $\sigma(M,F)$-continuous. Let p be the l.u.b of $\{p_\alpha\}$, then p is also a projection by Theorem 3.4. Therefore,

$$|\varphi(x(p-p_\alpha))|^2 \leq \varphi(x(p-p_\alpha)x^*)\,\varphi(p-p_\alpha) \leq \varphi(1)\,\varphi(p-p_\alpha) ;$$

hence $\varphi(xp)$ is a uniform limit of $\{\varphi(xp_\alpha)\}$ on S, so that $\varphi(xp)$ is also $\sigma(M,F)$-continuous on S and so on M.

Therefore there is a maximal projection p_0 such that $x \longrightarrow \varphi(xp_0)$ is $\sigma(M,F)$-continuous.

Suppose $p_0 < I$, and we take a $\psi \in T$ such that $\varphi(I-p_0) < \psi(I-p_0)$, then there is a non-zero porjection p_1 such that $p_1 \leq I-p_0$ and

$$\varphi(p) < \psi(p) \quad \text{if} \quad p \leq p_1 .$$

In fact, assume that this is negative, then for every non-zero projection p ($\leq 1-p_0$) there is a non-zero projection q such that

$$q < p \quad \text{and} \quad \varphi(q) \geq \psi(q) \ ;$$

hence take a maximal projection q_0 satisfying such condition, we obtain $q_0 = 1 - p_0$, this is a contradiction; hence

$$\varphi(p) < \psi(p) \quad \text{if } 0 < p \leq p_1 \ .$$

Since any maximal commutative subalgebra of a W^*-algebra p_1Mp_1 is stonean,

$$\varphi(a) \leq \psi(a) \quad \text{if} \quad a \in P \cap p_1Mp_1 \ ;$$

hence

$$|\varphi(x(p_0+p_1))| \leq |\varphi(xp_0)| + |\varphi(xp_1)| \leq |\varphi(xp_0)| + \varphi(I)^{1/2}\varphi(p_1x^*xp_1)^{1/2}$$
$$\leq |\varphi(xp_0)| + \varphi(I)^{1/2}\psi(p_1x^*xp_1)^{1/2} \ ,$$

so that $x \longrightarrow \varphi(x(p_0+p_1))$ is $S(M,F)$-continuous and so $\sigma(M,F)$-continuous, this contradicts the maximality of p_0; hence $p_0 = I$. This completes the proof.

The above theorem has an important meaning as follows:

Let M^* be the dual of M and we shall canonically imbed the F into M^*, then F is a norm-closed subspace of M^* generated by T. On the other hand, by the above theorem, T is the totality of normal positive linear functionals. Since the normality is determined by the order properties on M only, the space F is the unique subspace of M^*; therefore if $F_1^* = F_2^* = M$ for two Banach spaces F_1 and F_2, F_1 coincides with F_2, when they are canonically imbedded into M^*.

This important property is not true in general Banach spaces. For instance, let $\Omega = \{1,2,3,\ldots,n,\ \ldots\}$ be a discrete space, then $C_0^*(\Omega) = \ell_1(\Omega)$, $C_0(\Omega\times\Omega)^* = \ell_1(\Omega\times\Omega)$.

Though $\ell_1(\Omega)$ is isometrically isomorph to $\ell_1(\Omega\times\Omega)$, $C_0(\Omega)$ is not so to $C_0(\Omega\times\Omega)$, for if $C_0(\Omega)$ is isometrically isomorph to $C_0(\Omega\times\Omega)$, Ω is homeomorphic to $\Omega\times\Omega$.

Hence, let ρ be an isometry of $\ell_1(\Omega)$ onto $\ell_1(\Omega\times\Omega)$, and ρ^* the dual of ρ, then

$$\begin{array}{ccc} \ell_1(\Omega) & \xrightarrow{\rho} & \ell_1(\Omega\times\Omega) \\ \ell_\infty(\Omega) & \xleftarrow{\rho^*} & \ell_\infty(\Omega\times\Omega) \\ \cap & & \cap \\ C_0(\Omega) & & C_0(\Omega\times\Omega) \end{array}$$

Put $V = \rho^*(C_0(\Omega\times\Omega))$, then $V^* = \ell_1(\Omega)$, but V is not $C_0(\Omega)$.

Hence we conclude the following important theorem.

Theorem 5.2. Let M be a W^*-algebra such that $F^* = M$, then the $\sigma(M,F)$-topology is the strongest topology in locally convex topologies in which the unit sphere of M are compact. In particular, if $M = F_1^* = F_2^*$, then $F_1 = F_2$, where F, F_1 and F_2 are Banach spaces.

Since the uniqueness in this sense is assured, we shall simply call the topology $\sigma(M,F)$ on M the σ-topology, and the unique Banach space F is called the associated space with M, denoted by M_*. Moreover the topologies $S(M,F)$ and $\tau(M,F)$, defined by the unique associated space, are simply denoted by the S and τ topologies.

Notices of §5.

Let $\{e_\alpha \mid \alpha \varepsilon I\}$ be an infinite family of orthogonal projections, then the sum $\sum_\alpha e_\alpha$ is defined as follows: let J be any finite subset of I, and put $p_J = \sum_{\alpha\varepsilon J} e_\alpha$, then $\{p_J\}$ is a uniformly bounded increasing directed set under an order defined by the inclusion of subsets J; hence it converges to $\text{l.u.b.}_J\ p_J$ in the σ-topology; moreover

$$\alpha_\varphi(\text{l.u.b.}_J\ p_J-p_J)^2 = \varphi((\text{l.u.b.}_J\ p_J-p_J)^{1/2}(\text{l.u.b.}_J\ p_J-p_J)^{3/2})$$
$$\leq \varphi(\text{l.u.b.}_J\ p_J-p_J)^{1/2}\ \|\varphi\|^{1/2} \quad \text{for all} \quad \varphi \varepsilon T\ ;$$

hence $\{p_J\}$ converges to $\text{l.u.b}_J\ p_J$ in the S-topology, so that $\text{l.u.b}_J\ p_J$ is also a projection; now we define $\sum_{\alpha\varepsilon I} e_\alpha = \text{l.u.b}_J\ p_J$.

Then, form the proof of Theorem, it is easily seen that the normality is equivalent to the complete additivity; namely let $(e_\alpha \mid \alpha \varepsilon I)$ be any family of orthogonal projections, then $\varphi(\sum_{\alpha\varepsilon I} e_\alpha) = \sum_{\alpha\varepsilon I}\varphi(e_\alpha)$.

One, who found the final significance of normality, seems to be Dixmier [15].

In this section, the reader knows a quite new class of Banach spaces; namely the existence of a dual space E which has the unique Banach space F such that $F^* = E$. Of course, the reflexivity implies such property, but we can easily show that a reflexive W^*-algebra is finite-dimensional.

It seems to be interesting to seek a characterization of dual Banach spaces having such property.

§6. Polar decomposition of functionals

Definition 6.1. Let φ_1, φ_2 be positive linear functionals on M. We say that φ_1 and φ_2 mutually orthogonal and denote by $\varphi_1 \perp \varphi_2$ if they satisfy $\|\varphi_1 - \varphi_2\| = \|\varphi_1\| + \|\varphi_2\|$.

Let f be a σ-continuous self-adjoint linear functional $(f^* = f)$ on M. Suppose $\|f\| = 1$, then by the compactness of $A \cap S$ there is an element x of A such that $f(x) = 1$, $\|x\| = 1$.

Put $\mathcal{E} = \{x \mid f(x) = 1, \|x\| = 1, x \in A\}$, then $\mathcal{E}$ is σ-compact, convex and an extreme point of $\mathcal{E}$ is also extreme on S; hence there is a self-adjoint unitary element u such that $u = e - e'$, where e is a projection and $e' = 1-e$, and moreover $f(e-e') = 1$; hence put $f_1(x) = f(ex)$ and $f_2(x) = -f(e'x)$, then

$$(f_1+f_2)(1) = f_1(e) + f_2(e') = f(e) - f(e') = 1$$

and

$$|(f_1+f_2)(x)| = |f(ex) - f(e'x)| = |f((e-e')x)| \leq \|e-e'\| \, \|f\| \, \|x\| ;$$

hence $\|f_1+f_2\| = 1$, so that $f_1 + f_2$ is positive.

The norm of f_1 on eMe is $f_1(e)$, for supposing $f_1(x) > f_1(e)$ for some $x \in eMe$ $(\|x\| \leq 1)$, then $\|x+e'\| \leq 1$ and $f(x-e') = f(x) - f(e') > f(e) - f(e')$, this is a contradiction; hence $f_1(e) = \|f_1\|$ on eMe, so that f_1 is positive on eMe; moreover $f_1 = \frac{1}{2}(f+f_1+f_2)$ and so f_1 is self-adjoint; hence $f_1(x^*) = f(ex^*) = \overline{f(xe)} = \overline{f_1(x)} = \overline{f(ex)}$, so that $f(xe) = f(xe)$ for any $x \in M$. Therefore

$$f(e(xe)) = f((xe)e) = f(xe) = f(ex) ;$$

hence

$$f_1(exe) = f(exe) = f(ex) = f_1(x) ,$$

so that f_1 is positive on M, analogously f_2 is also positive on M, and $f_2(e) = \| f_2\|$.

Hence

Theorem 6.1. Let f be a σ-continuous self-adjoint linear functional on M, then it is a sum of normal positive functionals f_1 and f_2 on M as follows: $f = f_1 - f_2$ and $\| f\| = \| f_1\| + \| f_2\|$. Moreover such decomposition is unique.

We call such decomposition the orthogonal decomposition of f and denote $f_1 = f^+$ and $f_2 = f^-$.

Now it is enough to prove the uniqueness only. For this, we introduce a definition.

Definition 6.2. Let φ be a normal positive linear functional on M. Put $\mathcal{F} = \{e \mid \varphi(e) = 0, \ e \text{ projection}\}$, then there is the greatest projection e_0 such that $e_0 \in \mathcal{F}$ and $e \leq e_0$ for all $e \in \mathcal{F}$. Then the support of φ is $1 - e_0$ and denote by $S(\varphi)$.

The proof of uniqueness. Suppose that $f = f_1 - f_2 = f_1' - f_2'$. Then

$$f(S(f_1)) = f_1(S(f_1)) = \| f_1\| = \| f_1'\| = f_1'(S(f_1)) - f_2'(S(f_1));$$

hence $f_2'(S(f_1)) = 0$ and so $f_1'(S(f_1)) = \| f_1'\|$; therefore $S(f_1') \leq S(f_1)$. Therefore

$$f_1(x) = f(S(f_1)x) = f_1'(S(f_1)x) - f_2'(S(f_1)x) = f_1'(S(f_1)x) .$$

Moreover

$$f_1^{\varphi}(S(f_1)x) = f_1^{\varphi}(S(f_1^{\varphi})x) + f_1^{\varphi}\{(S(f_1) - S(f_1^{\varphi}))x\} = f_1^{\varphi}(S(f_1^{\varphi})x) = \\ = f_1^{\varphi}(x) ;$$

hence $f_1(x) = f_1^{\varphi}(x)$ and analogously $f_2(x) = f_2^{\varphi}(x)$.

This completes the proof.

Finally we shall show a structural theorem concerning general σ-continuous linear functionals, which we shall call the polar decomposition of linear functionals.

Theorem 6.2. Let g be a σ-continuous linear functional on M, then it can be written under $g = R_V\varphi$, where φ is a normal positive functional, $\|g\| = \|\varphi\|$ and V is a partial isometry of M having the support $S(\varphi)$ of φ as the initial projection. Moreover such decomposition is unique.

We call the above φ the absolute value of g and denote it by $|g|$. Then the final projection of the above V is $S(|g^*|)$.

Proof. It is enough to suppose $\|g\| = 1$. Let u be a partial isometry of M such that $g(u) = 1$, then R_ug is positive. Since $uu^*u = u$, $g(u) = g(uu^*u) = R_ug(uu^*) = 1$; hence $uu^* \geq S(R_ug)$.

Put $w = u^*S(R_ug)$, then $w^*w = S(R_ug)$; hence w is a partially isometry having $S(R_ug)$ as the initial projection. Moreover

$$R_ug(x) = R_ug(xS(R_ug)) = g(xS(R_ug)u) = g(xw^*) = R_{w^*}g(x)$$

for all $x \in M$; hence $R_ug = R_{w^*}g$.

Lemma 6.1. Let p and q be projections such that $p = ww^*$ and $q = w^*w$, then $g(x) = g(xp)$ and $g(x) = g(qx)$ for all $x \in M$.

Proof. Suppose that for some x_0 ($\| x_0\| \leq 1$), $g(x_0(I-p)) = \beta > 0$, then

$$\| nw^* + x(I-p)\| = \| \{nw^* + x(I-p)\} \{nw + (I-p)x^*\}\|^{1/2}$$
$$= \| n^2q + x(I-p)x^*\|^{1/2} \leq (1+n^2)^{1/2} .$$

On the other hand,

$$g(nw^* + x_0(I-p)) = ng(w^*) + g(x_0(I-p)) = n+\beta > (n^2+1)^{1/2} \geq$$
$$\geq \| nw^* + x_0(I-p)\|$$

for a suitable large number n.

This is a contradiction; hence $g(x(I-p)) = 0$ for all $x \in M$; hence $g(x) = g(xp)$ and analogously $g(x) = g(qx)$, so that $g(x) = g(xp) = g(qxp)$. This completes the proof.

Therefore

$$g(x) = g(xp) = g(xww^*) = (R_{w^*}g)(xw) .$$

Since $R_{w^*}g$ is positive, $g = R_w\varphi$, where $\varphi = R_{w^*}g$ and moreover $\| g\| = \| R_{w^*}g\| = \|\varphi\|$.

Finally we shall prove the uniqueness. Let $g = R_w\varphi = R_{w'}\varphi'$. Put $ww^* = p$, $w'w'^* = \bar{p}$, $w^*w = q$ and $w'^*w' = \bar{q}$, then

$$g(x) = \varphi(xw) = \varphi'(xw') \text{ and}$$
$$\varphi(x) = \varphi(xq) = \varphi(xw^*w) = \varphi'(xw^*w') = \varphi'(\bar{q}xw^*w') ;$$

hence $\varphi(I-\bar{q}) = 0$, so that $q = S(\varphi) \leq \bar{q}$; analogously $\bar{q} = S(\varphi') \leq q$ and so $q = \bar{q}$.

Since $w'^*w = qw'^*wq$, put $w'^*w = h+ik$ ($h,k \in qMq$), then

$$\varphi(w'^*w) = \varphi'(qw'^*ww^*w') = \varphi'(qw'^*w'w'^*w')$$

$$= \varphi'(q) = 1 = \varphi(h) + i\varphi(k) ;$$

hence $\varphi(h) = 1$; since $\|h\| \leq 1$, $h = q$; $\|w'^*w\| \leq 1$ implies $k = 0$, so that $w'^*w = q$ and so $w'w'^*w = pw = w = w'q = w'$; finally $\varphi = L_{w^*}g = \varphi'$. This completes the proof.

Notices of § 6.

The existence of the greatest projection e_0 in Definition 6.2 is shown as follows: let a be a positive element ($\|a\| \leq 1$) such that $\varphi(a) = 0$, then by Schwartz' inequality $\varphi(Ma) = 0$; let $\mathcal{J}$ be a σ-closure of Ma, then it is a σ-closed left ideal; hence $\mathcal{J} \cap \mathcal{J}^*$ is a W^*-algebra, so that it has a unit e in $\mathcal{J} \cap \mathcal{J}^*$ and e is a projection in M; since $a \in \mathcal{J} \cap \mathcal{J}^*$, $ae = ea = eae = a$; hence $a = eae \leq e$ and $\varphi(e) = 0$; therefore if $e_1, e_2 \in \mathcal{F}$, $\varphi(\frac{e_1 + e_2}{2}) = 0$, so that there is a projection e such that $\frac{e_1+e_2}{2} \leq e$ and $e \in \mathcal{F}$ and so $e_1, e_2 \leq e \in \mathcal{F}$; therefore the normality and Zorn's lemma assure the existence of the greatest projection e_0. Theorem 6.1 is due to Grothendieck [40].

§7. The polar decomposition of operators

Let a be an element of M and put $h(n) = (a^*a + \frac{1}{n}I)^{1/2}$ (n positive integer) and $a(n) = a(a^*a + \frac{1}{n}I)^{-1/2}$, then $a(n)^*a(n) = (a^*a + \frac{1}{n}I)^{-1/2}a^*a(a^*a + \frac{1}{n}I)^{-1/2} = \frac{a^*a}{a^*a + \frac{1}{n}I}$; hence $\|a(n)\| < 1$ and moreover $a(n)(a^*a + \frac{1}{n}I)^{1/2} = a$.

Since $\lim_{n \to \infty} \|h(n) - (a^*a)^{1/2}\| = 0$, there is an n_0 for arbitrary $\epsilon\ (>0)$ such that $\|h(n) - (a^*a)^{1/2}\| < \epsilon\ (n > n_0)$; hence $\|a(n)h(n) - a(n)(a^*a)^{1/2}\| = \|a - a(n)(a^*a)^{1/2}\| < \epsilon$ $(n > n_0)$. By the compactness of S, there is an accumulate point b of $\{a(n)\}$, and since $\{a(n)(a^*a)^{1/2}\}$ belongs to $a + \epsilon S$, $b(a^*a)^{1/2} \varepsilon\ a + \epsilon S$; hence $\|a - b(a^*a)^{1/2}\| \leq \epsilon$.

Since ϵ is arbitrary, $\|a - b(a^*a)^{1/2}\| = 0$; hence $a = b(a^*a)^{1/2}$.

Let e and f be the range projection of $(a^*a)^{1/2}$ and $(aa^*)^{1/2}$ respectively, then $a = fa = fbe\ (a^*a)^{1/2}$; hence $a^*a = (a^*a)^{1/2}eb^*fbe(a^*a)^{1/2}$ and so $(a^*a)^{1/2}\ (e-eb^*fbe)(a^*a)^{1/2} = 0$. Since $\|b\| \leq 1$, we can conclude $e = eb^*fbe$; hence put $u = fbe$, then u is a partial isometry having the initial projection e. Moreover $aa^* = u(a^*a)u^*$; therefore the final projection $ueu^* = f$.

Now suppose that $a = u_1|a| = u_2|a|$ satisfies the above conditions ($|a| = (a^*a)^{1/2}$), then $u_1^*a = |a| = u_1^*u_2|a|$. Since $eu_1^*u_2e = u_1^*u_2$ and $\|u_1^*u_2\| \leq 1$, we have $e = u_1^*u_2$; hence $u_1 = u_2$.

Hence we obtain the following theorem.

Theorem 7.1. Let M be a W^*-algebra and a an element of M, then it can be written under $a = v|a|$, where $|a| = (a^*a)^{1/2}$ and v is a partial isometry of M having the range projection of $(a^*a)^{1/2}$ as the initial projection and the range projection of $(aa^*)^{1/2}$ as the final projection. Moreover such decomposition is unique.

We call this decomposition the polar decomposition of operators.

Notices of §7.

The range projection of a positive element h of M is defined as follows: put $\mathcal{F} = \{e \mid eh = 0,\ e \text{ projection}\}$, then by the same method used in the notices of §6, we can show that there is the greatest projection e_0 in $\mathcal{F}$; then we call $1 - e_0$ the range projection of h; therefore $he_0 = 0$ implies $h = h(1-e_0) = h(1-e_0) = (1-e_0)h(1-e_0)$ and moreover $kh \neq 0$ for any $0 < h \leq 1 - e_0$; in fact, if $kh = 0$, by considering the left ideal $[Mk]$, we can easily show that there is a projection e such that $eh = 0$ and $\frac{k}{\|k\|} \leq e \leq 1-e_0$. a contradiction.

The polar decomposition of operators is a theorem which has been very often used in the field of functional analysis. The proof given here is new. Ti Yen [41] showed that the polar decomposition is true in AW^*-algebras.

§8. Spectral decompositions of operators.

Let h be a positive element of M and put $e(\lambda) = S((\lambda 1-h)^+)$, where $S(\cdot)$ is the range projection of $(\cdot)$ for non-negative number λ, then it is clear that $e(\lambda) \leq e(\mu)$ for $\lambda \leq \mu$.

Lemma 8.1. $\lambda_n \leq \lambda$ $(n=1,2,\ldots\ )$ and $\lambda_n \to \lambda$ imply $e(\lambda_n) \to e(\lambda)$ (S).

Proof. Let $p = \text{l.u.b. } e(\lambda_n)$. $\lambda_n 1 - h \to \lambda 1 - h$ (unif.) implies $(\lambda_n 1-h)^+ \to (\lambda 1-h)^+$ (unif.). Since $(\lambda_n 1-h)^+(1-p) = 0$, $(\lambda 1-h)^+(1-p) = 0$; hence $e(\lambda) \leq p$, so that $p = e(\lambda)$.

Lemma 8.2. $\lambda\{e(\mu)-e(\lambda)\} \leq \{\mu e(\mu)-(\mu 1-h)^+\} - \{\lambda e(\lambda)-(\lambda 1-h)^+\} \leq \mu\{e(\mu)-e(\lambda)\}$ for $\mu \geq \lambda$.

Proof.
$$(\lambda 1-h)^+ = (\lambda 1-h)e(\lambda)$$
$$(\mu 1-h)^+ = (\mu 1-h)e(\mu)\ ;$$

hence $\mu e(\mu) - (\mu 1-h)^+ = he(\mu)$ and $\lambda e(\lambda) - (\lambda 1-h)^+ = he(\lambda)$, so that $\{\mu e(\mu) - (\mu 1-h)^+\} - \{\lambda e(\lambda) - (\lambda 1-h)^+\} = h(e(\mu) - e(\lambda))$. Then

$$\lambda(e(\mu)-e(\lambda)) \leq h(e(\mu)-e(\lambda)) \leq \mu(e(\mu)-e(\lambda)).$$

This completes the proof.

For any division Δ and $\delta > 0$: $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \|h\| + \delta$ of the interval $[0, \|h\| + \delta]$ with $0 < \lambda_i - \lambda_{i-1} < \epsilon$ $(i=1,2,\ldots,n)$, we have

$$m(\Delta) = \sum_{i=1}^{n} \lambda_{i-1}(e(\lambda_i)-e(\lambda_{i-1}))$$
$$\leq \sum_{i=1}^{n} \{\lambda_i e(\lambda_i)-(\lambda_i 1-h)^+\} - \{\lambda_{i-1} e(\lambda_{i-1})-(\lambda_{i-1})h)^+\} =$$

$$= \{\lambda_0 e(\lambda_0) - (\lambda_0 1-h)^+\} + \{\lambda_n e(\lambda_n) - (\lambda_n 1-h)^+\}$$

$$= (\|h\|+\delta)\, e(\|h\| + \delta) - ((\|h\| + \delta)1 - h)^+$$

$$\leq \sum_{i=1}^{n} \lambda_i (e(\lambda_i) - e(\lambda_{i-1})) \equiv M(\Delta)$$

and

$$M(\Delta) - m(\Delta) = \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})(e(\lambda_i) - e(\lambda_{i-1}))$$

$$\leq \epsilon \sum_{i=1}^{n} (e(\lambda_i) - e(\lambda_{i-1}))$$

$$= \epsilon \{e(\|h\| + \delta) - e(0)\} \leq \epsilon I ;$$

hence by making $\epsilon \longrightarrow 0$, $(\|h\| + \delta) e(\|h\| + \delta) - \{(\|h\| + \delta)1 - h\}^+ = \int_0^{\|h\| + \delta} \lambda de(\lambda)$.

Since $e(\|h\| + \delta) = 1$ and $\{(\|h\| + \delta)1 - h\}^+ = \{\|h\| + \delta\}1 - h$, we have $(\|h\| + \delta) e(\|h\| + \delta) - \{(\|h\| 1 + \delta)1 - h\}^+ = (\|h\| + \delta)1 - \{(\|h\| + \delta)1 - h\} = h$; hence we have $h = \int_0^{\|h\| + \delta} \lambda de(\lambda)$ for $\delta > 0$, so we have $h = \int_0^{\|h\| + 0} \lambda de(\lambda)$.

Now we obtain

Theorem 8.1. For any self-adjoint x of M, there is a system of projections $\{e(\lambda)\}$ $(-\infty < \lambda < \infty)$, called the resolution of unity such that

(i) $\lambda \leq \mu$ implies $e(\lambda) \leq e(\mu)$

(ii) $\lambda_n \uparrow \lambda$ implies $e(\lambda_n) \longrightarrow e(\lambda)$ (s)

(iii) $e(\|h\| + 0) = 1$ and $e(-\|h\|) = 0$

(iv) $x = \int_{-\|h\|}^{\|h\| + 0} \lambda de(\lambda)$,

where the integration is of abstract Radon-Stielfjes type under the s-topology. Moreover such representation is unique.

The unicity of the representation can be easily proved from our construction.

Notices of §8.

A method, which is used in this section, is due to the method of Kakutani in AL-spaces.

§9. Spectral decomposition of functionals (Radon-Nikodym's theorem)

The above method of giving the spectral decomposition of operators is applicable to spectral decomposition of functionals. However, for this we need some assumptions.

Definition 9.1. Let M be a W^*-algebra, P the positive portion of M. We call trace on P a functional $\bar{x}$ defined on P, with values ≥ 0, finite or infinite, having the following properties

(i) If $a, b \in P$, $\bar{x}(a+b) = \bar{x}(a) + \bar{x}(b)$;

(ii) If $a \in P$ and λ is number ≥ 0, $\bar{x}(\lambda a) = \lambda\bar{x}(a)$ (we define $0 + \infty = 0$).

(iii) If $a \in P$ and u is unitary, $\bar{x}(u^{-1}au) = \bar{x}(a)$.

We say $\bar{x}$ is faithful if $\bar{x}(a) = 0$ implies $a = 0$.

We say that $\bar{x}$ is finite if $\bar{x}(a) < +\infty$ for all $a \in P$.

We say that $\bar{x}$ is semi-finite if, for every non zero $a \in P$, there exists a non-zero b of P majorate by a such that $\bar{x}(b) < +\infty$.

We say that $\bar{x}$ is normal if, for every uniformly bounded increasing directed set $(a_\alpha) \subset P$, $\bar{x}(\text{l.u.b } a_\alpha) = \text{l.u.b } \bar{x}(a_\alpha)$.

Proposition 9.1. Let M be a W^*-algebra, $\bar{x}$ a trace on the positive portion P. The set of $a \in P$ such that $\bar{x}(a) < +\infty$ is the positive portion of a two-sided ideal $\mathfrak{M}$ of M. There is a unique linear functional $\bar{\bar{x}}$ on $\mathfrak{M}$ which coincides with $\bar{x}$ on $\mathfrak{M} \cap P$, and one has $\bar{\bar{x}}(ax) = \bar{\bar{x}}(xa)$ for $a \in \mathfrak{M}$; $x \in M$. Finally, let $a \in \mathfrak{M}$; if $\bar{x}$ is normal, the linear functional

$x \longrightarrow \bar{x}(ax)$ is σ-continuous.

Proof. Put $\mathcal{F} = \{a \mid \bar{x}(a) < +\infty\}$, and $\mathcal{M} = \{ b \mid b^*b \in \mathcal{F}\}$. Let $b \in \mathcal{M}$ and u unitary of M, we have

$$(bu)^*(bu) = u^*b^*bu \in \mathcal{F},$$
$$(ub)^*(ub) = b^*b \in \mathcal{F},$$

hence $bu, ub \in \mathcal{M}$; since any element of M is linear combination of unitary elements, $bx, xb \in \mathcal{M}$ for $x \in M$, $b \in \mathcal{M}$.

Moreover, if $b,c \in \mathcal{M}$, $(b+c)^*(b+c) \leq 2(b^*b+c^*c) \in \mathcal{F}$; hence $b+c \in \mathcal{M}$, so that $\mathcal{M}$ is a two-sided ideal and so $\mathcal{M}\cdot\mathcal{M}$ is also a two-sided ideal.

Let $a \in \mathcal{F}$, then $a^{1/2} \in \mathcal{M}$, so $a \in \mathcal{M}\mathcal{M}$; hence $\mathcal{F} \subset (\mathcal{M}\cdot\mathcal{M}) \cap P$.

Conversely, let $d \in \mathcal{M}\cdot\mathcal{M}$; d is a sum of elements a^*b with $a,b \in \mathcal{M}$. (In general, a two-sided ideal $\mathcal{J}$ is self-adjoint; in fact let $a \in \mathcal{J}$ and put $a = V|a|$ the polar decomposition, then $|a| = V^*a$ and $a^* = |a|V^* = V^*aV^*$ belong to $\mathcal{J}$).

From the identity $4a^*b = (a+b)^*(a+b) - (a-b)^*(a-b) + i(a+ib)^*(a+ib) - i(a-ib)^*(a-ib)$,

if $d \geq 0$, it is majorated by an element of the form $\sum_{i=1}^{n} a_i^*a_i$ ($a_i \in \mathcal{M}$); hence $d \in \mathcal{F}$, so that $(\mathcal{M}\cdot\mathcal{M}) \cap P = \mathcal{F}$.

Let $a \in \mathcal{M}\cdot\mathcal{M}$. We have $a = a_1 + ia_2$ with $a_1, a_2 \in \mathcal{M}\cdot\mathcal{M}$, a_1 and a_2 self-adjoint. There are spectral projections p, q of a_1 such that $p+q = 1$, $pa_1 \geq 0$, $qa_1 \leq 0$. Then pa_1, $-qa_1 \in \mathcal{M}\cdot\mathcal{M} \cap P = \mathcal{F}$ and so $a_1 = (p+q)a_1 =$ linear combination of elements of $\mathcal{F}$; hence we obtain $\mathcal{M}\cdot\mathcal{M} =$ the totality of linear combinations of elements of $\mathcal{F}$. Therefore take $\mathfrak{m} = \mathfrak{n}\cdot\mathfrak{n}$, we conclude the first part of proposition 9.1.

Every element of $\mathfrak{m}$ is combination of elements of $\mathfrak{m} \cap P$, and the properties of $\bar{x}$ imply that there exists a linear functional $\bar{\bar{x}}$ on $\mathfrak{m}$ which coincides with $\bar{x}$ on $\mathfrak{m} \cap P$. If $a \in \mathfrak{m}$ and if u is unitary, then $\bar{\bar{x}}(u^*au) = \bar{\bar{x}}(a)$ by the properties of $\bar{\bar{x}}$; therefore $\bar{\bar{x}}(au) = \bar{\bar{x}}(u^*auu^*) = \bar{\bar{x}}(au)$; since every element of M is linear combination of unitary elements, we obtain $\bar{\bar{x}}(ab) = \bar{\bar{x}}(ba)$ for $a \in \mathfrak{m}$, $b \in M$. Finally, let $a \in \mathfrak{m}$ and put $\varphi(x) = \bar{\bar{x}}(ax)$ for $x \in M$. We shall show that φ is σ-continuous if $\bar{x}$ is normal.

It is enough to assume $a \geq 0$. Then for $x \geq 0$

$$\varphi(x) = \bar{\bar{x}}(ax^{1/2}x^{1/2}) = \bar{\bar{x}}(x^{1/2}ax^{1/2}) \geq 0 ;$$

therefore φ is positive.

Let (x_α) be a uniformly bounded increasing set of positive elements, then $\underset{\alpha}{\text{l.u.b}}\, a^{1/2}x_\alpha a^{1/2} = a^{1/2}(\underset{\alpha}{\text{l.u.b}}\, x_\alpha)a^{1/2}$; since $a^{1/2} \in \mathfrak{n}$, $x_\alpha^{1/2}a^{1/2} \in \mathfrak{n}$; hence $a^{1/2}x_\alpha a^{1/2} \in \mathfrak{m}$; put $x_\alpha^{1/2}a^{1/2} = V|x_\alpha^{1/2}a^{1/2}|$, then $a^{1/2}x_\alpha a^{1/2} = |x_\alpha^{1/2}a^{1/2}|^2$ and $x_\alpha^{1/2}ax_\alpha^{1/2} = V|x_\alpha^{1/2}a^{1/2}|^2V^* = V(a^{1/2}x_\alpha a^{1/2})V^*$; hence $\bar{\bar{x}}(a^{1/2}x_\alpha a^{1/2}) = \bar{\bar{x}}(V(a^{1/2}x_\alpha a^{1/2})V^*) = \bar{\bar{x}}(x_\alpha^{1/2}ax_\alpha^{1/2}) = \bar{\bar{x}}(ax_\alpha)$; therefore by the normality of $\bar{x}$, $\underset{\alpha}{\text{l.u.b}}\, \bar{\bar{x}}(ax_\alpha) = \bar{\bar{x}}(a(\underset{\alpha}{\text{l.u.b}}\, x_\alpha))$, so that φ is σ-continuous. This completes the proof.

The identically zero functional on M and the identically infinite functional on M with an exception of 0 are trivially traces, but such traces have nearly no meaning. One, which is interesting, is a W^*-algebra having a non-trivial, normal semi-trace.

In commutative algebras, any normal positive functional is a finite normal trace. In chapter II, we shall show there is a non-trivial normal semi-finite trace in a fairly wide class of W^*-algebras (semi-finite type).

Henceforward, in this section, we shall consider a W^*-algebra M which has a non-trivial, normal semi-finite trace.

At first, let M be a W^*-algebra with a normal finite trace $\bar{x}$ (therefore we can assume $\bar{x}$ to be a linear functional) and φ be a σ-continuous positive linear functional such that $\bar{x}(p) = 0$ implies $\varphi(p) = 0$ for projection $p \in M$, so that $s(\varphi) \leq s(\bar{x})$.

Let $e(\lambda) = s((\lambda\bar{x}-\varphi)^+)$ for non-negative number λ, then $|(\lambda\bar{x}-\varphi)^+(1-s(\bar{x}))| = |(\lambda\bar{x}-\varphi)(e(\lambda)(1-s(\bar{x})))| \leq \lambda\bar{x}(e(\lambda)(1-s(\bar{x}))) + \varphi(e(\lambda)(1-s(\bar{x}))) = 0$; hence $e(\lambda) \leq s(\bar{x})$.

Lemma 9.1. $e(\lambda) \leq e(\mu)$ for $\lambda \leq \mu$.

Proof. $\mu\bar{x}-\varphi = (\mu-\lambda)\bar{x} + (\lambda\bar{x}-\varphi) = (\mu-\lambda)R_{e(\lambda)}\bar{x} + (\lambda\bar{x}-\varphi)^+ + (\mu-\lambda)R_{1-e(\lambda)}\bar{x} - (\lambda\bar{x}-\varphi)^-$.

Since $R_{e(\lambda)}\bar{x}$, $R_{1-e(\lambda)}\bar{x} \geq 0$ and $\{(\mu-\lambda)R_{e(\lambda)}\bar{x} + (\lambda\bar{x}-\varphi)^+\} \perp |(\mu-\lambda)R_{1-e(\lambda)}\bar{x} - (\lambda\bar{x}-\varphi)^-|$, by the uniqueness of orthogonal decomposition $(\mu\bar{x}-\varphi)^+ \geq (\mu-\lambda)R_{e(\lambda)}\bar{x} + (\lambda\bar{x}-\varphi)^+ \geq (\lambda\bar{x}-\varphi)^+$; hence $e(\lambda) \leq e(\mu)$.

Lemma 9.2. $\lambda_n \leq \lambda$ $(n=1,2,\ldots)$ and $\lambda_n \Longrightarrow \lambda$ imply $e(\lambda_n) \Longrightarrow e(\lambda)(S)$.

Proof. $(\lambda\bar{x}-\varphi) - (\lambda_n\bar{x}-\varphi) = (\lambda-\lambda_n)\bar{x}$

$$= \{(\lambda\bar{x}-\varphi)^+ - (\lambda_n\bar{x}-\varphi)^+\} + \{(\lambda_n\bar{x}-\varphi)^- - (\lambda\bar{x}-\varphi)^-\} .$$

Since $(\lambda\bar{x}-\varphi)^+ \geq (\lambda_n\bar{x}-\varphi)^+$ and analogously $(\lambda_n\bar{x}-\varphi)^- \geq$

$(\lambda\bar{x}-\varphi)^-$, $\|(\lambda-\lambda_n)\bar{x}\| \geq \|(\lambda\bar{x}-\varphi)^+ - (\lambda_n\bar{x}-\varphi)^+\|$; hence $(\lambda_n\bar{x}-\varphi)^+ \to (\lambda\bar{x}-\varphi)^+$ (in norm).

Suppose that $e(\lambda_n) \not\to e(\lambda)$ (S), then there is a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \uparrow \lambda$ and $e = \text{l.u.b}_{n_j}\, e(\lambda_{n_j}) < e(\lambda)$.

On the other hand, let $p \leq 1-e$, then $(\lambda_{n_j}\bar{x}-\varphi)^+(p) = 0$, so that $(\lambda\bar{x}-\varphi)^+(p) = 0$; hence $p \leq 1-e(\lambda)$, a contradiction.

Lemma 9.3. $e(0) = 0$ and $\lim_{\lambda\to\infty} e(\lambda) = e(\infty) = s(\bar{x})$.

Proof. It is clear $e(0) = 0$. Put $\text{l.u.b}_{\lambda\geq 0}\, e(\lambda) = e(\infty)$ and let $p \leq s(\bar{x}) - e(\infty)$, then $(\lambda\bar{x}-\varphi)(p) = -(\lambda\bar{x}-\varphi)^-(p) \leq 0$ for all λ; hence $\lambda\bar{x}(p) \leq \varphi(p)$ for all $\lambda \geq 0$; hence $\bar{x}(p) = 0$ and so $p = 0$.

Lemma 9.4. φ is representable as follows: $\varphi = \int_0^\infty \lambda\, dR_{e(\lambda)}\bar{x}$. under the norm of M_*.

Proof. $(\lambda\bar{x}-\varphi)^+ = R_{e(\lambda)}(\lambda\bar{x}-\varphi) = \lambda R_{e(\lambda)}\bar{x} - R_{e(\lambda)}\varphi$

$$= L_{e(\lambda)}(\lambda\bar{x}-\varphi) = \lambda L_{e(\lambda)}\bar{x} - L_{e(\lambda)}\varphi .$$

Since $R_{e(\lambda)}\bar{x} = L_{e(\lambda)}\bar{x}$, $R_{e(\lambda)}\varphi = L_{e(\lambda)}\varphi = R_{e(\lambda)}L_{e(\lambda)}\varphi$ for all $\lambda \geq 0$.

Therefore, for $\mu \geq \lambda$

$$R_{e(\mu)}\varphi - R_{e(\lambda)}\varphi = R_{e(\mu)-e(\lambda)}\varphi = R_{e(\mu,\lambda)}L_{e(\mu,\lambda)}\varphi, \quad \text{where}$$

$$e(\mu,\lambda) = e(\mu)-e(\lambda) .$$

On the other hand,

$$L_{e(\mu,\lambda)}R_{e(\mu,\lambda)}(\mu\bar{x}-\varphi)^+ = L_{e(\mu,\lambda)}R_{e(\mu,\lambda)}R_{e(\mu)}(\mu\bar{x}-\varphi) =$$

$$L_{e(\mu,\lambda)}R_{e(\mu,\lambda)}(\mu\bar{x}-\varphi) = \mu R_{e(\mu,\lambda)}\bar{x} - R_{e(\mu,\lambda)}\varphi \geq 0$$

and,

$$L_{e(\mu,\lambda)}R_{e(\mu,\lambda)}(\lambda\bar{x}-\varphi) = -L_{e(\mu,\lambda)}R_{e(\mu,\lambda)}(\lambda\bar{x}-\varphi)^{-} \leq 0; \quad \text{hence}$$

$$\lambda(R_{e(\mu)}\bar{x} - R_{e(\lambda)}\bar{x}) \leq R_{e(\mu)}\varphi - R_{e(\lambda)}\varphi \leq \mu(R_{e(\mu)}\bar{x} - R_{e(\lambda)}\bar{x}) .$$

Hence, for any division $\Delta : 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \Lambda < \infty$ of the interval $(0,\Lambda)$ with $0 < \lambda_i - \lambda_{i-1} < \epsilon$ $(i=1,2,\ldots,n)$, we have

$$m(\Delta) \equiv \sum_{i=1}^{n} \lambda_{i-1} (R_{e(\lambda_i)}\bar{x} - R_{e(\lambda_{i-1})}\bar{x}) \leq \sum_{i=1}^{n} (R_{e(\lambda_i)}\varphi - R_{e(\lambda_{i-1})}\varphi)$$

$$= R_{e(\Lambda)}\varphi$$

$$\leq \sum_{i=1}^{n} \lambda_i (R_{e(\lambda_i)}\bar{x} - R_{e(\lambda_{i-1})}\bar{x}) \equiv M(\Delta)$$

and

$$M(\Delta) - m(\Delta) = \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})(R_{e(\lambda_i)}\bar{x} - R_{e(\lambda_{i-1})}\bar{x})$$

$$\leq \epsilon \sum_{i=1}^{n} (R_{e(\lambda_i)}\bar{x} - R_{e(\lambda_{i-1})}\bar{x})$$

$$= \epsilon (R_{e(\Lambda)}\bar{x} - R_{e(0)}\bar{x}) \leq \epsilon x \quad ;$$

hence by making $\epsilon \to 0$, $R_{e(\Lambda)}\varphi = \int_0^{\Lambda} \lambda \, dR_{e(\lambda)}\bar{x}$.

Moreover, $\lim_{\Lambda \to \infty} R_{e(\Lambda)}\varphi = R_{s(\bar{x})}\varphi = \varphi = \int_0^{\infty} \lambda \, dR_{e(\lambda)}\bar{x}$.

Therefore we obtain

Theorem 9.1. Let $\bar{x}$ be a normal finite trace on M, f a σ-continuous self-adjoint functional such that $\bar{x}(p) = 0$ implies $|f|(p) = 0$ for projection p. Then there is a system $(e(\lambda))$ of projections of M $(-\infty < \lambda < \infty)$ called the resolution of unity such that

(i) $\lambda \leq \mu$ implies $e(\lambda) \leq e(\mu)$

(ii) $\lambda_n \leq \lambda$ and $\lambda_n \to \lambda$ imply $e(\lambda_n) \to e(\lambda)$ (S)

(iii) $\lim_{\lambda \to -\infty} e(\lambda) = 0$ and $\lim_{\lambda \to \infty} e(\lambda) = s(\bar{x})$

(iv) $f = \int_{-\infty}^{\infty} \lambda dR_{e(\lambda)}\bar{x}$,

where the integration is of abstract Radon-Stieltjes type and

(v) $\|f\| = \int_{-\infty}^{\infty} |\lambda| \; d\|R_{e(\lambda)}\bar{x}\| = \int_{-\infty}^{\infty} |\lambda| \; d\bar{x}(e(\lambda))$.

Moreover such representation is unique.

The unicity of the representation can be easily proved from our construction.

Corollary. Let f be a σ-continuous self-adjoint linear functional which satisfies the above assumption, then

$$f(x) = \int_{-\infty}^{\infty} \lambda d\bar{x}(e(\lambda)x) \quad \text{for all} \quad x \in M.$$

Next we shall consider the case of having a normal semi-finite trace. Let $\bar{x}$ be a normal semi-finite trace of a W^*-algebra M and ψ be a σ-continuous positive linear functional such that $\bar{x}(p) = 0$ implies $\psi(p) = 0$ for projection $p \in M$. Then by Proposition 9.1, $\bar{x}$ is extendable to a unique linear functional $\bar{\bar{x}}$ on a two-sided ideal $\mathfrak{m}$ and $\mathfrak{m} \cap P =$ $\{ a \mid \bar{x}(a) < +\infty\}$.

Let P_1 be the set $h \in M$ such that $0 \leq h \leq 1$, and r_λ be the l.u.b. of $\{(\varphi-\lambda\bar{x})(h) \mid h \in P_1\}$ for positive number λ. Then there is a sequence (h_n) of P such that $r_\lambda = \lim_n$ $(\varphi-\lambda\bar{x})(h_n)$.

On the other hand, by the σ-compactness of P_1 there exists an accumulate point h_0 of (h_n) in P_1; hence we can construct a directed set (h_{n_α}) as follows: for any natural number m, we can take an α_m such that $n_\alpha \geq m$ if $\alpha \geq \alpha_m$ and $\lim_\alpha h_{n_\alpha} = h_0$.

Suppose that $\gamma_\lambda > \varphi(h_0) - \lambda\bar{x}(h_0) + \varepsilon$ for some $\varepsilon > 0$, then there is a projection p belonging to $\mathcal{M}$ as follows: $\gamma_\lambda > \varphi(h_0) - \lambda\bar{x}(h_0^{1/2}ph_0^{1/2}) + \frac{\varepsilon}{2} = \varphi(h_0) - \lambda\bar{x}(h_0 p) + \frac{\varepsilon}{2}$.

Since $\lim_\alpha \{\varphi(h_{n_\alpha}) - \lambda\bar{x}(h_{n_\alpha}p)\} = \varphi(h_0) - \lambda\bar{x}(h_0 p)$, we have $\gamma_\lambda > \varphi(h_{n_\alpha}) - \lambda\bar{x}(h_{n_\alpha}p) + \frac{\varepsilon}{4} \geq \varphi(h_{n_\alpha}) - \lambda\bar{x}(h_{n_\alpha}) + \frac{\varepsilon}{4}$ for $\alpha \geq \alpha'$; hence $\gamma_\lambda \geq \underset{\alpha \geq \alpha'}{\text{l.u.b}} (\varphi(h_{n_\alpha}) - \lambda\bar{x}(h_{n_\alpha})) + \frac{\varepsilon}{4} = \gamma_\lambda + \frac{\varepsilon}{4}$, this is a contradiction, so that $\gamma_\lambda = \varphi(h_0) - \lambda\bar{x}(h_0)$.

Let e be a projection such that $h_0 e > \mu e$ with a $\mu > 0$, then $\varphi(e) - \lambda\bar{x}(e) \geq 0$, for if $\varphi(e) - \lambda x(e) < 0$, $(\varphi - \lambda x)(h_0 - \mu e) > (\varphi - \lambda x)(h_0)$ and $h_0 - \mu e \in P_1$, a contradiction.

Since the range projection $s(h_0)$ of h_0 is a l.u.b of an increasing directed set of projections satisfying the above conditions $(\varphi - \lambda\bar{x})(s(h_0)) = \gamma_\lambda$.

Let p_1, p_2 be projections such that $(\varphi - \lambda\bar{x})(p_1) = (\varphi - \lambda\bar{x})(p_2) = \gamma_\lambda$, then $(\varphi - \lambda\bar{x})(\frac{p_1+p_2}{2}) = \gamma_\lambda$; hence $(\varphi - \lambda\bar{x})(s(\frac{p_1+p_2}{2})) = \gamma_\lambda$; therefore there is the greatest projection $p(\lambda)$ such that $(\varphi - \lambda\bar{x})(p(\lambda)) = \gamma_\lambda$.

Since $\frac{\varphi(p(\lambda))}{\lambda} \geq \bar{x}(p(\lambda))$, $p(\lambda) \in \mathcal{M}$. Let h be a self-adjoint element of M, then

$$0 \leq (\varphi - \lambda\bar{x})\{p(\lambda) - (1+\varepsilon ih)(1-\varepsilon ih)^{-1}p(\lambda)(1+\varepsilon ih)^{-1}(1-\varepsilon ih)\}$$

$$= (\varphi - \lambda\bar{x})(2\varepsilon i(p(\lambda)h - hp(\lambda)) + \varepsilon^2 p(\lambda)g_1 + \varepsilon^2 hp(\lambda)g_2 + \varepsilon^2 g_3 p(\lambda) + $$

$$+ \varepsilon^2 g_4 p(\lambda)h + \varepsilon^2 g_5 p(\lambda) g_6) =$$

$= 2\varepsilon i(\varphi-\lambda\bar{x})(p(\lambda)h-hp(\lambda)) + O(\varepsilon^2)$ for all small $|\varepsilon|$;

hence $(\varphi-\lambda\bar{x})(p(\lambda)h-hp(\lambda)) = 0$, so that $(\varphi-\lambda\bar{x})(p(\lambda)x) = (\varphi-\lambda\bar{x})(xp(\lambda)) = (\varphi-\lambda\bar{x})(p(\lambda)xp(\lambda))$ for all $x \in M$.

Now put $e(\lambda) = 1-p(\lambda)$, then $(\lambda\bar{x}-\varphi)(a) > 0$ for $0 < a \leq e(\lambda)$. Moreover

$$(\lambda\bar{x}-\varphi)(x) = (\lambda\bar{x}-\varphi)(e(\lambda)xe(\lambda)) + (\lambda\bar{x}-\varphi)(p(\lambda)xp(\lambda)) \quad \text{for} \quad x \in \mathfrak{M}.$$

Lemma 9.5. $e(\lambda) \leq e(\mu)$ for $\lambda < \mu$.

Proof. $\varphi-\lambda\bar{x} = \varphi-\mu\bar{x} + (\mu-\lambda)\bar{x} = R_{e(\mu)}(\varphi-\mu\bar{x}) + R_{p(\mu)}(\varphi-\mu\bar{x}) + (\mu-\lambda)R_{e(\mu)}\bar{x} + (\mu-\lambda)R_{p(\mu)}\bar{x} = R_{p(\mu)}(\varphi-\mu\bar{x}) + (\mu-\lambda)R_{p(\mu)}\bar{x} + R_{e(\mu)}(\varphi-\mu\bar{x}) + (\mu-\lambda)R_{e(\mu)}\bar{x}$ on $\mathfrak{M}$. Then,

$$(\varphi-\lambda\bar{x})(p(\lambda)) = (\varphi-\mu\bar{x})(p(\mu)p(\lambda)p(\mu)) + (\mu-\lambda)\bar{x}(p(\mu)p(\lambda)p(\mu))$$
$$+ (\varphi-\mu\bar{x})(e(\mu)p(\lambda)e(\mu)) + (\mu-\lambda)\bar{x}(e(\mu)p(\lambda)e(\mu)) .$$

Put $h = p(\mu) + e(\mu)p(\lambda)e(\mu)$, then $h \in P_1$ and $(\varphi-\lambda\bar{x})(h) = (\varphi-\mu\bar{x})(p(\mu)) + (\mu-\lambda)\bar{x}(p(\mu)) + (\varphi-\mu\bar{x})(e(\mu)p(\lambda)e(\mu)) + (\mu-\lambda)\bar{x}(e(\mu)p(\lambda)e(\mu)) \geq (\varphi-\lambda\bar{x})(p(\lambda))$; hence $\bar{x}(p(\mu)(1-p(\lambda))p(\mu)) = \bar{x}((1-p(\lambda))p(\mu)(1-p(\lambda))) = \bar{x}(e(\lambda)p(\mu)e(\lambda)) = 0$; on the other hand, $(\lambda\bar{x}-\varphi)(e(\lambda)p(\mu)e(\lambda)) \geq 0$; hence $(\lambda\bar{x}-\varphi)(e(\lambda)p(\mu)e(\lambda)) = 0$ and so $e(\lambda)p(\mu)e(\lambda) = 0$, so that $p(\mu) \leq 1-e(\lambda) = p(\lambda)$. This completes the proof.

Lemma 9.6. $\lambda_n \leq \lambda$ $(n=1,2,\ldots)$ and $\lambda_n \to \lambda$ imply $e(\lambda_n) \to e(\lambda)(s)$.

Proof. Suppose that $e(\lambda_n) \not\to e(\lambda)(s)$, then there is a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \uparrow \lambda$ and $p = \text{g.l.b}_{n_j}\, p(\lambda_{n_j}) > p(\lambda)$.

Let $0 < q < p-p(\lambda)$, then $(\lambda\bar{x}-\varphi)(q) > 0$ and $(\lambda_{n_j}\bar{x}-\varphi)(q) \leq 0$, so that $0 \geq \lim_{n_j} (\lambda_{n_j}\bar{x}-\varphi)(q) = (\lambda\bar{x}-\varphi)(q) > 0$; this is a contradiction.

Lemma 9.7. $\lim_{\lambda \to +0} e(\lambda) = e(+0) = s(\bar{x})-s(\varphi)$ and $\lim_{\lambda\to\infty} e(\lambda) = e(+\infty) = s(\bar{x})(s)$.

Proof. $(\lambda\bar{x}-\varphi)(e(\lambda)s'(\bar{x})e(\lambda)) = \lambda\bar{x}(e(\lambda)s'(\bar{x})e(\lambda)) - \varphi(e(\lambda)s'(\bar{x})e(\lambda)) = \lambda\bar{x}(e(\lambda)s'(\bar{x})) - \varphi(e(\lambda)s'(\bar{x})) = 0$, where $s'(\bar{x}) = 1 - s(\bar{x})$; hence $e(\lambda)s'(\bar{x})e(\lambda) = 0$ and $s'(\bar{x})e(\lambda) = 0$, so that $e(\lambda) \leq s(\bar{x})$; hence $e(+\infty) \leq s(\bar{x})$.

Conversely let $p \leq s(\bar{x})-e(\infty)$, then $p \leq s(\bar{x})-e(\lambda)$ for all $\lambda > 0$; hence $\lambda\bar{x}(p) \leq \varphi(p)$ and so $\bar{x}(p) = 0$; this means $p = 0$, so that $e(+\infty) = s(\bar{x})$.

Next let e be a projection belonging to $\mathfrak{M}$ such that $0 \leq e \leq e(+0)$, then $e \leq e(\mu)$ for all $\mu > 0$; hence $\mu\bar{x}(e) \geq \varphi(e)$ and so $e \leq s(\bar{x}) - s(\varphi)$; this means $e(+0) \leq s(\bar{x})-s(\varphi)$.

Conversely let p be a projection belonging to $\mathfrak{M}$ such that $p \leq s(\bar{x})-s(\varphi)$. Then,

$$(\lambda\bar{x}-\varphi)(p) = (\lambda\bar{x}-\varphi)(e(\lambda)pe(\lambda)) + (\lambda\bar{x}-\varphi)(p(\lambda)pp(\lambda)) = \lambda\bar{x}(p) \;;$$

hence $\lambda\bar{x}(e(\lambda)pe(\lambda)) = \lambda\bar{x}(p)$ and so $\bar{x}(p(1-e(\lambda))p) = 0$; therefore $p(1-e(\lambda))p = 0$ and $p = e(\lambda)p$, so that $s(\bar{x}) - s(\varphi) \leq e(\lambda)$ for all $\lambda > 0$; finally $s(\bar{x})-s(\varphi) \leq e(+0)$.

Lemma 9.8. $\varphi = \int_{+0}^{\infty} \lambda dR_{e(\lambda)}\bar{x}$ under the norm of M_*.

Proof. Let $\lambda > 0$, then $1-e(\lambda) = p(\lambda) \in \mathfrak{M}$, so that $e(\mu)-e(\lambda) \in \mathfrak{M}$ $(\mu \geq \lambda > 0)$. Since $R_{e(\lambda)}\varphi = L_{e(\lambda)}\varphi =$

$= R_{e(\lambda)}L_{e(\lambda)}\varphi$ for all $\lambda > 0$,

$$R_{e(\mu)-e(\lambda)}(\mu\bar{x}-\varphi) = R_{e(\mu)-e(\lambda)}R_{e(\mu)}(\mu\bar{x}-\varphi) \geq 0$$

$$R_{e(\mu)-e(\lambda)}(\lambda\bar{x}-\varphi) = R_{e(\mu)-e(\lambda)}R_{p(\lambda)}(\lambda\bar{x}-\varphi) \leq 0 \ ;$$

hence $\lambda(R_{e(\mu)-e(\lambda)}\bar{x}) \leq R_{e(\mu)}\varphi - R_{e(\lambda)}\varphi \leq \mu\ (R_{e(\mu)-e(\lambda)}\bar{x})$.

Let $\delta > 0$ and for any division Δ: $\delta = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \Lambda < +\infty$ of the interval (δ, Λ) with $0 < \lambda_i - \lambda_{i-1} < \varepsilon$ $(i=1,2,\ldots,n)$, we have

$$m(\Delta) \equiv \sum_{i=1}^{n} \lambda_{i-1}(R_{e(\lambda_i)-e(\lambda_{i-1})}\bar{x}) \leq \sum_{i=1}^{n} (R_{e(\lambda_i)}\varphi - R_{e(\lambda_{i-1})}\varphi)$$

$$= R_{e(\Lambda)}\varphi - R_{e(\delta)}\varphi \leq \sum_{i=1}^{n} \lambda_i(R_{e(\lambda_i)-e(\lambda_{i-1})}\bar{x}) \equiv M(\Delta)$$

and

$$M(\Delta) - m(\Delta) = \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})(R_{e(\lambda_i)-e(\lambda_{i-1})}\bar{x})$$

$$\leq \varepsilon\ (R_{e(\Lambda)-e(\delta)}\bar{x}) \ ;$$

hence making $\varepsilon \longrightarrow 0$, $R_{e(\Lambda)}\varphi - R_{e(\delta)}\varphi = \int_{\delta}^{\Lambda} \lambda dR_{e(\lambda)}\bar{x}$;

therefore by Lemma 9.7, $\lim_{\substack{\Lambda \to \infty \\ \delta \to +0}} (R_{e(\Lambda)}\varphi - R_{e(\delta)}\varphi) =$

$R_{e(+\infty)}\varphi - R_{e(+0)}\varphi = \varphi - 0 = \varphi = \int_{+0}^{\infty} \lambda dR_{e(\lambda)}\bar{x}$. Therefore we obtain

Theorem 9.2. Let $\bar{x}$ be a normal semi-finite trace on a W^*-algebra M, φ a σ-continuous positive linear functional such that $\bar{x}(p) = 0$ implies $\varphi(p) = 0$. Then there is a system $(e(\lambda))$ of projections of M $(0 < \lambda < +\infty)$, called the resolution of unity such that

(i) $\lambda \leq \mu$ implies $e(\lambda) \leq e(\mu)$

(ii) $\lambda_n \leq \lambda$ and $\lambda_n \longrightarrow \lambda$ imply $e(\lambda_n) \longrightarrow e(\lambda)$ (s)

(iii) $\lim_{\lambda \to +0} e(\lambda) = e(+0) = s(\bar{x}) - s(\varphi)$ and $\lim_{\lambda \to \infty} e(\infty) = s(\bar{x})$

(iv) $\bar{x}(1-e(\lambda)) < +\infty$ for $\lambda > 0$

(v)
$$\varphi = \int_{+0}^{\infty} \lambda dR_{e(\lambda)}\bar{x} = \lim_{\delta \to +0} \int_{\delta}^{\infty} \lambda dR_{e(\lambda)-e(\delta)}\bar{x},$$

where the integration is of abstract Radon-Stieltjes type under the norm of M_*

(vi) $\varphi(x) = \lim_{\delta \to +0} \int_{\delta}^{\infty} \lambda dR_{e(\lambda)-e(\delta)}\bar{x}(x)$ for all $x \in M$

(vii) $\varphi(a) = \int_{+0}^{\infty} \lambda d\bar{x}(e(\lambda)a)$ for $a \in \mathfrak{m}$

(viii) $\|\varphi\| = \lim_{\delta \to +0} \int_{\delta}^{\infty} \lambda d\|R_{e(\lambda)-e(\delta)}\bar{x}\| = \lim_{\delta \to +0} \int_{\delta}^{\infty} \lambda d\bar{x}(e(\lambda)-e(\delta)).$

Moreover such decomposition is unique.

Remark. It is an interesting, but very difficult problem to formulate a theorem of Radon-Nikodym's type without the assumption of semi-finite trace. If there is a normal semi-finite trace $\bar{x}$, we can prove the following: let φ and ψ be two σ-continuous positive linear functionals such that $\varphi \leq \psi$ and $s(\varphi), s(\psi) \leq s(\bar{x})$, then there exists an element t_0 of M with $\|t_0\| \leq 1$ such that $\varphi(x) = \psi(t_0 x t_0^*)$.

This form seems to be the simplest one in general W^*-algebras, which we can call a theorem of Radon-Nikodym; therefore it is interesting to study the following question: let φ and ψ two σ-continuous positive linear functionals such that $\varphi \leq \psi$. Then, is there an element t_0 of M with $\|t_0\| \leq 1$ such that $\varphi(x) = \psi(t_0 x t_0^*)$? We can show a more weak form as follows: Under the above assumptions we can take an element h_0 of M with $0 \leq h_0 \leq 1$ such that $\varphi(x) = \frac{1}{2}\psi(h_0 x + x h_0)$.

Proof. Let $A_1 = \{h \mid \|h\| \leq 1$ and $h^* = h\}$, then A_1 is σ-compact convex; hence under the mapping $h \longrightarrow \frac{1}{2}(L_h\psi + R_h\psi)$, the image $\mathcal{F}$ of P_1 is $\sigma(M_*, M)$-compact convex. Suppose that $\varphi \notin \mathcal{F}$, then $\varphi^o \not\supseteq \mathcal{F}^o$, where φ^o and $\mathcal{F}^o$ are polars of φ and $\mathcal{F}$ in the self-adjoint portion of M; hence there is an element x_0 of M with $x_0^* = x_0$ such that $|f(x_0)| \leq 1$ for all $f \in \mathcal{F}$ and $|\varphi(x_0)| > 1$. On the other hand, let $x_0 = x_0^+ - x_0^-$, then $\frac{1}{2}\psi\{(e-(1-e))x_0+x_0(e-(1-e))\} = \psi(x_0^+ + x_0^-) \geq \varphi(x_0^+ + x_0^-)$, where e is the range projection of x_0^+; hence $\varphi(x_0^+ + x_0^-) \leq 1$ and analogously $\frac{1}{2}\psi\{((1-e)-e)x_0 + x_0((1-e)-e)\} = -\psi(x_0^+ + x_0^-) \leq -\varphi(x_0^+ + x_0^-)$; hence $-\varphi(x_0^+ + x_0^-) \geq -1$, so that $-1 \leq -\varphi(x_0^+ + x_0^-) \leq \varphi(x_0) = \varphi(x_0^+) - \varphi(x_0^-) \leq \varphi(x_0^+ + x_0^-) \leq 1$; hence $|\varphi(x_0)| \leq 1$, a contradiction; therefore $\varphi(x) = \frac{1}{2}\psi(h_1 x + x h_1)$ for some $h_1 \in A_1$.

Moreover, since $\varphi \geq 0$, $\varphi(1) = \psi(h_1^+ - h_1^-) \geq \frac{1}{2}\psi(h_1(p-(1-p)) + (p-(1-p))h_1) = \psi(h_1^+ + h_1^-)$, where p is the range projection of h_1^+; hence $\psi(h_1^-) = 0$, so that $\varphi(x) = \frac{1}{2}\psi(h_1^+ x + x h_1^+)$. This completes the proof.

Notices of §9

The theorem of Radon-Nikodym tells us one aspect of W^*-algebras (at least, semi-finite ones); namely, a W^*-algebra is a non-commutative extension of L^{∞}-spaces and the associated space is also one of L^1-spaces.

From such point of view, Dixmier and Segal developed a non-commutative theory of L^p-spaces.

Dixmier constructed L^p-spaces by the abstract completion and showed L^p $(p > 1)$ is reflexive. On the other hand, Segal [33] realized L^p $(p=1, 2)$ by closed operators on hilbert spaces. From the results of this section and the polar decomposition of functionals, the reader can easily obtain the results of Segal. The method used here is analogous with one of Pukansky [28] in some parts.

§10. Continuity of Isomorphisms

Let D be a B^*-algebra, D^* the dual of D.

Definition 10.1. A subspace V of D^* is said invariant if $f \varepsilon V$ implies $L_a f$, $R_a f \varepsilon V$ for any $a, b \varepsilon D$.

Then,

Proposition 10.1. Let V be an invariant subspace of B^* which is $\sigma(D^*,D)$-dense in D^*, then $V \cap S^*$ is $\sigma(D^*,D)$-dense in S^*, where S^* is the unit sphere of D^*.

Proof. R_a is a linear operator on the normed space V and moreover $\| R_a f \| = \sup_{\|x\| \leq 1} |f(xa)| \leq \|f\| \, \|a\|$; hence $\|R_a\| \leq \|a\|$, where $\|R_a\|$ is the operator norm of R_a on V.

Suppose that $R_a = 0$, then $(R_a f)(x) = f(xa) = 0$ for all $f \varepsilon V$ and $x \varepsilon D$. Since V is $\sigma(D^*,D)$-dense in D^*, $ax = 0$ for all $x \varepsilon D$; hence $a = 0$. Moreover $R_{ab} = R_a R_b$ and so the mapping $a \longrightarrow R_a$ is an algebraic isomorphism; hence by the minimality of B^*-norm, $\|R_a\| = \|a\|$ for all $a \varepsilon D$. Therefore

$$\|a\| = \sup_{\substack{\|x\| \leq 1 \\ f \varepsilon V \cap S^*}} |f(xa)| = \sup_{\substack{\|x\| \leq 1 \\ f \varepsilon V \cap S^*}} |L_x f(a)| \leq \sup_{f \varepsilon V \cap S^*} |f(a)| \;;$$

so that $\|a\| = \sup_{f \varepsilon V \cap S} |f(a)|$ for all $a \varepsilon D$; hence the bipolar of $V \cap S^*$ in D^* is S^*, that is, $V \cap S^*$ is $\sigma(D^*,D)$-dense in S^*. This completes the proof.

In general, let E be a Banach space, E^* the dual of E and V be a subspace which is norm-closed and $\sigma(E^*,E)$-dense in E^*. Then, if any norm-closed, proper subspace of V is not $\sigma(E^*,E)$-dense in E^*, V is said to be minimal.

Then the following lemma is known.

Lemma 10.1. If there is a minimal subspace V in E^* such that $V \cap S^*$ is $\sigma(E^*, E)$-dense in S^*, where S^* is the unit sphere of E^*, then E is the dual of V.

From the above proposition and lemma, we can easily obtain

Proposition 10.2. For a given B^*-algebra D, if there is an invariant minimal subspace F in its dual D^*, it is a W^*-algebra and $\sigma(D,F)$ is the σ-topology of D.

Now we shall show the following theorem

Theorem 10.1. Let M be a W^*-algebra, N a B^*-algebra and ρ be an algebraic isomorphism (not necessarily adjoint preserving) of M onto N, then N is a W^*-algebra and ρ is σ-bicontinuous.

Proof. By Rickart's theorem, ρ is uniformly continuous. Let M^* and N^* be the duals of M and N respectively, and M_* the associated space of M. Then,

for any $f \in M^*$ and $x \in N$

$$< \rho^{-1}(x), f >_M = < x, (\rho^{-1})^*(f) >_N ,$$

where $<a,b>_M$ (resp. $<a',b'>_N$) is the value at a (resp. a') of a linear functional b of M^* (resp. b' of N^*), and $(\rho^{-1})^*$ is the dual of ρ^{-1}.

Since ρ^{-1} is uniformly bicontinuous, $(\rho^{-1})^*$ is a bi-continuous mapping of M^* with the topology $\sigma(M^*, M)$ onto N^* with topology $\sigma(N^*, N)$ and so $(\rho^{-1})^*(M_*)$ is a minimal sub-space of N^*. Moreover, if $(\rho^{-1})^*(\eta) \in (\rho^{-1})^*(M_*)$,

$$R_a L_b (\rho^{-1})^*(\eta)(x) = (\rho^{-1})^*(\eta)(bxa) = \langle bxa, (\rho^{-1})^*(\eta)\rangle_N$$
$$= \langle \rho^{-1}(bxa), \eta\rangle_M = \langle \rho^{-1}(b)\rho^{-1}(x)\rho^{-1}(a), \eta\rangle_M$$
$$= \langle \rho^{-1}(x), R_{\rho^{-1}(a)} L_{\rho^{-1}(b)} \eta\rangle_M$$
$$= \langle x, (\rho^{-1})^*(R_{\rho^{-1}(a)} L_{\rho^{-1}(b)} \eta)\rangle_N .$$

Since $R_{\rho^{-1}(a)} L_{\rho^{-1}(b)} \eta$ belongs to M_*, $(\rho^{-1})^*(M_*)$ is an invariant subspace of N^*; hence by Proposition 10.2, N is a W^*-algebra. As $\sigma(N, (\rho^{-1})^*(M_*))$ is the σ-topology of N, ϕ is σ-continuous. This completes the proof.

Corollary 10.1. Under the assumption of Theorem 10.1, ρ is τ-bicontinuous.

Corollary 10.2. Under the assumption of Theorem 10.1, ρ is s-bicontinuous on bounded spheres.

Proof. Let $\{x_\alpha\}$ $(\| x_\alpha \| \leq 1)$ be a directed set of M converging to 0 under the s-topology, then for any $f \in M_*$

$$|f(xx_\alpha)| = |R_V|f|(xx_\alpha)| = ||f|(xx_\alpha V)|$$
$$\leq |f|(x^* x)^{1/2} \; |f|(V^* x_\alpha^* x_\alpha V)^{1/2}$$
$$\leq \| f\|^{1/2} \; (L_{V^*} R_V |f|)(x_\alpha^* x_\alpha)^{1/2} \longrightarrow 0$$ (uniformly with respect to x $(\| x\| \leq 1)$).

For any $g \in N_*$,

$$|g(y\rho(x_\alpha))| = |\langle y\rho(x_\alpha), g\rangle_N|$$
$$= |\langle \rho(\rho^{-1}(y)x_\alpha), g\rangle_N| = \langle \rho^{-1}(y)x_\alpha, \rho^*(g)\rangle_M .$$

Since $\rho^*(g) \in M_*$, $g(y\rho(x_\alpha)) \longrightarrow 0$ (uniformly with respect to y such that $\|\rho^{-1}(y)\| \leq K$); hence if g is positive, $g(\rho(x_\alpha)^* \rho(x_\alpha)) \longrightarrow 0$, for $\{\rho(x_\alpha)\}$ are bounded in M. This completes the proof.

Corollary 10.3. Under the assumption of Theorem 10.1, and moreover if ρ is an $*$-isomorphism, ρ is s-bicontinuous.

Proof. Let φ be a positive functional on N, then $\rho^*(\varphi)$ is also positive on M and moreover

$<\rho(x^*)\rho(x), \varphi>_N = <\rho(x^*x), \varphi>_N = <x^*x, \rho^*(\varphi)>_M$; hence ρ is s-bicontinuous.

Remark. It is an open question whether the assumption of "adjoint preserving" can be removed in the above corollary.

Corollary 10.4. Let M be a W^*-algebra and ρ be an automorphism (not necessarily adjoint preserving) on M, then ρ is σ- and τ-bicontinuous, and it is s-bicontinuous on bounded spheres.

Remark. In Theorem 10.1, the assumption of "isomorphism" is essential -- in fact there is an $*$-homomorphism of a W^*-algebra onto another W^*-algebra which is not σ-continuous, we shall show this in Chapter II.

In order that a homomorphism of a W^*-algebra onto another W^*-algebra be σ-continuous, it is necessary and sufficient that its kernel is σ-closed.

Definition 10.2. Let ξ be a linear mapping of a W^*-algebra M into another W^*-algebra N. We call normal if it satisfies the following conditions

(i) $\xi(a) \geq 0$, if $a \geq 0$

(ii) $\xi(\underset{\alpha}{\text{l.u.b}}\, a_\alpha) = \underset{\alpha}{\text{l.u.b}}\, \xi(a_\alpha)$ for any uniformly bounded increasing directed set (a_α) of M.

Then

Proposition 10.3. Let ξ be a normal linear mapping of M into N, then ξ is σ-continuous.

Proof. Let $g \in N_*$ and positive, then

$$\begin{aligned}\xi^*(g)(\underset{\alpha}{\text{l.u.b}}\ a_\alpha) &= g(\xi(\underset{\alpha}{\text{l.u.b}}\ a_\alpha)) \\ &= g(\underset{\alpha}{\text{l.u.b}}\ \xi(a_\alpha)) = \underset{\alpha}{\text{l.u.b}}\ g(\xi(a_\alpha)) \\ &= \underset{\alpha}{\text{l.u.b}}\ \xi^*(g)(a_\alpha)\end{aligned}$$

for any uniformly bounded increasing directed set; hence $\xi^*(g) \in M_*$, so that ξ is σ-continuous.

Corollary 10.5. Let ρ be a normal *-homomorphism between W^*-algebras, then ρ is σ, τ and s-continuous.

Remark. In studying the structure of W^*-algebras, it is necessary to introduce some equivalent relation. The most natural equivalence relation is defined by *-isomorphism; if there is an *-isomorphism between two W^*-algebra M and N, we say M is equivalent to N and denote by $M \sim N$. Then we have important unsolved problems.

(i) Suppose that M is algebraically isomorphic to N. Then can we conclude $M \sim N$?

(ii) Suppose that M is anti-*-isomorphic to N. Then can we conclude $M \sim N$? (We mean by an anti-isomorphism ρ: $\rho(\lambda x+\mu y) = \lambda\rho(x)+\mu\rho(y)$, $\rho(x^*) = \rho(x)^*$ and $\rho(xy) = \rho(y)\rho(x)$). In all known examples of W^*-algebras, the problem (ii) is positive.

From the proof of Theorem 10.1, it is easy to show that an anti-algebraic isomorphism is σ-bicontinuous.

Notices of §10.

The continuity of *-isomorphism has been studied by a number of authors. Dixmier [3], using the normality, gave an elegant proof.

Here, we shall give the proofs of two lemmas used in this section.

1. The minimality of B^*-norm

(i) (Kaplansky) Let $C(\Omega)$ be a commutative B^*-algebra, $\|\cdot\|$ the B^*-norm and $\|\cdot\|_1$ another norm under which $C(\Omega)$ becomes a normed algebra, then $\|\cdot\| \leq \|\cdot\|_1$.

Proof. Let $t \in \Omega$, then $x \longrightarrow x(t)$ is a character; since from the property of $C(\Omega)$, $\|\cdot\|_1$-continuous characters are dense in Ω: therefore we have $|x(t)| \leq \|x\|_1$; hence $\|x\| = \sup_{t\in\Omega} |x(t)| \leq \|x\|_1$.

(ii) (Bonsall) Let D be a B^*-algebra, $\|\cdot\|$ the B^*-norm and $\|\cdot\|_1$ another norm under which D becomes a normed algebra. Then if $\|\cdot\|_1 \leq \|\cdot\|$, $\|\cdot\|_1 = \|\cdot\|$.

Proof. By the above lemma, $\|x\|^2 = \|x^*x\| \leq \|x^*x\|_1 \leq \|x^*\|_1 \|x\|_1 \leq \|x^*\|_1 \|x\| \leq \|x^*\| \|x\| = \|x\|^2$; hence $\|x\|_1 = \|x\|$.

2. Lemma 10.1 (Dixmier).

Proof. Let f be a bounded functional on V and put $\mathcal{N}_f = \{x \mid f(x) = 0, \ x \in V\}$, then by the minimality of V, $\mathcal{N}_f$ is not $\sigma(E^*, E)$-dense in E^*; hence there is an element g of E such that $g(\mathcal{N}_f) = 0$, so that $f = \lambda g$. This implies the dual of V coincides with E as the set.

Moreover $(V \cap S^*)^{00} = S^0$; hence $(V \cap S^*)^0 =$ the unit sphere of E, so that E is the dual of V.

§11. The continuity of Derivations.

A derivation of an algebra is a linear transformation $x \to x'$ of the algebra such that $(xy)' = x'y + xy'$. If a belongs to the algebra, the application $x \to xa - ax$ is the inner derivation defined by a.

Put $D(x) = xa-ax$, then $\rho_t(x) = (\exp tD)(x)$ is an inner automorphism and $\lim_{t \to 0} \frac{\rho_t(x)-\rho_0(x)}{t} = D(x)$.

Proposition 11.1. Let B be a commutative B^*-algebra, then every derivation $x \to x'$ of B is identically zero.

Proof. It is enough to prove that $x' = 0$ for all self-adjoint element x of B. Let $C(\Omega)$ be the function-representation of B and λ be any point of Ω, then $\{x - x(\lambda)I\}' = x'-x(\lambda)I'$; since $I' = (I\cdot I)' = I'\cdot I + I\cdot I' = 2I'$; hence $I' = 0$, so that $x' = \{x-x(\lambda)I\}'$.

Write $x-x(\lambda)I$ as the difference of two positive elements, $x-x(\lambda)I = x_1-x_2$ with $\{x-x(\lambda)I\}(\lambda) = x_1(\lambda) = x_2(\lambda) = 0$. We have $x_1 = h^2$, and so $x_1' = h'h + hh' = 2hh'$; hence $x_1'(\lambda) = 2h(\lambda)h'(\lambda) = 0$ and analogously $x_2'(\lambda) = 0$; therefore $x'(\lambda) = x_1'(\lambda)-x_2'(\lambda) = 0$ for all $\lambda \in \Omega$, so that $x' = 0$. This completes the proof.

Theorem 11.1. Every derivation of a B^*-algebra is automatically continuous.

Proof. Let $\mathfrak{A}$ be a B^*-algebra, $'$ a derivation of $\mathfrak{A}$. It is enough to show that the derivation is continuous on the self-adjoint portion $\mathfrak{A}^s$ of $\mathfrak{A}$. Therefore if it is not continuous, by the closed graph theorem there is a sequence $\{x_n\}$

$(x_n \neq 0)$ in $\mathfrak{A}^s$ such that $x_n \longrightarrow 0$ and $x_n' \longrightarrow a+ib$ $(\neq 0)$, where a and b are self-adjoint. First, suppose that $a \neq 0$ and there exists a positive number $\lambda(>0)$ in the spectrum of a (otherwise consider $\{-x_n\}$). It is enough to assume that $\lambda=1$.

Then there is a positive element h $(\|h\|=1)$ of $\mathfrak{A}$ such that $hah \geq \frac{1}{2}h^2$. Put $y_n = x_n+3 \cdot \|x_n\| \cdot I$, then $y_n \longrightarrow 0$, $y_n' = x_n'$ and $(hy_nh)' = h'y_nh + hy_n'h + hy_nh'$; hence $(hy_nh)' \longrightarrow h(a+ib)h$.

Therefore

$$\|(hy_{n0}h)' - h(a+ib)h\| < \frac{1}{8} \text{ for some } n_0 \cdots (1)$$

On the other hand

$$hy_nh \leq 4\|x_n\|h^2 \text{ and } \frac{1}{2} \cdot \frac{hy_nh}{4\|x_n\|} \leq hah \cdots (2)$$

Since $\|x_n\| \cdot I + x_n \geq 0$, $\dfrac{hy_nh}{4\|x_n\|} \geq \frac{1}{2}h^2$.

Hence

$$\left\| \frac{hy_nh}{4\|x_n\|} \right\| \geq \frac{1}{2}\|h\|^2 = \frac{1}{2} \cdots (3)$$

Let C be a B^*-subalgebra of $\mathfrak{A}$ generated by $hy_{n0}h$ and I, then by the (3) there is a character φ of C such that $\varphi\left(\dfrac{hy_{n0}h}{4\|x_{n0}\|}\right) \geq \frac{1}{2}$.

Let $\bar{\varphi}$ be an extended positive linear functional of φ on $\mathfrak{A}$, and $\mathfrak{M} = \{x \mid \bar{\varphi}(x^*x) = 0, x \in \mathfrak{A}\}$, then $C \cap \mathfrak{M}$ is a maximal ideal of C; it can be written $hy_{n0}h - \varphi(hy_{n0}h) \cdot I = u^2 - v^2$ with $u, v \in C \cap \mathfrak{M}$ $(u, v \geq 0)$; hence $(hy_{n0}h)' = u'u + uu' - v'v - vv'$, so that by Schwartz's inequality

$$\bar{\varphi}((hy_{n0}h)') = 0 \cdots (4)$$

Then by the (1) and (4)

$$|\bar{\varphi}(h(a+ib)h)| < \frac{1}{8} \quad \cdots \quad (5)$$

On the other hand by the (2)

$$|\bar{\varphi}(h(a+ib)h)| \geq \bar{\varphi}(hah) = \frac{1}{2}\bar{\varphi}\left(\frac{hy_{n0}h}{4\|x_{n0}\|}\right) \geq \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

This contradicts the above inequality (5), so that $a = 0$.

Next suppose that $b \neq 0$ and there exists a positive number $\mu(>0)$ in the spectrum of b (otherwise consider $\{-x_n\}$). It is enough to assume that $\mu = 1$. Then there is a positive element k ($\|k\| = 1$) of $\mathfrak{A}$ such that $kbk \geq \frac{1}{2}k^2$; moreover $\|(ky_{n1}k)' - k(a+ib)k\| < \frac{1}{8}$ for some n_1.

Let C_1 be a B^*-subalgebra of $\mathfrak{A}$ generated by $ky_{n1}k$ and I, then there is a character φ_1 of C_1 such that $\varphi_1\left(\frac{ky_{n1}k}{4\|x_{n1}\|}\right) \geq \frac{1}{2}$. Let $\bar{\varphi}_1$ be an extended positive linear functional of φ_1 on $\mathfrak{A}$, then $\bar{\varphi}_1((ky_{n1}k)') = 0$; hence $|\bar{\varphi}_1(k(a+ib)k)| < \frac{1}{8}$.

On the other hand

$$|\bar{\varphi}_1(k(a+ib)k)| \geq \bar{\varphi}_1(kbk) \geq \bar{\varphi}_1(\tfrac{1}{2}k^2) \geq \frac{1}{2}\bar{\varphi}_1\left(\frac{ky_{n1}k}{4\|x_{n1}\|}\right) \geq \frac{1}{4};$$

hence $|\bar{\varphi}_1(k(a+ib)k)| \geq \frac{1}{4}$.

This contradicts the above inequality; hence $b = 0$, so that $a+ib = 0$. Now we obtain a contradiction and this completes the proof.

Remark 1. In Proposition 11.1 and Theorem 11.1, the assumption that an algebra has unit I is unnecessary. Indeed, if the algebra has no unit, we may adjoin I and define $I' = 0$;

then the extended '-operation is also a derivation on the extended algebra; hence our proof is available for any case.

Remark 2. It is not known, even in commutative semi-simple Banach algebras, whether every derivation of semi-simple Banach algebras is automatically continuous. This open question is very important. If we can solve this positively, we can assert that a derivation in commutative semi-simple Banach algebras is identically zero, because every continuous derivation in the algebras is identically zero by Theorem of Singer and Wermer.

Finally we show,

Theorem 11.2. Let M be a W^*-algebra, $'$ a derivation on M, then it is σ-continuous and τ-continuous, and moreover s-continuous on bounded spheres.

Proof. Put $D(x) = x'$, then by the above theorem, D is bounded; therefore we can consider linear transformations $\exp t D$; since $\exp t D$ are automorphisms on D for t real number, it is σ and τ-continuous. Moreover $\lim_{t \to 0} \| \frac{\exp(tD) - II}{t} - D \| = 0$, where II is the identity automorphism on M.

Hence for any $f \in M_*$

$$|f(D(x)) - f((\frac{\exp tD - II}{t})(x))| = |f((D - \frac{\exp tD - II}{t})(x))|$$

$$\leq \|f\| \, \|D - \frac{\exp tD - II}{t}\| \, \|x\| \underset{t \to 0}{\longrightarrow} 0 \quad \text{(uniformly for}$$

$f \in M_*$ $(\|f\| \leq 1)$ and $x \in M$ $(\|x\| \leq 1)$).

Since $g(x) = f(\frac{\exp tD - II}{t} x)$ belongs to M_* for each t, $g_1(x) = f(Dx)$ belongs also to M_*; hence D is σ-continuous.

Next let G be a relatively $\sigma(M_*, M)$-compact set of M_*, then $D^*(G)$ is also relatively $\sigma(M_*, M)$-compact, for D^* is continuous under the topology $\sigma(M_*, M)$; hence D is τ-continuous.

Finally let $\{x_\alpha\}$ $(\| x_\alpha \| \leq 1)$ be a directed set converging to 0 under the $s(M, M_*)$, then $\{ \frac{\exp tD - II}{t} x_\alpha \}$ converges to 0 in the s-topology. Then for any $\varphi \in T$

$$\left| \varphi\{(Dx_\alpha)^*(Dx_\alpha)\} - \varphi\{(\frac{\exp tD - II}{t} x_\alpha)^* (\frac{\exp tD - II}{t} x_\alpha)\} \right|$$

$$\leq \| (Dx_\alpha)^*(Dx_\alpha) - (\frac{\exp tD - II}{t} x_\alpha)^* (\frac{\exp tD - II}{t}) x_\alpha \|$$

$$\leq \| Dx_\alpha \| \| Dx_\alpha - \frac{\exp tD - II}{t} x_\alpha \| +$$

$$+ \| \frac{\exp tD - II}{t} x_\alpha \| \| Dx_\alpha - \frac{\exp tD - II}{t} x_\alpha \|$$

$$\leq M \| D - \frac{\exp tD - II}{t} \| \longrightarrow 0 \quad (t \longrightarrow 0) ;$$

hence $\varphi((Dx_\alpha)^*(Dx_\alpha)) \longrightarrow 0$, so that D is s-continuous on bounded spheres.

Notices of §11

Concerning the continuity of a derivation of commutative semi-simple Banach algebras, a partial answer was given by Curtis, Jr. [45].

Here, to state some question, we use the notion of type of W^*-algebras. The reader may refer to Chapter II and the Book of Dixmier. Kaplansky showed that every automorphism of a W^*-algebra of type I which fixes elements of the center is inner and he also did that every derivation of a W^*-algebra of type I is inner. On the other hand, Dixmier [5] and Singer [47] showed that there are outer automorphisms in a W^*-algebra of type II_1. Therefore, we have the following natural question: Is there an outer derivation in a W^*-algebra?

§12. Isometry between W^*-algebras

Let M be a W^*-algebra, N a B^*-algebra and ρ be a linear isometry of M onto N, then N is a dual space; hence it is also a W^*-algebra. Moreover $\rho(1)$ is an extreme point of the unit sphere of N, so that $\rho(1)$ is a partial isometry.

Lemma 12.1. $\rho(1)$ is unitary in N.

Proof. Suppose that $e = \rho(1)^*\rho(1) < 1$, then

$$\|\rho(1) + \lambda(1-e)\| = \|(\rho(1) + \lambda(1-e))(\rho(1) + \lambda(1-e)^*)^*\|^{1/2}$$
$$= \|\rho(1)\rho(1)^* + \lambda^2(1-e)\|^{1/2} \leq (1+\lambda^2)^{1/2} \quad \text{for } \lambda \geq 0 .$$

On the other hand, let $\rho^{-1}(1-e) = a_1 + ia_2$ (a_1, a_2 self-adjoint), then it is enough to assume that there is a positive number α in the spectrum of a_1 (otherwise consider $\pm i(1-e)$ or $-(1-e)$).

Then

$$\|1 + \rho^{-1}(\lambda(1-e))\| = \|1 + \lambda\rho^{-1}(1-e)\| \geq (1+\lambda\alpha) \quad \text{for } \lambda \geq 0.$$

Hence

$$(1+\lambda^2)^{1/2} \geq (1+\lambda\alpha) \quad \text{for } \lambda \geq 0 ;$$

therefore $(1+\lambda^2) - (1+\lambda\alpha)^2 = (1-\alpha^2)\lambda^2 - 2\lambda\alpha \geq 0$ and so $(1-\alpha^2)\lambda - 2\alpha \geq 0$ for $\lambda \geq 0$; this is a contradiction, since $0 \leq \alpha \leq 1$, so that $\rho(1)^*\rho(1) = 1$. Analogously we obtain $\rho(1)\rho(1)^* = 1$. Therefore $\xi = \rho(1)^*\rho$ is an isometry of M onto N which takes 1 into 1.

Lemma 12.2 $\xi(a)$ is self-adjoint for self-adjoint a.

Proof. Put $\xi(a) = b_1 + ib_2$ (b_1, b_2 self-adjoint). Suppose that $b_2 \neq 0$, then it is enough to assume that there is a positive number β in the spectrum of b_2 (otherwise consider $-a$).

Then

$$(\|a\|^2 + n^2)^{1/2} \geq \|\xi(a+in1)\| \geq n+\beta \quad \text{for all positive } n.$$

This is a contradiction; hence $\xi(a)$ is self-adjoint.

Lemma 12.3. $\xi(a)$ is a projection for projection p.

Proof. $2p-1$ is self-adjoint and unitary, so that $2\xi(p)-1$ is also self-adjoint, unitary; hence $\xi(p)$ is a projection.

Therefore ξ preserves the orthogonality of projections, and so we obtain

Theorem 12.1. Let ρ be a linear isometry of a W^*-algebra M onto a B^*-algebra N, then N is a W^*-algebra, ρ is σ-bicontinuous, $\rho(1)$ is unitary, and $\xi = \rho(1)^*\rho$ is a linear isometry satisfying $\xi(ab) = \xi(a)\xi(b)$, where a and b are mutually commuting self-adjoint elements of M.

Remark. Using the structure theorem of W^*-algebras, we shall prove that the above ξ is a sum of *-isomorphism and *-anti-isomorphism in Chapter II.

Notices of § 12

Theorem 12.1 is due to Kadison [8]. For a commutative B^*-algebra, Theorem 12.1 is the classical theorem of Banach and Stone. In general, it is meaningful to find Banach algebras in which the isometry induces the isomorphism.

Nagasawa [50] showed that a Banach algebra $H^\infty(\Omega)$ of analytic functions has such property.

§13. Representation theorem.

Let $\mathfrak{h}$ be a complex hilbert space, $B(\mathfrak{h})$ the set of all bounded operators on $\mathfrak{h}$. Such set is a B^*-algebra.

Let $f,g \in \mathfrak{h}$. The function $x \longrightarrow |\langle xf,g\rangle|$ is a semi-norm on $B(\mathfrak{h})$. The set of these semi-norms defines a separate locally convex topology, called the weak operator topology.

Theorem 13.1. Let M be a W^*-algebra. Then it is faithfully representable as a weakly closed $*$-subalgebra of $B(\mathfrak{h})$ on some hilbert space $\mathfrak{h}$, and moreover under any such representation the σ-topology is equivalent to the weak operator topology on bounded spheres.

Proof. Let T_1 be a complete set of σ-continuous positive linear functionals such that $\varphi(1) = 1$. Let $\{\pi_\varphi, \mathfrak{h}_\varphi\}$ be the $*$-representation of M on a hilbert space $\mathfrak{h}_\varphi$, constructed via the element φ of T_1. Let $\mathfrak{h}$ be the direct sum of the $\mathfrak{h}_\varphi$; $\mathfrak{h} = \sum_{\varphi \in T_1} \oplus \mathfrak{h}_\varphi$. We shall consider a representation π of M on $\mathfrak{h}$ defined as follows: $\pi(x) = \sum_{\varphi \in T_1} \oplus \pi_\varphi(x)$. Then it is a faithful B^*-representation of M. Let $B(\mathfrak{h})$ be the algebra composed of all bounded operators on $\mathfrak{h}$.

Let $\mathfrak{A}$ be the weak closure of $\pi(M)$ in $B(\mathfrak{h})$, then by Kaplansky's theorem (Theorem 4.1), the unit sphere of $\pi(M)$ is dense in the unit sphere of $\mathfrak{A}$; therefore for any $A \in \mathfrak{A}$ $(\|A\| \leq 1)$ there is a directed set $\{\pi(x_\alpha)\}$ $(x_\alpha \in S)$ such that weak $\lim_\alpha \pi(x_\alpha) = A$.

Therefore

$$(\pi(x_\alpha)b_\varphi, a_\varphi)_\varphi = \varphi(a^* x_\alpha b) \longrightarrow (Ab_\varphi, a_\varphi)_\varphi \quad \text{for any } a,b \in M,$$

where $(\; , \;)_\varphi$ is the inner product of $\mathfrak{h}_\varphi$, and d_φ, b_φ are images of a, b in $\mathfrak{h}_\varphi$.

Since $R_b L_{a^*}\varphi \in M_*$ and T_1 is complete, $\{R_b L_{a^*}\varphi \mid a, b \in M, \varphi \in T_1\}$ is total in M_*; hence the bounded set $\{x_\alpha\}$ converges to some x_0 in the σ-topology. Hence

$$(\pi(x_0)b_\varphi, a_\varphi)_\varphi = \varphi(a^* x_0 b) = \lim_\alpha \varphi(a^* x_\alpha b) = \lim_\alpha (\pi(x_\alpha)b_\varphi, a_\varphi)$$

$$= (Ab_\varphi, a_\varphi)_\varphi .$$

Since linear combinations of the images M_φ of M in $\mathfrak{h}_\varphi$ are dense in $\mathfrak{h}$, $A = \pi(x_0)$, so that $\pi(M) = \mathfrak{A}$.

Moreover, since the unit sphere of $\mathfrak{A}$ is weakly compact, by the uniqueness of the σ-topology, the σ-topology is equivalent to the weak operator topology on bounded spheres.

This completes the proof.

Conversely,

Theorem 13.2. A weakly closed *-subalgebra on a hilbert space is a W^*-algebra.

Proof. Let $\mathfrak{A}$ be a weakly closed *-subalgebra on a hilbert space, then its unit sphere is weakly compact; hence by the well known theorem of Banach spaces, $\mathfrak{A}$ is a dual space. This completes the proof.

Notices of §13

Hitherto, regarding the following result as a well known one, we have developed our discussions: Let B be a B^*-algebra (that is, $\|x^*x\| = \|x\|^2$), then x^*x is positive. This important theorem is not classical.

Here, we shall sketch topics concerning that theorem.

Gelfand and Naimark [6] gave intrinsic three postulates for C^*-algebras as follows: (1) $\|x^*x\| = \|x^*\|\,\|x\|$, (2) $\|x^*\| = \|x\|$, (3) $x^*x + 1$ has an inverse.

(Of course, (1) and (2) can be replaced by $\|x^*x\| = \|x\|^2$.) The first conjecture of Gelfand and Naimark is that the condition (3) is unnecessary -- namely x^*x is always positive. This conjecture has been solved by M. Fukamiya, Kelley-Vaught and Kaplansky for B^*-algebras with unit, and Rickart for ones without unit.

The second conjecture is that the condition (2) is also unnecessary. This conjecture has recently been solved by Ono, Kadison-Glimm and Rickart.

The reader shall refer to the book of Rickart and Naimark.

§14. Extension of functionals

Definition 14.1. Let M be a W^*-algebra, N a σ-closed $*$-subalgebra of M, then we call N a W^*-subalgebra of M.

Let N be a W^*-subalgebra of M, φ be a σ-continuous positive linear functional on N. Then, since N is the dual of M_*/V, where V is the polar of N in M_*, there is a self-adjoint element f of M_* such that $f = \varphi$ on N.

Let $f = f^+ - f^-$ and $\psi = f^+ + f^-$, then ψ is positive on M and $\varphi \leq \psi$ on N. Let $\{\pi_\psi, \mathfrak{H}_\psi\}$ be the representation of M constructed by ψ, then

$$0 \leq \varphi(a^*a) \leq \psi(a^*a) = (\pi_\psi(a)\psi, \pi_\psi(a)\psi) \quad \text{for } a \in N ;$$

hence define $L(\pi_\psi(a)\psi, \pi_\psi(b)\psi) = \varphi(b^*a)$ for $a, b \in N$, then L is a bounded bilinear functional on the pre-hilbert space $\pi_\psi(N)\psi$, so that it is extended to a bounded bilinear functional on the hilbert space $[\pi_\psi(N)\psi]$. Let ξ, η be arbitrary elements of $\mathfrak{H}_\psi$ and let $\xi = \xi_1 + \xi_2$ and $\eta = \eta_1 + \eta_2$ ($\xi_1, \eta_1 \in [\pi_\psi(N)\psi]$, and $\xi_2, \eta_2 \in [\pi_\psi(N)\psi]^\perp$). Then we define $\widetilde{L}(\xi, \eta) = L(\xi_1, \eta_1)$.

$\widetilde{L}$ is a bounded bilinear functional on $\mathfrak{H}$, so that there is a bounded operator A on $\mathfrak{H}$ as follows:

$$\widetilde{L}(\xi, \eta) = (A\xi, \eta) \quad \text{for } \xi, \eta \in \mathfrak{H}.$$

Moreover since $\widetilde{L}(\xi, \xi) \geq 0$, $A \geq 0$, and

$$(A\pi_\psi(ac)\psi, \pi_\psi(b)\psi) = \widetilde{L}(\pi_\psi(ac)\psi, \pi_\psi(b)\psi) = \varphi(b^*ac) = \varphi((a^*b)^*c)$$
$$= (A\pi_\psi(c)\psi, \pi_\psi(a^*b)\psi) = (\pi_\psi(a)A\pi_\psi(c)\psi, \pi_\psi(b)\psi) \quad \text{for } a, b, c \in N ;$$

hence $EA\pi_\psi(a)E = E\pi_\psi(a)AE$, where E is the orthogonal projection of $\mathfrak{h}$ onto $[\pi_\psi(N)\psi]$. Since $[\pi_\psi(N)\psi]$ is invariant under $\pi_\psi(N)$, $EAE\pi_\psi(a) = \pi_\psi(a)EAE$ for $a \varepsilon N$. Since $A \geq 0$, $EAE \geq 0$; put $A_1 = (EAE)^{1/2}$, then

$$\begin{aligned}\varphi(a) &= L(\pi_\psi(a)\psi,\psi) = (A\pi_\psi(a)\psi,\psi)\\ &= (EAE\pi_\psi(a)\psi,\psi) = (A_1\pi_\psi(a)\psi, A_1\psi)\\ &= (\pi_\psi(a)A_1\psi, A_1\psi) \quad \text{for } a \varepsilon N.\end{aligned}$$

Define $\tilde{\varphi}(x) = (\pi_\psi(x)A_1\psi, A_1\psi)$ for $x \varepsilon M$, then $\tilde{\varphi} = \varphi$ on N and $\tilde{\varphi} \geq 0$ on M. Hence we obtain

Lemma 14.1. Let φ be a σ-continuous positive linear functional on a W^*-subalgebra of M, then it can be extensible to a σ-continuous positive linear functional on M.

Now we shall show

Theorem 14.1. Let f be a σ-continuous linear functional on a W^*-subalgebra of M, then it can be extensible to a σ-continuous linear functional $\tilde{f}$ on M such that $\|\tilde{f}\| = \|f\|$.

Proof. Let $f = R_V|f|$ be the polar decomposition of f, then by Lemma 14.1, $|f|$ can be extensible to a σ-continuous positive linear functional φ on M, and put $\tilde{f} = R_V\varphi$, then

$$\tilde{f}(a) = \varphi(aV) = |f|(aV) \quad \text{for } a \varepsilon N$$

and $\|\tilde{f}\| = \|R_V\varphi\| \leq \|\varphi\| = \varphi(1) = |f|(1) = \|f\|$; hence $\|\tilde{f}\| = \|f\|$. This completes the proof.

Notices of § 14

Theorem 14.1 is a theorem of Hahn-Banach type. We may find a special property of W^*-algebras as Banach spaces in the theorem. In fact, this theorem is negative for vector spaces. Here, we shall show a counter example.

Suppose that there is a positive element h with continuous spectrum in a W^*-algebra M, and let $C(\Omega)$ be the function representation of the B^*-algebra generated by h and 1, then we can take a positive element k such that $k(t_0) = 1$ and $k(t) < 1$ for $t \neq t_0$. Let V be a two-dimensional subspace of M such that $V = \{\lambda 1 + \mu k\}$, then V is $\sigma(M, M_*)$-closed. Put $\varphi(\lambda 1 + \mu k = \lambda + \mu k(t_0)$, then φ is a linear functional on V with norm 1.

If we can extend φ to $\bar{\varphi}$ on M such that $\bar{\varphi} \in M_*$ and $\|\bar{\varphi}\| = \|\varphi\|$, $\bar{\varphi}$ is a σ-continuous positive functional.

Since $\bar{\varphi}(1-k) = 0$, $\bar{\varphi}(S(1-k)) = 0$. On the other hand, $S(1-k) = 1$, a contradiction.

§15. Examples

1. Let B be a Banach algebra with unit 1, B^* the dual of B, and B^{**} the second dual of B. For any $a, b, x \varepsilon B$, put $L_a x = ax$ and $R_b x = xb$, then L_a and R_b are mutually commuting bounded operators. Let L_a^*, R_b^* be the duals of L_a and R_b, and L_a^{**} and R_b^{**} be the second duals of them. Then we have $R_b^{**} L_a^{**} = L_a^{**} R_b^{**}$.

Let $\mathfrak{A}$ be an algebra of $\sigma(B^{**}, B^*)$-continuous operators on B^{**} commuting with R_b^{**} (all $b \varepsilon B$). Since $B \subset B^{**}$, we can consider $L1$ for $L \varepsilon \mathfrak{A}$. Let L_1 and L_2 be elements belonging to $\mathfrak{A}$ such that $L_1 1 = L_2 1$, then

$$R_a^{**} L_1 1 = L_1 a = R_a^{**} L_2 1 = L_2 a \quad \text{for all } a \varepsilon B.$$

Since L_1, L_2 are $\sigma(B^{**}, B^*)$-continuous, so that by the correspondence $L \longrightarrow L1$ of $\mathfrak{A}$ into B^{**} is one-to-one.

Moreover

$$\| L \| = \sup_{\substack{\|y\| \leq 1 \\ y \varepsilon B^{**}}} \| Ly \| = \sup_{\substack{\|a\| \leq 1 \\ a \varepsilon B}} \| La \| = \sup \| R_a^{**} L1 \| \leq \| a \| \, \| L1 \|$$

$$\leq \| L \| ;$$

hence $\| L \| = \| L1 \|$, so that the mapping is isometric.

Let S be the unit sphere of $\mathfrak{A}$, then elements of $\mathfrak{A}$ are $\sigma(B^{**}, B^*)$-continuous, so that there is a bounded operator P on B^* for any $L \varepsilon S$ such that $P^* = L$.

Put $\mathfrak{S} = \{ P \mid \text{all } P^* \varepsilon S \}$ and $\mathfrak{S}_f = \{ Pf \mid P^* \varepsilon \mathfrak{S} \}$ for any $f \varepsilon B^*$, then $\mathfrak{S}_f$ is relatively $\sigma(B^*, B)$-compact in B^*.

Let $\bar{\sigma}_f$ be the $\sigma(B^*,B)$-closure of σ_f in B^*, then $\bar{\sigma}_f$ is $\sigma(B^*,B)$-compact, so that by Tchiconoff's theorem

$$\Omega = \prod_{f\varepsilon B^*} \bar{\sigma}_f$$

is weakly compact. By the mapping $P \xrightarrow{\rho} \{P_f\}$ of σ into Ω, we introduce a topology on σ. Let $\{g_f\}$ be an element of the $\overline{\rho(\sigma)}$, then there is a directed set $\{P_\alpha\}$ such that

$$\lim_\alpha P_\alpha f = g_f \quad \text{for all} \quad f \varepsilon B^* ;$$

hence when we put $P_0 f = g_f$, then P_0 is a bounded operator on B^*. Moreover

$$< P_0^* y, f > = < y, P_0 f > = \lim_\alpha < y, P_\alpha f > \quad \text{for} \quad y \varepsilon B \quad \text{and} \quad f \varepsilon B^*.$$

Therefore

$$<R_a^{**} P_0^* x, f > = < P_0^* x, R_a^* f > = < x, P_0 R_a^* f > = \lim_\alpha < x, P_\alpha R_a^* f >$$
$$= \lim_\alpha < R_a^{**} P_\alpha^* x, f > = \lim_\alpha < P_\alpha^* R_a^{**} x, f >$$
$$= \lim_\alpha < xa, P_\alpha f > = < P_0^* R_a^{**} x, f > \quad \text{for } x \varepsilon B \text{ and } f \varepsilon B^*;$$

hence $R_a^{**} P_0^* = P_0^* R_a^{**}$ for all $x \varepsilon B$.

Therefore $\rho(\sigma)$ is compact. Let $\{P_\alpha\}$ be a directed set convergent to P_0 in the topology, then $\sigma(B^*,B) - \lim_\alpha P_\alpha f = P_0 f$ for all $f \varepsilon B^*$. Hence

$$\lim_\alpha < P_\alpha^* 1, f > = \lim_\alpha < 1, P_\alpha f > = < 1, P_0 f > = < P_0^* 1, f > ;$$

hence $S \cdot 1$ is $\sigma(A^{**}, A^*)$-compact in B^{**}, so that the mapping $L \longrightarrow L \cdot 1$ of $\mathfrak{A}$ into B^{**} is an onto-mapping, so that B^{**}

is isometrically isomorphic to a Banach algebra $\mathfrak{A}$, that is, the second dual of B^{**} of a Banach algebra B is also a Banach algebra, and moreover when B is canonically imbedded into B^{**}, it is a subalgebra of B^{**}, for $(L_1 1)\cdot(L_2 1) = L_1 L_2 \cdot 1$ is defined, so that $(L_a^{**}1)(1_b^{**}1) = a\cdot b = L_a^{**}L_b^{**}1 = ab$.

Hence we obtain

Theorem 15.1. Let B be a Banach algebra, then the second dual B^{**} of B is also a Banach algebra, and when B is canonically imbedded into B^{**}, B is a subalgebra of B.

Remark. Even if B is commutative, B^{**} is not necessarily commutative, and also there is a semi-simple Banach algebra of which the second dual is not semi-simple [cf. 44].

Remark. The assumption of unit in the above theorem is not essential. Indeed, if B has no unit, we consider $A + (\alpha 1)$, then $(A+\alpha 1)^{**}$ is a Banach algebra and $(A^{*}+(\alpha)1)^{**} = A^{**}+(\alpha)1$, so that A^{**} is a Banach algebra.

In case that B is a B^*-algebra, the situation is more exact. Suppose that B is a B^*-algebra, then it can be represented as a B^*-algebra B_α on a hilbert space $\mathfrak{H}_\alpha$. We denote the representation by π_α. Let π be the direct sum representation of all representations $\pi\alpha$, $\overline{\pi(B)}$ the weak closure of $\pi(B)$, then $\overline{\pi(B)}$ is a W^*-algebra. Let F be the associated space of $\overline{\pi(B)}$, then by Kaplansky's density theorem

$$\|f\|_{\pi(B)} = \|f\| \quad \text{for any } f \in F .$$

Therefore

$$B \xrightarrow{\pi} \pi(B)$$

$$B^* \xleftarrow{\pi^*} \pi(B)^* \supset F$$

$$B^{**} \xrightarrow{\pi^{**}} \pi(B)^{**} \supset V : F^* = \overline{\pi(B)} = \pi(B)^{**}/V, \text{ where } V = \text{the polar of } F.$$

Lemma 15.1. Let g be a continuous linear functional on $\pi(B)$, then it is a linear combination of positive functional on $\pi(B)$.

Proof. It is enough to suppose that g is self-adjoint and $\|g\| = 1$. Let $\pi(B)_0$ be the self-adjoint portion of $\pi(B)$, and σ be the totality of positive functionals of norm ≤ 1 on $\pi(B)$, then the polar of σ = the unit sphere of $\pi(B)_0$; hence $g \in$ the $\sigma(\pi(B)^*, \pi(B))$-closure of the convex hull of σ and $-\sigma$; since σ is $\sigma(\pi(B)^*, \pi(B))$-compact convex set, we obtain $g = \lambda_1\psi_1 - \lambda_2\psi_2$, where $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, $\psi_1, \psi_2 \in \sigma$. This completes the proof.

Since any positive linear functional φ constructs a representation $\{\pi_\varphi, \mathfrak{h}_\varphi\}$, $\varphi \in F$; hence all linear functionals of $\pi(B)$ belongs to F, that is, $\pi(B)^* = F$; therefore $V = 0$.

Hence,

$$B \xrightarrow{\pi} \pi(B)$$

$$B^* \xleftarrow{\pi^*} F$$

$$B^{**} \xrightarrow{\pi^{**}} \overline{\pi(B)}$$

This means,

Theorem 15.2. Let B be a B^*-algebra, then the second dual B^{**} is a W^*-algebra, and when B is canonically imbedded into B^{**}, B is a $*$-subalgebra of B^{**}.

Remark 1. Using this theorem, we can reduce many problems concerning B^*-algebras to ones concerning W^*-algebras. In general, such reduction shall make their studies easier, though it may miss elegancy. For example, Theorem 12.1 concerning a linear isometry between W^*-algebras is extended to C^*-algebras by the above theorem.

Corollary 15.1. Let A be a B^*-algebra, A^* the dual of A and V be an invariant closed subspace of A^* under R_a, L_b $(a, b \in A)$, then V is algebraically spanned by positive linear functionals belonging to itself.

Proof. Let A^{**} be the second dual of A and V^0 the polar of V in A^{**}, that is, $V^0 = \{a \mid |\langle a, f\rangle| \leq 1,\ a \in A^{**}$ and $f \in V\}$, then it is a $\sigma(A^{**}, A^*)$-closed ideal of A^{**}, for $|\langle bac, V\rangle| = |\langle a, L_b R_c V\rangle| \leq |\langle a, V\rangle| \leq 1$ for $a \in V^0$ and $b, c \in A$; hence $bac \in V^0$.

Since A is $\sigma(A^{**}, A)$-dense in A^{**}, this means $bac \in V^0$ for $b, c \in A^{**}$, so that V^0 is an ideal. Since $V^* = A^{**}/V^0$ and A^*/V^0 is a C^*-algebra, it is a W^*-algebra; hence by Theorem 6.1, V is algebraically spanned by positive linear functionals belonging to V. This completes the proof.

2. The dual of W^*-algebras. Let M be a W^*-algebra, M_* the associated space, M^* the dual of M and M^{**} the second dual. Let M_*^0 be the polar of M_* in M^{**}; since M_* is invariant, M_*^0 is a $\sigma(M^{**}, M^*)$-closed ideal; therefore M_*^0 is also a W^*-algebra; hence M_*^0 has unit z in itself and so $M_*^0 = M_*^0 z = z M_*^0 = M^{**} z = z M^{**}$; therefore for arbitrary $a \in M^{**}$,

$az = az \cdot z = zaz$ and $za = z \cdot za = zaz$, so that $az = za$; hence z is a projection belonging to the center of M^{**}, which we call a central projection.

Therefore $M^{**} = M^{**}(1-z) \oplus M^{**}z = M \oplus M^{**}z$ and $M \cong M^{**}(1-z)$; hence $M^{*} = R_{(1-z)}M^{*} \oplus R_z M^{*} = M_{*} \oplus R_z M^{*}$ and $M^{*} = R_{(1-z)}M^{*}$.

Hence we have

Theorem 15.3. Let M be a W^{*}-algebra, M_{*} the associated space of M and M^{*} the dual of M, then there is a linear mapping $R_{(1-z)}$ of M^{*} onto M_{*} satisfying the following

(i) $R^2_{1-z} = R_{1-z}$

(ii) $\|R_{1-z}f\| \leq \|f\|$ for $f \in M^{*}$

(iii) $R_{1-z}f \geq 0$, if $f \geq 0$

(iv) M_{*} is an invariant closed subspace of M^{*} under R_a, L_b $(a, b \in M^{**})$.

Theorem 15.4. Let M be a W^{*}-algebra, M^{**} the second dual of M, then there is a σ-continuous $*$-homomorphism ρ of the W^{*}-algebra M^{**} onto the W^{*}-algebra M.

Proof. $M^{**} = M^{**}(1-z) \oplus M^{**}z = M \oplus M^{**}z$ and $M \cong M^{**}(1-z)$. The mapping ρ_1 $(x \to x(1-z))$ of M^{**} onto $M^{**}(1-z)$ is a σ-continuous $*$-homomorphism and the mapping ρ_2 $(y \to y(1-z))$ of M onto $M^{**}(1-z)$ is a $*$-isomorphism; hence $\rho = \rho_2^{-1}\rho_1$ is a σ-continuous $*$-homomorphism of M^{**} onto M. This completes the proof.

Remark 2. We call singular a positive linear functional belonging to $R_z M^*$. Then, it is known that a positive linear functional φ on M is singular if and only if for any non-zero projection $p \in M$ there is a projection q such that $\varphi(q) = 0$ and $0 < q \leq p$ [cf. 36].

3. Let (Ω, μ) be a measure space, $L^\infty(\Omega, \mu)$ the B^*-algebra of all essentially bounded μ-measurable functions on Ω, then $L^\infty(\Omega, \mu)$ is the dual space of $L^1(\Omega, \mu)$; hence $L^\infty(\Omega, \mu)$ is a W^*-algebra and $L^1(\Omega, \mu)$ is the associated space of $L^\infty(\Omega, \mu)$.

Conversely, let M be a commutative W^*-algebra and M_* the associated space, then M is of type $C(K)$, where we mean the Banach algebra of all continuous functions on a compact space K and so the dual space M^* is an (AL)-space; hence by Theorem 15.3 M_* is also an (AL)-space; hence M is a $L^\infty(\Omega, \mu)$ on some measure space (Ω, μ).

Hence we obtain

Theorem 15.5. A commutative B^*-algebra is a W^*-algebra if and only if it is isomorphic to an $L^\infty(\Omega, \mu)$ on some measure space (Ω, μ).

Moreover by Theorem 5.2, we obtain a new Banach space-like characterization of the space $L^1(\Omega, \mu)$ as follows:

Theorem 15.6. A Banach space is of type $L^1(\Omega, \mu)$, if and only if its dual is of type $C(K)$.

4. Let $\mathbb{C}$ be an operator algebra of all completely continuous operators on a hilbert space $\mathfrak{h}$, $\mathbb{C}^*$ and $\mathbb{C}^{**}$ be the dual and second dual of $\mathbb{C}$ respectively, then $\mathbb{C}^{**}$ is a

W^*-algebra.

Let 1 be the unit of $\mathbb{C}^{**}$, $\{e_\alpha\}$ a maximal family of orthogonal projections in $\mathbb{C}$. Put $p_\beta = \sum_{\alpha \varepsilon \Omega_\beta} e_\alpha$, where Ω_β is a finite set of $\{\alpha\}$, then $\{p_\beta\}$ is an increasing directed set of projections, so that there is a projection p such that $p = \sigma\text{-}\lim_\beta p_\beta$.

Since a is a c.c. operator, $\lim_\beta \| p_\beta a p_\beta - a \| = 0$ for any $a \varepsilon \mathbb{C}$; hence $a \varepsilon p\mathbb{C}^{**}p$ and so $\mathbb{C} \subset p\mathbb{C}^{**}p$; since $p\mathbb{C}^{**}p$ is σ-closed, $p\mathbb{C}^{**}p = \mathbb{C}^{**}$; hence $p = 1$.

Let φ be a positive functional with norm 1 on $\mathbb{C}$, then $\lim_\beta \varphi(p_\beta) = \varphi(1) = 1$; hence there is an index β_n such that $\varphi(1-p_{\beta_n}) < \frac{1}{n}$. Then

$$\sup_{\substack{\|a\| \leq 1 \\ a \varepsilon \mathbb{C}}} |\varphi(a) - \varphi(p_{\beta_n} a p_{\beta_n})| \leq \sup |\varphi((I-p_{\beta_n}) a p_{\beta_n}) + \varphi(p_{\beta_n} a (1-p_{\beta_n})) + \varphi(I-p_{\beta_n}) a (I-p_{\beta_n}))|$$

$$\leq 3\varphi(1-p_{\beta_n})^{1/2} \longrightarrow 0 \; ;$$

hence $\lim_n \|\varphi - L_{p_{\beta_n}} R_{p_{\beta_n}} \varphi \| = 0$.

Since $L_{p_{\beta_n}} R_{p_{\beta_n}} \varphi$ is zero on $(I-p_{\beta_n}) \mathbb{C} p_{\beta_n} + p_{\beta_n} \mathbb{C} (1-p_{\beta_n}) + (1-p_{\beta_n}) \mathbb{C} (1-p_{\beta_n})$, $L_{\beta_n} R_{\beta_n} \varphi$ can be considered a positive functional on $p_{\beta_n} \mathbb{C} p_{\beta_n}$. Since $p_{\beta_n} \mathbb{C} p_{\beta_n}$ is finite dimensional, there is a positive element $p_{\beta_n} a_n p_{\beta_n}$ of $p_{\beta_n} \mathbb{C} p_{\beta_n}$ as follows:

$$\varphi(p_{\beta_n} a p_{\beta_n}) = \mathrm{Tr}(p_{\beta_n} a_n p_{\beta_n} p_{\beta_n} a p_{\beta_n})$$

$$= \mathrm{Tr}(p_{\beta_n} a_n p_{\beta_n} a) \; .$$

Moreover

$$\| L_{p_{\beta_n}} R_{p_{\beta_n}} \varphi - L_{p_{\beta_m}} R_{p_{\beta_m}} \varphi \| = Tr(\{(p_{\beta_n} a_n p_{\beta_n} - p_{\beta_m} a_m p_{\beta_m})^2\}^{1/2})$$

$$= \| p_{\beta_n} a_n p_{\beta_n} - p_{\beta_m} a_m p_{\beta_m} \|_1 .$$

Therefore $\{p_{\beta_n} a_n p_{\beta_n}\}$ is a Cauchy sequence with trace-norm; hence there is a trace-class operator a_0 such that $\lim_n \varphi(p_{\beta_n} a p_{\beta_n}) = Tr(a_0 a)$; hence $\varphi(a) = Tr(a_0 a)$. Since the trace-class $\mathbb{T} \subset \mathfrak{C}^*$ is trivial, $\mathbb{T} = \mathfrak{C}^*$. Since $\mathbb{T}^*$ is $B(\mathfrak{h})$, $\mathfrak{C}^{**} = B(\mathfrak{h})$.

Therefore we obtain the following theorem

Theorem 15.7. Let $\mathfrak{C}$ be the operator algebra of all completely continuous operators on a hilbert space $\mathfrak{h}$, then $\mathfrak{C}^* = \mathbb{T}$ and $\mathfrak{C}^{**} = B(\mathfrak{h})$, where $\mathbb{T}$ is the Banach space of all trace-class operators on $\mathfrak{h}$ and $B(\mathfrak{h})$ is the operator algebras of all bounded operators on $\mathfrak{h}$.

Remark 3. The problem whether this theorem can be extended to general Banach spaces is an important one. A Grothendieck has solved this problem for some special cases [49].

Remark 4. The relation between a W^*-algebra M and its associated space M_* can be considered a non-commutative extension of the relations between L^∞ and L^1. Therefore we have many extension problems for the classical theorems in L^∞ and L^1-spaces. It seems to be valuable that the reader will try such plan. Here we shall show some examples of the problems. It is known that L^1-spaces are $\sigma(L^1, L^\infty)$-sequentially complete.

This fact is extendable to general W^*-algebras. On the other hand, let Ω be a discrete space, then $c_0(\Omega)^* = \ell^1(\Omega)$ and $\ell^1(\Omega)^* = \ell^\infty(\Omega)$; analogously by Theorem 15.7, $\mathbb{C}^* = \mathbb{T}$ and $\mathbb{T}^* = B(\mathfrak{h})$. Moreover $\sigma(\ell^1, \ell^\infty)$-sequential convergence in ℓ^1 is equivalent to norm-convergence; but this fact is not true in the space $\mathbb{T}$.

Notices of 15

Theorem 15.2 is the theorem of Sherman, and the proof was firstly given by Takeda [35].

Theorem 15.7 is the theorem of von Neumann-Schatten and Dixmier.

Chapter III The theory of representations

§1. The commutation theorem of von Neumann:

In §13, Chapter I, we gave a short-cut proof for the representation theorem of W^*-algebras. For the completeness, at first, we shall show a fundamental theorem concerning weakly closed *-subalgebras on a hilbert space.

For this, we shall provide some tools.

Let $\mathfrak{H}$ be a hilbert space, $B(\mathfrak{H})$ the set of all bounded operators on $\mathfrak{H}$ and $\mathfrak{M}$ be a subset of $B(\mathfrak{H})$.

We denote by $\mathfrak{M}'$ the set of elements of $B(\mathfrak{H})$ commuting with all elements of $\mathfrak{M}$ and call $\mathfrak{M}'$ the commutant of $\mathfrak{M}$. Put $(\mathfrak{M}')' = \mathfrak{M}''$ (bicommutant of $\mathfrak{M}$), $(\mathfrak{M}'')' = \mathfrak{M}'''$,... . It is clear that $\mathfrak{M}'$ is a subalgebra of $B(\mathfrak{H})$ containing the identity operator; $\mathfrak{M}'' \supset \mathfrak{M}$ and $\mathfrak{M} \subset \mathfrak{N}$ implies $\mathfrak{M}' \supset \mathfrak{N}'$ and so $\mathfrak{M}'' \subset \mathfrak{N}''$; therefore $\mathfrak{M}' \supset (\mathfrak{M}'')' = \mathfrak{M}'''$; on the other hand $\mathfrak{M}' \subset (\mathfrak{M}')'' = \mathfrak{M}'''$; hence $\mathfrak{M}' = \mathfrak{M}''' = \mathfrak{M}^{(5)} = \ldots$

$\mathfrak{M} \subset \mathfrak{M}'' = \mathfrak{M}^{(4)} = \ldots$

If $\mathfrak{M}$ is a self-adjoint set, $\mathfrak{M}'$ is a *-subalgebra of $B(\mathfrak{H})$.

Next, let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be hilbert spaces, and $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ be the algebraic tensor product of $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Then there is a unique pre-hilbert structure on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ such that

$$(\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2) = (\xi_1, \eta_1)(\xi_2, \eta_2)$$

for $\xi_1, \eta_1 \varepsilon \mathfrak{H}_1$ and $\xi_2, \eta_2 \varepsilon \mathfrak{H}_2$,

where (,) is the inner product of $\mathfrak{H}_1$, $\mathfrak{H}_2$ and $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ respectively.

The hilbert space obtained by the completion of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$

is called the tensor product of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, and denoted by $\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$.

Let $a_1 \in B(\mathfrak{H}_1)$ and $a_2 \in B(\mathfrak{H}_2)$ then the algebraic tensor product $a_1 \odot a_2$ defines a continuous linear operator on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$; therefore it can be uniquely extended to a bounded operator on $\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$; we denote by $a_1 \otimes a_2$ that operator. Then $a_1 \otimes a_2$ is bilinear for a_1 and a_2, $(a_1 b_1) \otimes (a_2 b_2) = (a_1 \otimes b_1)(a_2 \otimes b_2)$ and $(a \otimes b)^* = a^* \otimes b^*$.

Let $(\eta_\alpha)_{\alpha \in \mathbb{I}}$ be a complete ortho-normal system of $\mathfrak{H}_2$, then the mapping $\xi_1 \longrightarrow \xi_1 \otimes \eta_\alpha$ is an isometry u_α of $\mathfrak{H}_1$ onto a closed subspace $\mathfrak{H}^\alpha$ of $\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$; $\mathfrak{H}^\alpha$ are mutually orthogonal; the vector subspace generated by $\{\mathfrak{H}^\alpha \mid \alpha \in \mathbb{I}\}$ is dense in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$; hence

$$\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2 = \sum_{\alpha \in \mathbb{I}} \mathfrak{H}^\alpha .$$

u_α^* is a linear mapping of $\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$ onto $\mathfrak{H}_1$ such that $u_\alpha^* (\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2 \ominus \mathfrak{H}^\alpha) = 0$; it is also an isometry on $\mathfrak{H}^\alpha$; $u_\alpha^* u_\alpha$ is the identity operator on $\mathfrak{H}_1$ and $u_\alpha u_\alpha^*$ is the projection e_α of $\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$ onto $\mathfrak{H}^\alpha$.

Let $a \in B(\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2)$, then $u_\alpha^* a u_\beta \in B(\mathfrak{H}_1)$; put $a_{\alpha\beta} = u_\alpha^* a u_\beta$, then a is perfectly determined by the matrix $(a_{\alpha\beta})$ -- in fact, if $a_{\alpha\beta} = b_{\alpha\beta}$ for $\alpha, \beta \in \mathbb{I}$, $u_\alpha^* a u_\beta = u_\alpha^* b u_\beta$; therefore $e_\alpha a e_\beta = e_\alpha b e_\beta$ and so $a = b$; moreover $(\lambda a)_{\alpha\beta} = \lambda a_{\alpha\beta}$, $(a+b)_{\alpha\beta} = a_{\alpha\beta} + b_{\alpha\beta}$, $(a^*)_{\alpha\beta} = (a_{\beta\alpha})^*$ and $(ab)_{\alpha\beta}\xi_1 = u_\alpha^*(ab)u_\beta \xi_1 = u_\alpha^* a (\sum_{\gamma \in \mathbb{I}} u_\gamma u_\gamma^*) b u_\beta \xi_1 = \sum_{\gamma \in \mathbb{I}} a_{\alpha\gamma} b_{\gamma\beta} \xi_1$ (in norm of $\mathfrak{H}_1$) for $\xi_1 \in \mathfrak{H}_1$.

Moreover,

$$(a_1 \otimes 1)_{\alpha\beta}\xi_1 = u_\alpha^*(a_1 \otimes 1)u_\beta \xi_1 = u_\alpha^*(a_1 \otimes 1)\xi_1 \otimes \eta_\beta$$
$$= u_\alpha^*(a_1\xi_1 \otimes \eta_\beta) = \delta_{\alpha\beta} a \xi_1 \quad \text{for} \quad a_1 \in B(\mathfrak{h}_1) \quad \text{and}$$

$\xi_1 \in \mathfrak{h}_1$; hence $(a_1 \otimes 1)_{\alpha\beta} = \delta_{\alpha\beta} a_1$, where $\delta_{\alpha\beta}$ is the symbol of Kronecker.

Lemma 1.1. If $a \in B(\mathfrak{h}_1 \bar{\otimes} \mathfrak{h}_2)$ commutes with $u_\alpha u_\beta^*$, a is of the form $a_1 \otimes 1$ with $a_1 \in B(\mathfrak{h}_1)$.

Proof. $a_{\alpha\beta} = u_\alpha^* a u_\beta = u_\gamma^* u_\gamma u_\alpha^* a u_\beta = u_\gamma^* a u_\gamma u_\alpha^* u_\beta$. Since $u_\alpha^* u_\beta = 0$ for $\alpha \neq \beta$ and $u_\alpha^* u_\alpha = 1$, $a_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $a_{\alpha\alpha} = u_\gamma^* a u_\gamma$ for all γ; hence $a_{\alpha\beta} = \delta_{\alpha\beta} a_1$ with $a_1 \in B(\mathfrak{h}_1)$. This completes the proof.

Lemma 1.2. For a subset D of $B(\mathfrak{h}_1)$ containing 0, let $\mathcal{M}_D$ be the set of $B(\mathfrak{h}_1 \bar{\otimes} \mathfrak{h}_2)$ such that $a_{\alpha\beta} \in D$, then $(D \otimes 1)' = \mathcal{M}_{D'}$ and $(D \otimes 1)'' = D'' \otimes 1$; moreover if D contains the identity, $(\mathcal{M}_D)' = D' \otimes 1$ and $(\mathcal{M}_D)'' = \mathcal{M}_{D''}$.

Proof. Let $a_1 \otimes 1$ $(a_1 \in D)$ and $b \in (D \otimes 1)'$, then

$$\{(a_1 \otimes 1)b\}_{\alpha\beta} = a_1 b_{\alpha\beta}$$
$$\{b(a_1 \otimes 1)\}_{\alpha\beta} = b_{\alpha\beta} a_1 \;;$$

hence $(D \otimes 1)' \subset \mathcal{M}_{D'}$ and the converse is clear; hence $(D \otimes 1)' = \mathcal{M}_{D'}$.

Moreover, $(u_\gamma u_\delta^*)_{\alpha\beta} = u_\alpha^* u_\gamma u_\delta^* u_\beta = 0 \quad (\alpha \neq \gamma \text{ or } \beta \neq \delta)$
$= 1 \quad (\alpha = \gamma \text{ and } \beta = \delta)$;

therefore $u_\gamma u_\delta^* \in \mathcal{M}_{D'}$; hence $(D \otimes 1)'' = (\mathcal{M}_{D'})' \subset B(\mathfrak{h}_1) \otimes 1$ by Lemma 1 and $(D \otimes 1)'' \subset (D' \otimes 1)' = \mathcal{M}_{D''}$, so that $(D \otimes 1)'' = D'' \otimes 1$.

Finally, suppose $1 \in D$, then $u_\gamma u_\delta^* \in \mathcal{M}_D$;

$(\mathcal{M}_D)' \subset B(\mathfrak{H}_1)\otimes 1$ and $\mathcal{M}_D \supset D\otimes 1$; hence $(\mathcal{M}_D)' = D'\otimes 1$; $(\mathcal{M}_D)'' = (D'\otimes 1)' = \mathcal{M}_D''$. This completes the proof.

In particular, $\{B(\mathfrak{H}_1)\otimes 1\}' = \mathcal{M}_{(\lambda 1)}$; on the other hand, if $a_{\alpha\beta} = \lambda_{\alpha\beta}1$ for $\alpha, \beta \in \mathbb{I}$, where $\lambda_{\alpha\beta}$ are complex numbers, then $a_{\alpha\beta} = u_\alpha^* a u_\beta = \lambda_{\alpha\beta}1$; hence $e_\alpha a e_\beta = \lambda_{\alpha\beta} u_\alpha u_\beta^*$; therefore $a\xi_1\otimes\eta_\beta = ae_\beta\xi_1\otimes\eta_\beta = \sum_{\alpha\in\mathbb{I}} e_\alpha a e_\beta \xi_1\otimes\eta_\beta = \sum_{\alpha\in\mathbb{I}} \lambda_{\alpha\beta}\xi_1\otimes\eta_\alpha =$ $\xi_1\otimes(\sum_{\alpha\in\mathbb{I}}\lambda_{\alpha\beta}\eta_\alpha)$ for all $\beta\in\mathbb{I}$ (in norm of $\mathfrak{H}_1\bar{\otimes}\mathfrak{H}_2$); therefore $a = 1\otimes a_2$ with $a_2 \in B(\mathfrak{H}_2)$; hence we have $(B(\mathfrak{H}_1)\otimes 1)' = \mathcal{M}_{(\lambda 1)} = 1\otimes B(\mathfrak{H}_2)$, and $(B(\mathfrak{H}_1)\otimes 1)'' = (1\otimes B(\mathfrak{H}_2))' = B(\mathfrak{H}_1)''\otimes 1$ $= B(\mathfrak{H}_1)\otimes 1$.

Now, let $B(\mathfrak{H})$ be the set of all bounded operators on a hilbert space, $B(\mathfrak{H})$ is a W^*-algebra, and its associated space is considered the space $\mathbb{I}$ of all trace-class operators on $\mathfrak{H}$. We can consider three topologies σ, s and τ on $B(\mathfrak{H})$; moreover we shall consider the following two topologies:

(i) The strong operator topology (the so-topology)

Let $\xi \in \mathfrak{H}$. The function $x \longrightarrow \|x\xi\|$ is a semi-norm on $B(\mathfrak{H})$. The set of such semi-norms defines a separate locally convex topology on $B(\mathfrak{H})$. We call this the strong operator topology and denote by the so-topology.

Since $\varphi(x) = (x\xi, \xi)$ is a σ-continuous positive functional, where $(\ , \ )$ is the inner product of $\mathfrak{H}$, s $\gtrsim$ so.

(ii) The weak operator topology (the wo-topology). Let $\xi, \eta \in \mathfrak{H}$. Then the function $|(x\xi, \eta)|$ is a semi-norm on $B(\mathfrak{H})$. The set of such semi-norms defines a separate locally convex topology on $B(\mathfrak{H})$. We call this topology the weak operator topology and denote by the wo-topology. Then clearly $\sigma \precsim$ wo; therefore we have

$$\tau \precsim s \precsim \sigma \precsim wo$$

$$\tau \precsim s \precsim so \precsim wo .$$

Let $\mathcal{M}$ be a *-subalgebra of $B(\mathfrak{h})$. If $\mathcal{M}$ is wo-closed, it is clearly closed under the remained four topologies.

One of the purposes of this section is to show that conversely if $\mathcal{M}$ is τ-closed, it is also wo-closed.

Let $a \in \mathcal{T}$ and $a = v|a|$ the polar decomposition of a, then $|a| = \sum_{i=1}^{\infty} \lambda_i e_i$ and $\sum_{i=1}^{\infty} \lambda_i = Tr(|a|)$, where (e_i) is a sequence of orthogonal one-dimensional projections of $B(\mathfrak{h})$, (λ_i) is a sequence of positive numbers and Tr is the trace on $B(\mathfrak{h})$. Let (ξ_i) be a sequence of elements of $\mathfrak{h}$ such that $e_i\xi_i = \xi_i$ and $\|\xi_i\| = 1$, then

$$Tr(xa) = Tr(xv|a|) = \sum_{i=1}^{\infty} (xv|a|\xi_i, \xi_i)$$

$$= \sum_{i=1}^{\infty} \lambda_i (xv\xi_i, \xi_i) = \sum_i (xv\sqrt{\lambda_i}\,\xi_i, \sqrt{\lambda_i}\,\xi_i) \qquad (x \in B(\mathfrak{h})).$$

Put $\sqrt{\lambda_i}\, v\xi_i = \xi_i'$ and $\sqrt{\lambda_i}\,\xi_i = \xi_i''$, then

$$Tr(xa) = \sum_{i=1}^{\infty} (x\xi_i', \xi_i'') ,$$

where $\sum_{i=1}^{\infty} \|\xi_i'\|^2 = \sum_{i=1}^{\infty} \lambda_i < +\infty$ and $\sum_{i=1}^{\infty} \|\xi_i''\|^2 < +\infty$.

Conversely, let (ξ_i) (η_i) be two sequences of elements of $\mathfrak{h}$ such that $\sum_{i=1}^{\infty} \|\xi_i\|^2 < +\infty$, $\sum_{i=1}^{\infty} \|\eta_i\|^2 < +\infty$, and put $f(x) = \sum_{i=1}^{\infty} (x\xi_i, \eta_i)$ $(x \in B(\mathfrak{h}))$.

Then $\sum_{i=1}^{\infty} |(x\xi_i, \eta_i)| \leq \|x\| \sum_{i=1}^{\infty} \|\xi_i\| \, \|\eta_i\| \leq$

$\| x \| \left(\sum_{i=1}^{\infty} \| \xi_i \|^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} \| \eta_i \|^2 \right)^{1/2}$; hence f is a linear functional on $B(\mathfrak{H})$; $\{ \sum_{i=1}^{n} (x\xi_i, \eta_i) \mid n=1,2,\dots \}$ converges uniformly to $f(x)$ on the unit sphere of $B(\mathfrak{H})$; since $\{ \sum_{i=1}^{n} (x\xi_i, \eta_i) \} \in \Pi$, $f \in \Pi$. Hence we obtain

Proposition 1.1. The following conditions for a linear functional f on $B(\mathfrak{H})$ are equivalent

(i) f is σ-continuous

(ii) $f(x) = \sum_{i=1}^{\infty} (x\xi_i, \eta_i)$, where $\sum_{i=1}^{\infty} \| \xi_i \|^2 < +\infty$, $\sum_{i=1}^{\infty} \| \eta_i \|^2 < +\infty$.

In particular, from the above considerations, we obtain

Corollary 1.1. The following conditions for a positive linear functional φ are equivalent.

(i) φ is normal

(ii) $\varphi(x) = \sum_{i=1}^{\infty} (x\xi_i, \xi_i)$, where $\sum_{i=1}^{\infty} \| \xi_i \|^2 < +\infty$.

Now let $\mathcal{M}$ be a τ-closed *-subalgebra of $B(\mathfrak{H})$, then it is σ-closed by the corollary of Theorem 3.3 in chapter I.

Since $\mathcal{M}$ is a dual space, it is a W^*-algebra; hence it has a unit p, which is a projection in $B(\mathfrak{H})$; $pB(\mathfrak{H})p = B(p\mathfrak{H})$, $pB(\mathfrak{H})p$ is wo-closed in $B(\mathfrak{H})$; therefore, to show that $\mathcal{M}$ is also wo-closed, it is enough to assume $p = 1$.

Then, we shall show the following fundamental theorem concerning weakly closed *-subalgebras.

Theorem 1.1. (the commutation theorem of von Neumann). Let $\mathcal{M}$ be a *-subalgebra of $B(\mathfrak{H})$ containing 1, then the following conditions are equivalent,

(i) $\mathcal{M}$ is τ-closed

(ii) $\mathcal{M}$ is wo-closed

(iii) $\mathcal{M}'' = \mathcal{M}$.

Proof. It is clear that (iii) $\longrightarrow$ (ii) $\longrightarrow$ (i); we shall show (i) $\longrightarrow$ (iii). Let K be a $\aleph_0$-dimensional hilbert space, $\mathfrak{H} \bar{\otimes} K$ the tensor product of $\mathfrak{H}$ and K; we consider a mapping $a \longrightarrow a\otimes 1$ of $B(\mathfrak{H})$ onto $B(\mathfrak{H})\otimes 1$.

Suppose that $\mathcal{M} \subsetneqq \mathcal{M}''$; since $\mathcal{M}$ is τ-closed, there is a σ-continuous linear functional f such that $f(\mathcal{M}) = 0$ and $f(a) \neq 0$ for some $a \in \mathcal{M}''$.

By Proposition 1.1,

$$f(x) = \sum_{i=1}^{\infty} (x\xi_i, \xi_i'), \quad \text{where} \quad \sum_{i=1}^{\infty} \|\xi_i\|^2 < +\infty, \ \sum_{i=1}^{\infty} \|\xi_i'\|^2 < +\infty.$$

Let $(\eta_n \mid n=1,2,\dots)$ be a complete ortho-normal system of K, $\mathfrak{H}^n = \mathfrak{H}\otimes\eta_n$ and u_n be an isometry of $\mathfrak{H}$ onto $\mathfrak{H}^n$ such that $\xi \longrightarrow \xi\otimes\eta_n$ $(\xi \in \mathfrak{H})$. Then,

$$f(x) = \sum_{i=1}^{\infty} (x\xi_i, \xi_i') = \sum_{i=1}^{\infty} (xu_i^* u_i\xi_i, u_i^* u_i\xi_i')$$

$$= \sum_{i=1}^{\infty} (u_i x u_i^* u_i\xi_i, u_i\xi_i') = ((x\otimes 1)\xi, \xi') \quad (x \in B(\mathfrak{H})),$$

where $\xi = \sum_{i=1}^{\infty} u_i\xi_i$, $\xi' = \sum_{i=1}^{\infty} u_i\xi_i' \in \mathfrak{H}\bar{\otimes}K$; hence $((\mathcal{M}\otimes 1)\xi, \xi') = 0$ and $((a\otimes 1)\xi, \xi') \neq 0$.

Let $\mathcal{K}$ be the closed subspace of $\mathfrak{H}\bar{\otimes}K$ generated by the set $\{(\mathcal{M}\otimes 1)\xi\}$ and e be the projection of $\mathfrak{H}\bar{\otimes}K$ onto $\mathcal{K}$; since $\mathcal{K}$ is invariant under the *-algebra $\mathcal{M}\otimes 1$, the projection e belongs to $(\mathcal{M}\otimes 1)'$; by Lemma 2.1, $\mathcal{M}''\otimes 1 = (\mathcal{M}\otimes 1)''$ and $\mathcal{M}\otimes 1$ contains the identity operator $1\otimes 1$, $e(a\otimes 1)\xi$ =

$(a\otimes 1)e\xi = (a\otimes 1)\xi$; hence $(a\otimes 1)\xi \in \mathcal{K}$; therefore there is a sequence (a_n) of $\mathcal{M}$ such that $\| (a_n\otimes 1)\xi - (a\otimes 1)\xi \| \longrightarrow 0$ $(n \longrightarrow \infty)$; hence $((\mathcal{M}\otimes 1)\xi, \xi') = 0$ implies $((a\otimes 1)\xi, \xi') = 0$, a contradiction. This completes the proof.

Corollary 1.2. Let $\mathcal{O}$ be a *-subalgebra of $B(\mathfrak{h})$ and $\bar{\mathcal{O}}^{\tau}$ (resp. $\bar{\mathcal{O}}^{s}$, $\bar{\mathcal{O}}^{\sigma}$, $\bar{\mathcal{O}}^{so}$ and $\bar{\mathcal{O}}^{wo}$) be the τ (resp. s, σ, so and wo) -- closure of $\mathcal{O}$, then $\bar{\mathcal{O}}^{\tau} = \bar{\mathcal{O}}^{s} = \bar{\mathcal{O}}^{\sigma} = \bar{\mathcal{O}}^{so} = \bar{\mathcal{O}}^{wo}$.

Proof. By Theorem 3.2, chapter I, $\bar{\mathcal{O}}^{\tau}$ is a *-subalgebra of $B(\mathfrak{h})$; hence by the above theorem, $\bar{\mathcal{O}}^{\tau} = \bar{\mathcal{O}}^{wo}$.

Corollary 1.3. Let $\mathcal{O}, \mathcal{L}$ be two *-subalgebras of $B(\mathfrak{h})$ such that $\mathcal{O} \subset \mathcal{L}$. If $\mathcal{O}$ is wo-dense in $\mathcal{L}$, $\mathcal{O} \cap \mathcal{S}$ is τ-dense in $\mathcal{L} \cap \mathcal{S}$, where $\mathcal{S}$ is the unit sphere of $B(\mathfrak{h})$.

Proof. Let $\bar{\mathcal{O}}^{\sigma}$ be the σ-closure of $\mathcal{O}$, then $\bar{\mathcal{O}}^{\sigma}$ is a W^*-algebra and its σ-topology is equivalent, on $\bar{\mathcal{O}}^{\sigma}$, to the σ-topology of $B(\mathfrak{h})$; hence by Theorem 4.1, Chapter I, $\mathcal{O} \cap \mathcal{S}$ is σ-dense in $\mathcal{L} \cap \mathcal{S}$; since $\mathcal{O} \cap \mathcal{S}$ is convex, the τ-closure of $\mathcal{O} \cap \mathcal{S}$ in $B(\mathfrak{h})$ is σ-closed; therefore we can conclude that $\mathcal{O} \cap \mathcal{S}$ is τ-dense in $\mathcal{L} \cap \mathcal{S}$.

Corollary 1.4. Let $\mathcal{M}$ be a wo-closed *-subalgebra of $B(\mathfrak{h})$, then the so-topology (resp. the wo-topology) is equivalent to the s-topology (resp. the σ-topology) on bounded spheres of $\mathcal{M}$.

Proof. Since the unit sphere of $\mathcal{M}$ is wo-compact, $\sigma \precsim$ wo by Theorem 5.2, Chapter I; therefore they are equivalent on bounded spheres. Since "$x_\alpha \longrightarrow 0(\mathcal{S})$" is equivalent to "$x_\alpha^* x_\alpha \longrightarrow 0\ (\sigma)$", the s-topology is equivalent to the so-topology on bounded spheres.

Henceforward, a wo-closed *-subalgebra of $B(\mathfrak{H})$ is called a weakly closed *-subalgebra, as, by Theorem 1.1, the anxiety of misunderstanding vanishes.

Corollary 1.5. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two weakly closed *-subalgebras of $B(\mathfrak{H})$ containing the identity operator, then $R(\mathfrak{M},\mathfrak{N}) = (\mathfrak{M}' \cap \mathfrak{N}')'$, where $R(\mathfrak{M},\mathfrak{N})$ is the weakly closed *-subalgebra of $B(\mathfrak{H})$ generated by $\mathfrak{M}$ and $\mathfrak{N}$.

Notices of §1.

Dixmier [4] took the property (iii) of Theorem 1.1 as the definition of von Neumann algebras.

Dieudonné [Portugal. Math. 14(1955) 35-38] showed that there is an example of the algebra $\mathfrak{A}$ such that $\mathfrak{A}'' = \mathfrak{A}' \neq \mathfrak{A}$ in a reflexive Banach space.

Henceforward, we shall use the notation $R(\mathfrak{M},\mathfrak{N})$ in Corollary 1.5 without notices.

§2. Tensor products of W^*-algebras.

Firstly we shall state some facts concerning the tensor products of Banach spaces.

Let E and F be two Banach spaces, $E \otimes F$ the algebraic tensor product of E and F. A norm α on $E \otimes F$ is said to be a cross norm if for every $x \in E$ and $y \in F$, $\alpha(x \otimes y) = \|x\| \|y\|$. $E \otimes_\alpha F$ denotes the completion of $E \otimes F$ with respect to α. The "least cross norm" λ is obtained by the natural algebraic imbedding of $E \otimes F$ into $\mathcal{L}(E^*, F)$, where $\mathcal{L}(E^*, F)$ is the Banach space of all bounded linear transformations of E^* into F. If under this mapping, $T^u \in \mathcal{L}(E^*, F)$ corresponds to a tensor $u = \sum_{j=1}^{n} x_j \otimes y_j$, then for $x^* \in E^*$

$$T^u x^* = \sum_{j=1}^{n} < x_j, x^* > y_j \ .$$

We define $\lambda(u) = \| T^u \|$; hence $\lambda(u) = \sup_{\substack{y^* \in F^* \\ x^* \in E^* \\ \|x^*\| = \|y^*\| = 1}} \left| \sum_{j=1}^{n} <x_j, x^*><y_j, y^*> \right|$

λ is the least cross norm of all cross norms having cross norms as dual norms.

The greatest cross norm γ is defined by $\gamma(u) = \inf \sum_{j=1}^{n} \|x_j\| \|y_j\|$, where the inf is taken over all representations of u. γ is also a cross norm and $\gamma \geq \lambda$. In general, a norm α is of interest as follows: $\lambda \leq \alpha \leq \gamma$. If $\lambda \leq \alpha \leq \gamma$, $\alpha(x \otimes y) = \|x\| \|y\|$; α is also a cross norm.

Analogously, we consider the algebraic tensor product $E^* \otimes F^*$, where E^* and F^* are the duals of E and F

respectively; elements of $E^*\otimes F^*$ can be considered linear functionals on $E\otimes F$. Let β be a cross norm on $E\otimes F$ and put

$$\beta^*(f) = \sup_{\beta(x)\leq 1} |\langle x,f\rangle| \quad \text{for} \quad f \in E^*\otimes F^*.$$

If β^* is finite on $E^*\otimes F^*$, it defines a norm on $E^*\otimes F^*$; we call β^* the dual norm of β; if $\beta \geq \lambda$, β^* is also a cross norm on $E^*\otimes F^*$. Moreover $\gamma^* = \lambda$; therefore $\lambda^* \geq \alpha^* \geq \lambda$ if $\lambda \leq \alpha \leq \gamma$. Concerning these things, we shall refer "A theory of cross-spaces" by Schatten.

Next, let A and B be two B^*-algebras with units, then the algebraic tensor product $A\otimes B$ can be considered an $*$-algebra; the algebraic tensor product $A^*\otimes B^*$ of duals A^* and B^* can be considered a set of linear functionals on $A\otimes B$; then positive elements Φ of $A^*\otimes B^*$ are algebraically defined as follows: $\Phi(x^*x) \geq 0$ for $x \in A\otimes B$.

Lemma 2.1. Let φ and ψ be positive elements of A^* and B^* respectively, then $\varphi\otimes\psi$ is a positive linear functional on $A\otimes B$.

Proof. Let $x = \sum_{i=1}^{n} a_i\otimes b_i \in A\otimes B$, then

$$\varphi\otimes\psi(x^*x) = \varphi\otimes\psi((\sum_{i=1}^{n} a_i^*\otimes b_i^*)(\sum_{i=1}^{n} a_i\otimes b_i))$$

$$= \sum_{i,j=1}^{n} \varphi(a_i^*a_j)\psi(b_i^*b_j) .$$

$$\sum_{i,j=1}^{n} \psi(b_i^*b_j)\bar{\lambda}\lambda_j = \psi((\sum_{i=1}^{n}\lambda_i b_i)^* (\sum_{i=1}^{n}\lambda_i b_i)) \geq 0 \quad \text{for any}$$

complex numbers $\lambda_1,\lambda_2,\ldots,\lambda_n$; hence a matrix $(\psi(b_i^*b_j))_{i,j=1,2,\ldots n}$ is positive; therefore we can write $(\psi(b_i^*b_j))_{i,j=1,2,\ldots,n} =$

$(\bar{\alpha}_i \alpha_j)_{i,j=1,2,\ldots,n}$.

Then,

$$\sum_{i,j=1}^{n} \varphi(a_i^* a_j)\, \psi(b_i^* b_j) = \sum_{i,j=1}^{n} \varphi(a_i^* a_j)\bar{\alpha}_i \alpha_j = \varphi\{(\sum_{i=1}^{n} \alpha_i a_i)^* (\sum_{i=1}^{n} \alpha_i a_i)\}$$
$$\geq 0.$$

This completes the proof.

Our interesting cross norm α on $A \otimes B$ is as follows: α is a B^*-norm and positive elements of $A^* \otimes B^*$ are bounded under the α.

Since elements of A^* and B^* are linear combination of positive elements; if positive elements of $A^* \otimes B^*$ are bounded under the α, all elements are also so; hence our interesting cross norm α on $A \otimes B$ is a B^*-cross norm such that α^* is finite.

Then we shall show

Theorem 2.1. There is the least B^*-cross norm α_0 in all B^*-norms α on $A \otimes B$ such that α^* is finite, and moreover $\alpha_0 \geq \lambda$.

Proof. Let α be a B^*-norm such that α^* is finite, and $\mathfrak{S}$ be the totality of all positive elements of $A^* \otimes B^*$, then

$$\Phi(\alpha(x^* x) 1 \otimes 1 - x^* x) \geq 0 \quad \text{for} \quad x \in A \otimes B \quad \text{and} \quad \Phi \in \mathfrak{S};$$

hence $\alpha(x^* x) = \alpha(x)^2 \geq \dfrac{\Phi(x^* x)}{\Phi(1 \otimes 1)}$ for $(\Phi \neq 0)$.

On the other hand, put $$\alpha_0(x) = \sup_{\substack{\Phi \in \mathfrak{S} \\ \Phi \neq 0}} \left(\frac{\Phi(x^* x)}{\Phi(1 \times 1)} \right)^{1/2},$$

then $\alpha_0(x)$ is a B^*-norm on $A \otimes B$.

$$\alpha_0(x)^2 \geq \sup_{\substack{\varphi,\psi>0 \\ \varphi(1)=\psi(1)=1}} |<x^* x, \varphi\otimes\psi>| \geq |<x,\varphi\otimes\psi>|^2 \quad ;$$

therefore if $\alpha_0(x) = 0$, $|<x,\varphi\otimes\psi>| = 0$; hence $<x, f\otimes g> = 0$ for $f \in A^*$ and $g \in B^*$; $\lambda(x) = 0$ and so $x = 0$; hence α_0 is a B^*-norm.

$$\alpha_0(x_1 \otimes y_1)^2 \geq \sup_{\substack{\varphi,\psi>0 \\ \varphi(1)=\psi(1)=1}} \varphi\otimes\psi(x_1^* x_1 \otimes y_1^* y_1) = \| x_1 \|^2 \, \| y_1 \|^2$$

On the other hand, since $(\| x_1^* x_1 \| 1 - x_1^* x_1) \otimes 1 \geq 0$,

$$\Phi((\| x_1^* x_1 \| 1 - x_1^* x_1) \otimes 1) \geq 0 \quad ;$$

hence $\| x_1^* x_1 \| \geq \alpha_0(x_1 \otimes 1)^2$ and analogously $\| y_1 \| \geq \alpha_0(y_1 \otimes 1)$, so that $\| x_1 \| \, \| y_1 \| \geq \alpha_0(x_1 \otimes 1) \alpha_0(1 \otimes y_1) \geq \alpha_0(x_1 \otimes y_1) \geq \| x_1 \| \, \| y_1 \|$; hence α_0 is a cross norm.

Moreover

$$\alpha_0(x) \geq \sup_{\substack{\|a\| \leq 1, \|b\| \leq 1 \\ \varphi,\psi>0 \\ \varphi(1)=\psi(1)=1}} |\varphi\otimes\psi(xa\otimes b)|$$

$$= \sup |R_a\varphi \otimes R_b\psi(x)|$$

$$= \sup_{\substack{\|f\| \leq 1 \\ \|g\| \leq 1}} |<x, f\otimes g>| = \lambda(x) \quad ,$$

for A^{**} and B^{**} are W^*-algebras, and A^* and B^* are their associated spaces; by the polar decomposition of f and g, $f = R_V|f|$ and $g = R_U|g|$ $(V \in A^{**}, U \in B^{**})$; by the density theorem of Kaplansky, there are two directed sets (a_α) and (b_β) such that $a_\alpha \longrightarrow v(\tau)$ and $b_\beta \longrightarrow u(\tau)$, and $\|a_\alpha\|, \|b_\alpha\| \leq 1$.

Therefore we have $\alpha_0 \geq \lambda$; hence α_0^* is also a cross norm. This completes the proof.

Definition 2.1. Let A and B be two B^*-algebras with units, then we shall call $A \otimes_{\alpha_0} B$ the B^*-tensor product of A and B.

Now we shall consider A and B the operator algebras on hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively.

Let $\mathfrak{H}_1 \overline{\otimes} \mathfrak{H}_2$ be the tensor product of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, then $A \otimes 1_{\mathfrak{H}_2}$ and $1_{\mathfrak{H}_1} \otimes B$ are operator algebras on $\mathfrak{H}_1 \overline{\otimes} \mathfrak{H}_2$, where $1_{\mathfrak{H}_1}$, $1_{\mathfrak{H}_2}$ denote the identity operators on $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. Let $\mathfrak{A}$ be the B^*-algebra on $\mathfrak{H}_1 \overline{\otimes} \mathfrak{H}_2$ generated by $A \otimes 1_{\mathfrak{H}_2}$ and $1_{\mathfrak{H}_1} \otimes B$, then we obtain

Theorem 2.2. $A \otimes_{\alpha_0} B$ is $*$-isomorphic to $\mathfrak{A}$.

Proof. We consider a mapping $\rho : \sum_{i=1}^{n} a_i \otimes b_i \longrightarrow \sum_{i=1}^{n} (a_i \otimes 1_{\mathfrak{H}_2})(1_{\mathfrak{H}_1} \otimes b_i)$ of $A \otimes B$ into $\mathfrak{A}$; then it is easily shown that ρ is an $*$-isomorphism; put $\alpha(x) = \|\rho(x)\|$ for $x \in A \otimes B$, then α is a B^*-norm on $A \otimes B$; moreover $\alpha^*(\varphi \otimes \psi) = \|\varphi\| \ \|\psi\|$, where $\varphi(x) = (x\xi_1, \xi_1)$ and $\psi(y) = (y\xi_2, \xi_2)$ for $x \in A$, $y \in B$ and $\xi_1 \in \mathfrak{H}_1$, $\xi_2 \in \mathfrak{H}_2$.

Let V_1 = the convex span of $\{\varphi \mid \varphi(\cdot) = (\cdot\xi_1, \xi_1)$, $\xi_1 \in \mathfrak{H}_1 \ \ \|\xi_1\| = 1\}$ and V_2 = the convex span of $\{\psi \mid \psi(\cdot) = (\cdot\xi_2, \xi_2)$, $\xi_2 \in \mathfrak{H}_2$, $\|\xi_2\| = 1\}$ and $\overline{V}_1$ (resp. $\overline{V}_2$) the $\sigma(A^*, A)$ (resp. $\sigma(B^*, B)$)-closure in A^* (resp. B^*), then $\overline{V}_1 = \mathfrak{S}_1$ (resp. $\overline{V}_2 = \mathfrak{S}_2$), where $\mathfrak{S}_1$ (resp. $\mathfrak{S}_2$) is the totality of all positive functionals f on A (resp. B) such that $f(1) = 1$ -- in fact, if $\overline{V}_1 \subsetneqq \mathfrak{S}_1$, there is a self-adjoint element a of A such that $|\langle a, \overline{V}_1\rangle| \leq 1-\varepsilon$ $(\varepsilon > 0)$ and $\sup_{f \varepsilon \mathfrak{S}_1} |\langle a,f\rangle| = 1$; on the other hand $|\langle a, V_1\rangle| \leq 1-\varepsilon$ implies $|(a\xi_1, \xi_1)| \leq 1-\varepsilon$ for ξ_1 $(\|\xi_1\| = 1)$ ε $\mathfrak{H}_1$; hence $\|a\| \leq 1-\varepsilon$, so that $\sup_{f \varepsilon \mathfrak{S}_1} |\langle a,f\rangle| = \|a\| \leq 1-\varepsilon$, a contradiction.

Therefore for any $f_1 \varepsilon \mathfrak{S}_1$, $f_2 \varepsilon \mathfrak{S}_2$, there are two directed sets (φ_γ) and (ψ_δ) such that $\varphi_\gamma \longrightarrow f_1(\sigma(A^*, A))$ and $\psi_\delta \longrightarrow f_2(\sigma(B^*, B))$; for $\alpha(x) \leq 1$, $\varphi_\gamma \otimes \psi_\delta(x) \longrightarrow f_1 \otimes f_2(x)$; since $|\varphi_\gamma \otimes \psi_\delta(x)| \leq 1$, $|f_1 \otimes f_2(x)| \leq 1$; hence $\alpha^*(f_1 \otimes f_2) \leq \|f_1\| \, \|f_2\|$; this implies that α^* is finite; by Theorem 2.1, $\alpha \geq \alpha_0$.

Conversely, for any elements $\xi_1^i \varepsilon \mathfrak{H}_1$, $\xi_2^i \varepsilon \mathfrak{H}_2$ $(i=1,2,\ldots,n)$ put $\Phi(x) = (\rho(x) \sum_{i=1}^n \xi_1^i \otimes \xi_2^i, \sum_{i=1}^n \xi_1^i \otimes \xi_2^i)$ for $x \varepsilon A \otimes B$, then clearly $\Phi \varepsilon A^* \otimes B^*$ and it is positive; therefore $\alpha_0(x^*x) \geq$

$$\frac{\Phi(x^*x)}{\Phi(1\times 1)} = \frac{\|\rho(x) \sum_{i=1}^n \xi_1^i \otimes \xi_2^i\|^2}{\|\sum_{i=1}^n \xi_1^i \otimes \xi_2^i\|^2};$$

hence we have $\alpha_0(x^*x) \geq \alpha(x)^2$, so that $\alpha_0(x) = \alpha(x)$ for $x \varepsilon A \otimes B$; this implies that $A \otimes_{\alpha_0} B$ is $*$-isomorphic to $\mathfrak{A}$, and completes the proof.

Finally we shall define the tensor product of W^*-algebras. Let M and N be two W^*-algebras, M_* and N_* be the associated spaces of N and N respectively.

Firstly we consider the B^*-tensor product $M \otimes_{\alpha_0} N$. Then $M_* \otimes_{\alpha_0^*} N_*$ is an invariant subspace of the dual $(M \otimes_{\alpha_0} N)^*$ of $M \otimes_{\alpha_0} N$; the polar of $M_* \otimes_{\alpha_0^*} N_*$ in the second dual of $M \otimes_{\alpha_0} N$ is an ideal and we can consider $(M_* \otimes_{\alpha_0^*} N_*)^* =$

$(M \otimes_{\alpha_0} N)^{**}/\mathcal{J}$; the canonical mapping $M \otimes_{\alpha_0} N \longrightarrow M \otimes_{\alpha_0} N/\mathcal{J}$ is an $*$-isomorphism; therefore it is an isometry; by this mapping, we can consider the B^*-algebra $M \otimes_{\alpha_0} N$ as a B^*-subalgebra of a W^*-algebra $(M \otimes_{\alpha_0} N)^{**}/\mathcal{J}$; then we obtain a W^*-algebra P such that $P = (M_* \otimes_{\alpha_0^*} N_*)^*$ and the B^*-algebra $M \otimes_{\alpha_0} N$ is σ-dense in P.

Definition 2.2. Let M and N be two W^*-algebras, M_* and N_* their associated spaces, then the above W^*-algebra P is called the W^*-tensor product of M and N, and denoted by $M \bar{\otimes} N$.

Now we shall faithfully represent M and N as the weakly closed $*$-algebras on hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively.

Let $\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$ be the tensor product of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, then $M \otimes 1_{\mathfrak{H}_2}$ and $1_{\mathfrak{H}_1} \otimes N$ are weakly closed $*$-algebras on $\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$. Let $\mathcal{L}$ be the weakly closed $*$-algebra on $\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$ generated by $M \bar{\otimes} 1_{\mathfrak{H}_2}$ and $1_{\mathfrak{H}_1} \bar{\otimes} N$, then we obtain

Theorem 2.3. $M \bar{\otimes} N$ is $*$-isomorphic to $\mathcal{L}$.

Proof. Let $\mathfrak{A}$ be the B^*-algebra on $\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$ generated by $M \otimes 1_{\mathfrak{H}_2}$ and $1_{\mathfrak{H}_1} \otimes N$, then by Theorem 2.2 there is an $*$-isomorphism ρ of $M \otimes_{\alpha_0} N$ onto $\mathfrak{A}$.

Let $(x_\alpha) \subset M \bar{\otimes} N$ be a cauchy directed set in the s-topology of $M \bar{\otimes} N$ such that $\alpha_0(x_\alpha) \leq 1$, then for any $\xi_1 \in \mathfrak{H}_1$, $\xi_2 \in \mathfrak{H}_2$,

$$(\rho(x_\alpha)\xi_1 \otimes \xi_2, \xi_1 \otimes \xi_2) = \varphi_1 \otimes \varphi_2(x_\alpha) ,$$

where $\varphi_1(x) = (x\xi_1, \xi_1)$ $(x \in M)$ and $\varphi_2(y) = (y\xi_2, \xi_2)$ $(y \in N)$. From the unicity of the σ-topology, the s-topology coincides with the so-topology on the unit sphere of M (resp. N), for the unit sphere of M (resp. N) is wo-compact in $B(\mathfrak{H}_1)$ (resp. $B(\mathfrak{H}_2)$); hence $\varphi_1 \in M_*$ and $\varphi_2 \in N_*$; since linear combinations of $\{\xi_1 \otimes \xi_2 \mid \xi_1 \in \mathfrak{H}_1, \xi_2 \in \mathfrak{H}_2\}$ are dense in $\mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$ and the operator norms of $(\rho(x_\alpha))$ are bounded, we can conclude that $(\rho(x_\alpha))$ is a cauchy-directed set in the so-topology of $B(\mathfrak{H}_1 \otimes \mathfrak{H}_2)$; therefore by the density theorem of Kaplansky, ρ can be uniquely extended to a *-homomorphism $\tilde{\rho}$ of $M \bar{\otimes} N$ into $\mathcal{B}$, which is continuous with respect to the topologies s and so; therefore the kernel $\mathcal{J}$ of $\tilde{\rho}$ is σ-closed, so that $\tilde{\rho}$ is a *-homomorphism of $M \bar{\otimes} N$ onto $\mathcal{B}$. Let $x(>0) \in \mathcal{J}$, then ($(\tilde{\rho}(x)\xi_1 \otimes \xi_2, \xi_1 \otimes \xi_2) = 0$ for any $\xi_1 \in \mathfrak{H}_1$ and $\xi_2 \in \mathfrak{H}_2$; hence $(\tilde{\rho}(x)\xi, \eta) = 0$ for any $\xi, \eta \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$. On the other hand, let φ be any positive element of M_*, then by Corollary 1.1, there is a sequence $(\xi_1^i) \subset \mathfrak{H}$, such that $\varphi(x_1) = \sum_{i=1}^{\infty} (x_1 \xi_1^i, \xi_1^i)$, $\sum_{i=1}^{\infty} \|\xi_1^i\|^2 < +\infty$, and analogously for any $\psi(>0) \in M_*$, there is a sequence $(\xi_2^i) \subset \mathfrak{H}_2$ such that $\psi(x_2) = \sum_{i=1}^{\infty} (x_2 \xi_2^i, \xi_2^i)$, $\sum_{i=1}^{\infty} \|\xi_2^i\|^2 < +\infty$; therefore $\varphi \otimes \psi(x) = \sum_{i,j=1}^{\infty} (\rho(x)\xi_1^i \otimes \xi_2^j, \xi_1^i \otimes \xi_2^j) = 0$; so that $f \otimes g(x) = 0$ for $f \in M_*$ and $g \in N_*$; since $M_* \otimes N_*$ is norm-dense in $M_* \otimes_{\alpha_0^*} N_*$; hence $\Psi(x) = 0$ for $\Psi \in M_* \otimes_{\alpha_0^*} N_*$; hence we have $x = 0$, so that $\mathcal{J} = (0)$ this implies that $\tilde{\rho}$ is an *-isomorphism of $M \bar{\otimes} N$ onto $\mathcal{B}$. This completes the proof.

Theorem 2.4. Let $C(\Omega)$ be the commutative B^*-algebra of all continuous functions on a compact space Ω, A a B^*-algebra, then $C(\Omega) \otimes_{\alpha_0} A = C(A, \Omega)$, where $C(A, \Omega)$ is the B^*-algebra of all A-valued continuous functions on Ω.

Proof. By the theorem of Grothendieck [49], $C(\Omega)\otimes_\lambda A = C(A,\Omega)$; since A is a B^*-algebra, $C(A,\Omega)$ becomes naturally a B^*-algebra; a mapping $f\otimes a \xrightarrow{\rho} f(t)a\ (t\in\Omega)$ is uniquely extended to an $*$-isomorphism of $C(\Omega)\otimes A$ into $C(A,\Omega)$; hence by Theorem 2.1, $\alpha_0 = \lambda$ and the above ρ is uniquely extended to an $*$-isomorphism of $C(\Omega)\otimes_{\alpha_0} A$ onto $C(A,\Omega)$. This completes the proof.

Theorem 2.5. Let $L^\infty(\Omega,\mu)$ be the commutative W^*-algebra of all essentially bounded measurable functions on a measure space (Ω,μ), M a W^*-algebra, then $L^1(\Omega,\mu)\otimes_{\alpha_0^*} M_* = L^1(M_*,\Omega,\mu)$, where $L^1(M_*,\Omega,\mu)$ is the Banach space of all M_*-valued Bochner-integrable functions on the measure space (Ω,μ).

Proof. $L^\infty(\Omega,\mu)\otimes_{\alpha_0} M = L^\infty(\Omega,\mu)\otimes_\lambda M$; since $\gamma^* = \lambda$, we have $L^\infty(\Omega,\mu)\otimes_{\alpha_0} M = L^\infty(\Omega,\mu)\otimes_{\gamma^*} M$, where $L^1(\Omega,\mu)\otimes_\gamma M_*$; since $L^1(\Omega,\mu)$ has the metric approximation property, by Corollary 1 of Proposition 40, [49], the canonical mapping of $L^1(\Omega,\mu)\otimes_\gamma M_*$ into the Banach space $J(L^\infty(\Omega,\mu),M)$ of all integrable bilinear forms is an isometry; the dual of $L^\infty(\Omega,\mu)\otimes_\lambda M$ is $J(L^\infty(\Omega,\mu),M)$; hence we have $\lambda^* = \gamma$ on $L^1(\Omega,\mu)\otimes M_*$; therefore we obtain $L^1(\Omega,\mu)\otimes_{\alpha_0^*} M_* = L^1(M_*,\Omega,\mu)$. This completes the proof.

Now suppose that M_* is separable, then by Dunford-Pettis' theorem, for any $x\in L^\infty(\Omega,\mu)\overline{\otimes} M$, there is a unique M-valued essentially bounded weakly $*$ measurable function $f^x(t)$ on Ω such that $x(\xi\otimes\eta) = \int_\Omega \langle f^x(t),\xi\rangle\eta(t)\,d\mu(t)$ and ess. sup $\|f^x\| = \|x\|$, where $\xi\in M_*$ and $\eta\in L^1(\Omega,\mu)$.

Under such mapping $x\xrightarrow{\rho} f^x$, $L^\infty(\Omega,\mu)\overline{\otimes} M$ is isometric to $L^\infty(M,\Omega,\mu)$, where $L^\infty(M,\Omega,\mu)$ is the Banach space of all

M-valued essentially bounded weakly * measurable functions on Ω.

In general, let $f = \sum_{i=1}^{\infty} f_i \otimes \varphi_i \in L^1(\Omega,\mu) \otimes_\gamma M_*$, where $\sum_{i=1}^{\infty} \|f_i\| \, \|\varphi_i\| < +\infty$, $f_i \in L^1(\Omega,\mu)$ and $\varphi_i \in M_*$, then $R_g \otimes R_a f = \sum_{i=1}^{\infty} f_i g \otimes R_a \varphi_i \in L^1(\Omega,\mu) \bar{\otimes} M_*$, where $g \in L^\infty(\Omega,\mu)$, $a \in M$; therefore for $a \in M$, $g \in L^\infty(M,\Omega)$ and $x \in L^\infty(\Omega,\mu) \bar{\otimes} M$, $\rho(xa \otimes g) = \rho(x)\rho(a \otimes g)$; hence $\rho(xy) = \rho(x)\rho(y)$ for $x \in L^\infty(\Omega,\mu) \bar{\otimes} M$ and $y \in L^\infty(\Omega,\mu) \otimes M$.

Therefore,

$$\langle xy, \rho^*(f)\rangle = \int_\Omega \langle \rho(xy)(t), f(t)\rangle d\mu(t) = \int_\Omega \langle \rho(x)(t)\rho(y)(t), f(t)\rangle d\mu(t)$$
$$= \int_\Omega \langle \rho(y)(t), L_{\rho(x)(t)} f(t)\rangle \, d\mu(t) \quad \text{for } f \in L^1(M_*,\Omega,\mu) \; ;$$

hence $\langle y, L_x \rho^*(t)\rangle = \langle \rho(y), \rho^{*-1} L_x \rho^*(f)\rangle$

$$= \int_\Omega \langle \rho(y)(t), \{\rho^{*-1} L_x \rho^*(f)\}(t)\rangle d\mu(t) \; .$$

Since $\rho(L^\infty(\Omega,\mu) \otimes M)$ contains $L^\infty(\Omega,\mu) \cdot 1$, we have

$$\int_\Omega \langle \rho(y)(t), L_{\rho(x)(t)} f(t)\rangle \, g(t) d\mu(t)$$
$$= \int_\Omega \langle \rho(y)(t), \{\rho^{*-1} L_x \rho^*(f)\}(t)\rangle \, g(t) d\mu(t) \quad \text{for } g \in L^\infty(\Omega,\mu);$$

hence we have $\langle \rho(y)(t), L_{\rho(x)(t)} f(t)\rangle = \langle \rho(y)(t), \{\rho^{*-1} L_x \rho^*(f)\}(t)\rangle$ for $t \notin N_{x,y}$, where $N_{x,y}$ is a null set which depends on (x,y).

Since M_* is separable, we can take a sequence (a_n) from M such that (a_n) is σ-dense in M; put $y_n = 1 \otimes a_n$, then $\rho(y_n)(t) = a_n$; therefore

$$\langle \rho(y_n)(t), L_{\rho(x)(t)} f(t)\rangle = \langle a_n, L_{\rho(x)(t)} f(t)\rangle$$
$$= \langle a_n, \{\rho^{*-1} L_x \rho^*(f)\}(t)\rangle \quad \text{for all } n \text{ and}$$

$t \notin \bigcup_{n=1}^{\infty} N_{x,y_n}$; hence $L_{\rho(x)(t)}f(t) = \{\rho^{*-1}L_x\rho^*(f)\}(t)$ for $t \notin \bigcup_{n=1}^{\infty} N_{x,y_n}$; since $\bigcup_{n=1}^{\infty} N_{x,y_n}$ is a null set, we have $L_{\rho(x)(t)}f(t) = \{\rho^{*-1}L_x\rho^*(f)\}(t)$ in $L^1(M,\Omega,\mu)$.

Then for any $x_1 \in L^\infty(M,\Omega)\bar{\otimes}M$

$$\begin{aligned}
<\rho(xx_1),f> &= <xx_1,\rho^*(f)> = <x_1,L_x\rho^*(f)> \\
&= <\rho(x_1),\rho^{*-1}L_x\rho^*(f)> \\
&= \int_\Omega <\rho(x_1)(t),\{\rho^{*-1}L_x\rho^*(f)\}(t)> \, d\mu(t) \\
&= \int_\Omega <\rho(x_1)(t) \; L_{\rho(x)(t)}f(t)> \, d\mu(t) \\
&= \int_\Omega <\rho(x_1)(t)\rho(x)(t), \; f(t)> \, d\mu(t) \\
&= \int_\Omega <\rho(xx_1)(t), \; f(t)> \, d\mu(t);
\end{aligned}$$

hence we have $\rho(xx_1) = \rho(x)\rho(x_1)$ in $L^\infty(M,\Omega,\mu)$.

Now we obtain the following theorem.

Theorem 2.5. Let M be a W^*-algebra, $L^\infty(\Omega,\mu)$ a commutative W^*-algebra. If the associated space M_* of M is separable, the space $L^\infty(M,\Omega,\mu)$ of all M-valued essentially bounded, weakly * measurable functions is a W^*-algebra and $L^\infty(\Omega,\mu)\bar{\otimes}M = L^\infty(M,\Omega,\mu)$.

Remark 2.1. This theorem is important. By this, we can approach to the reduction theory of von Neumann through a quite different method [cf. § 5]; therefore it is a very important problem whether the condition of separability in Theorem 2.5 can be dropped; recently, Tomita [37, cf. 19] extended some parts of the reduction theory of von Neumann; however the author thinks that many results obtained by von Neumann remain unsolvedly in the non-separable case; also we have many unsolved problems even in the separable case [cf. 4]; therefore the author thinks that the new approach has much significance even in the separable case.

Now, using the tensor product of W^*-algebras, we shall show the structure theorem of type I.

Proposition 2.1. Let M be a W^*-algebra, $(e_\beta \mid \beta \in \mathbb{I})$ be a family of orthogonal equivalent projections of M such that $\sum_{\alpha \in \mathbb{I}} e_\alpha = 1$. For a fixed $\alpha \in \mathbb{I}$, let $(v_{\alpha,\beta} \mid \beta \in \mathbb{I})$ be a family of partial isometries of M such that $v_{\alpha,\alpha} = e_\alpha$ and $v_{\alpha,\beta}^* v_{\alpha,\beta} = e_\alpha$, $v_{\alpha,\beta} v_{\alpha,\beta}^* = e_\beta$, and put $N = \{ x \mid x = \sum_{\beta \in \mathbb{I}} v_{\alpha,\beta} x_\alpha v_{\alpha\beta}^*$, $x_\alpha \in e_\alpha M e_\alpha\}$ and let B be a W^*-subalgebra of M generated by $(v_{\alpha,\beta}, v_{\alpha,\beta}^* \mid \beta \in \mathbb{I})$, then N is a W^*-subalgebra of M composed of all elements of M commuting with B and $M = N \overline{\otimes} B$; moreover B is the factor of type I_n, where $n = \text{card}(\mathbb{I})$.

Proof. We shall faithfully represent M as a weakly closed *-subalgebra on a hilbert space $\mathfrak{H}$ such that 1 is the identity operator on $\mathfrak{H}$.

Put $\mathfrak{H}_1 = e_\alpha \mathfrak{H}$ and $\mathfrak{H}_2 = L^2(\mathbb{I})$, where $L^2(\mathbb{I})$ is the hilbert space of square integrable complex valued functions on a discrete space $\mathbb{I}$. Let u_β be the isomorphism of $\mathfrak{H}_1$ onto $e_\beta \mathfrak{H}$ defined by $v_{\alpha,\beta}$, then we can define canonically an isomorphism u of $\mathfrak{H}$ onto $\mathfrak{H}_1 \overline{\otimes} \mathfrak{H}_2$; by this isomorphism, we shall identify two hilbert spaces; then from the considerations of §1, the weakly closed *-subalgebra generated by $\{u_\beta u_\gamma^* \mid \beta, \gamma \in \mathbb{I}\}$ is the algebra $1_{\mathfrak{H}_1} \otimes B(\mathfrak{H}_2)$, so that $B = 1_{\mathfrak{H}_1} \otimes B(\mathfrak{H}_1)$; since N commutes clearly with B, $N \subset B(\mathfrak{H}_1) \otimes 1_{\mathfrak{H}_2}$; moreover for $a \in M$,

$$a_{\beta\gamma} = u_\beta^* a u_\gamma = v_{\alpha\beta}^* a v_{\alpha\gamma};$$

hence $M \subset \{(e_\alpha M e_\alpha)' \otimes 1\}' \subset \{(e_\alpha M e_\alpha \otimes 1)' \cap (1_{\mathfrak{H}_1} \otimes B(\mathfrak{H}_2))'\}' = R(e_\alpha M e_\alpha \otimes 1, 1_{\mathfrak{H}_1} \otimes B(\mathfrak{H}_2))$, and $N = (e_\alpha M e_\alpha) \otimes 1$; therefore we have $B = 1_{\mathfrak{H}_1} \otimes B(\mathfrak{H}_2)$, $N = (e_\alpha M e_\alpha) \otimes 1$ and $M = R((e_\alpha M e_\alpha) \otimes 1, 1_{\mathfrak{H}_1} \otimes B(\mathfrak{H}_2))$; therefore by the theorem 2.3, $M = N \bar{\otimes} B$; moreover since B is *-isomorphic to $B(\mathfrak{H}_2)$, it is a factor of type I_n where $n = \text{card}\ \mathbb{I}$. This completes the proof.

Corollary 2.1. Let M be a homogeneous algebra of type I_n, then $M = Z \bar{\otimes} B$, where Z is the center of M and B is the factor of type I_n.

Proof. Let $(e_\beta \mid \beta \in \mathbb{I})$ be a family of orthogonal maximal abelian projections such that $\sum_{\beta \in \mathbb{I}} e_\beta$, then $\text{card}\ \mathbb{I} = n$. For a fixed $\alpha \in \mathbb{I}$, let $(v_{\alpha,\beta} \mid \beta \in \mathbb{I})$ be a family of partial isometries of M such that $v_{\alpha,\alpha} = e_\alpha$ and $v_{\alpha,\beta}^* v_{\alpha,\beta} = e_\alpha$, $v_{\alpha,\beta} v_{\alpha,\beta}^* = e_\beta$; since $e_\alpha M e_\alpha = e_\alpha Z e_\alpha$, $x = \sum_{\beta \in \mathbb{I}} v_{\alpha,\beta} x_\alpha v_{\alpha\beta}^* = \sum_{\beta \in \mathbb{I}} v_{\alpha,\beta} e_\alpha z_\alpha e_\alpha v_{\alpha\beta}^* = \sum_{\beta \in \mathbb{I}} z_\alpha v_{\alpha,\beta} v_{\alpha\beta}^* = \sum_{\beta \in \mathbb{I}} z_\alpha e_\beta = z_\alpha$ for $x_\alpha \in e_\alpha M e_\alpha$; hence $N = Z$, so that we have $M = Z \bar{\otimes} B$, where B is the factor of type I_n. This completes the proof.

Remark 2.2. In proposition 2.1, we show B is *-isomorphic to $B(\mathfrak{H})$, where $\dim(\mathfrak{H}) = n$; therefore the above corollary implies that a factor of type I_n is *-isomorphic to $B(\mathfrak{H})$, with $\dim(\mathfrak{H}) = n$.

Corollary 2.2. Let M be a homogeneous algebra of type I_n $(n \leq \aleph_0)$, then $M = Z \bar{\otimes} B = L^\infty(B, \Omega, \mu)$, where $Z = L^\infty(\Omega, \mu)$.

Proof. B is considered $B(\mathfrak{H})$ with $\dim(\mathfrak{H}) \leq \aleph_0$; therefore $B_* = \mathbb{T}(\mathfrak{H})$, where $\mathbb{T}(\mathfrak{H})$ is the Banach space of all traces-class operators on $\mathfrak{H}$; hence B_* is separable, so that by Theorem 2.5 we have the conclusion

Corollary 2.3. Let M be a W^*-algebra of type I which has a faithful W^*-representation on a separable hilbert space, then

$M = \sum_{n \leq \aleph_0} \oplus M_{z_n} = \sum_{n \leq \aleph_0} \oplus L^\infty(B_n, \Omega_n, \mu_n)$, where z_n is a central projection of M, M_{z_n} is a homogeneous algebra of type I_n and $M_{z_n} = L^\infty(B_n, \Omega_n, \mu_n)$ and $Z_{z_n} = L^\infty(\Omega_n, \mu_n)$ and B_n is the factor of type I_n.

Remark 2.3. This corollary gives a reduction theory a new approach [cf. §5]; therefore in non-separable cases, the following problem is important; it is possible $Z \bar{\otimes} B = L^\infty(B, \Omega, \mu)$ with $Z = L^\infty(\Omega, \mu)$ and $B = B(\mathfrak{H})$, where $B(\mathfrak{H})$ is the second dual of the Banach space of all completely continuous operators on $\mathfrak{H}$.

Notices of §2

If the measure space (Ω, μ) is not σ-finite, the measurability is interpreted in the meaning of local measurability; therefore "a null set" is interpreted in the meaning of "a locally null set". The tensor product of B^*-algebras was introduced by Turumaru. Theorem 2.3 is due to Turumaru [38].

§3. Standard representations.

In this note, we shall not give the complete explanation concerning the representation theory of W^*-algebras into the operator algebras on hilbert spaces. In this section, we shall state standard representations only for later discussions. The reader is referred for further information on the representation theory to the book of Dixmier.

At first we shall introduce the notion of hilbert algebras.

Definition 3.1. Let $\mathfrak{A}$ be an algebraic $*$-algebra. We say that $\mathfrak{A}$ is a hilbert algebra, if it satisfies the following axioms

(i) $\mathfrak{A}$ is a pre-hilbert space with an inner product $(\ ,\ )$.

(ii) $(x,y) = (y^*,x^*)$ for $x,y \in \mathfrak{A}$

(iii) $(xy,z) = (y,x^*z)$ for $x,y,z \in \mathfrak{A}$

(iv) the mapping $y \longrightarrow xy$ is continuous for $x,y \in \mathfrak{A}$

(v) the elements xy $(x,y \in \mathfrak{A})$ are dense in $\mathfrak{A}$.

Let $\mathfrak{A}$ be a hilbert algebra, $\mathfrak{h}$ the hilbert space obtained by the completion.

By the axiom (ii), the mapping $x \longrightarrow x^*$ is uniquely extended to a conjugate linear isometry J such that $J^2 = 1$.

Since the mapping $y \longrightarrow xy$ is continuous, $(yx,yx) = (x^*y^*,x^*y^*) \leq K(y^*,y^*) = K(y,y)$ for some $K > 0$; hence $y \longrightarrow yx$ is also continuous; the mappings $y \longrightarrow xy$, $y \longrightarrow yx$ are uniquely extended to elements U_x, V_x of $B(\mathfrak{h})$; then we have:

$$U_{(\lambda x+\mu y)} = \lambda U_x + \mu U_y,\quad U_{xy} = U_x U_y,\quad U_{x^*} = U_x^*,$$

$$V_{(\lambda x+\mu y)} = \lambda V_x + \mu V_y,\quad V_{xy} = V_y V_x,\quad V_{x^*} = V_x^*,$$

$$JU_xJ = V_{x^*},\qquad JV_xJ = U_{x^*}.$$

Therefore $\{U_x \mid x \in \mathfrak{A}\}$, $\{V_x \mid x \in \mathfrak{A}\}$ are $*$-algebras on $\mathfrak{h}$. Let $\mathcal{U}(\mathfrak{A})$ (resp. $\mathcal{V}(\mathfrak{A})$) be the weakly closed $*$-algebra generated by $\{U_x \mid x \in \mathfrak{A}\}$ (resp. $\{V_x \mid x \in \mathfrak{A}\}$). We call $\mathcal{U}(\mathfrak{A})$ (resp. $\mathcal{V}(\mathfrak{A})$) the associated left algebra (resp. the associated right algebra) of $\mathfrak{A}$.

Then it is easy that $\mathcal{U}(\mathfrak{A}) \subset \mathcal{V}(\mathfrak{A})'$ and $J\mathcal{U}(\mathfrak{A})J = \mathcal{V}(\mathfrak{A})$.

Definition 3.2. An element $a \in \mathfrak{h}$ is said left-bounded (resp. right-bounded) if there is an element U_a (resp. V_a) of $B(\mathfrak{h})$ such that $U_a x = V_x a$ (resp. $V_a x = U_x a$) for $x \in \mathfrak{A}$.

Since $U_a x = V_x a$ (resp. $V_a x = U_x a$), $a \in [U_a(\mathfrak{h})]$ (resp. $a \in [V_a(\mathfrak{h})]$; hence $a \longrightarrow U_a$ (resp. $a \longrightarrow V_a$) is one-to-one.

Lemma 3.1. An element $a \in \mathfrak{h}$ is left-bounded if and only if it is right-bounded; moreover, if a is left-bounded, Ja is also so, and we have $U_{Ja} = U_a^* = JV_aJ$, $V_{Ja} = V_a^* = JU_aJ$.

Proof. For $x \in \mathfrak{A}$ and $a \in \mathfrak{h}$, $U_{Jx}a = JJU_{Jx}JJa = JV_xJa$; therefore if a is right-bounded, $JV_aJx = JU_{Jx}a = V_xJa$, so that Ja is left-bounded and $U_{Ja} = JV_aJ$. Conversely if Ja is left-bounded, $U_{Jx}a = JU_{Ja}x$, so that $U_ya = JU_{Ja}Jy$ for $y \in \mathfrak{A}$; therefore a is right-bounded and $V_a = JU_{Ja}J$.

Suppose that a is right bounded, then

$$(U_xJa,y) = (Ja,x^*y) = (y^*x,a) = (x,Uya) = (x,V_ay)$$
$$= (V_a^*x,y) \quad \text{for } x,y \in \mathfrak{A};$$

hence $U_xJa = V_a^*x$; therefore Ja is right-bounded and $V_{Ja} = V_a^*$. We obtained: a is right-bounded $\Longleftrightarrow$ Ja is left-bounded $\Rightarrow$ Ja is right-bounded $\Longleftrightarrow$ $JJa = a$ is left-bounded; moreover

$V_{Ja} = V_a^* = JU_{J\cdot Ja}J = JU_aJ$ and $U_{Ja} = JV_aJ = JV_{Ja}^*J = U_a^*$. This completes the proof.

Henceforward, we shall call a right and left-bounded element a bounded element.

Lemma 3.2. Let a be bounded, $T \in \mathcal{V}(\mathfrak{A})'$ and $S \in \mathcal{U}(\mathfrak{A})'$, then Ta, Sa are bounded, and $TU_a = U_{Ta}$ and $SV_a = V_{Sa}$; moreover the $\{U_a \mid a \text{ bounded}\}$ (resp. $\{V_a \mid a \text{ bounded}\}$) is a two-sided ideal of $\mathcal{V}(\mathfrak{A})'$ (resp. $\mathcal{U}(\mathfrak{A})'$).

Proof. $U_aV_xy = U_a(yx) = V_{yx}a = V_xV_ya = V_xU_ay$ for $x, y \in \mathfrak{A}$; hence $U_a \in \mathcal{V}(\mathfrak{A})'$. If $T \in \mathcal{V}(\mathfrak{A})'$,

$$TU_ax = TV_xa = V_xTa \quad \text{for } x \in \mathfrak{A};$$

hence Ta is bounded and $U_{Ta} = TU_a$; moreover $U_aT = (T^*U_a^*)^* = (T^*U_{Ja})^* = (U_{TJa})^* = U_{JTJa}$: hence $\{U_a \mid a \text{ bounded}\}$ is a two-sided ideal of $\mathcal{V}(\mathfrak{A})'$. It is analogous concerning the Va. This completes the proof.

Lemma 3.3. Let $\mathfrak{M} = \{U_a \mid a \text{ bounded}\}$, $\mathfrak{N} = \{V_a \mid a \text{ bounded}\}$, then $\mathfrak{M}'' = \mathcal{V}(\mathfrak{A})'$ and $\mathfrak{N}'' = \mathcal{U}(\mathfrak{A})'$.

Proof. By Lemma 3.2, $\mathfrak{M}'' \subset \mathcal{V}(\mathfrak{A})'$ and $\mathfrak{N}'' \subset \mathcal{U}(\mathfrak{A})'$. Let $T \in \mathcal{V}(\mathfrak{A})'$, then $TU_x \in \mathfrak{M}$ for $x \in \mathfrak{A}$; therefore $TU_xT_1 = T_1TU_x$ for $T_1 \in \mathfrak{M}'$; taking $\{U_{x_\alpha}\}$ such that $U_{x_\alpha} \longrightarrow 1(wo)$ $(x_\alpha \in \mathfrak{A})$, then we have $TU_{x_\alpha}T_1 = T_1TU_{x_\alpha} \longrightarrow TT_1 = T_1T$ (wo), so that $T \in \mathfrak{M}''$, so that $\mathcal{V}(\mathfrak{A})' = \mathfrak{M}''$. Analogously we have $\mathfrak{N}'' = \mathcal{U}(\mathfrak{A})'$. This completes the proof.

Lemma 3.4. $\mathfrak{M} \subset \mathfrak{N}'$.

Proof. Let $U_a \in \mathfrak{M}$, $V_b \in \mathfrak{N}$, then

$$(U_aV_bx,y) = (V_bx,U_a^*y) = (V_bx,U_{Ja}y)$$
$$= (U_xb,V_yJa) = (U_xV_{Jy}b,Ja) = (V_{Jy}b,U_{Jx}Ja)$$
$$= (U_bJy,V_{Ja}Jx) = (JV_{Ja}Jx,JU_bJy) = (U_ax,V_b^*y)$$
$$= (V_bU_ax,y) \quad \text{for } x,y \in \mathfrak{A};$$

hence $U_aV_b = V_bU_a$. This completes the proof.

Now we shall show

Theorem 3.1. $\mathcal{U}(\mathfrak{A})' = \mathcal{V}(\mathfrak{A})$ and $\mathcal{V}(\mathfrak{A})' = \mathcal{U}(\mathfrak{A})$.

Proof. $\mathcal{V}(\mathfrak{A})' = \mathfrak{M}'' \subset \mathfrak{N}''' = \mathcal{U}(\mathfrak{A})'' = \mathcal{U}(\mathfrak{A})$; $\mathcal{V}(\mathfrak{A})' \supset \mathcal{U}(\mathfrak{A})$ is clear; this completes the proof.

Now, let $\mathfrak{A}$ be a hilbert algebra, $\mathfrak{A}_1$ a *-subalgebra of $\mathfrak{A}$ which is dense in $\mathfrak{A}$, then $\{x_1y_1 \mid x_1,y_1 \in \mathfrak{A}_1\}$ is dense in $\{xy \mid x,y \in \mathfrak{A}\}$, so that it is also dense in $\mathfrak{h}$; therefore $\mathfrak{A}_1$ is also a hilbert algebra.

Corollary. Let $\mathfrak{A}$ be a hilbert algebra, $\mathfrak{A}_1$ a *-subalgebra of $\mathfrak{A}$ which is dense in $\mathfrak{A}$, then $\mathcal{U}(\mathfrak{A}_1) = \mathcal{U}(\mathfrak{A})$ and $\mathcal{V}(\mathfrak{A}_1) = \mathcal{V}(\mathfrak{A})$.

Proof. Clearly $\mathcal{U}(\mathfrak{A}_1) \subset \mathcal{U}(\mathfrak{A})$, $\mathcal{V}(\mathfrak{A}_1) \subset \mathcal{V}(\mathfrak{A})$. By Theorem 3.1, $\mathcal{U}(\mathfrak{A}_1) = \mathcal{V}(\mathfrak{A}_1)' \supset \mathcal{V}(\mathfrak{A})' = \mathcal{U}(\mathfrak{A})$, so that $\mathcal{U}(\mathfrak{A}_1) = \mathcal{U}(\mathfrak{A})$ and $\mathcal{V}(\mathfrak{A}_1) = \mathcal{V}(\mathfrak{A})$.

Next we shall consider the tensor product of hilbert algebras. Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two hilbert algebras and $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ be the algebraic tensor product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, then $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is an *-algebra; moreover there is a unique pre-hilbert structure on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ such that $(x_1 \otimes y_1, x_2 \otimes y_2) = (x_1,x_2)(y_1,y_2)$; then $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ becomes a hilbert algebra with this inner product. We shall call this the tensor product of hilbert algebras and denote by $\mathfrak{A}_1 \otimes \mathfrak{A}_2$. Let $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{h}$ be the completions of $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, then we have $\mathfrak{h} = \mathfrak{h}_1 \bar{\otimes} \mathfrak{h}_2$.

Proposition 3.1 $\mathcal{U}(\mathfrak{A}_1 \otimes \mathfrak{A}_2) = \mathcal{R}(\mathcal{U}(\mathfrak{A}_1) \otimes 1, 1 \otimes \mathcal{U}(\mathfrak{A}_2))$
$\mathcal{V}(\mathfrak{A}_1 \otimes \mathfrak{A}_2) = \mathcal{R}(\mathcal{V}(\mathfrak{A}_1) \otimes 1, 1 \otimes \mathcal{V}(\mathfrak{A}_2))$

The proof is very easy.

Proposition 3.2. Let $\mathfrak{A}$ be a hilbert algebra, then $\mathcal{U}(\mathfrak{A})$ (resp. $\mathcal{V}(\mathfrak{A})$) is a semi-finite W^*-algebra.

Proof. Let $\mathfrak{M} = \{U_a \mid a \text{ (bounded) } \varepsilon\ \mathfrak{h}\}$, then $\mathfrak{M}$ is an ideal of $\mathcal{U}(\mathfrak{A})$ and it is s-dense in $\mathcal{U}(\mathfrak{A})$. Let P be a non-zero central projection of $\mathcal{U}(\mathfrak{A})$, then there is a positive element U_h of $\mathfrak{M}$ such that $PU_h \neq 0$, so that there is a non-zero projection E of $\mathcal{U}(\mathfrak{A})$ such that $E \leq \lambda PU_h$ for some positive number λ; then $E \,\varepsilon\, \mathfrak{M}$; therefore $\mathcal{U}(\mathfrak{A})E \subset \mathfrak{M}$; let $\{U_\alpha E\}$ be a directed set of $\mathcal{U}(\mathfrak{A})$ such that $U_\alpha E \longrightarrow 0(s)$ and $\| U_\alpha E \| \leq 1$; put $E = U_a$ ($a \,\varepsilon\, \mathfrak{h}$, bounded), then $U_\alpha E = U_\alpha EE = U_{U_\alpha Ea}$ and

$$((U_\alpha E)^* x, (U_\alpha E)^* x) = (U_{JU_\alpha E_a} x, U_{JU_\alpha E_a} x)$$
$$= (V_x JU_\alpha E_a, V_x JU_\alpha Ea)$$
$$\leq \| V_x \| \, \| U_\alpha Ea \| \longrightarrow 0 \quad \text{for } x \,\varepsilon\, \mathfrak{A};$$

hence $\{(U_\alpha E)^*\}$ is s-convergent to zero, so that by Theorem 5.3, chapter II, E is a finite projection; this implies that $\mathcal{U}(\mathfrak{A})$ is semi-finite; moreover since $\mathcal{V}(\mathfrak{A}) = J\mathcal{U}(\mathfrak{A})J$, $\mathcal{V}(\mathfrak{A})$ is also semi-finite. This completes the proof.

Definition 3.3. Let M be a W^*-algebra. If there exists a faithful *-representation ρ of M onto the associated left algebra $\mathcal{U}(\mathfrak{A})$ of a hilbert algebra $\mathfrak{A}$, ρ is said a standard representation of M.

By Proposition 3.2, if M has a standard representation, it is semi-finite. Conversely,

Theorem 3.2. Let M be a semi-finite algebra, then it has a standard representation.

Proof. Since M is semi-finite, there is a family of semi-finite normal traces (τ_α) such that their supports $\{s(\tau_\alpha)\}$ are mutually orthogonal central projections and $\sum_\alpha s(\tau_\alpha) = 1$. Now put $\tau(h) = \sum_\alpha \tau_\alpha(h)$ for $(h \geq 0) \varepsilon M$, then τ is also a semi-finite normal trace on M; moreover suppose that $\tau(h) = 0$, then $\tau_\alpha(h) = 0$ and so $hs(\tau_\alpha) = 0$ for all α; hence $h = 0$; τ is faithful.

Let $\mathcal{F} = \{a \mid \tau(a) < +\infty\}$, then by Proposition 9.1, chapter I, $\mathcal{F}$ is the positive portion of an ideal $\mathfrak{A}$, and there is a unique linear functional $\bar{\tau}$ on $\mathfrak{A}$ coincides with τ on $\mathcal{F}$; moreover the linear functional $x \longrightarrow \bar{\tau}(ax)$ is σ-continuous. Put $(a,b) = \bar{\tau}(b^*a)$ for $a,b \,\varepsilon\, \mathfrak{A}$, then $\mathfrak{A}$ becomes a pre-hilbert space; it is easily shown that the pre-hilbert space is a hilbert algebra and M is *-isomorphic to $\mathcal{U}(\mathfrak{A})$.

This completes the proof.

Here, we shall state somes concerning general representations of W^*-algebras into the operator algebras on hilbert spaces.

Definition 3.4. Let M be a W^*-algebra, then a W^*-representation of M is a continuous *-homomorphism π of M with the σ-topology into the algebra $B(\mathfrak{H})$, with the wo-topology, of all bounded operators on a hilbert space $\mathfrak{H}$ such that $\pi(1)$ is the identity operator on $\mathfrak{H}$.

We shall denote by $\{\pi, \mathfrak{H}\}$ a W^*-representation π of M into $B(\mathfrak{H})$. Let $\{\pi, \mathfrak{H}\}$ be a W^*-representation of a W^*-algebra M, then the kernel $\mathcal{J}$ of π is a σ-closed ideal; there is a central projection z such that $\mathcal{J} = Mz$; the restriction π on $M(1-z)$ is an *-isomorphism; by the well known theorem of B^*-algebras, it is an isometry; $\pi(S)$ is the unit sphere of $\pi(M)$, where S is the unit sphere of M; the unit sphere of $\pi(M)$ is wo-compact; by Corollary 1.3, $\pi(M)$ is weakly closed.

Hence we obtain

Proposition 3.3. Let $\{\pi, \mathfrak{H}\}$ be a W^*-representation of M, then the image $\pi(M)$ is a weakly closed *-subalgebra of $B(\mathfrak{H})$.

Definition 3.45. Let $\{\pi_1, \mathfrak{H}_1\}$, $\{\pi_2, \mathfrak{H}_2\}$ be two W^*-representations of M. If there is an isometry U of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that $U\pi_1(x) = \pi_2(x)U$ for $x \in M$, it is said that $\{\pi_1, \mathfrak{H}_1\}$ is equivalent to $\{\pi_2, \mathfrak{H}_2\}$ and denote by $\{\pi_1, \mathfrak{H}_1\} \sim \{\pi_2, \mathfrak{H}_2\}$.

It is clear that the above equivalence satisfies a usual equivalent relations. By this equivalence, classifying representations of M, we shall identify representations which belong to the same class.

Definition 3.6. Let $\{\pi, \mathfrak{H}\{$ be a W^*-representation of M. If there is an element ξ of $\mathfrak{H}$ such that $[\pi(M)\xi] = \mathfrak{H}$, π is said a cyclic W^*-representation and ξ is said a cyclic vector, where $[\pi(M)\xi]$ is the closed subspace of $\mathfrak{H}$ generated $\pi(M)\xi$.

Let $\{\pi, \mathfrak{H}\}$ be a cyclic W^*-representation of M, ξ $(\|\xi\| = 1)$ a cyclic vector of π. Put $\varphi(x) = (\pi(x)\xi, \xi)$ for

$x \in M$, where $(\ ,\ )$ is the inner product of $\mathfrak{H}$, then φ is a σ-continuous positive functional such that $\varphi(1) = 1$; it is easy that $\{\pi, \mathfrak{H}\}$ is equivalent to the W^*-representation $\{\pi_\varphi, \mathfrak{H}_\varphi\}$ on a hilbert space $\mathfrak{H}_\varphi$, constructed via φ.

Conversely, let φ be a σ-continuous positive functional of M, $\{\pi_\varphi, \mathfrak{H}_\varphi\}$ a B^*-representation of M on a hilbert space $\mathfrak{H}_\varphi$; let 1_φ be the image of 1 in $\mathfrak{H}_\varphi$, then $[\pi(M)1_\varphi] = \mathfrak{H}$; $f(x) = (\pi(x)\pi(a)1_\varphi, \pi(b)1_\varphi) = (\pi(b^*xa)1_\varphi, 1_\varphi) = \varphi(b^*xa)$; for any $\xi_1, \xi_2 \in \mathfrak{H}$ there are two sequences (a_n) and (b_n) of M such that $\| \pi(a_n)1_\varphi - \xi_1 \| \longrightarrow 0$ and $\| \pi(b_n)1_\varphi - \xi_2 \| \longrightarrow 0$; hence $(\pi(x)\pi(a_n)1_\varphi, \pi(b_n)1_\varphi) \longrightarrow (\pi(x)\xi_1, \xi_2)$ (uniformly on the unit sphere of M); this implies $(\pi(x), \xi_1, \xi_2) \in M_*$; $\{\pi_\varphi, \mathfrak{H}_\varphi\}$ is a W^*-representation.

Hence we obtain

Proposition 3.4. Every cyclic W^*-representation is equivalent to a W^*-representation $\{\pi_\varphi, \mathfrak{H}_\varphi\}$ constructed via φ, where φ is a σ-continuous positive functional of M.

We shall introduce some fundamental operations of constructing W^*-representations.

Definition 3.7. Let $\{\pi, \mathfrak{H}\}$ be a W^*-representation of M. Let K be a hilbert space, $\mathfrak{H} \bar{\otimes} K$ the tensor product of $\mathfrak{H}$ and K. Then a mapping $x \longrightarrow \pi(x) \otimes 1_K$ is a W^*-representation of M. We call this an amplification of $\{\pi, \mathfrak{H}\}$ and denote by $\{\pi \otimes 1_K, \mathfrak{H} \bar{\otimes} K\}$.

Definition 3.8. Let $\{\pi, \mathfrak{H}\}$ be a W^*-representation of M. Let E be a projection of $\{\pi(M)\}'$, then a mapping $x \longrightarrow \pi(x)E$ is considered a W^*-representation of M into $B(E\mathfrak{H})$.

We call this an induction of π and denote by $\{\pi E, E\mathfrak{H}\}$.

Definition 3.9. Let $\{\pi_\alpha, \mathfrak{H}_\alpha\}_{\alpha\varepsilon\Pi}$ be a family of W^*-representations of M, $\mathfrak{H}$ be the direct sum of $\{\mathfrak{H}_\alpha\}_{\alpha\varepsilon\Pi}$. We consider a mapping of M into $B(\mathfrak{H})$ as follows: $x \longrightarrow \pi(x) = \sum_{\alpha\varepsilon\Pi} \oplus \pi_\alpha(x)$; then π is also a W^*-representation. We call this a sum of W^*-representations $\{\pi_\alpha, \mathfrak{H}_\alpha\}_{\alpha\varepsilon\Pi}$ and denote by $\{\sum_{\alpha\varepsilon\Pi} \oplus \pi_\alpha, \sum_{\alpha\varepsilon\Pi} \oplus \mathfrak{H}_\alpha\}$.

Proposition 3.5. Let $\{\pi, \mathfrak{H}\}$ be a W^*-representation of M, and E_1, E_2 be two projections of $\{\pi(M)\}'$, then two inductions $\{\pi E_1, E_1\mathfrak{H}\}$ and $\{\pi E_2, E_2\mathfrak{H}\}$ are equivalent if and only if $E_1 \sim E_2$ in the W^*-algebra $\{\pi(M)\}'$.

Proof. Suppose that $E_1 \sim E_2$; let v be a partial isometry such that $v^*v = E_1$, $vv^* = E_2$ and $v \varepsilon \{\pi(M)\}'$, then $\pi(x)E_1 = \pi(x)v^*v = v^*\pi(x)v = v^*\pi(x)E_2v$; hence $\pi E_1 \sim \pi E_2$. Conversely suppose that $\pi E_1 \sim \pi E_2$; let U be an isometry of $E_1\mathfrak{H}$ onto $E_2\mathfrak{H}$ which gives the equivalence $\pi E_1 \sim \pi E_2$; then by defining $U(1-E_1)\mathfrak{H} = 0$, U can be extended to a bounded operator $\tilde{U}$ on $\mathfrak{H}$; then $U\pi(x) = \pi(x)U$ for $x \varepsilon M$ implies $\tilde{U} \varepsilon \{\pi(M)\}'$ and $\tilde{U}^*\tilde{U} = E_1$, $\tilde{U}\tilde{U}^* = E_2$, so that $E_1 \sim E_2$. This completes the proof.

Now let $\{\pi, \mathfrak{H}\}$ be a W^*-representation of M, and ξ_1 be a vector of $\mathfrak{H}$ such that $\|\xi_1\| = 1$; $[\pi(M)\xi_1]$ is an invariant subspace of $\mathfrak{H}$; let E_1 be the projection of $\mathfrak{H}$ onto $[\pi(M)\xi_1]$, then $E_1 \varepsilon \{\pi(M)\}'$. If $E_1 \neq 1$, we take a vector ξ_2 such that $\xi_2 \varepsilon (1-E_1)\mathfrak{H}$ and $\|\xi_2\| = 1$; then the projection E_2 of $\mathfrak{H}$ onto $[\pi(M)\xi_2]$ belongs to $\{\pi(M)\}'$ and $E_1 \cdot E_2 = 0$; therefore by continuing such process transfinitely, we obtain a family of orthogonal projections $(E_\alpha)_{\alpha \varepsilon \mathbb{I}}$ of $\{\pi(M)\}'$ such that $\sum_{\alpha \varepsilon \mathbb{I}} E_\alpha = 1$ and πE_α is cyclic; therefore any W^*-representation is equivalent to a sum of cyclic W^*-representations $\{\pi_\alpha\}_{\alpha \varepsilon \mathbb{I}}$, where $\pi E_\alpha = \pi_\alpha$.

Now let $\{\tilde{\pi}, \tilde{\mathfrak{H}}\}$ be a faithful W^*-representation of M, and K be a $\aleph_0$-dimensional hilbert space, then from the considerations in the proof of Theorem 1.1, every cyclic W^*-representation of M is equivalent to an induction of the W^*-representation $\{\tilde{\pi} \otimes 1_K, \tilde{\mathfrak{H}} \bar{\otimes} K\}$; let K' be a n-dimensional hilbert space with $n = \text{card}\,(\mathbb{I})$, then clearly π is equivalent to an induction of $\{\tilde{\pi} \otimes 1_K \otimes 1_{K'}, \tilde{\mathfrak{H}} \bar{\otimes} K \bar{\otimes} K'\}$.

Hence we obtain

Theorem 3.3. Let $\{\pi, \mathfrak{H}\}$ be a W^*-representation of M and $\{\tilde{\pi}, \tilde{\mathfrak{H}}\}$ be a faithful W^*-representation of M, then $\{\pi, \mathfrak{H}\}$ is equivalent to an induction of an amplification of $\{\tilde{\pi}, \tilde{\mathfrak{H}}\}$.

In the theory of W^*-representation, the following problem is important; let $\{\pi_1, \mathfrak{H}_1\}$, $\{\pi_2, \mathfrak{H}_2\}$ be two W^*-representations of a W^*-algebra M having the same kernel. Then, under any additional conditions, can we conclude that $\{\pi_1, \mathfrak{H}_1\}$ is equivalent to $\{\pi_2, \mathfrak{H}_2\}$?

This problem has been studied by a number of authors. Nowadays, the problem has almost completely been solved. The reader is referred for this information to the book of Dixmier.

Notices of §3

Theorem 3.1 is called the commutation theorem of hilbert algebras. The *-algebra L(G) of continuous functions with compact supports on a unimodular locally compact group G is a hilbert algebra; therefore the left-regular representation and the right regular representation of G satisfy the commutation theorem; Dixmier [cf. 4] introduced the notion of quasi-hilbert algebras and he succeeded in proving that the left-regular representation and the right-regular representation of all locally compact groups satisfy the commutation theorem.

§4. Types of tensor products

Let M, N be two W^*-algebras, and $M \bar{\otimes} N$ be the tensor product of M and N. The purpose of this section is to show the types of $M \bar{\otimes} N$.

Let M (resp. N) be the direct sum of a family of W^*-algebras of $\{M_\alpha\}_{\alpha \varepsilon \mathbb{I}}$ (resp. $\{N_\beta\}_{\beta \varepsilon \mathbb{J}}$), then, using Theorem 2.3, we can easily conclude that $M \bar{\otimes} N = \sum_{\substack{\alpha \varepsilon \mathbb{I} \\ \beta \varepsilon \mathbb{J}}} \oplus (M_\alpha \bar{\otimes} N_\beta)$.

Now let M and N be two finite algebras with faithful normal finite traces φ, ψ respectively. Then by the inner products: $(a,b) = \varphi(b^* a)$, $(c,d) = \psi(d^* c)$ and $(a \otimes c, b \otimes d) = \varphi \otimes \psi((b \otimes d)^* (a \otimes c))$ for $a, c \varepsilon M$, $b, d \varepsilon N$ and $a \otimes b$, $c \otimes d \varepsilon M \otimes N$, M, N and $M \otimes N$ become hilbert algebras respectively. By Proposition 3.1, $\mathcal{U}(M \otimes N) = R(\mathcal{U}(M) \otimes 1, 1 \otimes \mathcal{U}(N))$, so that by Theorem 2.3, $M \bar{\otimes} N$ is *-isomorphic to $\mathcal{U}(M \otimes N)$. Let $\mathfrak{h}$ be the hilbert space obtained by the completion of the pre-hilbert space $M \otimes N$ and $\mathfrak{M} = \{U_a \mid a \text{ (bounded) } \varepsilon \mathfrak{h}\}$, then $\mathfrak{M}$ is an ideal of $\mathcal{U}(M \otimes N)$; since the identity operator $U_{1 \otimes 1}$ belongs to $\mathfrak{M}$, $\mathfrak{M} = \mathcal{U}(M \otimes N)$; therefore from the proof of Proposition 3.2, $U_{1 \otimes 1}$ is a finite projection, so that $\mathcal{U}(M \otimes N)$ and so $M \bar{\otimes} N$ are finite.

Next let M and N be general finite algebras, then there are families of orthogonal central projections $(z_\alpha)_{\alpha \varepsilon \mathbb{I}}$ and $(z_\beta)_{\beta \varepsilon \mathbb{J}}$ of M and N respectively such that $\sum_{\alpha \varepsilon \mathbb{I}} \oplus z_\alpha = 1$, $\sum_{\beta \varepsilon \mathbb{J}} \oplus z_\beta = 1$, and Mz_α and Nz_β have faithful normal finite traces respectively; moreover $M \bar{\otimes} N = \sum_{\substack{\alpha \varepsilon \mathbb{I} \\ \beta \varepsilon \mathbb{J}}} \oplus (Mz_\alpha \otimes Nz_\beta)$; hence $M \bar{\otimes} N$ is finite.

Now let M and N be two W^*-algebras, and e and f be projections of M and N respectively, then by Theorem 2.3, $(e\otimes f)(M\bar{\otimes}N)(e\otimes f) = (eMe)\bar{\otimes}fMf$; therefore, if e and f are finite projections, $e\otimes f$ is also finite in $M\bar{\otimes}N$.

Let M and N be two semi-finite W^*-algebras, then there are two increasing directed sets (e_α) and (f_β) of projections in M and N, respectively such that $e_\alpha \to 1(s)$ in M, $f_\beta \to 1(s)$ in N, and e_α, f_β are finite; since $e_\alpha\otimes f_\beta$ is finite and $\{e_\alpha\otimes f_\beta\}$ is an increasing directed set of projections such that $e_\alpha\otimes f_\beta \to 1(s)$, we can conclude that $M\bar{\otimes}N$ is semi-finite.

Hence we obtain

Theorem 4.1. $M\bar{\otimes}N$ is finite if and only if M and N are finite; $M\bar{\otimes}N$ is semi-finite if M and N is semi-finite.

It is clear that if $M\bar{\otimes}N$ is finite, M and N are finite.

Next, let M (resp. N) be a homogeneous algebra of type I_m (resp. type I_n) then there is a family $(e_\alpha)_{\alpha\in\mathbb{I}}$ (resp. $(f_\beta)_{\beta\in\mathbb{J}}$) of orthogonal equivalent, maximal abelian projections in M (resp. N) such that $\sum_{\alpha\in\mathbb{I}} e_\alpha = 1$, $\sum_{\beta\in\mathbb{J}} f_\beta = 1$, card $(\mathbb{I}) = m$, card $(\mathbb{J}) = n$; therefore $\{e_\alpha\otimes f_\beta\}_{\substack{\alpha\in\mathbb{I}\\ \beta\in\mathbb{J}}}$ is a family of orthogonal equivalent projections such that $\sum_{\substack{\alpha\in\mathbb{I}\\ \beta\in\mathbb{J}}} e_\alpha\otimes f_\beta = 1$; moreover $(e_\alpha\otimes f_\beta)(M\bar{\otimes}N)(e_\alpha\bar{\otimes} f_\beta) = (e_\alpha Me_\alpha)\bar{\otimes}(f_\beta Nf_\beta)$, so that $e_\alpha\otimes f_\beta$ is abelian and clearly maximal; hence we obtain that $M\bar{\otimes}N$ is a homogeneous algebra of type mn; therefore, by Proposition 3.2, chapter II, we obtain

Theorem 4.2. $M\bar{\otimes}N$ is of type I, if M and N are of type I. Next, let M (resp. N) be a finite algebra (resp. a continuous finite algebra) having a faithful normal finite trace φ(resp. ψ), then $M\bar{\otimes}N$ is continuous; in fact, suppose that $M\bar{\otimes}N$ contains a direct summand of type I; since N is continuous, there is a decreasing sequence (e_n) of projections such that $e_n - e_{n+1} \sim e_{n+1}$, $z(e_n) = 1$ for $n=1,2,\ldots$; then $\varphi\otimes\psi(1\otimes e_n) = \psi(e_n) = 2^{-n}\psi(e_1) \to 0$; on the other hand, let p be an abelian projection of $M\bar{\otimes}N$, then $p \prec 1\otimes e_n$; hence $\varphi\otimes\psi(p) \leq \varphi(1\otimes e_n) \to 0$, a contradiction; therefore $M\bar{\otimes}N$ is continuous.

Next, let M(resp. N) be a semi-finite algebra (resp. a continuous semi-finite algebra), then $M\bar{\otimes}N$ is a continuous; in fact, let e (resp. f) be a finite projection of M (resp. N) such that $Z(e) = 1$ (resp. $Z(f) = 1$), then $(e\otimes f)(M\bar{\otimes}N)(e\otimes f) = (eMe)\bar{\otimes}(fNf)$ is continuous; hence $M\bar{\otimes}N$ is continuous from the comparability theorem.

Hence we obtain

Theorem 4.3. $M\bar{\otimes}N$ is of type II, if M and N are semi-finite and one of them is continuous.

Finally we shall show

Theorem 4.4. Let M and N be two W^*-algebras, one of which is of type III, then $M\bar{\otimes}N$ is of type III.

To prove the theorem, we shall provide some considerations. Let M and N faithfully represent weakly closed *-algebras $\mathfrak{M}_1$ and $\mathfrak{M}_2$ on hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively; then by the considerations of §1, an element a of $B(\mathfrak{H}_1\otimes\mathfrak{H}_2)$ can be represented by an operator matrix $(a_{\alpha,\beta})_{\alpha,\beta\in\Pi}$

$(a_{\alpha,\beta} \in B(\mathfrak{H}_2))$; since $\mathcal{R}(B(\mathfrak{H}_1)\otimes 1, 1\otimes\mathcal{M}_2)' = 1\otimes\mathcal{M}_2'$, $\mathcal{R}(B(\mathfrak{H}_1)\otimes 1, 1\otimes\mathcal{M}_2) = (1\otimes\mathcal{M}_2')' = M_{\mathcal{M}_2}$; therefore the element b in $\mathcal{R}(B(\mathfrak{H}_1)\otimes 1, 1\otimes\mathcal{M}_2)$ is expressed by $(b_{\alpha,\beta})$ $(b_{\alpha,\beta} \in \mathcal{M}_2)$ under the above representation. Now put $P_\gamma(b_{\alpha,\beta}) = (\delta_{\alpha\beta}b_{\gamma,\gamma})$ for all γ, then P_γ are considered as linear mappings of $\mathcal{R}(B(\mathfrak{H}_1)\otimes 1, 1\otimes\mathcal{M}_2)$ onto $1\otimes\mathcal{M}_2$; we can easily show the following properties:

(1) $P_\gamma(1\otimes 1) = 1\otimes 1$, (2) $\|P_\gamma(b)\| \leq \|b\|$

(3) $P_\gamma(h) \geq 0$ for $h \geq 0$, (4) $P_\gamma(ubv) = uP_\gamma(b)v$

for $u, v \in 1\otimes\mathcal{M}_2$. (5) $P_\gamma(b)^*P_\gamma(b) \leq P_\gamma(b^*b)$

(6) P_γ are wo and so-continuous on bounded spheres, and (7) $P_\gamma(b^*b) = 0$ for all γ imply $b = 0$, for $b, h \in \mathcal{R}(B(\mathfrak{H}_1)\otimes 1, 1\otimes\mathcal{M}_2)$.

Since $\mathcal{R}(\mathcal{M}_1\otimes 1, 1\otimes\mathcal{M}_2)$ is a subalgebra of $\mathcal{R}(B(\mathfrak{H}_1)\otimes 1, 1\otimes\mathcal{M}_2)$, the restriction of P_γ on $\mathcal{R}(\mathcal{M}_1\otimes 1, 1\otimes\mathcal{M}_2)$ defines a linear mapping $\widetilde{P_\gamma}$ of $\mathcal{R}(\mathcal{M}_1\otimes 1, 1\otimes\mathcal{M}_2)$ onto $1\otimes\mathcal{M}_2$, so that by Theorem 2.3, $\widetilde{P_\gamma}$ can be considered a linear mapping of $M\bar{\otimes}N$ onto $1\otimes N$.

Proof of Theorem 4.4. Suppose that N is of type III and that there is a non-zero central projection z of $M\bar{\otimes}N$ such that $(M\bar{\otimes}N)z$ is semi-finite; let e be a non-zero finite projection of $(M\bar{\otimes}N)z$, by the above consideration, there is a mapping $\widetilde{P}_{\gamma_0}$ such that $\widetilde{P}_{\gamma_0}(e) \neq 0$; since $\widetilde{P}_{\gamma_0}(e) > 0$, there is a non-zero projection p $(\in 1\otimes N)$ such that $\lambda p \leq \widetilde{P}_{\gamma_0}(e)$ for some positive number λ (>0). Suppose that (x_α) $(\|x_\alpha\| \leq 1,\ x_\alpha \in p(1\otimes N)p)$ is s-convergent to 0, then $(x_\alpha e)$ is s-convergent to 0;

hence by Theorem 5.3, chapter II, (ex_α^*) is s-convergent to 0; by the so-continuity of $\widetilde{P_{\gamma_0}}$ on bounded spheres, $\widetilde{P_{\gamma_0}}(ex_\alpha^*) = \widetilde{P_{\gamma_0}}(e)x_\alpha^*$ is s-convergent to 0, so that $\{p\widetilde{P_{\gamma_0}}(e)p + (1-p)\}^{-1}.$ $p\widetilde{P_{\gamma_0}}(e)x_\alpha^* = x_\alpha^*$ is s-convergent to 0; therefore the *-operation is s-continuous on bounded spheres of $p(1\otimes N)p$; by Theorem 5.3, chapter II, p is a finite projection of $1\otimes N$; hence $1\otimes N$ and so N is not of type III, a contradiction. This completes the proof.

Now we obtain the following diagram concerning the type of tensor products

(Type I_m) $\overline{\otimes}$ (Type I_n) = (Type I_{mn})

(Type I) $\overline{\otimes}$ (Type I) = (Type I)

(Type I or Type II) $\overline{\otimes}$ (Type II) = (Type II)

(any W^*-algebra) $\overline{\otimes}$ (Type III) = (Type III)

(finite) $\overline{\otimes}$ (finite) = (finite)

(semi-finite) $\overline{\otimes}$ (semi-finite) = (semi-finite)

(any W^*-algebra) $\overline{\otimes}$ (properly infinite) = (properly infinite)

(any W^*-algebra) $\overline{\otimes}$ (purely infinite) = (purely infinite)

(discrete) $\overline{\otimes}$ (discrete) = (discrete)

(any W^*-algebra) $\overline{\otimes}$ (continuous) = (continuous)

Finally we shall show some facts concerning the commutant of a weakly closed *-algebra on a hilbert space.

Proposition 4.1. Let $\mathfrak{M}$ be a weakly closed *-algebra containing the identity operator on a hilbert space $\mathfrak{H}$, and E_1 (resp. E_2) be a projection of $\mathfrak{M}$ (resp. $\mathfrak{M}'$), then we have: $(E_1\mathfrak{M}E_1)' = \mathfrak{M}'E_1$ and $(\mathfrak{M}E_2)' = E_2\mathfrak{M}'E_2$ on the hilbert space $E_1\mathfrak{H}$.

Proof. Clearly $(\mathcal{M}'E_1)' \supset (E_1\mathcal{M}E_1)$; suppose that $T \varepsilon B(E_1\mathfrak{H})$ and $T \varepsilon (\mathcal{M}'E_1)'$, then $T_1 = TE_1$ belongs to $B(\mathfrak{H})$ and $T_1 \varepsilon \mathcal{M}'' = \mathcal{M}$; hence $T = E_1T_1E_1 \varepsilon E_1\mathcal{M}E_1$; by symmetry, we have $(\mathcal{M}E_2)' = E_2\mathcal{M}'E_2$.

Proposition 4.2. Let $\mathcal{M}$ be a weakly closed *-subalgebra of $B(\mathfrak{H})$ containing the identity operator on a hilbert space $\mathfrak{H}$, then we have: $\mathcal{M}'$ is of type I (resp. type II, type III) if and only if $\mathcal{M}$ is of type I (resp. type II, type III.)

Proof. Suppose that $\mathcal{M}$ is a semi-finite W^*-algebra, then $\mathcal{M}$ has a standard representation $\{\pi_1, \mathfrak{H}_1\}$ such that $\pi_1(\mathcal{M}) = \mathcal{U}(\mathfrak{A})$, where $\mathfrak{A}$ is a hilbert algebra and $\mathfrak{H}_1$ is the completion of $\mathfrak{A}$.

Since $\mathcal{M}$ on $\mathfrak{H}$ is considered a W^*-representation of $\mathcal{M}$, by Theorem 3.3, it can be considered a W^*-representation $\{\pi_2, \mathfrak{H}_2\}$ such that $\pi_2 = (\pi_1 \otimes 1_K)E$, where K is a hilbert space and E is a projection of $\{\pi_1(\mathcal{M}) \otimes 1_K\}'$; since $\{\pi_1(\mathcal{M}) \otimes 1_K\}' = \{\mathcal{U}(\mathfrak{A}) \otimes 1_K\}' = \mathcal{R}(\mathcal{U}(\mathfrak{A})' \otimes 1_K, 1_{\mathfrak{H}_1} \otimes B(K)) \sim \mathcal{U}(\mathfrak{A})' \bar{\otimes} B(K)$; therefore $\{\pi_1(\mathcal{M}) \otimes 1_K\}'$ is of type I (resp. type II) $\Longleftrightarrow$ $\mathcal{U}(\mathfrak{A})'$ is of type I (resp. type II); since $\mathcal{U}(\mathfrak{A})'$ is conjugate linear *-isomorphic to $\mathcal{U}(\mathfrak{A})$, $\{\pi_1(\mathcal{M}) \otimes 1_K\}'$ is of type I (resp. type II) $\Longleftrightarrow$ $\mathcal{U}(\mathfrak{A})$ is of type I (resp. type II); easily $E\{\pi_1(\mathcal{M}) \otimes 1_K\}'E$ is of type I (resp. type II) $\Longleftarrow$ $\{\pi_1(\mathcal{M}) \otimes 1_K\}'$ is of type I (resp. type II); $\mathcal{M}' = \{(\pi_1(\mathcal{M}) \otimes 1_K)E\}' = E\{\pi_1(\mathcal{M}) \otimes 1_K\}'E$; hence we have : $\mathcal{M}'$ is of type I (resp. type II) $\Longleftrightarrow$ $\mathcal{M} \sim \mathcal{U}(\mathfrak{A})$ is of type I (resp. type II); therefore, easily, $\mathcal{M}'$ is of type III $\Longleftrightarrow$ $\mathcal{M}$ is of type III. This completes the proof.

Proposition 4.3. Let $\mathcal{M}_1, \mathcal{M}_2$ be two semi-finite weakly closed *-algebras containing the identities on hilbert space $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively, then $R(\mathcal{M}_1 \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \mathcal{M}_2)' = R(\mathcal{M}_1' \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \mathcal{M}_2')$.

Proof. Let $\{\pi_1, K_1\}$, $\{\pi_2, K_2\}$ be two standard representations of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

Put $\pi_1(\mathcal{M}_1) = \mathcal{U}(\mathfrak{A}_1)$ and $\pi_2(\mathcal{M}_2) = \mathcal{U}(\mathfrak{A}_2)$, then we can consider as follows:

$$\mathcal{M}_1 = (\mathcal{U}(\mathfrak{A}_1) \otimes 1_{K_3})E_1$$

$$\mathcal{M}_2 = (\mathcal{U}(\mathfrak{A}_2) \otimes 1_{K_4})E_2 ,$$

where $E_1 \in (\mathcal{U}(\mathfrak{A}_1) \otimes 1_{K_3})'$ and $E_2 \in (\mathcal{U}(\mathfrak{A}_2) \otimes 1_{K_4})'$.

Put $E = E_1 \otimes E_2$, then we have

$$R(\mathcal{M}_1 \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \mathcal{M}_2) = R(\mathcal{U}(\mathfrak{A}_1) \otimes 1_{K_2} \otimes 1_{K_3} \otimes 1_{K_4},$$

$$1_{K_1} \otimes \mathcal{U}(\mathfrak{A}_2) \otimes 1_{K_3} \otimes 1_{K_4})E = R(\mathcal{U}(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \otimes 1_{K_3} \otimes 1_{K_4})E ;$$

On the other hand

$$\mathcal{M}_1' = E_1(\mathcal{U}(\mathfrak{A}_1) \otimes 1_{K_3})'E_1 = E_1 R(\mathcal{U}(\mathfrak{A}_1)' \otimes 1_{K_3}, 1_{K_1} \otimes B(K_3))E_1$$

$$\mathcal{M}_2' = E_2(\mathcal{U}(\mathfrak{A}_2) \otimes 1_{K_4})'E_2 = E_2 R(\mathcal{U}(\mathfrak{A}_2)' \otimes 1_{K_4}, 1_{K_2} \otimes B(K_4))E_2 ;$$

therefore

$$R(\mathcal{M}_1' \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \mathcal{M}_2') = E R(\mathcal{U}(\mathfrak{A}_1)' \otimes 1_{K_2} \otimes 1_{K_3} \otimes 1_{K_4} ,$$

$$1_{K_1} \otimes 1_{K_2} \otimes B(K_3) \otimes 1_{K_4}, 1_{K_1} \otimes \mathcal{U}(\mathfrak{A}_2)' \otimes 1_{K_3} \otimes 1_{K_4},$$

$$1_{K_1} \otimes 1_{K_2} \otimes 1_{K_3} \otimes B(K_4))E =$$

$= E\, \mathcal{R}(\mathcal{U}(\mathcal{O}_1 \otimes \mathcal{O}_2)' \otimes 1_{K_3} \otimes 1_{K_4}, 1_{K_1} \otimes 1_{K_2} \otimes B(K_3 \bar{\otimes} K_4))E$

$= E\, \mathcal{R}(\mathcal{U}(\mathcal{O}_1 \otimes \mathcal{O}_2) \otimes 1_{K_3} \otimes 1_{K_4})' E$

$= \{\mathcal{R}(\mathcal{U}(\mathcal{O}_1 \otimes \mathcal{O}_2) \otimes 1_{K_3} \otimes 1_{K_4})E\}' = \mathcal{R}(\mathcal{M}_1 \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \mathcal{M}_2)'$.

This completes the proof.

Corollary 4.1. Let M_1, M_2 be two W^*-algebras, Z_1 and Z_2 be the centers of M_1 and M_2, respectively, then the centers of $M_1 \bar{\otimes} M_2$ is the σ-closure of $Z_1 \otimes Z_2$ in $M_1 \bar{\otimes} M_2$. In particular, if M_1 and M_2 are factors, $M_1 \bar{\otimes} M_2$ is a factor.

Proof. Let $\{\pi_1, \mathfrak{H}_1\}$, $\{\pi_2, \mathfrak{H}_2\}$ be a faithful W^*-representation of M_1 and M_2 respectively, then

$\mathcal{R}(\pi_1(Z_1) \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \pi_2(Z_2)) \subset$ the center $\mathfrak{Z}$ of $\mathcal{R}(\pi_1(M_1) \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \pi_2(M_2))$.

On the other hand,

$$\mathcal{R}(\pi_1(M_1) \otimes 1_{\mathfrak{H}_2}) \subset \mathcal{R}(\pi_1(M_1) \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \pi_2(M_2)) \subset \mathfrak{Z}'$$

$$\mathcal{R}(\pi_1(M_1)' \otimes 1_{\mathfrak{H}_2}) \subset \mathcal{R}(\pi_1(M_1) \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \pi_2(M_2))' \subset \mathfrak{Z}' ;$$

hence $\mathcal{R}(\pi_1(M_1) \otimes 1_{\mathfrak{H}_2}, \pi_1(M_1)' \otimes 1_{\mathfrak{H}_2})$

$$= \mathcal{R}(\pi_1(Z_1)' \otimes 1_{\mathfrak{H}_2}) \subset \mathfrak{Z}'$$

and analogously $\mathcal{R}(1_{\mathfrak{H}_1} \otimes \pi_2(Z_2)') \subset \mathfrak{Z}'$; hence $\mathcal{R}(\pi_1(Z_1)' \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \pi_2(Z_2)') \subset \mathfrak{Z}'$; therefore $\mathfrak{Z} \subset \mathcal{R}(\pi_1(Z_1)' \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \pi_2(Z_2)')'$; since $\pi_1(Z_1)' \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \pi_2(Z_2)'$ are semi-finite, $\mathfrak{Z} \subset \mathcal{R}(\pi_1(Z_1) \otimes 1_{\mathfrak{H}_2}, 1_{\mathfrak{H}_1} \otimes \pi_2(Z_1))$. This completes the proof.

Remark. It is an open question whether the assumption of semi-finiteness can be dropped in Proposition 4.3.

Notices of §4.

In the diagram concerning the type of tensor products, we do not give the proof of that (any W^*-algebra) $\bar{\otimes}$ (properly infinite) = (properly infinite); however the reader can easily do it.

Many problems, which are particularly interesting for factors with separable associated spaces, are unsolved -- for instance,

(i) Are there two factors M_1 and M_2 of type II_1 (resp. type $II\infty$, type III) with separable associated spaces such that $M_1 \bar{\otimes} M_2 \not\sim M_1$ and $M_1 \bar{\otimes} M_2 \not\sim M_2$ (that is, not *-isomorphic)?

(ii) Is there an example of a continuous factor M with a separable associated space such that $M \bar{\otimes} M \not\sim M$? (not *-isomorphic).

Concerning these problems, the reader should also be referred for the section 6.

§5. On the reduction theory.

One of the purposes in this section is to state the reduction theory of von Neumann, using a new method obtained in §2. Here we will not try to reappear all results obtained by von Neumann, because it needs fairly many pages; also the reader can easily do that work by himself in referring to the book of Dixmier; it is the coming problem whether our approach is useful for remained problems in the reduction theory; at least, it is certain that our method simplifies the discussions concerning the construction of the reduction theory. Also we want to state somes concerning the algebraic reduction theory shown in chapter II. Therefore, at first, we shall consider the situation of commutative W^*-algebras.

Let C be a commutative W^*-algebra, then it can be considered the B^*-algebra $L^\infty(\Omega,\mu)$ on a measure space (Ω,μ). On the other hand, by the representation theorem of Gelfand, it can be also considered the B^*-algebra $C(K)$ of all continuous functions on a compact space K.

In many cases, the latter of these two representation theorems is more useful than the former -- in fact, the simplest representation of C corresponds to every point of K, and elements of $C(K)$ are continuous functions; moreover $C(K)$ can also be considered the L^∞-algebra on a measure space as follows: let $\{\varphi_\alpha\}_{\alpha\varepsilon\Pi}$ be a family of σ-continuous positive functionals on $C(K)$ such that the supports $\{s(\varphi_\alpha)\}$ are orthogonal and $\Sigma\ s(\varphi_\alpha) = 1$; since φ_α is bounded, by the theorem of Riesz, there is a finite measure μ_α such that $\varphi_\alpha(f) = \int_K f(t)d\mu_\alpha(t)$; since $s(\varphi_\alpha)$ is the characteristic function of an open, closed subset G_α of K, the support of μ_α is G_α. Put $G = \bigcup_{\alpha \varepsilon \Pi} G_\alpha$, then G is an open set of K, so that it is locally compact and G is dense in K; for a subset E of K, if $E \cap G_\alpha$ is μ_α-measurable for all $\alpha \varepsilon \Pi$, we say that E is locally measurable and define $\widetilde{\mu}(E) = \sum_{\alpha \varepsilon \Pi} \mu_\alpha(E \cap K_\alpha)$, then $\widetilde{\mu}$ is a measure on K and $\widetilde{\mu}(K-G) = 0$; moreover by the theorem of Radon-Nikodym, we can easily obtain that $C = L^\infty(G, \widetilde{\mu})$; the utility of this measure space is that every essentially bounded locally measurable function is locally equivalent to a bounded continuous function on a locally compact space G; therefore a non-dense set E is $\widetilde{\mu}(E) = 0$, so that the first category set is also a null set; of course, the representation $L^\infty(G, \widetilde{\mu})$ has also weak points -- for instance, let $C_1 = L^\infty(\Omega_1, \mu_1)$, where Ω_1 is the interval $[0,1]$ of the real line and μ_1 is the Lebesgue measure, then Ω_1 is a compact space satisfying the second countability axiom; on the other hand, using the representation theorem of Gelfand, let C_1 represent as $L^\infty(G_1, \widetilde{\mu}_1)$, then G_1 satisfies no the axiom of second countability -- in fact, let t be a point of G_1; if $\widetilde{\mu}(t) \neq 0$, the characteristic function x of t is an element of $L^\infty(G_1, \widetilde{\mu}_1)$, so that it can be considered an element of $L^\infty(\Omega_1, \mu_1)$; since the Lebesgue measure is continuous, we can easily have a contradiction; therefore, any sequence (t_n) of G_1 is a null set; hence (t_n) is not dense in G_1.

This weak point might be not essential in the commutative case (however in the non-commutative case, we shall show that this has essential influence on the reduction theory).

To avoid this weak point, we have a suitable method for a separable case; we shall state that method; for simplicity, we assume that C has a faithful normal functional φ; let C_1 be a B^*-subalgebra of C which is σ-dense in C and $C_1 = C(K_1)$ be the function representation of C_1, where K_1 is a locally compact space; then we have: $\varphi(f) = \int_{K_1} f(t)\,d\mu(t)$ for $f \in C(K_1)$, where μ is a bounded Radon measure on K_1; such equality induces a $*$-isomorphism ρ of C_1 into $L^\infty(K_1,\mu)$; suppose that $\{f_\alpha\}$ $(\subset C_1)$ is a s-Cauchy directed set in C such that $\|f_\alpha\| \leq 1$, then

$$\varphi((f_\alpha - f_\beta)^*(f_\alpha - f_\beta)) = \int |f_\alpha(t) - f_\beta(t)|^2 d\mu(t) \longrightarrow 0 ,$$

so that $\int |f_\alpha(t) - f_\beta(t)|^2 g(t)\,d\mu(t) \longrightarrow 0$ for $g \in L^\infty(K_1,\mu)$; since $L^\infty(K_1,\mu)$ is L^1-norm-dense in $L^1(K_1,\mu)$, and $|f_\alpha(t)| \leq 1$, we have $\int |f_\alpha(t) - f_\beta(t)|^2 f(t)\,d\mu(t) \longrightarrow 0$ for $f \in L^1(K_1,\mu)$; hence (f_α) is s-Cauchy in $L^\infty(K_1,\mu)$ and moreover equivalent s-Cauchy directed sets are transformed into equivalent s-Cauchy directed sets, so that ρ can be uniquely extended to a σ-continuous $*$-homomorphism $\tilde{\rho}$ of C into $L^\infty(K_1,\mu)$; moreover $\int \tilde{\rho}(x^*x)(t)\,d\mu(t) = \varphi(x^*x)$, so that $\tilde{\rho}$ is an isomorphism; since $C(K_1)$ is σ-dense in $L^\infty(K_1,\mu)$ we have $\tilde{\rho}(C) = L^\infty(K_1,\mu)$; therefore if C is separable in the topology $\sigma(C, C_*)$, we can take a B^*-subalgebra C_1 such that C_1 is uniformly separable and σ-dense in C, then the spectrum space K_1 satisfies the second countability axiom — in fact, let (f_i) be a sequence of C_1 which is uniformly dense in C_1; put $U_i = \{\xi \mid |f_i(\xi)| > 1,\ \xi \in K_1\}$, then it is very easy that $\{U_i\}$ is a base of the topology on K_1. In this case, every point of $L^\infty(K_1,\mu)$ gives the simplest representation of C_1, but it can not do for C.

Now we shall pass our subject to the non-commutative case. The purpose of the reduction theory is to construct the representation theorems of W^*-algebras similar with $L^\infty(\Omega, \mu)$ or $C(K)$.

We shall call the reduction theory of von Neumann (resp. the algebraic reduction theory) the reduction theory of constructing the representation theorem of L^∞-type (resp. $C(K)$-type) in general W^*-algebras.

Even in W^*-algebras with separable associated spaces, these two reduction theories have quite different features.

To develop the algebraic reduction theory for a W^*-algebra M, at first we should construct the following situation: let Z be the center of M, K the spectrum space of Z, $\mathcal{N}_t$ the maximal ideal of Z for $t \in K$, then there is a uniformly closed ideal $\mathcal{M}_t$ (in general, $\mathcal{M}_t$ may not be maximal and not σ-closed) such that $\mathcal{N}_t \subset \mathcal{M}_t$ and $M/\mathcal{M}_t = N_t$ is a factor and moreover $\bigcap_{t \in K} \mathcal{M}_t = (0)$.

We could construct such situation for finite W^*-algebras in chapter II; therefore for them, we can consider the algebraic reduction theory.

Even if we can construct such one for other W^*-algebras M, there is an essential weak point in this reduction theory; in fact, even if M has the separable associated space, in general, N_t has not the property [cf. 54]; therefore in this theory, the W^*-algebra M which has a faithful W^*-representation on a separable hilbert space, is reduced to the study of W^*-factors on non-separable hilbert spaces; this is certainly a weak point;

for example, we shall consider a W^*-algebra $Z\bar{\otimes}B$, where $Z = L^\infty(0,1)$ with the Lebesgue measure and B is a factor of type $I_{\aleph_0}$; for such simple form, we can not expect that N_t has a separable faithful W^*-representation.

However, this reduction theory has also much utility, if possible -- in fact, a simplest representation corresponds to every point of K; moreover the strongest point of it can eliminate the pathology of "almost everywhere" for which the reduction theory of von Neumann can not essentially separate from the separability; therefore the author thinks that the problem of constructing the algebraic reduction theory is very important, though it is very difficult; We have some tools as follows: Let M be a W^*-algebra and we shall faithfully represent M as a weakly closed $*$-algebra on a hilbert space, then $\mathcal{R}(M,M')' = M' \cap M$, so that $\mathcal{R}(M,M') = Z'$, where Z is the center of M; therefore $\mathcal{R}(M,M')$ is of type I; let e be a maximal abelian projection of $\mathcal{R}(M,M')$, then $e\mathcal{R}(M,M')e = eze$; since the mapping $ze \xrightarrow{\theta} z$ of eZe onto Z is an $*$-isomorphism; we have a linear mapping $x \rightarrow \rho(x)$ of M onto Z by defining $\rho(x) = \theta(exe)$ for $x \in M$, then ρ has the following properties: (i) $\rho(x^*) = \rho(x)^*$, $\rho(h) \geq 0$, (ii) $\rho(zx) = z\rho(x)$, (iii) $\rho(1) = 1$, (iv) $\rho(x^*)\rho(x) \leq \rho(x^*x)$ and ρ is σ and s-continuous, where $x \in M$, $h(\geq 0) \in M$ and $z \in Z$; therefore for any $t \in K$, where K is the spectrum space of Z, put $\varphi_t(x) = \rho(x)(t)$, then φ_t is a positive linear functional on M; let $\{\pi_{\varphi_t}, \mathfrak{H}_{\varphi_t}\}$ be the B^*-representation of M on a hilbert space $\mathfrak{H}_{\varphi_t}$ constructed via φ_t, then the problems are as follows:

(i) Is the representation $\pi = \sum_{t \varepsilon K} \pi_{\varphi_t}$ faithful? (ii) Is $\pi_{\varphi_t}(M)$ a W^*-algebra? (iii) Is $\pi_{\varphi_t}(M)$ a factor? (iv) At least, if M is of type III and has a faithful W^*-representation on a separable hilbert space, is the kernel of π_{φ_t} maximal and π faithful, so that M semi-simple? (v) At any rate, should these representations $\{\pi_{\varphi_t}\}$ be studied? (vi) For finite algebras, does $\{\pi_{\varphi_t}\}$ coincide with the results in chapter II?

Now we shall consider the reduction theory of von Neumann. At first we shall state that theory in the separable case and later formulate the problem concerning non-separable cases.

Let M be a W^*-algebra with the separable associated space, then it can be faithfully represented as a weakly closed *-subalgebra on a separable hilbert space $\mathfrak{h}$; then $\mathcal{R}(M,M') = M \cap M' = \widetilde{Z}$, where $\widetilde{Z}$ is the center of M; therefore $\mathcal{R}(M,M') = \sum_{\alpha \varepsilon \Pi} \oplus N_\alpha$ where N_α is a homogeneous algebra of type I_{n_α} with $n_\alpha \leq \aleph_0$: therefore it is enough to assume that $\mathcal{R}(M,M') = N$, where N is a homogeneous algebra of type I_n with $n \leq \aleph_0$; from the result of §2, we can consider $N = Z\bar{\otimes}B = L^\infty(B,\Omega,\mu)$, where $Z \otimes 1$ is the center of N and B is the factor of type I_n; since $Z \otimes 1$ has the separable associated space, we can assume that Ω is a compact space satisfying the second countability axiom and $\mu(\Omega) = 1$.

Let $a \varepsilon Z\bar{\otimes}B$, then a is considered an essentially bounded weakly *-measurable B-valued function on Ω; we express by $a = \int a(t)d\mu(t)$ such situation, then $\|a\| = \text{ess. sup} \|a(t)\|$; for $a_1, a_2 \varepsilon Z\bar{\otimes}B$, $a_1 + a_2 = \int (a_1(t) + a_2(t))d\mu(t)$, $a_1 a_2 =$

$\int a_1(t)a_2(t)d\mu(t)$ $\lambda a_1 = \int \lambda a_1(t)d\mu(t)$ and $a^* = \int a(t)^* d\mu(t)$.

Proposition 5.1. Let $a_i = \int a_i(t)d\mu(t)$ $(i=1,2,\ldots)$ and $a = \int a(t)d\mu(t)$.

(i) If (a_i) is s-convergent to a, there is a subsequence (a_{i_j}) such that $(a_{i_j}(t))$ is s-convergent to $a(t)$ for almost all $t \in \Omega$.

(ii) If $(a_i(t))$ is s-convergent to $a(t)$ for almost all $t \in \Omega$ and if $\sup_i \| a_i \| < +\infty$, then (a_i) is s-convergent to a.

Proof. Since (a_i) is s-convergent to a, $\sup_i \| a_i \| \leq k$; therefore $\| a_i(t) \| \leq k$ a.e. for all i; let (ξ_i) be a dense subset in the positive portion of B_* and put $f_n = 1 \otimes \xi_n$ $(1 \in L^\infty(\Omega,\mu) \subset L^1(\Omega,\mu))$, then

$$<(a_i - a)^*(a_i - a), f_n> = \int <\{a_i(t) - a(t)\}^*\{a_i(t) - a(t)\}, \xi_n> d\mu(t)$$

$\longrightarrow 0$ $(i \longrightarrow \infty)$ for all n; there is a subsequence (a_{i_j}) of (a_i) such that $\sum_{j=1}^{\infty} <(a_{i_j} - a)^*(a_{i_j} - a), f_n> < +\infty$; then $\sum_{j=1}^{\infty} <\{a_{i_j}(t) - a(t)\}^*\{a_{i_j}(t) - a(t)\}, \xi_n> < +\infty$ for $t \notin N_n$, where $\mu(N_n) = 0$; therefore $\lim_j <\{a_{i_j}(t) - a(t)\}^*\{a_{i_j}(t) - a(t)\}, \xi_n> = 0$ for $t \notin N_n$; by the application of diagonal process, we can assume that such (i_j) is independent on n; then

$$\lim_j <\{a_{i_j}(t) - a(t)\}^*\{a_{i_j}(t) - a(t)\}, \xi_n> = 0$$

for $t \notin \bigcup_{n=1}^{\infty} N_n$ and all n; since $\| a_{i_j}(t) \| \leq k$ (a.e.), we can conclude that $a_{i_j}(t) \longrightarrow a(t)$ (s) a.e.

Conversely, suppose that $a_i(t) \longrightarrow a(t)$ (s) a.e. and $\sup_i \| a_i \| < +\infty$, then $\| a_i(t) \| \leq k$ a.e., so that $\| a(t) \| \leq k$

a.e.; for any n, $<\{a_i(t)-a(t)\}^*\{a_i(t)-a(t)\},\xi_n> \longrightarrow 0$ a.e. $(i \longrightarrow \infty)$; moreover

$$<\{a_i(t)-a(t)\}^*\{a_i(t)-a(t)\},\xi_n> \ \leq 4k^2 \| f_n \| ;$$

therefore we have

$$\lim_{n \to \infty} < (a_i-a)^*(a_i-a), g\otimes\xi_n> = \lim_{i \to \infty} \int_\Omega <\{a_i(t)-a(t)\}^*\{a_i(t)-a(t)\},$$

$\xi_n > g(t)d\mu(t) = 0$ for all n and $g \in L^1(\Omega,\mu)$; since $\sup \| a_i \| < +\infty$, this implies that $a_i \longrightarrow a(s)$. This completes the proof.

Proposition 5.2. Let $(a_\alpha)_{\alpha \in \mathbb{I}}$ be a family of elements of $L^\infty(B,\Omega,\mu)$ containing the unit, $\mathfrak{A}(t)$ the W^*-subalgebra of B generated by $\{a_\alpha(t)\}_{\alpha\in\mathbb{I}}$, $\mathfrak{A}$ the W^*-subalgebra of $L^\infty(B,\Omega,\mu)$ generated by $\{a_\alpha\}$ and $Z\otimes 1$, and let $b \in L^\infty(B,\Omega,\mu)$. Then,

(i) If $b \in \mathfrak{A}$, $b(t) \in \mathfrak{A}(t)$ a.e.

(ii) If $b(t) \in \mathfrak{A}(t)$ a.e. and if $\mathbb{I}$ is enumerable, then $b \in \mathfrak{A}$.

Proof. Let A be an *-algebra generated by $(a_\alpha)_{\alpha\in\mathbb{I}}$ and $Z\otimes 1$ then if $b \in A$, it is clear that $b(t) \in \mathfrak{A}(t)$ a.e.; if $b \in \mathfrak{A}$, there is a sequence (b_n) of A such that $b_n \longrightarrow b\ (s)$; hence by Proposition 5.1 $b(t) \in \mathfrak{A}(t)$ a.e.

Conversely suppose that $b(t) \in \mathfrak{A}(t)$ a.e. and $\mathbb{I}$ is enumerable; since $\mathfrak{A}' \subset (Z\otimes 1)' = Z\bar{\otimes}B$, for any $a' \in \mathfrak{A}'$, we have $a' = \int a'(t)d\mu(t)$; since $a'a_\alpha = a_\alpha a'$ and $a'a_\alpha^* = a_\alpha^* a'$, $a'(t)a_\alpha(t) = a_\alpha(t)a'(t)$, $a'(t)a_\alpha^*(t) = a_\alpha^*(t)a'(t)$ for $t \notin N_\alpha$, where $\mu(N_\alpha) = 0$; hence $a'(t) \in \mathfrak{A}(t)'$ a.e. so that $a'(t)b(t) = b(t)a'(t)$ and $a'(t)^*b(t) = b(t)a'(t)^*$ a.e.; hence $b \in \mathfrak{A}'' = \mathfrak{A}$.

This completes the proof.

Definition 5.1. A family $\{\mathfrak{A}(t) \mid t \in \Omega\}$ of W^*-subalgebras of B is said measurable if there is a sequence (a_n) of $L^\infty(B,\Omega,\mu)$ containing the unit such that $\mathfrak{A}(t)$ is generated by $\{a_n(t)\}$ for almost all $t \in \Omega$.

Proposition 5.3. Let a family $\{\mathfrak{A}(t) \mid t \in \Omega\}$ be measurable.

(i) the totality $\mathfrak{A}$ of elements a of $L^\infty(B,\Omega,\mu)$ such that $a(t) \in \mathfrak{A}(t)$ a.e. is a W^*-subalgebra of $L^\infty(B,\Omega,\mu)$ containing $Z \otimes 1$.

(ii) If a family $\{\mathfrak{B}(t) \mid t \in \Omega\}$ is measurable and it defines a same W^*-algebra $\mathfrak{A}$ of $L^\infty(B,\Omega,\mu)$, then $\mathfrak{B}(t) = \mathfrak{A}(t)$ a.e.

Proof (i) is clear from Proposition 5.2.

(ii) Let (a_n) be a sequence such that $\{a_n(t)\}$ generates $\mathfrak{A}(t)$ a.e. since $a_i \in \mathfrak{A}$ $a_i(t) \in \mathfrak{B}(t)$ a.e.; hence $\mathfrak{A}(t) \subset \mathfrak{B}(t)$ a.e.; analogously we have $\mathfrak{A}(t) \supset \mathfrak{B}(t)$ a.e. This completes the proof.

By Propositions 5.2 and 5.3, there is a one-to-one correspondence between a W^*-subalgebra $\mathfrak{A}$ containing $Z \otimes 1$ of $L^\infty(B,\Omega,\mu)$ and a measurable family $\{\mathfrak{A}(t) \mid t \in \Omega\}$ of W^*-subalgebras of B; we express by $\mathfrak{A} = \int \mathfrak{A}(t)\,d\mu(t)$ such situation.

Proposition 5.4. If a family $\{\mathfrak{A}(t) \mid t \in \Omega\}$ is measurable, a family $\{\mathfrak{A}(t)' \mid t \in \Omega\}$ is also measurable.

The proof of this Proposition will be omitted, because it needs a long discussion; the reader should be referred to the book of Dixmier (p. 184, Lemma 1 and Appendix V).

Now we shall state a fundamental theorem.

Theorem 5.1. Let $M = \int M(t)\,d\mu(t)$ and $M' = \int M'(t)\,d\mu(t)$, then $\{M(t)\}' = M'(t)$ a.e. and moreover $M(t)$, $M'(t)$ are factors for almost all $t \in \Omega$.

Proof. Let (a_n) (resp. (b_m)) be a sequence of M (resp. M') containing the unit such that (a_n) (resp. (b_m)) with $Z \otimes 1$ generates M (resp. M'), then $\{a_n(t)\}$ (resp. $\{b_m(t)\}$) generates $M(t)$ (resp. $M'(t)$) for almost all $t \in \Omega$; since $a_i b_j = b_j a_i$ and $a_i b_j^* = b_j^* a_i$, we have $M'(t) \subset \{M(t)\}'$ a.e. On the other hand $\{a_n, b_m\}$ with $Z \otimes 1$ generates $L^\infty(B, \Omega, \mu)$, so that $\{a_n(t)\ b_m(t)\}$ generates B for almost all $t \in \Omega$; hence we have $\mathcal{R}(M(t), M'(t)) = B$ a.e.; therefore $M(t) \cap \{M(t)\}' \subset \{M'(t) \cup M(t)\}' = B' = (\lambda 1)$ (λ complex number); hence $M(t)$ is a factor; moreover since $\{M(t)' \mid t \in \Omega\}$ is measurable, there is a W^*-subalgebra N such that $N = \int \{M(t)\}' d\mu(t)$; then $N \subset M'$, so that $N = M'$; hence we have $M'(t) = \{M(t)\}'$ a.e. This completes the proof.

Finally we shall state the problems concerning the reduction theory of von Neumann; since we can find unsolved problems for separable cases in the book of Dixmier, they will be omitted.

Problem 1. Is it possible that $Z \bar{\otimes} B = L^\infty(B, \Omega, \mu)$ for a general factor B of type I?

Problem 2. Is it possible that $Z \bar{\otimes} M = L^\infty(M, \Omega, \mu)$ for a general finite factor?

Problem 3. Is it possible to formulate the reduction theory in the form of Theorem 5.1? At least, is it possible to formulate the reduction theory for general W^*-algebras such that there is a one-to-one correspondence between a W^*-subalgebra of $Z\bar{\otimes}B$ containing $Z\otimes 1$ and a measurable family $\{M(t) \mid t \in \Omega\}$ (in a suitable sense)?

Remark. The reader should be referred to the book of Naimark concerning the reduction theory of Tomita which extended some parts of the reduction theory of von Neumann to general cases; in that theory, it is as yet unsolved that the problem $\{M(t)\}' = M'(t)$ a.e.?

Problem 4. In separable case, is there an example as follows? $M = \int M(t)\,d\mu(t)$ and $M(t)$ is $*$-isomorphic to a W^*-factor N for almost all $t \in \Omega$, but $Z\bar{\otimes}N$ is not $*$-isomorphic to M.

Notices of §5

Concerning the representation $L^\infty(G,\tilde{\mu})$ in page 3.46, the reader should be referred to the paper of Dixmier [2]; in that measure space, non-dense sets, the first category sets and null sets coincide.

§ 6. Examples.

In this section, we shall state various topics and problems concerning examples of W^*-algebras. We have classified W^*-algebras into ones of types I, II and III in Chapter II. Then, the problem is that do these all types really exist? This problem is particularly interesting for factors - in fact, if we have a factor, we can easily construct a general W^*-algebra with the same type, using the tensor product. Concerning the factor of type I_n (n, cardinal), we can easily construct as follows: let $\mathfrak{h}$ be a hilbert space with dimension n, $B(\mathfrak{h})$ the W^*-algebra of all bounded operators on $\mathfrak{h}$, then $B(\mathfrak{h})$ is the factor of type I_n; moreover all factors of type I_n are *-isomorphic to $B(\mathfrak{h})$ [cf. Theorem 2.1, chapter III].

Next, let M be a factor of type II_∞ (namely, type II and infinite) e be a non-zero finite projection of M and $(e_\alpha)_{\alpha \in \mathbb{I}}$ be a maximal family of orthogonal equivalent projections of M such that $e_\alpha \sim e$ for $\alpha \in \mathbb{I}$, then $p = 1 - \sum_{\alpha \in \mathbb{I}} e_\alpha \prec e$; suppose that $p \not\sim e$, then we choose a sequence (α_n) of $\mathbb{I}$; since $p + \sum_{i=1}^{\infty} e_{\alpha_{2i}} \sim \sum_{i=1}^{\infty} e_{\alpha_{2i+1}}$, by this equivalence, the projection $p + \sum_{i=1}^{\infty} e_{\alpha_{2i}}$ is decomposed as follows: $p + \sum_{i=1}^{\infty} e_{\alpha_{2i}} = \sum_{i=1}^{\infty} e_{\beta_i}$ and $e_{\beta_i} \sim e$; then,

$$1 = \sum_{\alpha \in \mathbb{I}-(\alpha_i)} e_\alpha + \sum_{i=1}^{\infty} e_{\alpha_{2i+1}} + \sum_{i=1}^{\infty} e_{\beta_i} \quad \text{and} \quad e_\alpha \sim e$$

for $\alpha \in \mathbb{I}-(\alpha_i)$, $e_{\alpha_{2i+1}} \sim e$ for all i and $e_{\beta_i} \sim e$

for all i; hence there is a family of orthogonal equivalent projections $(p_\gamma)_{\gamma \in \mathbb{J}}$ such that $p_\gamma \sim e$ and $\sum_{\gamma \in \mathbb{J}} p_\gamma = 1$.

Now let f be a non-zero finite projection of M and $(f_\alpha)_{\alpha \in \mathbb{H}}$ be a family of orthogonal equivalent projections such that $f_\alpha \sim f$ for $\alpha \in \mathbb{H}$ and $\sum_{\alpha \in \mathbb{H}} f_\alpha = 1$; let τ_α be a normal finite trace on $f_\alpha M f_\alpha$ such that $\tau_\alpha(f_\alpha) = 1$; since $f_\alpha M f_\alpha$ is a factor (cf. § 4), by the uniqueness of the $\natural$ -operation, such τ_α is unique and faithful; $\tau_\alpha(f_\alpha) = \tau_\alpha(f_\alpha(\sum_{\gamma \in \mathbb{J}} p_\gamma) f_\alpha) = \sum_{\gamma \in \mathbb{J}} \tau_\alpha(f_\alpha p_\gamma f_\alpha)$, so that there is an enumerable subset $\mathbb{J}_\alpha$ of $\mathbb{J}$ such that $\tau_\alpha(f_\alpha p_\gamma f_\alpha) = 0$ for $\gamma \in \mathbb{J} - \mathbb{J}_\alpha$; hence $p_\gamma f_\alpha = 0$ for $\gamma \in \mathbb{J} - \mathbb{J}_\alpha$; let $\gamma \in \mathbb{J} - \bigcup_{\alpha \in \mathbb{H}} \mathbb{J}_\alpha$, then $p_\gamma f_\alpha = 0$ for all $\alpha \in \mathbb{H}$; hence $p_\gamma = \sum_{\alpha \in \mathbb{H}} p_\gamma f_\alpha = 0$, so that $\mathrm{Card}(\mathbb{J}) \leqq \aleph_0 \mathrm{Card}(\mathbb{H}) = \mathrm{Card}(\mathbb{H})$, and by the analogous method, we have $\mathrm{Card}(\mathbb{J}) = \mathrm{Card}(\mathbb{H})$. Therefore, by Proposition 2.1, $M = N \overline{\otimes} B$, where B is the factor of type I_n ($n = \mathrm{Card}(\mathbb{J})$(unique)); since M is a factor, N is a factor; moreover $N \otimes 1 \sim eMe$, so that N is a factor of type II_1; it is an open question whether N is unique [cf. 18].

Conversely, let N be a II_1-factor, B the I_n-factor ($n \geq \aleph_0$), then $N \overline{\otimes} B$ is a II_∞-factor; therefore, at any way, the studies of II_∞-factors can be reduced the ones of II_1-factors. Therefore, our problem can be reduced to the constructions of type II_1 and III-factors. Then, the essential point of that problem is to construct factors of type II_1 (resp. III) with separable associated spaces - in fact, if we have an example of II_1-factor (resp. III) and if we do not put the restriction of the separability, we can easily construct infinite many examples of II_1-factors (resp. III), using the motion of incomplete infinite direct product (cf. [23]) (resp. the tensor product). Therefore, in this section, we shall always consider the constructions under the restriction of separability.

Murray and von Neumann [16], at first, gave an example of II_1-factor, and next, in [24], von Neumann did an example of type III and finally in [18], they showed that are two examples of II_1-factors which are mutually not *-isomorphic. In 1955, Pukansky [43] also showed that there are two examples of III-factors which are mutually not *-isomorphic; therefore we had two examples of II_1-(resp.III) factors respectively; it is worthy to note that these facts do not unconditionally imply the existence of two II_∞-factors - in fact, the following question is open: let M_1 and M_2 be two II_1-factors which are mutually not *-isomorphic, and B be the I_∞-factor, then can we conclude that $M_1 \overline{\otimes} B$ is not *-isomorphic to $M_2 \overline{\otimes} B$?; however by the considerations of Murray and von Neumann, we can assert that there are also two examples of type II_∞-factors which are mutually not *-isomorphic. Many specialists believe the existence of infinitely many factors of type II_1 (resp. II_∞ and III), which are mutually not *-isomorphic, but we could not have even one more example; however, according to the communication of Professor Kadison, very recently, J. Schwartz has shown the existence of the third II_1-factor by proving the problem (i) for II_1-factors in §4; this result is very significant; the author expects that the appearance of the third II_1-factor will inspire the appearance of more other examples.

Now we shall state the construction of examples.

(α) The considerations of general situations.

Let $\mathfrak{H}$ be a hilbert space, $\mathfrak{A}$ a weakly closed *-subalgebra of $B(\mathfrak{H})$ containing the identity operator, $\mathfrak{G}$ a discrete group and $s \longrightarrow U_s$ a unitary representation of $\mathfrak{G}$ in $\mathfrak{H}$. We suppose that $U_s^{-1} \mathfrak{A} U_s = \mathfrak{A}$ for every $s \in \mathfrak{G}$; then $T \longrightarrow U_s^{-1} T U_s = T^s$ is an automorphism of $\mathfrak{A}$. For every $s \in \mathfrak{G}$, let $\mathfrak{H}_s$ a hilbert space which is isomorphic to $\mathfrak{H}$ and J_s an isometric linear mapping of $\mathfrak{H}$ onto $\mathfrak{H}_s$. Let $\widetilde{\mathfrak{H}} = \sum_{s \in \mathfrak{G}} \oplus \mathfrak{H}_s$; then the considerations of § 1, chap. III, we can respresent each element R of $B(\widetilde{\mathfrak{H}})$ by the matrix $(R_{s,t})_{s,t \in \mathfrak{G}}$, where $R_{s,t} = J_s^* R J_t \in B(\mathfrak{H})$.

For $T \in \mathfrak{A}$, let $\Phi(T)$ be the element of $B(\widetilde{\mathfrak{H}})$ such that $R_{s,t} = 0$ if $s \neq t$; $R_{s,s} = T$ for $s \in \mathfrak{G}$; Φ is an *-isomorphism of $\mathfrak{A}$ onto a weakly closed *-subalgebra $\widetilde{\mathfrak{A}}$ of $B(\widetilde{\mathfrak{H}})$ For $y \in \mathfrak{G}$, let $\widetilde{U}_y$ be the element of $B(\widetilde{\mathfrak{H}})$ defined by the matrix $(R_{s,t})$ such that $R_{s,t} = 0$ if $st^{-1} \neq y$; $R_{yt,t} = U_y$ for $t \in \mathfrak{G}$; then we have $\widetilde{U}_x \widetilde{U}_y = \widetilde{U}_{xy}$; moreover

$J_s^*(\tilde{U}_{y^{-1}} \Phi(T) \tilde{U}_y) J_s = (J_s^* \tilde{U}_{y^{-1}} J_{ys}) (J_{ys}^* \Phi(T) J_{ys})(J_{ys}^* \tilde{U}_y J_s) =$ $U_{y^{-1}} T U_y = T^y$; hence $\tilde{U}_{y^{-1}} \Phi(T) \tilde{U}_y = \Phi(T^y)$ The operators of the forms $\Phi(T_1) \tilde{U}_{y_1} + \Phi(T_2) \tilde{U}_{y_2} + \ldots + \Phi(T_n) \tilde{U}_{y_n}$ make a *-sub algebra $\mathcal{L}_o$ of $B(\tilde{\mathfrak{H}})$ - in fact, $(\Phi(T) \tilde{U}_y)^* = \tilde{U}_y^* \Phi(T) =$ $\Phi(T^{*y}) \tilde{U}_{y^{-1}}$ and $\Phi(T_1) \tilde{U}_{y_1} \Phi(T_2) \tilde{U}_{y_2} = \Phi(T_1) \Phi(T_2^{y_1^{-1}}) \tilde{U}_{y_1} \tilde{U}_{y_2}$ $= \Phi(T_1 T_2^{y_1^{-1}}) \tilde{U}_{y_1 y_2}$.

Let $\mathcal{L}$ be a weakly closed *-subalgebra of $B(\tilde{\mathfrak{H}})$ generated by $\mathcal{L}_o$. It is easily seen that the matrix $(R_{s,t})$ of $\Phi(T)\tilde{U}_y$ satisfies : $R_{s,t} = 0$, if $st^{-1} \neq y$ and $R_{yt,t} = T U_y$; therefore there is a family $(T_y)_{y \in \mathfrak{G}}$ of $\mathfrak{A}$ such that $R_{s,t} =$ $T_{st^{-1}} U_{st^{-1}}$. Such properties are preserved for the elements of $\mathcal{L}_o$ and their weak operator limits; hence every element $S \in \mathcal{L}$ is represented by the matrix of the form $(T_{st^{-1}} U_{st^{-1}})$, with $T_y \in \mathfrak{A}$ for $y \in \mathfrak{G}$.

Proposition 6.1 Let φ be a faithful semi-finite normal trace on $\mathfrak{A}$ which is invariant under $\mathfrak{G}$ (namely $\varphi(T^y) =$ $\varphi(T)$ for $y \in \mathfrak{G}$) For every $S = (T_{st^{-1}} U_{st^{-1}})(S \geq 0) \in \mathcal{L}$, put $\psi(S) = \varphi(T_e)$ where e is the unit of $\mathfrak{G}$. Then ψ is a semi-finite faithful normal trace on $\mathcal{L}$; moreover ψ is finite if and only if φ is finite; finally for $T(\geq 0) \in \mathfrak{A}$, $\psi(\Phi(T)) = \varphi(T)$

Proof. Since $T_e = R_{e,e} = J_e^* S J_e$, the linear mapping $S \to T_e$ of $\mathcal{L}$ onto $\mathfrak{A}$ is σ-continuous; moreover clear $T_e \geq 0$ for $S \geq 0$, so that χ is normal. Let $S = (T_{st^{-1}} U_{st^{-1}})$, then $S^* = (U^*_{ts^{-1}} T^*_{ts^{-1}})$; therefore, put $SS^* = (R_{st^{-1}} U_{st^{-1}})$ and $S^*S = (R'_{st^{-1}} U_{st^{-1}})$, then we have

$$R_e = \sum_{t \in \mathfrak{G}} T_{t^{-1}} U_{t^{-1}} U^*_{t^{-1}} T^*_{t^{-1}} = \sum_{t \in \mathfrak{G}} T_t T_t^*$$

$$R'_e = \sum_{t \in \mathfrak{G}} U_t^* T_t^* T_t U_t$$

where the sums are taken, using the strong operator topology; therefore we have $\chi(S^*S) = \chi(SS^*)$, so that χ is a trace; moreover $\chi(S^*S) = 0$ implies $\varphi(T_t^* T_t) = 0$ for all $t \in \mathfrak{G}$ and so $T_t = 0$ for all $t \in \mathfrak{G}$, this implies $S = 0$.

Since the matrix $(R_{s,t})$ of $\Phi(T)$ is; $R_{e,e} = T$, $\chi(\Phi(T)) = \varphi(T)$; since φ is semi-finite, there is an increasing directed set (T_α) of $\mathfrak{A}$ such that $\mathrm{lub}_\alpha\, T_\alpha = 1_{\mathfrak{H}}$ and $\varphi(T_\alpha) < +\infty$; therefore $\mathrm{lub}_\alpha\, \Phi(T_\alpha) = 1_{\tilde{\mathfrak{H}}}$ and $\chi(\Phi(T_\alpha)) < +\infty$, this implies the semi-finiteness of χ.; it is clear that χ is finite if and only if φ is finite. This completes the proof.

Proposition 6.2 (i) Suppose that $\tilde{\mathfrak{A}}$ is a maximal commutative *-subalgebra of $\mathcal{L}$. In order that $\mathcal{L}$ is semi-finite, it is necessary and sufficient that there is a faithful normal semi-finite trace of $\mathfrak{A}$ which is invariant under $\mathfrak{G}$.

(ii) If $\mathfrak{A}$ is purely infinite, $\mathfrak{L}$ is also purely infinite

Proof The proof of (i) The condition is sufficient by Proposition 6.1. Conversely suppose that $\mathfrak{L}$ is semi-finite. Let χ be a faithful normal semi-finite trace of $\mathfrak{L}$ and put $\varphi(T) = \chi(\Phi(T))$ for $T(\geq 0) \in \mathfrak{A}$, then clearly φ is a faithful normal trace of $\mathfrak{A}$; the problem is to show that φ is semi-finite.

Let $T_1 = \Phi(T)$ be a unitary element of $\tilde{\mathfrak{A}}$ and put $ST_1^{-1} = (T_{st^{-1}} U_{st^{-1}})$, then

$$J_s^*(T_1 S T_1^{-1}) J_s = J_s^* T_1 J_s J_s^* S T_1^{-1} J_s = T T_e = T_e T$$

$$= J_s^* S T_1^{-1} J_s T = J_s^* S T_1^{-1} T_1 J_s = J_s^* S J_s \quad ;$$

therefore, put K_S = the convex span of $\{T_1 S T_1^{-1} | T_1 \in \tilde{\mathfrak{A}}$ and unitary$\}$, then $J_s^* K_S J_s$ is one point, so that $J_s^* \bar{K}_S J_s$ is also one point, where $\bar{K}_S$ is the σ-closure of K_S.

On the other hand, $\bar{K}_S$ is σ-compact and invariant under the mappings $R \longrightarrow T_\alpha R T_\alpha^{-1}$, where T_α (unitary) $\in \tilde{\mathfrak{A}}$; since $\tilde{\mathfrak{A}}$ is commutative, by the fixed point theorem of Markoff and Kakutani (cf. Appendix, Bourbaki, Espace vectriels topologiques), there is a fixed point S_o in $\bar{K}_S$, then $S_o \in \tilde{\mathfrak{A}}' \cap \mathfrak{L} = \tilde{\mathfrak{A}}$;

hence we have: $J_s^* S J_s = J_s^* S_o J_s = T_e$, where $\Phi(T_e) = S_o$.

Now let $S(\geq o) \in \mathcal{L}$ and $\varphi(S) < +\infty$, then $\varphi(R) = \varphi(S) < +\infty$ for all $R \in K_S$; let $R_1 \in \overline{K}_S$, then there is a directed set (S_α) in K_S such that $\sigma\text{-}\lim_\alpha S_\alpha = R_1$. Put $\mathcal{F} = \{V \mid 1 \geq V \geq o, \varphi(V) < +\infty$, and $V \in \mathcal{L}\}$, then

$\lim_\alpha \varphi(VS_\alpha) = \varphi(VR_1)$ for $V \in \mathcal{F}$; since $\varphi(VS_\alpha) \leq \varphi(S_\alpha) = \varphi(S)$, $\varphi(VR_1) \leq \varphi(S)$; since $\ell.u.b_{V \in \mathcal{F}} \varphi(VR_1) = \varphi(R_1)$, we have $\varphi(R_1) \leq \varphi(S)$; therefore for $S_o \in \overline{K}_S \cap \tilde{\mathfrak{A}}' \subset \tilde{\mathfrak{A}}$, $\varphi(S_o) = \varphi(T_e) < +\infty$, where $\Phi(T_e) = S_o$; moreover since the identity operator $1_{\mathfrak{H}}$ is the strong limit of operators $\{S_\alpha\}$ of $\mathcal{F}$, $\{J_s^* S_\alpha J_s\}$ converges strongly to $1_{\mathfrak{H}}$; this implies φ is semi-finite.

The proof of (ii). We consider a linear mappping P of $\mathcal{L}$ onto $\tilde{\mathfrak{A}}$ as follows: $P(S) = \Phi(T_e)$, where $S = (T_{st^{-1}} U_{st^{-1}})$, then P satisfies the conditions (i) $P(1_{\mathfrak{H}}) = 1_{\mathfrak{H}}$, (ii) $||P(S)|| \leq ||S||$, (iii) $P(H) \geq o$ for $H \geq o$, (iv) $P(USV) = UP(S)V$, where $U, V \in \tilde{\mathfrak{A}}$ (v) $P(S)^* P(S) \leq P(S^* S)$, (vi) $P(S^* S) = o$ implies $S = o$; therefore by the same method with the proof of Theorem 4.4 in chapter III, we have that $\mathcal{L}$ is purely infinite This completes the proof.

Lemma 6.1 Let $\mathfrak{A}$ be a maximal commutative *-subalgebra of $B(\mathfrak{H})$. Suppose that $\mathfrak{A} \cap \mathfrak{A} U_y = o$ for $y \neq e$. Then, $\tilde{\mathfrak{A}}$ is a maximal commutative *-subalgebra of $\mathcal{L}$.

Proof. Let $S = (T_{st}U_{st^{-1}})$ be an element of $\mathcal{L}$ which commutes with $\tilde{\mathfrak{A}}$. For $T \in \mathfrak{A}$, we have $S\Phi(T) = \Phi(T)S$, this implies $TT_{st^{-1}}U_{st^{-1}} = T_{st^{-1}}U_{st^{-1}}T$. Since $\mathfrak{A}$ is maximal commutative, $T_sU_s \in \mathfrak{A} \cap \mathfrak{A}_s$; hence we have $T_sU_s = o$ for $s \neq e$, so that $S \in \tilde{\mathfrak{A}}$. This completes the proof.

Proposition 6.3. Suppose that $\tilde{\mathfrak{A}}$ is a maximal commutative *-subalgebra of $\mathcal{L}$ and elements of $\mathfrak{A}$ which are invariant under $\mathfrak{G}$ are scalar operators, then $\mathcal{L}$ is a factor.

Proof. Let S be an element of the center of $\mathcal{L}$, then $S \in \tilde{\mathfrak{A}}$; therefore $S = \Phi(T)$ for some $T \in \mathfrak{A}$; moreover $\tilde{U}_{s^{-1}}\Phi(T)\tilde{U}_s = \Phi(T^s) = \Phi(T)$; hence $T^s = T$, so that $T = \lambda 1_{\mathfrak{H}}$ and so $\Phi(T) = \lambda 1_{\tilde{\mathfrak{H}}}$. This completes the proof.

(β) The considerations of concrete situations.

Let Ω be a locally compact space satisfying the second countability axiom, μ a positive Radon measure on Ω and $\mathfrak{G}$ be a countable discrete group of homeomorphisms on Ω. For $\zeta \in \Omega$ and $a \in \mathfrak{G}$, we denote the effect of the mapping corresponding to a on ζ by ζa. The measure μ is said to be quasi-invariant under $\mathfrak{G}$ if $\mu(E) = o$ for a measurable set E implies $\mu(Ea) = o$ for every $a \in \mathfrak{G}$.

In this case, the translated measure μ_a defined for measurable sets F by $\mu_a(F) = \mu(F_a)$ is absolutely continuous with respect to μ, thus we can form the Radon-Nikodym's derivative $V_a(\zeta)$; since $V_{ab}(\zeta)d\mu(\zeta) = d\mu_{ab}(\zeta) = d\mu(\zeta_{ab}) = V_b(\zeta_a)d\mu(\zeta_a) = V_b(\zeta_a)V_a(\zeta)d\mu(\zeta)$; hence

$\nu_{ab}(\xi) = \nu_b(\xi a)\nu_a(\xi)$ a.e.

The group $\mathfrak{G}$ is said to be free (i) if for $a \in \mathfrak{G}$, $a \neq e$ the set of points satisfying the condition $\xi = \xi a$ ($\xi \in \Omega$) is of a μ-measure 0, (ii) ergodic, if $\mu((E \cup Ea) - (E \cap Ea)) = o$ for a measurable set E and every $a \in \mathfrak{G}$ implies either $\mu(E) = o$ or $\mu(\Omega - E) = o$, (iii) measurable, if there exists an invariant measure ν (namely $\nu(Ea) = \nu(E)$) which is equivalent to μ, (iv) non-measurable, if it is not measurable.

Suppose we are given a measure μ on a locally compact space Ω and a countable discrete group $\mathfrak{G}$ under which μ is quasi-invariant.

We formulate the hilbert space $L^2(\Omega, \mu)$ of complex valued square-integrable functions on Ω. For $f \in L^\infty(\Omega, \mu)$ and $g \in L^2(\Omega, \mu)$, put $(T_f g)(\xi) = f(\xi)g(\xi)$, then $L^\infty(\Omega, \mu)$ is considered a weakly closed *-subalgebra of $B(\mathfrak{H})$, where $\mathfrak{H} = L^2(\Omega, \mu)$.

Let C be the algebra of all continuous functions on Ω with compact supports, then $C \subset L^2(\Omega, \mu)$ and C is a hilbert algebra; moreover $\mathcal{U}(C) = \mathcal{V}(C) = L^\infty(\Omega, \mu)$; since $\mathcal{U}(C)' = \mathcal{V}(C)$, $L^\infty(\Omega, \mu)$ is a maximal commutative *-subalgebra of $B(\mathfrak{H})$. For $a_o \in \mathfrak{G}$, put $(Ua_o g)(\xi) = \sqrt{\nu_a(\xi)}\, g(\xi a)$ for $g \in L^2(\Omega, \mu)$, then Ua_o is unitary and moreover $U_a U_b = U_{ab}$, so that $a \longrightarrow U_a$ is a unitary representation of $\mathfrak{G}$ in $L^2(\Omega, \mu)$.

$$(U_a^{-1} T_f U_a g)(\xi) = \sqrt{\nu_{a^{-1}}(\xi)}\; f(\xi a^{-1})\; \sqrt{\nu_a(\xi a^{-1})}\, g(\xi) = f(\xi a^{-1}) g(\xi);$$

hence $U_a^{-1} T_f U_a = T_{f_a}$, where $f_a(\xi) = f(\xi a^{-1})$.

Now put $L^2(\Omega, \mu) = \mathfrak{H}$ and $L^\infty(\Omega, \mu) = \mathfrak{A}$, then we obtain the situations in the (α); we shall construct a weakly closed *-subalgebra $\mathfrak{L}$ of $B(\tilde{\mathfrak{H}})$ according to the processes of the (α). Then,

Proposition 6.4. If the group $\mathfrak{G}$ is free and ergodic, then $\mathfrak{L}$ is a factor.

Proof. At first, we shall show that $\mathfrak{A} \cap \mathfrak{A} U_y = 0$ for $y(\neq e) \in \mathfrak{G}$. Let $T_{f_1} = T_{f_2} U_y \in \mathfrak{A} \cap \mathfrak{A} U_y$, where $f_1, f_2 \in L^\infty(\Omega, \mu)$ and $E = \{\xi \mid f_1(\xi) \neq 0\}$. Since $\mathfrak{G}$ is free and y is a homeomorphism, $K_y = \{\xi \mid \xi y = \xi, \xi \in \Omega\}$ for $y(\neq e) \in \mathfrak{G}$ is a closed null set; therefore it is sufficient to assume that $K_y = (\phi)$; for any $\xi \in E$, there is a compact neighborhood V_ξ of ξ such that $V_\xi \cap V_\xi y = (\phi)$; put $U_\xi = E \cap V_\xi$, then $U_\xi \cap U_\xi y = (\phi)$ and $U_\xi \subset E$; if $\mu(E) \neq 0$, there is a set U_{ξ_0} such that $\mu(U_{\xi_0}) \neq 0$; let χ be the characteristic function of U_{ξ_0}, then $f_1\chi = T_{f_1}\chi = T_{f_2} U_y \chi = f_2(U_y \chi)$; $(U_y \chi)(\xi) = \sqrt{\nu_y(\xi)}\,\chi(\xi y) = 0$ a.e. for $\xi \in U_{\xi_0}$; therefore $f_1(\xi)\chi(\xi) = f_1(\xi) = 0$ a.e. $\xi \in U_{\xi_0}$, this is a contradiction; hence we have $\mathfrak{A} \cap \mathfrak{A} U_y = 0$ $(y \neq e)$; therefore by Lemma 6.1, $\tilde{\mathfrak{A}}$ is a maximal commutative *-subalgebra of $\mathfrak{L}$.

Next, let T_f be an positive element of $\mathfrak{A}$ which is invariant under $\mathfrak{G}$, then $U_s^{-1} T_f U_s = T_{f_s} = T_f$; hence $f_s(\xi) =$

$f(\xi s^{-1}) = f(\xi)$ a.e. for $s \in \mathfrak{G}$; let $E_{p,q} = \{\xi \mid o \leq p \leq f(\xi) \leq q,$ $p < q,\ \xi \in \Omega\}$, then $M\{(E_{p,q} \cup E_{p,q}y) - (E_{p,q} \cap E_{p,q}y)\} = o$ for $y \in \mathfrak{G}$; hence $\mu(E_{p,q}) = o$ or $\mu(\Omega - E_{p,q}) = o$, this implies that $f(\xi)$ is a constant function, so that by Proposition 6.3, $\mathcal{L}$ is a factor.

Proposition 6.5. Suppose that the group $\mathfrak{G}$ is free, ergodic, measurable. Then if the invariant measure ν has the following properties: $\nu(\{\xi\}) = o$ for $\xi \in \Omega$ and $o < \nu(\Omega) < +\infty$ (resp. $\nu(\Omega) = +\infty$), $\mathcal{L}$ is a factor of type II_1 (resp. type II_∞).

Proof. For $T_f\ (\geq o) \in \mathfrak{A}$, put $\varphi(T_f) = \int_\Omega f(\xi)d\nu(\xi)$, then φ is a faithful semi-finite normal trace on $\mathfrak{A}$; moreover $\varphi(U_s^* T_f U_s) = \varphi(T_{f_s}) = \int_\Omega f(\xi s^{-1})d\nu(\xi) =$ $\int_\Omega f(\xi)d\nu(\xi) = \varphi(T_f)$; therefore by Proposition 6.2, $\mathcal{L}$ is a semi-finite factor; moreover if $o < \nu(\Omega) < +\infty$, $\mathcal{L}$ is a finite factor, and if $\nu(\Omega) = +\infty$, $\mathcal{L}$ is a semi-finite, properly infinite factor, since $\mathcal{L}$ is a factor; moreover since $\nu(\{\xi\}) = o$, there is a decreasing sequence $\{E_n\}$ of measurable sets such that $\nu(E_n) > \nu(E_{n+1}) >$ and $\lim_n \nu(E_n) = o$; $\psi(\Phi(T_{\chi_{E_n}})) = \varphi(T_{\chi_{E_n}})$ for the trace ψ of $\mathcal{L}$, where χ_{E_n} is the characteristic function of En, so that $\mathcal{L}$ is continuous This completes the proof

Proposition 6.6. Suppose that $\mathfrak{G}$ is free, ergodic and non-measurable, then $\mathcal{L}$ is a factor of type III.

From Proposition 6.2, (i), this is clear.

For later use, we shall mention a lemma.

Lemma 6.2 Let $\mathfrak{G}_o = \{y \mid {}_y(\xi) = 1$ a.e. $y \in \mathfrak{G}\}$, then $\mathfrak{G}_o$ is a subgroup of $\mathfrak{G}$. If $\mathfrak{G}_o$ is ergodic and $\mathfrak{G}_o \subsetneqq \mathfrak{G}$, $\mathfrak{G}$ is not measurable.

Proof. Suppose that $\mathfrak{G}$ is measurable, then there is a positive Radon measure ν which is equivalent to μ and invariant under $\mathfrak{G}$; then $d\mu(\xi) = (\frac{d\mu}{d\nu})(\xi)d\nu(\xi)$; $d\mu_a(\xi) = d\mu(\xi) = (\frac{d\mu}{d\nu})(\xi a)d\nu_a(\xi) = (\frac{d\mu}{d\nu})(\xi a)d\nu(\xi)$ for $a \in \mathfrak{G}_o$; since $\mathfrak{G}_o$ is ergodic, $(\frac{d\mu}{d\nu})(\xi) = \lambda =$ const.; therefore we have $\mathfrak{G}_o = \mathfrak{G}$, a contradiction.

Now we shall show concrete examples.

(A) Let Ω be the one-dimensional torus group, μ the Haar measure of Ω, $\mathfrak{G}$ a countable infinite subgroup of Ω which is dense in Ω; for $a \in \mathfrak{G}$ and $\xi \in \Omega$, we define a homeomorphism $\xi \longrightarrow \xi a$ by $\xi a = \xi + a$; clearly $\mathfrak{G}$ is free; now let E be a measurable set of Ω such that $\mu((E a \cup E) - (E \cap E a)) = o$ for all $a \in \mathfrak{G}$; let X_E be the characteristic function of E, then $X_E \in L^2(\Omega, \mu)$; $\varphi_n(\xi) = e^{2\pi i n \xi}$ $(n = o, \pm 1, \pm 2, \dots)$ are a complete orthonormal system of $L^2(\Omega, \mu)$; therefore we have $X_E(\xi) = \sum_{n=-\infty}^{\infty} \lambda_n e^{2\pi i n \xi}$, where $\sum_{n=-\infty}^{+\infty} |\lambda_n|^2 < +\infty$; then $X_E(\xi a) =$

$$\sum_{n=-\infty}^{\infty} \lambda_n e^{2\pi i n(\xi + a)} = \sum_{n=-\infty}^{\infty} \lambda_n e^{2\pi i n a} e^{2\pi i n \xi} = X_E(\xi) \quad \text{in}$$

$L^2(\Omega, \mu)$; hence $\lambda_n e^{2\pi i n a} = \lambda_n$ for all $a \in \mathfrak{G}$, so that $\lambda_n = 0$ for $|n| \geqq 1$, this implies $X_E(\xi) = \lambda_0 =$ const.; hence $\mu(E) = 0$ or $\mu(\Omega - E) = 0$; $\mathfrak{G}$ is ergodic; since μ is invariant under $\mathfrak{G}$, $\mu(\{\xi\}) = 0$ and $\mu(\Omega) = 1$, $\mathcal{L}$ is a factor of type II_1.

For instance we can take

(A_α): $\mathfrak{G} = \{n\,\Theta \pmod{1} \mid n = 0, \pm 1, \pm 2, \ldots;\ \Theta$ an irrational$\}$

(A_β): $\mathfrak{G} = \{\gamma \pmod{1} \mid \gamma$ all rational numbers$\}$

(A_γ): $\mathfrak{G} = \{ m/p^n \pmod{1} \mid m = 0, \pm 1, \pm 2, \ldots;\ n = 0, 1, 2, \ldots;\ p =$ any fixed number $2, 3, 4, \ldots \}$.

(B) Let Ω be the locally compact group of all real numbers, μ the Haar measure of Ω and $\mathfrak{G}$ a countably subgroup which is dense in Ω. Then, analogously, we can show that $\mathfrak{G}$ is free, ergodic and measurable; since $\mu(\Omega) = +\infty$, $\mathcal{L}$ is a factor II_∞

For instance.

(B_α): $\mathfrak{G} = \{m + n\,\Theta \mid m, n = 0, \pm 1, \pm 2, \ldots;\ \Theta$ an irrational$\}$

(B_β): $\mathfrak{G} = \{\gamma \mid \gamma$ all rational numbers$\}$

(B_γ): $\mathfrak{G} = \{m/p^n \mid m = 0, \pm 1, \pm 2, \ldots;\ n = 0, 1, 2, \ldots;\ p =$ any fixed number $2, 3, 4, \ldots\}$

(C_α): Let Ω be the locally compact group of all real numbers, μ the Haar measure of Ω; consider the following one-to-one mapping on Ω: $\alpha_1(\rho,\sigma): \eta \to \rho\eta + \sigma$ ($\rho > 0$, ρ, σ rational); put $\mathfrak{G} = \{\alpha_1(\rho,\sigma)\}$, then $\mathfrak{G}$ is a countable group; clearly μ is quasi-invariant and $\mathfrak{G}$ is free; $\mu(E\alpha) = \rho\mu(E)$ for measurable sets E, where $\alpha = \alpha_1(\rho,\sigma)$; $\mathfrak{G}_0 = \{\alpha \mid \mu_\alpha = \mu\} = \{\alpha_1(1,\sigma)\}$; since $\mathfrak{G}_0$ is ergodic, by Lemma 6.2, $\mathfrak{G}$ is non-measurable, so that $\mathcal{L}$ is a factor of type III.

(C_β): Let Ω be the one-dimensional torus group and now we consider Ω as the set of all complex numbers z with $|z| = 1$ and μ be the Haar measure of Ω. Consider the following one-to-one mappings on Ω: $\alpha_2(\theta,u): z \longrightarrow \theta\frac{z+u}{1+\bar{u}z}$ ($|\theta| = 1$, $|u| < 1$) Let $\mathfrak{G}$ be a countable group generated by $\{\alpha_2(\theta,u) \mid \theta = e^{2\pi i\rho}, \rho$ rational, $u = 0$ and $\theta = 1, u = \frac{1}{2}\}$: then, analogously, we can show that μ is quasi-invariant under $\mathfrak{G}$ and $\mathfrak{G}$ is free, ergodic and non-measurable; therefore $\mathcal{L}$ is a factor of type III

(C_γ): Let Ω_n ($n=1,2,\ldots$) be the additive groups of integers, reduced mod 2; therefore Ω_n is a compact group composed of two elements (0,1) as follows: $0+0 = 0$, $0+1 = 1$, $1+1=0$ Let Ω be the weakly infinite direct product $X_{n=1}^{\infty}\Omega_n$ of $\{\Omega_n\}$, then Ω is a compact group; let μ_n be a Radon measure on Ω_n such that $\mu_n(\{0\}) = \frac{1-\alpha_n}{2}$, $\mu(\{1\}) = \frac{1+\alpha_n}{2}$, where $0 < \delta \leqq \alpha_n \leqq 1 - \delta$ for some $\delta > 0$, and μ be the Radon measure on Ω defined by the infinite direct product $X_{n=1}^{\infty}\mu_n$ of $\{\mu_n\}$; Ω is the set of all sequences $\zeta =$

$(\xi_n; n=1,2,\ldots)$, where $\xi_n = 0,1$. Let $\mathcal{G}$ be the set of those $a = (a_n; n=1,2,\ldots) \in \Omega$ for which $a_n \neq 0$ occurs for finite number of n' only, then $\mathcal{G}$ is a countable group.

For $a \in \mathcal{G}$ and $\xi \in \Omega$, we define a homeomorphism $\xi \longrightarrow \xi a$ by $\xi a = \xi + a$. Now put $\frac{1-\alpha_n}{2} = p_n$ and $\frac{1+\alpha_n}{2} = q_n$, then

Lemma 6.3. The measure μ is quasi-invariant under $\mathcal{G}$.

Proof. For a fixed k, the function

$$f_k(\xi) = \begin{cases} \dfrac{p_k}{q_k} & \text{if} \quad \xi_k = 1 \\ \dfrac{q_k}{p_k} & \text{if} \quad \xi_k = 0 \end{cases}$$

and let $\gamma_k = (\gamma_{k,n}; \gamma_{k,n}=0 \text{ if } k \neq n, \gamma_{k,k}=1, n = 1,2,\ldots) \in \mathcal{G}$ and $E = \{\xi \mid \xi_{m_i} = 0, \xi_{n_j}=1; i = 1,2,\ldots,u; j = 1,2,\ldots,v\}$; suppose that k occurs among the numbers m_i, then

$$\int_E f_k(\xi)\,d\mu(\xi) = \frac{q_k}{p_k}\,\mu(E) = \mu(E\gamma_k);$$

analogously we can show that $\mu(E\gamma_k) = \int_E f_k(\xi)\,d\mu(\xi)$ for all k, so that $\mu(E\gamma_k) = \int_E f_k(\xi)\,d\mu(\xi)$ for $\gamma_k \in \mathcal{G}$; since the set of the form E is a fundamental family of neighborhoods of Ω, μ is quasi-invariant under $\mathfrak{g}$.

We can easily show $V_a^{\mu}(\xi) = f_{k_1}(\xi) f_{k_2}(\xi) \dots f_{k_n}(\xi) = \pi_{n=1}^{\omega} (\frac{p_n}{q_n})^{(2\xi_n - 1)a_n}$ for $a = \gamma_{k_1} + \gamma_{k_2} + \dots + \gamma_{k_n} \in \mathfrak{g}$.

Lemma 6.4. The system of functions

$$W_a(\xi) = (-1)^{\sum_{n=1}^{\infty} a_n \xi_n} \pi_{n=1}^{\infty} (\frac{p_n}{q_n})^{(\xi_n - \frac{1}{2}) a_n} \qquad (a \in \mathfrak{g})$$

forms a complete orthonormal system in $L^2(\Omega, \mu)$

Proof For $a, b \in \mathfrak{g}$,

$$\int W_a(\xi) W_b(\xi) d\mu(\xi) = \int W_{a+b}(\xi) W^2_{a \wedge b}(\xi) d\mu(\xi)$$

$$= \int W_{a+b}(\xi) d\mu(\xi) \int W^2_{a \wedge b}(\xi) d\mu(\xi), \text{ where } a \wedge b =$$

$$\{\beta_n; \beta_n = 1 \text{ if } a_n = b_n = 1; \text{ otherwise } \beta_n = o\};$$

suppose that $a = \gamma_{k_1} + \gamma_{k_2} + \dots + \gamma_{k_n}$, we have

$$\int W_a(\xi) d\mu(\xi) = \pi_{i=1}^{n} \int W_{\gamma_{k_i}}(\xi) d\mu(\xi)$$

and $\int W_a^2(\xi) d\mu(\xi) = \pi_{i=1}^{n} \int W^2_{\gamma_{k_i}}(\xi) d\mu(\xi)$.

But for every k, $\int W_{\gamma_k}(\xi) d\mu(\xi) = -q_k \sqrt{\frac{p_k}{q_k}} + p_k \sqrt{\frac{q_k}{p_k}} = o$, and $\int W^2_{\gamma_k}(\xi)d\mu(\xi) = q_k \frac{p_k}{q_k} + p_k \frac{q_k}{p_k} = 1$

To prove the completeness it suffices to show that the characteristic functions of the form $E = \{\xi \mid \xi m_i = o, \xi n_j = 1; i = 1,2, \ldots u; j = 1,2, \ldots, v\}$; such set is the intersection of a finite number of the sets $E_j = \{\xi ; \xi_j = o\}$ and $F_k = \{\xi \mid \xi_k = 1\}$ $(j,k = 1,2, \ldots)$; let e_j and f_k be the characteristic functions of E_j and F_k respectively, we have $e_j(\xi) = \sqrt{p_j q_j}\; W_{\gamma_j}(\xi) + p_j W_o(\xi)$ and $f_k(\xi) = -\sqrt{p_k q_k}\; W_{\gamma_k}(\xi) + q\, W_o(\xi)$. For $a,b \in \mathfrak{G}$, if $a \wedge b = o$, $W_{a+b}(\xi) = W_a(\xi)\, W_b(\xi)$, therefore we have the completeness of $\{W_a \mid a \in \mathfrak{G}\}$.

Lemma 6.5. The group $\mathfrak{G}$ is free, ergodic and non-measurable

Proof Clearly, $\mathfrak{G}$ is free. Now we shall show the ergodicity of $\mathfrak{G}$. Let $f(\xi)$ be a bounded measurable function on Ω such that $f(\xi\gamma_k) = f(\xi)$ a e $(k = 1,2, \ldots)$.

Since $V_{\gamma_k}(\xi) = \pi_{n=1}^{\infty} \left(\frac{p_n}{q_n}\right)^{(2\xi_n - 1)\gamma_{k,n}} = \left(\frac{p_k}{q_k}\right)^{(2\xi_k - 1)} = \frac{q_k - p_k}{\sqrt{p_k q_k}}\; W_{\gamma_k}(\xi) + 1$, if $a_k = o$ for $a = (a_n ; n = 1,2, \ldots), C_a = \int f(\xi)W_a(\xi)d\mu(\xi)$

$$\int f(\xi\gamma_k)W_a(\xi)d\mu(\xi) = \int f(\xi)W_a(\xi\gamma_k)\, V_{\gamma_k}(\xi)d\mu(\xi)$$

$$= \frac{q_k - p_k}{\sqrt{p_k q_k}} \int f(\xi)W_a(\xi\gamma_k)W_{\gamma_k}(\xi)d\mu(\xi) + \int f(\xi)W_a(\xi\gamma_k)d\mu(\xi)$$

$$= \frac{q_k - p_k}{\sqrt{p_k q_k}} \int f(\xi) W_{a+\gamma_k}(\xi) d\mu(\xi) + \int f(\xi) W_a(\xi) d\mu(\xi)$$

$$= \frac{q_k - p_k}{\sqrt{p_k q_k}} C_{a+\gamma_k} + C_a, \quad \text{where} \quad f(\xi) = \sum_{a \in \mathfrak{g}} C_a W_a(\xi);$$

therefore $C_{a+\gamma_k} = o$, so that $C_a = o$ for $a (\neq e) \in \mathfrak{g}$.

Finally we shall show the non-measurability of $\mathfrak{g}$. Suppose that there is a positive R don measure ν $(\nu(\Omega) = 1)$ on Ω which is equivalent to μ and invariant under $\mathfrak{g}$

Let C be the algebra of continuous functions on Ω, then a function $g(s) = \int f(\xi + s) d\mu(\xi)$ on Ω $(f \in C)$ is continuous and constant on $\mathfrak{g}$; since $\mathfrak{g}$ is dense in Ω, $g(s) = \text{const}$, so that ν is Haar measure on Ω; by the unicity of Haar measure, $\nu = \times_{n=1}^{\infty} \nu_n$, where ν_n is Haar measure on Ω such that $\nu_n(\{o\}) = \nu_n(\{1\}) = \frac{1}{2}$; therefore

$$\frac{d\mu_n}{d\nu_n}(\{o\}) = 2p_n \quad \text{and} \quad \frac{d\mu_n}{d\nu_n}(\{1\}) = 2q_n, \quad \text{so that}$$

$$\int_{\Omega_n} \sqrt{\frac{d\mu_n}{d\nu_n}(\xi)} \sqrt{\frac{d\nu_n}{d\nu_n}(\xi)}\, d\nu_n(\xi) = \frac{\sqrt{p_n} + \sqrt{q_n}}{\sqrt{2}} = \left(\frac{1 + 2\sqrt{p_n q_n}}{2}\right)^{1/2} =$$

$$\left(\frac{1}{2} + \sqrt{\left(\frac{1+\alpha_n}{2}\right)\left(\frac{1-\alpha_n}{2}\right)}\right)^{1/2} = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \alpha_n^2}\right)^{1/2} \leqq \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \delta^2}\right)^{1/2} = \lambda < 1;$$

therefore

$$\pi_{n=1}^{\infty} \int_{\Omega_n} \sqrt{\frac{d\mu_n}{d\nu_n}(\xi)} \sqrt{\frac{d\nu_n}{d\nu_n}(\xi)}\, d\nu_n(\xi) = o;$$

by the theorem of Kakutani [cf Ann. of Math., 49(1948)p.p.214 - 226], μ is orthogonal to ν, so that μ is not equivalent to ν, a contradiction. This completes the proof

Now we can conclude that a factor $\mathcal{L}$ obtained by the processes of the (α) is a factor of type III.

Remark If $\alpha_n = o$ for all n in the (C_γ), μ is Haar measure on Ω; therefore we can easily show that $\mathfrak{G}$ is free, ergodic and measurable, so that $\mathcal{L}$ is a factor of type II_1.

In spite of such many examples, we can not assert that there are substantially many examples, because somes of them may be mutually *-isomorphic (namely the same algebraical type)-in fact, Murray and von Neumann [cf. Lemma 5 2.3 in [18]) noticed that if the group $\mathfrak{G}$ is commutative, free, ergodic, measurable and $\nu(\Omega) = 1$ for an invariant measure ν which is equivalent to μ, the obtained II_1-factor $\mathcal{L}$ has the same algebraical type (hyper finite (cf. Theorem 6.)); therefore all II_1-factors, obtained in the (A) are mutually *-isomorphic.

To show the existence of different algebraical types, more other considerations are needed Now we shall state somes concerning them.

(γ) Other construction of II_1-factors. Let G be a countably infinite discrete group with unit e, $\mathfrak{C}$ be the set of all complex valued continuous functions on G with compact supports

For f, g $\in \mathfrak{C}$, we define the multiplication as follows: $f*g(a) = \sum_{b \in G} f(b)g(b^{-1}a)$. For $f \in \mathfrak{C}$, define $f^*(a) = \overline{f(a^{-1})}$;

for $f, g \in C$, put $(f,g) = \sum_{a \in G} f(a)\overline{g(a)}$, then $\mathcal{C}$ is a hilbert algebra; denote by ε_a the following function $\varepsilon_a(b) = o$ for $b \neq a$ and $\varepsilon_a(a) = 1$, then ε_e is the unit of the hilbert algebra $\mathcal{C}$; therefore the left associated algebra $\mathcal{U}(\mathcal{C})$ is finite and every element of $\mathcal{U}(\mathcal{C})$ can be written by U_f, where $f \in L^2(G)$.

Now suppose that G satisfies the following condition

(*) Every conjugate class of G with an exception of $\{e\}$, is infinite.

Then, let U_f be a central element of $\mathcal{U}(\mathcal{C})$, $U_{\varepsilon_a}^{-1} U_f U_{\varepsilon_a} =$

$U_{\varepsilon_a^{-1}} {*f*} \varepsilon_a = U_f$; hence $\varepsilon_a^{-1} * f * \varepsilon_a(b) = f(aba^{-1}) = f(b)$ for all

$a, b \in G$; since $\sum_{b \in G} |f(b)|^2 < +\infty$, $f(b) = o$ if $b \notin$ the class

$\{e\}$, so that $f(b) = o$ for $b \neq e$, this implies $U_f = \lambda 1$;

$\mathcal{U}(\mathcal{C})$ is a finite factor; moreover, since $\mathcal{C}$ is infinite-dimensional, $\mathcal{U}(\mathcal{C})$ is a II_1-factor.

Next, we shall consider general II_1-factors.

Lemma 6.6. Let M be a finite W*-algebra, $\{M_\alpha\}$ an increasing directed set of factors containing the unit of M such that $M_\alpha \subset M$, and M_1 be the σ-closure of $\{M_\alpha\}$ in M, then M is a finite factor

Proof It is clear that M_1 is finite Suppose that M_1 is not a factor, then there are two normal finite traces on M_1

such that $\varphi_1(1) = \varphi_2(1) = 1$ and $\varphi_1 \neq \varphi_2$.

On the other hand, by the unicity of the $\natural$-operation, $\varphi_1 = \varphi_2$ on M_α for all α, this implies $\varphi_1 = \varphi_2$ on M_1, a contradiction.

Definition 6.1 A finite factor M is said to be hyper finite if it satisfies one of the following conditions:

(i) M is of type I_p $(p < +\infty)$

(ii) M is generated by an increasing sequence of factors of type $I_1, I_2, \ldots I_{2^n}, \ldots,$ containing the unit of M.

Then,

Theorem 6.1. Every continuous finite factor contains a continuous hyper finite factor.

Proof. Let M be a continuous finite factor, then the unit 1 can be written in the form $1 = e_1 + e_2$ and $e_1 \sim e_2$, where e_1 is orthogonal to e_2; therefore we can write $M = B_1 \otimes M_1$, where B_1 is a factor of type I_2; since M_1 is also continuous, $M_1 = B_2 \otimes M_2$ (B_2 is of type I_2); therefore $M = B_1 \otimes B_2 \otimes M$, where $B_1 \otimes B_2 \otimes 1$ is of type I_4; by analogous methods, we can construct an increasing sequence $\{M_n\}$ of type I_{2^n}; let N be the σ-closure of $\{M_n\}$, then by Lemma 6.6, N is a factor; clearly N is a continuous hyper finite factor. This completes the proof.

Theorem 6.2. Two continuous hyper finite factors are mutually *-isomorphic.

Proof. Let M_1, M_2 be two continuous hyper finite factors, $(M_{1,n})$ (resp. $(M_{2,n})$) an increasing sequence of factors of type I_{2^n} containing the unit of M_1 (resp. M_2) which generates M_1 (resp. M_2) Since $M_{1,n+1} = M_{1,n} \otimes B_1$; $M_{2,n+1} = M_{2,n} \otimes B_2$, where B_i $(i = 1,2)$ is a factor of type I_2, an *-isomorphism of $M_{1,n}$ onto $M_{2,n}$ can be extended to an *-isomorphism of $M_{1,n+1}$ onto $M_{2,n+1}$; therefore there is an *-isomorphism Φ of $\bigcup_{n=1}^{\infty} M_{1,n}$ onto $\bigcup_{n=1}^{\infty} M_{2,n}$; from the unicity of the $\natural$-operations, we have $\varphi_2(\Phi(x)) = \varphi_1(x)$ for $x \in \bigcup_{n=1}^{\infty} M_{1,n}$, where φ_1 (resp. φ_2) is the unique finite trace of M_1 (resp. M_2) such that $\varphi_1(1) = 1$ (resp $\varphi_2(1) = 1$) and moreover Φ is isometric with respect to the uniform norm; therefore by the density theorem of Kaplansky, Φ is uniquely extended to an *-isomorphism of M_1 onto M_2. This completes the proof.

Definition 6.2 Let M be a factor. We say that M has the property L if there is a sequence $\{U_n\}$ of unitary elements of M such that $\sigma\text{-}\lim_n U_n = o$ and $s\text{-}\lim_n U^*_n \, a \, U_n = a$ for all $a \in M$.

The property L is invariant under *-isomorphisms; therefore if we can construct two factors M_1 and M_2 of the same type such that M_1 has the property L and M_2 has no the property L, we can assert that there are two factors of the same type which are not mutually *-isomorphic

Proposition 6.6. A continuous hyper finite factor has the property L.

Proof. Let $\{M_n\}$ be an increasing sequence of factors of type I_{2^n} $(n = 1, 2, \ldots)$ containing the unit of M, which generates M, then $M_{n+1} = M_n \otimes B_n$, where B_n is a factor of type I_2; let $\bar{U}_n$ be a unitary element of B_n such that $\varphi(\bar{U}_n) = 0$, and put $U_n = 1_{M_n} \otimes \bar{U}_n$, where 1_{M_n} is the unit of M_n, then

$\lim_n \varphi(a U_n) = \lim_n \varphi(a)\varphi(U_n) = 0$ for $a \in M_m$, where φ is the unique trace on M such that $\varphi(1) = 1$; since $\bigcup_{m=1}^{\infty} M_m$ is σ-dense, $\{R_a \varphi \mid a \in \bigcup_{n=1}^{\infty} M_m\}$ is total in M_*, so that $\sigma\text{-}\lim_n U_n = 0$.

On the other hand, put $||x||_2 = \varphi(x^* x)^{1/2}$ for $x \in M$, then

$$\lim_n ||U_n^* a U_n - a||_2 = ||a - a||_2 = 0 \quad \text{for } a \in \bigcup_{m=1}^{\infty} M_m ;$$

moreover, since $\bigcup_{m=1}^{\infty} M_m$ is s-dense in M, for any $b \in M$ and $\varepsilon > 0$, there is an element a of $\bigcup_{m=1}^{\infty} M_m$ such that $||a - b||_2 < \varepsilon$, then

$$||U_n^* b U_n - b||_2 \leqq ||U_n^* b U_n - U_n^* a U_n||_2 + ||U_n^* a U_n - a||_2$$

$+ ||a - b||_2 = ||b - a||_2 + ||U_n^* a U_n - a||_2 + ||a - b||_2$;

hence $\overline{\lim_n} ||U_n^* b U_n - b||_2 \leqq 2\varepsilon$, so that we have

$\lim_n ||U_n^* b U_n - b||_2 = 0$; since $|\varphi(y x^* x)| \leqq \varphi(y x^* x y^*)^{1/2} \cdot \varphi(x^* x)^{1/2}$ and $\{U_n^* b U_n - b\}$ is uniformly bounded, we have that $s\text{-}\lim_n U_n^* b U_n = b$ for all $b \in M$. This completes the proof

Let G be the free group of two generators a_1, a_2 then it is clear that G is a countable infinite discrete group satisfying (*), so that $\mathcal{U}(\mathcal{C})$ is a continuous finite factor. Let F be the set of $a \in G$ which when written as a power of a_1, a_2 of minimum length end with a a_1^n, $n = \pm 1, \pm 2, \ldots$. Then it is clear that $F \cup a_1 F a_1^{-1} \cup \{e\} = G$ Moreover, F, $a_2 F a_2^{-1}, a_2^{-1} F a_2$ are mutually disjoint.

Proposition 6.7. Let G be a countably infinite discrete group satisfying the condition (*) and moreover suppose that there is a set F of G such that

(i) there is an element a_1 of G such that $F \cup a_1 F a_1^{-1} \cup \{e\} = G$;

(ii) there are two elements a_2, a_3 of G such that F, $a_2 F a_2^{-1}$, $a_3 F a_3^{-1}$ are mutually disjoint. Then $\mathcal{U}(\mathcal{C})$ is a continuous finite factor having no the property L.

Proof Suppose that $\mathfrak{U}(\mathcal{C})$ has the property L Let U_{y_n} ($y_n \in L^2(G)$) be a sequence of unitary elements such that

$\sigma\text{-}\lim_n U_{y_n} = o$ and $s\text{-}\lim_n U_{y_n}^* A U_{y_n} = A$ for $A \in \mathfrak{U}(\mathcal{C})$.

$$\varphi(U_{y_n}) = (U_{y_n}, 1) = (U_{y_n}, \varepsilon_e) = y_n(e) \longrightarrow o$$

On the other hand,

$$||U_{\varepsilon_{a_i}} - U_{y_n}^* U_{\varepsilon_{a_i}} U_{y_n}||_2 = ||U_{y_n} - U_{\varepsilon_{a_i}}^* U_{y_n} U_{\varepsilon_{a_i}}||_2 =$$

$$||y_n - \varepsilon_{a_i^{-1}} * y_n * \varepsilon_{a_i}||_2 \longrightarrow o, \text{ and } ||y_n||_2 =$$

$$||U_{y_n}||_2 = 1 \quad (i = 1, 2, 3)$$

Therefore, for arbitrary $\varepsilon > o$,

$$1 \leqq \sum_{a \in F} |y_n(a)|^2 + \sum_{a \in a_1 F a_1^{-1}} |y_n(a)|^2 + |y_n(e)|^2$$

$$= \sum_{a \in F} |y_n(a)|^2 + \sum_{a \in F} |(\varepsilon_{a_1^{-1}} * y_n * \varepsilon_{a_1})(a)|^2 + |y_n(e)|^2$$

$$\leqq 2 \sum_{a \in F} |y_n(a)|^2 + 2\varepsilon^2 \quad (n \geqq n_o).$$

On the other hand,

$$1 \geqq \sum_{a \in F} |y_n(a)|^2 + \sum_{a \in F} |(\varepsilon_{a_2^{-1}} * y_n * \varepsilon_a)(a)|^2 + \sum_{a \in F}$$

$$|\varepsilon_{a_3^{-1}} * y_n * \varepsilon_{a_3}(a)|^2 \geqq 3 \sum_{a \in F} |y_n(a)|^2 - 2\varepsilon^2 \quad (n \geqq n_o);$$

hence $\frac{1}{2} - \varepsilon^2 \leqq \sum_{a \in F} |y_n(a)|^2 \leqq \frac{1}{3} + \frac{2}{3}\varepsilon^2$, a contradiction

This completes the proof.

Hence we have

Theorem 6.3. There are two II_1-factors $\mathcal{A}$ and $\mathcal{B}$ which are not mutually *-isomorphic as follows:

(i) $\mathcal{A}$ is hyper finite (therefore has the property L)

(ii) $\mathcal{B} = \mathcal{U}(\mathcal{C})$ for the free group G of two generators (therefore has no the property L)

Murray and von Neumann (Theorem XII and XV in [18]) mentioned the following two properties, each of which is equivalent to the hyper finiteness: 1. a finite factor M is generated by an increasing sequence of finite dimensional *-subalgebras; 2. a finite factor M is generated by enumerable elements and for arbitrary finite elements $a_1, a_2, \ldots, a_n$ and $\varepsilon > o$, there is a finite dimensional *-subalgebra N of M such that $||a_i - b_i||_2 \leqq \varepsilon$ $(i = 1, 2, \ldots m)$ for some $b_1, b_2, \ldots, b_m \in N$; consequently they showed that if $\mathcal{A}$ is a continuous hyper finite factor, the factor $e \mathcal{A} e$ for every non-zero projection

e of $\mathcal{A}$ is also so, so that by Theorem 6.2., $\mathcal{A}$ and $e\mathcal{A}e$ are mutually *-isomorphic; this is open for other II_1-factors- namely we have the following question

Question 1. Let M be a II_1-factor, e be a non-zero projection of M, then can we conclude that $M \sim eMe$? (that is, *-isomorphic)

The simplest form of this question is as follows: let B be a factor of type I_2, then can we conclude $M \sim M \otimes B$?

Explicit constructions of continuous hyper finite factors are obtained as follows: in the construction (γ), $G = \bigcup_{n=1}^{\infty} G_n$, where G_n is a finite subgroup and $G_m \supset G_n$ $(m>n)$; all examples in the construction (A); $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \ldots o$ in the construction (C_γ).

Murray and von Neumann (cf. § 6.3 [18]) gave many constructions of examples of type II_1 which have no the property L; however it is an open question whether there are different algebraical types having no the property L in the same type.

Question 2. Are there two examples of the same type having no the property L which are not mutually *-isomorphic?

Concerning the tensor product of factors, we have easily: (hyper finite) $\overline{\otimes}$ (hyper finite) = (hyper finite). Moreover,

Theorem 6.4 Let M be a factor, N a finite factor having the property L, then $M \overline{\otimes} N$ has the property L.

Proof. Let $\{U_n\}$ be a sequence of unitary elements of N such that $\sigma\text{-}\lim_n U_n = o$ and $s\text{-}\lim_n U_n^* b U_n = b$ for $b \in N$. Now put $\tilde{U}_n = 1 \otimes U_n$ in $M \overline{\otimes} N$, then $\tilde{U}_n$ is unitary; for

$f \in M_*$ and $g \in N_*$, $(f \otimes g)(\tilde{U}_n) = f(1)g(U_n) \to o$; since $\{\tilde{U}_n\}$ is uniformly bounded, we have σ-$\lim \tilde{U}_n = o$ in $M \bar{\otimes} N$.

Let φ be the unique trace on N such that $\varphi(1) = 1$ and ψ be arbitrary positive normal functional on M, then $\psi \otimes \varphi(\tilde{U}_n^* x \tilde{U}_n) = \psi \otimes R_{\tilde{U}_n} L_{\tilde{U}_n}^* \varphi(x) = \psi \otimes \varphi(x)$ for $x \in M \bar{\otimes} N$.; moreover we can easily show that s-$\lim_n \tilde{U}_n$ $\left(\sum_{i=1}^m a_i \otimes b_i\right) \tilde{U}_n = \sum_{i=1}^m a_i \otimes b_i$, where $a_i \in M$ and $b_i \in N$.

Now put $||x||_2 = \psi \otimes \varphi(x^* x)^{1/2}$ for $x \in M \bar{\otimes} N$; since $M \otimes N$ is s-dense in $M \bar{\otimes} N$, for $x \in M \bar{\otimes} N$ and $\varepsilon > o$, there is an element y of $M \otimes N$ such that $||x - y||_2 < \varepsilon$. Then

$$||\tilde{U}_n^* x \tilde{U}_n - x||_2 \leqq ||\tilde{U}_n^* x \tilde{U}_n - \tilde{U}_n^* y \tilde{U}_n||_2 + ||\tilde{U}_n^* y \tilde{U}_n - y||_2$$

$$+ ||y - x||_2 = 2||x - y||_2 + ||\tilde{U}_n^* y \tilde{U}_n - y||_2 ;$$

hence $\overline{\lim_n} ||\tilde{U}_n^* x \tilde{U}_n - x||_2 \leqq 2\varepsilon$, so that $\lim_n ||\tilde{U}_n^* x \tilde{U}_n - x||_2 = o$.; since $|(\psi \otimes L_b \varphi)(x^* x)| = |\psi \otimes \varphi(1 \otimes b\, x^* x)|$ $\leqq \psi \otimes \varphi(1 \otimes b\, x^* x\, 1 \otimes b^*)^{1/2} \cdot \psi \otimes \varphi(x^* x)^{1/2}$, we have $\{\psi \otimes L_b \varphi\}\{(\tilde{U}_n^* x \tilde{U}_n - x)^* (\tilde{U}_n^* x \tilde{U}_n - x)\} \to o$ $(n \to \infty)$; since $\{\psi \otimes L_b \varphi \mid \psi (> o) \in M_*,\ b \in N\}$ is total and $\{\tilde{U}_n^* x \tilde{U}_n - x\}$

is uniformly bounded, we have $s\text{-}\lim_n \tilde{U}_n^* x \tilde{U}_n = x$ for $x \in M \bar{\otimes} N$. This completes the proof.

Remark 6.1 Misonou [Tohoku Math. J 8 (1956), pp. 63-69] proved the above theorem under the assumption that M is also finite. Now let $\mathbb{A}$ and $\mathbb{B}$ be two factors in Theorem 6.3 then $\mathbb{A} \bar{\otimes} \mathbb{B}$ has the property L, so that $\mathbb{A} \bar{\otimes} \mathbb{B} \not\cong \mathbb{B}$; according to the communication of Kadison, J. Schwartz has shown that also $\mathbb{A} \bar{\otimes} \mathbb{B} \not\cong \mathbb{A}$; therefore $\mathbb{A} \bar{\otimes} \mathbb{B}$ is the third II_1-factor; the reader should be referred to the coming paper of Schwartz.

Remark 6.2. In the above theorem, it is an open question whether the finiteness of N can be also dropped.

Corollary 6.1. There is a II_∞-factor (resp III-factor) having the property L.

Proof. Let $\mathbb{A}$ be a continuous hyper finite factor, B (resp. M) a I_∞ (resp III)-factor, then $B \times \mathbb{A}$ (resp. $M \bar{\otimes} \mathbb{A}$) is a II_∞(resp. III)-factor; moreover by the above theorem, they have the property L. This completes the proof.

The following questions are interesting.

Question 3. (i) Is there a continuous factor which is not hyper finite and is contained in a continuous hyper finite factor?

(ii) Can we construct a hyper finite factor $\mathbb{A}$ for any finite factor M such that $M \subset \mathbb{A}$?

Proposition 6.8. There are two II_∞-factors which are not mutually *-isomorphic.

Proof. Let $\mathbb{A}$ be a continuous hyper finite factor, M a II_1-factor having no the property L and B be a $I_{\aleph_0}$-factor; consider two II_∞-factors $\mathcal{A}\bar{\otimes}B$ and $M\bar{\otimes}B$, then $\mathcal{A}\bar{\otimes}B$ is not *-isomorphic to $M\bar{\otimes}B$ in fact, suppose that $\mathcal{A}\bar{\otimes}B$ is *-isomorphic to $M\bar{\otimes}B$, then we can construct a II_∞-factor N as follows: $N = \mathcal{A}\bar{\otimes}B_1 = M\bar{\otimes}B_2$, where B_1, B_2 are I_∞-factors; let e_1 be an abelian projection of B_1, then $M\otimes e_1 \sim M$, so that $1_M \otimes e_1$ is a finite projection of N, where 1_M is the unit of M; analogously, let e_2 be an abelian projection of B_2, then $1_{\mathcal{A}} \otimes e_2$ is a finite projection of N, where $1_{\mathcal{A}}$ is the unit of $\mathcal{A}$; by the comparability theorem, there is a finite family (p_i) of orthogonal projections of M such that $p_i \sim 1_{\mathcal{A}} \otimes e_i$ and $1_M \otimes e_1 \leqq \sum_{i=1}^{n} p_i$; since $(1_{\mathcal{A}} \otimes e_2) N (1_{\mathcal{A}} \otimes e_2) =$

$$\mathcal{A} \otimes e_2 B e_2 = \mathcal{A} \otimes e_2, \ (\sum_{i=1}^{n} p_i) N (\sum_{i=1}^{n} p_i) = \mathcal{A}_1 \otimes B_0,$$

where B_0 is a I_n-factor, so that it is hyper finite; hence $(1\otimes e_1) N (1 \otimes e_1) = M \otimes e_2$ is also hyper finite, a contradiction. This completes the proof.

Now we shall show the existence of two III-factors which are not mutually *-isomorphic, according to the method of Pukansky [43]; for this it is enough to show that there is a III-factor which has no the property L

The method is a modification of the construction (C_γ)

Let G be the free group of two generators, Ω_a $(a \in G)$

be the additive group of two elements $(o,1)$ Now define a measure μ_a on Ω_a such that $\mu_a(\{o\}) = p$, $\mu_a(\{1\}) = q$, where $q > p > o$ and $p + q = 1$; let Ω be the compact group $X_{g \in G} \Omega_g$ and μ the Radon measure on Ω defined by $X_{g \in G} \mu_g$; every $\xi \in \Omega$ may be identified with a function (ξ_g) $(g \in G)$ defined on G taking the values o and 1 only. Let $\mathfrak{G}_1$ be the set of those $\alpha = (\alpha_g) \in \Omega$ for which $\alpha_g \neq o$ occurs for finite number of g only. We denote the set of pair (α, g) $(\alpha \in \mathfrak{G}_1, g \in G)$ by $\mathfrak{G}$. To an element $(\alpha, a) = \sigma$ of $\mathfrak{G}$ we associated the mapping $\xi \to \xi\sigma$ of Ω onto itself defined by $\xi\sigma = (\xi_{ag} + \alpha_g)$ $(g \in G)$. These mappings are one to one Introducing the notation $\alpha^a = (\alpha_{ag})(g \in G)$, we get $\{\xi(\alpha, a)\}(\beta, b) = (\xi_{abg} + \alpha_{bg} + \beta_g)$; hence $(\alpha, a)(\beta, b) = (\alpha^b + \beta, ab)$; moreover if $\xi(\alpha, a) = \xi(\alpha', a')$ for $\xi \in \Omega$, we have $\alpha = \alpha'$ and $a = a'$; therefore $\mathfrak{G}$ is a semi-group; observing $(\alpha, a)(o,e) = (\alpha^e + o, a\,e) = (\alpha, a)$, where o (resp e) is the unit of $\mathfrak{G}_1$ (resp. G_1).

$$(o,e)\ (\alpha, a) = (o^a + \alpha,\ e\ a) = (\alpha, a)$$

and

$$(\alpha, a)\ (\alpha^{a^{-1}}, a^{-1}) = (\alpha^{a^{-1}} + \alpha^{a^{-1}}, aa^{-1}) = (o,e)$$

$$(\alpha^{a^{-1}}, a^{-1})\ (\alpha, a) = (\alpha^{a^{-1}a} + \alpha, a^{-1}a) = (o,e)\ ;$$

therefore (o,e) is the unit and the inverse $(\alpha,a)^{-1} = (\alpha^{a^{-1}}, a^{-1})$, so that $\mathfrak{G}$ is a group. It is easily shown that the correspondence $\alpha \longrightarrow (\alpha, e)$ and $a \longrightarrow (o, a)$ define isomorphisms of the groups $\mathfrak{G}_1$ and G with subgroups of $\mathfrak{G}$. We denote these subgroups in the sequel again by $\mathfrak{G}_1$ and G_1.

Then by the analogous method with the (C_γ), we can easily show that μ is quasi-invariant and $\mathfrak{G}$ is free, ergodic and non-measurable

Now put $L^2(\Omega, \mu) = \mathfrak{H}$, $L^\infty(\Omega,\mu) = \mathfrak{A}$ and $U_{(\alpha, a)} f(\xi) = \nu_{(\alpha, a)}(\xi)\, f(\xi(\alpha, a))$ for $f \in L^2(\Omega, \mu)$ and $(\alpha, a) \in \mathfrak{G}$, and we shall construct a weakly closed $*$-subalgebra $\mathcal{L}$ of $B(\mathfrak{H})$ according to the method of the (α), then $\mathcal{L}$ is a III-factor. We shall denote this special W^*-algebra $\mathcal{L}$ by $\mathbb{M}$. Then

Theorem 6.5. The III-factor $\mathbb{M}$ has no the property L.

To prove this theorem, we shall provide some lemmas.

Lemma 6.7. Let $\mathfrak{G}_0$ be a group and E a subset of $\mathfrak{G}_0$. Suppose there exists a subset $\mathfrak{F} \subset E$ and two elements $g_1, g_2 \in \mathfrak{G}_0$ such that (i) $\mathfrak{F} \cup g_1 \mathfrak{F} g_1^{-1} = E$, (ii) $\mathfrak{F}$, $g_2^{-1} \mathfrak{F} g_2$ and $g_2 \mathfrak{F} g_2^{-1} \subset E$ are mutually disjoint.

Let $f(g)$ be a complex valued function on $\mathfrak{G}_0$ such that $\Sigma_{g \in \mathfrak{G}_0} |f(g)|^2 < +\infty$ and $(\sum_{g \in \mathfrak{G}_0} |f(g_i a g_i^{-1}) - f(g)|^2)^{1/2} < \varepsilon$ $(i=1,2)$. Then $(\sum_{g \in E} |f(g)|^2)^{1/2} < K_1 \varepsilon$, where K_1 does not depend on $\mathcal{E}$.

Proof. Put $\nu(F) = \sum_{g \in F} |f(g)|^2$ for a subset $F \subset \mathfrak{G}_0$, then

$$\mathcal{E} > \Big(\sum_{g \in \mathfrak{G}} |f(g_1 g g_1^{-1}) - f(g)|^2\Big)^{1/2} \geqq |\nu(g_1 \mathcal{F} g_1^{-1})^{1/2} - \nu(\mathcal{F})^{1/2}| .$$

Putting $\nu(E)^{1/2} = s$, then

$$|\nu(g_1 \mathcal{F} g_1^{-1}) - \nu(\mathcal{F})| = |\nu(g_1 \mathcal{F} g_1^{-1})^{1/2} + \nu(\mathcal{F})^{1/2}$$

$$||\nu(g_1 \mathcal{F} g_1^{-1})^{1/2} - \nu(\mathcal{F})^{1/2}| < 2\, s\, \mathcal{E},$$ and so

$\nu(g_1 \mathcal{F} g_1^{-1}) < \nu(\mathcal{F}) + 2\, s\, \mathcal{E}$; hence $s^2 \leqq \nu(g_1 \mathcal{F} g_1^{-1}) + \nu(\mathcal{F}) < 2(\nu(\mathcal{F}) + s\,\mathcal{E})$, so that $\nu(\mathcal{F}) > \frac{s^2}{2} - s\,\mathcal{E}$

Since $\Big(\sum_{g \in \mathfrak{G}_0} |f(g_2 g g_2^{-1}) - f(g)|^2\Big)^{1/2} = \Big(\sum_{g \in \mathfrak{G}_0} |f(g_2 g_2^{-1} g g_2 g_2^{-1}) - f(g_2^{-1} g g_2)|^2\Big)^{1/2}$, analogously we have $|\nu(g_2 \mathcal{F} g_2^{-1}) - \nu(\mathcal{F})| < 2\, s\, \mathcal{E}$, $|\nu(g_2^{-1} \mathcal{F} g_2) - \nu(\mathcal{F})| < 2\, s\,\mathcal{E}$, so that $\nu(g_2 \mathcal{F} g_2^{-1}) > \nu(\mathcal{F}) - 2\, s\,\mathcal{E} > \frac{s^2}{2} - 3\, s\, \mathcal{E}$ and $\nu(g_2^{-1} \mathcal{F} g_2) > \frac{s^2}{2} - 3\, s\, \mathcal{E}$.

Therefore,

$$s^2 = \nu(E) \geqq \nu(\mathcal{F}) + \nu(g_2^{-1} \mathcal{F} g_2) + \nu(g_2 \mathcal{F} g_2^{-1})$$

$> \frac{3}{2} s^2 - 7 s \varepsilon$, that is $s < 14 \varepsilon$

Lemma 6.8. Let G be the free group of two generators g_1, g_2. Suppose that a function $f(g)$ on G such that $\sum_{g \in G} |f(g)|^2 < +\infty$ and $(\sum_{g \in G} |f(gg_i) - f(g)|^2)^{1/2} < \varepsilon$ $(i = 1, 2)$. Then $(\Sigma_{g \in G} |f(g)|^2)^{1/2} < K_2 \varepsilon$; where K_2 does not depend on ε .

Proof Let F be the set of $g \in G$ when written as a power of g_1, g_2 of minimum length end with a g_1^n, $n = \pm 1, \pm 2, \ldots$. Then it is clear that $F \cup F g_1 = G$; moreover $F \cap F g_2 = (\phi)$, $F \cap F g_2^{-1} = (\phi)$ and $F g_2 \cap F g_2^{-1} = (\phi)$

Put $\nu(P) = \sum_{g \in P} |f(g)|^2$ for a subset P of G, then

$$\varepsilon > (\sum_{g \in G} |f(gg_i) - f(g)|^2)^{1/2} \geqq | \nu(F g_1)^{1/2} - \nu(F)^{1/2}| .$$

Putting $\nu(G) = s^2$, then

$$| \nu(Fg_1) - \nu(F)| = | \nu(Fg_1)^{1/2} + \nu(F)^{1/2}| \, | \nu(Fg_1)^{1/2} - \nu(F)^{1/2}| < 2 s \varepsilon \quad \text{and so} \quad \nu(Fg_1) < \nu(F) + 2 s \varepsilon ;$$

hence $s^2 \leqq \nu(Fg_1) + \nu(F) < 2(\nu(F) + s\varepsilon)$, so that $\nu(F) > \frac{s^2}{2} - s \varepsilon$. Analogously, we have

$$| \nu(Fg_2) - \nu(F)| < 2 s \varepsilon, \; | \nu(Fg_2^{-1}) - \nu(F) | < 2 s \varepsilon ;$$

therefore $\nu(Fg_2) > \nu(F) - 2 s \varepsilon > \frac{s^2}{2} - 3 s \varepsilon$ and $\nu(Fg_2^{-1}) > \frac{s^2}{2} - 3 s \varepsilon$. Hence finally we have

$$s^2 = \nu(G) \geqq \nu(F) + \nu(Fg_2) + \nu(Fg_2^{-1})$$

$$> \frac{3}{2} s^2 - 7 s \varepsilon, \quad \text{that is} \quad s < 14 \varepsilon$$

Lemma 6.9. For a function $f \in L^2(\Omega, \mu)$, suppose that $|\int_\Omega | f(\xi g_i) - f(\xi)|^2 d\mu)^{1/2} < \varepsilon \ (i = 1,2)$, then

$$|(\int |f(\xi)|^2 d\mu)^{1/2} - |\int_\Omega f(\xi) d\mu|| < K_3 \varepsilon,$$

where K_3 does not depend on ε

Proof. Let $f(\xi) = \sum_{\alpha \in \mathfrak{G}_1} C_\alpha W_\alpha(\xi)$ be the expansion of $f(\xi)$ in terms of the system $\{W_\alpha(\xi)\}$ (cf (C_γ)) Since the mapping $\xi \longrightarrow \xi g \ (g \in G)$ leave invariant the measure μ. and $W_\alpha(\xi g) = W_{\alpha g^{-1}}(\xi)$ for $\alpha \in \mathfrak{G}_1$ and $g \in G \subset \mathfrak{G}$, we have

$$\int_\Omega f(\xi g_i) W_\alpha(\xi) d\mu = \int_\Omega f(\xi) W_\alpha(\xi g_i^{-1}) d\mu =$$

$$\int f(\xi) W_{\alpha g_i}(\xi) d\mu = C_{\alpha g_i}; \quad \text{hence} \quad f(\xi g_i) =$$

$$\sum_{\alpha \in \mathfrak{G}_1} C_{\alpha g_i} W_\alpha(\xi) \quad \text{and so} \quad (\int_\Omega |f(\xi g_i) - f(\xi)|^2 d\mu)^{1/2}$$

$$= (\sum_{\alpha \in \mathfrak{G}_1} |C_{\alpha g_i} - C_\alpha|^2)^{1/2} \ (i = 1,2)$$

For $\alpha, \beta \in \mathfrak{G}_1$, we write $\alpha \sim \beta$ if there exists a $g \in G$

such that $\alpha^g = \beta$. Then the relation $\sim$ is a usual equivalent relation Let Δ be the totality of the equivalence classes not containing the null element o of $\mathfrak{O}_1$. If α_λ is an element of the class $\lambda \in \Delta$, then every element of λ can be written uniquely in the form α_λ^g $(g \in G)$.

Put $f^{(\lambda)}(g) = C_{\alpha_\lambda^g}$, $a_\lambda = (\sum_{g \in G} |f^{(\lambda)}(g)|^2)^{1/2}$, $b_\lambda =$

$\sup_{i=1,2} (\sum_{g \in G} |f^{(\lambda)}(gg_i) - f^{(\lambda)}(g)|^2)^{1/2}$; then $\sum_{\lambda \in \Delta} b_\lambda^2 =$

$$\sum_{\substack{\alpha \neq o \\ \alpha \in \mathfrak{O}_1}} |C_{\alpha^{g_i}} - C_\alpha|^2 + \sum_{\substack{\alpha \neq e \\ \alpha \in \mathfrak{O}_1}} |C_{\alpha^{g_i}} - C_\alpha|^2 < 2\varepsilon^2$$

and by Lemma 6.8, $a_\lambda \leqq K_2 b_\lambda$; hence $|\int |f(\xi)|^2 d\mu -$ $|\int f(\xi) d\mu|^2| = \sum_{\substack{\alpha \neq o \\ \alpha \in \mathfrak{O}_1}} |C_\alpha|^2 = \sum_{\lambda \in \Delta} a_\lambda^2 \leqq K_2^2 \sum_{\lambda \in \Delta} b_\lambda^2$ $< 2 K_2^2 \varepsilon^2$; therefore we have $|(\int |f(\xi)|^2 d\mu)^{1/2} -$ $|\int f(\xi) d\mu|| < K_3 \varepsilon$.

The proof of Theorem 6.5. Now let f_o be the function on Ω such that $f_o(\xi) = 1$. We denote by g_1, g_2 the generators of G. To prove the theorem, it is enough to show that in the notations in the (α) $||(\tilde{U}g_i - U \tilde{U}g_i U^*) J_{(o,e)} f_o|| <$ ε $(i = 1,2)$ for a unitary $U' \in \mathbb{M}$ and a sufficiently small ε $(> o)$ implies $|(UJ_{(o,e)} f_o, J_{(o,e)} f_o)| \geqq \frac{1}{2}$

Supposing $U = (\varphi_{st^{-1}} U_{st^{-1}})$, where $\varphi_t \in \mathfrak{A} = L^\infty(\Omega, \mu)$

and $s, t \in \mathfrak{g}$; put $(U)_t = \varphi_t(\xi)$, then $(U^*)_t = \overline{\varphi_{t^{-1}}(\xi\, t)}$;

therefore $(U^*\tilde{U}g_i)_t = \overline{\varphi_{g_i t^{-1}}(\xi\, t\, g_i^{-1})}$, $(\tilde{U}g_i U^*)_t = \overline{\varphi_{t^{-1}g_i}(\xi\, t)}$

and so $(U^*\tilde{U}g_i - \tilde{U}g_i U^*)_t = \overline{\varphi_{g_i t^{-1}}(\xi\, t\, g_i^{-1})} - \overline{\varphi_{t^{-1}g_i}(\xi\, t)}$; therefore

$$||\{\tilde{U}g_i - U\, \tilde{U}g_i U^*\}J_{(o,e)}f_o||^2 = ||\{U^*\tilde{U}g_i - \tilde{U}g_i U^*\}J_{(o,e)}f_o||^2$$

$$= \sum_{t\in\mathfrak{g}} \int |\varphi_{g_i t^{-1}}(\xi\, t g_i^{-1}) - \varphi_{t^{-1}g_i}(\xi\, t)|^2 |U_t f_o|^2 d\mu(\xi)$$

$$= \sum_{t\in\mathfrak{g}} \int |\varphi_{g_i t^{-1}}(\xi\, t g_i^{-1}) - \varphi_{t^{-1}g_i}(\xi\, t)|^2 \; \nu_t(\xi) d\mu(\xi)$$

$$= \sum_{t\in\mathfrak{g}} \int |\varphi_{g_i t^{-1}}(\xi\, g_i^{-1}) - \varphi_{t^{-1}g_i}(\xi)|^2 d\mu(\xi)$$

$$= \sum_{t\in\mathfrak{g}} \int |\varphi_{g_i t}(\xi\, g_i^{-1}) - \varphi_{t g_i}(\xi)|^2 d\mu(\xi)$$

$$= \sum_{t\in\mathfrak{g}} \int |\varphi_{g_i t g_i^{-1}}(\xi\, g_i^{-1}) - \varphi_t(\xi)|^2 d\mu(\xi).$$

Put $f(t) = \left(\int_\Omega |\varphi_t(\xi)|^2 d\mu(\xi)\right)^{1/2}$; since μ is invariant under G,

$$\left(\sum_{t\in\mathfrak{g}} |f(g_i t g_i^{-1}) - f(t)|^2\right)^{1/2} = \Big(\sum_{t\in\mathfrak{g}} \Big| \int |\varphi_{g_i t g_i^{-1}}(\xi)|^2 d\mu(\xi) -$$

$$\int_\Omega |\varphi_t(\xi)|^2 d\mu(\xi)\Big|^2\Big)^{1/2} = \Big(\sum_{t\in\mathfrak{g}} \Big| \int |\varphi_{g_i t g_i^{-1}}(\xi\, g_i^{-1})|^2 d\mu(\xi) -$$

$$\int_\Omega |\varphi_t(\xi)|^2 d\mu(\xi)|^2)^{1/2} \leq ||(U^*\tilde{U}g_i - \tilde{U}g_i U^*)J_{(o,e)}f_o|| < \mathcal{E}$$

We put $\mathcal{E} = \{(\alpha, g);\ g \neq e\}$ and $\mathcal{F} = \{(\alpha, g); g \in F\}$, where F is the subset of G defined in the proof of Lemma 6.8 ; then we can easily show that $\mathcal{F} \cup g_1 \mathcal{F} g_1^{-1} = \mathcal{E}$; and $\mathcal{F}, g_2 \mathcal{F} g_2^{-1}, g_2^{-1} \mathcal{F} g_2$ are mutually disjoint; therefore by Lemma 6.7 , $(\sum_{t \in \mathcal{E}} |f(t)|^2)^{1/2} < K_1 \mathcal{E}$.

Since $g_i \alpha g_i^{-1} = \alpha^{g_i^{-1}}$ $(\alpha \in \mathcal{O}_1 \subset \mathcal{O})$, by the analogous method in the proof of Lemma 6.9 , we have

$$(\sum_{\substack{\alpha \neq o \\ \alpha \in \mathcal{O}_1}} |f(\alpha)|^2)^{1/2} < K_3 \mathcal{E}$$

Since $1 = ||U^* J_{(o,e)} f_o||^2 = \sum_{t \in \mathcal{O}} \int |\varphi_t(\xi)|^2 d\mu(\xi) =$

$$\sum_{t \in \mathcal{O}} |f(t)|^2,$$

we have $(\int_\Omega |\varphi_e(\xi)|^2 d\mu(\xi))^{1/2} > 1 - K\mathcal{E}$, where K does not depend on $\mathcal{E}$.; moreover, $(\int_\Omega |\varphi_e(\xi g_i) - \varphi_e(\xi)|^2 d\mu))^{1/2}$ $= (\int_\Omega |\varphi_e(\xi) - \varphi_e(\xi g_i^{-1})|^2 d\mu(\xi))^{1/2} < \mathcal{E}$, so that by Lemma 6.9 , $|(\int |\varphi_e(\xi)|^2 d\mu(\xi))^{1/2} - |\int \varphi_e(\xi) d\mu(\xi)|| < K_3 \mathcal{E}$

and so

$$|(UJ_{(o,e)}f_o, J_{(o,e)}f_o)| = |\int \varphi_e(\xi) d\mu(\xi)| > 1 - K' \mathcal{E},$$

where K' does not depend on $\mathcal{E}$; therefore if $\mathcal{E} < \frac{1}{2K'}$, $|(UJ_{(o,e)}f_o, J_{(o,e)}f_o)| \geqq \frac{1}{2}$. This completes the proof

Finally we shall state an important question concerning examples. The question can simply be stated as follows:

Question 4. Is there a quite new construction of examples of factors? - that is, is there a new construction which is different from von Neumann's ones? The construction according to the quotient algebra (cf §7, chap II) is certainly a new one; however it can not give a new separable factor.

A concrete form of this question is as follows: Let $\mathfrak{H}$ be a hilbert space. Modifying the fundamental operations $\lambda f, f + g, (f,g)$ in $\mathfrak{H}$ by replacing them by $\bar{\lambda} f, f + g, \overline{(f,g)}$ Denote the set $\mathfrak{H}$ with the new definitions of its fundamental operations by $\mathfrak{H}_c$. Clearly $\mathfrak{H}_c$ is a hilbert space; every operator A in $\mathfrak{H}$ is also one in $\mathfrak{H}_c$ and every weakly closed *-subalgebra $\mathfrak{M}$ in $\mathfrak{H}$ is also one in $\mathfrak{H}_c$. But we shall denote the $A, \mathfrak{M}$ of $\mathfrak{H}$, when considered in $\mathfrak{H}_c$, by $A_c, \mathfrak{M}_c$.

Now consider $\mathfrak{M}$ in $\mathfrak{H}$. The identical mapping J^o then maps $\mathfrak{M}$ in $\mathfrak{H}$ on $\mathfrak{M}_c$ in $\mathfrak{H}_c$ and it is in this aspect a conjugate *-isomorphism of $\mathfrak{M}$ and $\mathfrak{M}_c$. Consider the mapping $J^*: A \longrightarrow A^*$ This maps $\mathfrak{M}$ in $\mathfrak{H}$ on $\mathfrak{M}$ in $\mathfrak{H}$ and it is a conjugate linear anti-*-isomorphism; therefore J^*J^o is a linear anti-*-isomorphism of $\mathfrak{M}$ and $\mathfrak{M}_c$.

In all examples of von Neumann (cf. (A) - (C) and (γ)), any specific non-real number is not mentioned; therefore we can always construct a conjugate linear *-automorphism on them, so that $\mathcal{M}$ and $\mathcal{M}_c$ are mutually * isomorphic.

Therefore we have the following question:

Question 5 Is there a factor $\mathcal{M}$ which is not *-isomorphic to $\mathcal{M}_c$?

If we can solve this question positively, it is the most fruitful solution to the question 4. Moreover, we shall give some remarks in excepting the appearance of next new examples.

Let G be the free group of two generators, G_i ($i = 1, 2, 3, \ldots$) be the groups which are isomorphic to G and $\mathcal{G}_n$ be the direct product group of $\{G_i | i = 1, 2, 3, \ldots, n\}$. Let $\mathbb{B}_n$ be the II_1-factor $\mathcal{U}(\mathbb{C})$ corresponding to the group $\mathcal{G}_n$, then clearly $\mathbb{B}_n = \mathbb{B} \bar{\otimes} \mathbb{B} \bar{\otimes} \ldots \bar{\otimes} \mathbb{B} = \overset{n}{\underset{i=1}{\bar{\otimes}}} \mathbb{B}$, where $\mathbb{B}$ is the II_1-factor in Theorem 6.3. Then,

Proposition 6.10. $\mathbb{B}_n$ for $n = 1, 2, \ldots$ has no the property L.

Proof. Let $g_{1,i}$, $g_{2,i}$ be two generators of G_i. We shall consider G_i as the subgroup of $\mathcal{G}_n$ for $i = 1, 2, \ldots n$ by the mapping $g_i \rightarrow (e_1, e_2, \ldots, g_i, \ldots, e_n)$ where e_i is the unit of G_i. For i, put $\mathcal{E}_i = \{(\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_n) | \alpha_i \neq e_i\}$, where $\alpha_j \in G_j (j = 1, 2, \ldots, n)$ and $\mathcal{F}_i = \{\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_n) | \alpha_i \in F_i\}$, where F_i is analogously defined with the set F in the proof of Lemma 6.8; then $\mathcal{F}_i \cup g_{1,i} \mathcal{F}_i g_{1,i}^{-1} = \mathcal{E}_i$ and $\mathcal{F}_i$, $g_{2i} \mathcal{F}_i g_{2i}^{-1}$, $g_{2,i}^{-1} \mathcal{F}_i g_{2,i}$ are mutually disjoint.

Now suppose that $\mathbb{B}_n$ has the property L, then there is a sequence of unitary elements (U_m) such that $\lim_m \varphi(U_m) = 0$

$$\lim_{m \to \infty} \|U_m^* U_{g_{k,i}} U_m - U_{g_{k,i}}\|_2 = 0$$

for $i = 1, 2, \ldots, n$; $k = 1, 2$, where φ is the unique trace of $\mathbb{B}_n$ such that $\varphi(1) = 1$ and $\|x\|_2 = \varphi(x^* x)^{1/2}$. Now put $U_m = U_{f_m}$ $(f_m \in L^2(\mathcal{G}_n))$

Then for $\varepsilon > 0$,

$$\|U_m^* U_{g_{k,i}} U_m - U_{g_{k,i}}\|_2$$

$$= \Big(\sum_{t \in \mathcal{G}_n} |f_m(g_{k,i}^{-1}\, t\, g_{k,i}) - f_m(t)|^2\Big)^{1/2} < \varepsilon$$

for $i = 1, 2, \ldots, n$; $m \geq m_0$.
$k = 1, 2$

Hence by Lemma 6.7

$$\Big(\sum_{t \in \mathcal{E}_i} |f_m(t)|^2\Big)^{1/2} < k_i \varepsilon$$

where k_i does not depend on ε ; therefore

$$\Big(\sum_{t \in \bigcup_{i=1}^{\infty} \mathcal{E}_i} |f_m(t)|^2\Big)^{1/2} \leq \sum_{i=1}^{n} \Big(\sum_{t \in \mathcal{E}_i} |f_m(t)|^2\Big)^{1/2} < \Big(\sum_{i=1}^{n} k_i\Big)\varepsilon.$$

Since $\mathcal{G}_n = (e) \cup \bigcup_{i=1}^{n} \mathcal{E}_i$, where e is the unit of $\mathcal{G}_n$,

$$|f_m(e)| \geq \|f_m\|_2 - \Big(\sum_{t \in \bigcup_{i=1}^{n} \mathcal{E}_i} |f_m(t)|^2\Big)^{1/2} > 1 - \Big(\sum_{i=1}^{n} k_i\Big)\varepsilon.$$

On the other hand,

$$f_m(e) = \varphi(U_{f_m}) \rightarrow 0$$

a contradiction. This completes the proof therefore the following question is important.

Question 6. If $m \neq n$, $\mathbb{B}_m \not\sim \mathbb{B}_n$?

Also we have the following question.

Question 7. Let M_1, M_2 be two II_1-factors. If M_1 and M_2 have no the property L, can we conclude that $M_1 \bar{\otimes} M_2$ has no the property L?

Notices of 6

To show the existence of different algebraical types in II_1-factors, Murray and von Neumann [18] defined the property Γ; Pukansky [43] replaced it by the property L which is available for any factor.

Pukansky [43] showed the existence of III-factors M_1 having the property L in the construction (C_γ); therefore, for the III-factor $\mathbb{M}$ in Theorem 6.5 and a continuous hyper finite factor $\mathbb{A}$, $\mathbb{M} \bar{\otimes} \mathbb{A}$ and the above M_1 are in a quite similar situation with the one of $\mathbb{A}$ and $\mathbb{A} \bar{\otimes} \mathbb{B}$, where $\mathbb{B}$ is the II_1-factor in Theorem 6.3; therefore, if we assume the result of Schwartz, it is almost certain that $\mathbb{M} \bar{\otimes} \mathbb{A} \not\sim M_1$ and so we have three examples of III-factors.

Moreover, let B_∞ be the restricted infinite direct product of II_1-factors $\{M_i \mid i = 1, 2, \ldots\}$, where $M_i \sim B$ (cf. Takeda, Tohohu Math. J.7 (1955) pp. 67-86, then we can write $\mathbb{B}_\infty = N \bar{\otimes} \mathbb{A}$, where N is a II_1-factor (cf. Nakamura, Tohohu Math. J. 6(1954) pp. 205-207); therefore B_∞ has the property L, and if $\mathbb{B}_\infty \sim \mathbb{A}$, $\mathbb{B} \bar{\otimes} \mathbb{A} \sim \mathbb{B} \bar{\otimes} \mathbb{B}_\infty \sim \mathbb{B}_\infty \sim \mathbb{A}$; therefore if we assume the result of Schwartz, $\mathbb{B}_\infty \not\sim \mathbb{A}$; hence $\mathbb{B}_\infty$ has the possibility of the fourth II_1-factor.

Concerning the construction (α), Suzuki (Tohohu Math J 11 (1959) pp. 113-124) showed the conditions in order that $\mathcal{B}$ is a II_1-factor, when $\mathfrak{A}$ is a II_1-factor.