SOME REMARKS ON APPROXIMATIVE COMPACTNESS

BY

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INTRODUCTION

The notion of approximative compactness has been introduced recently by N. V. Efimov and S. B. Stechkin [1]. Let E be a metric space and G a subset of E. A sequence $\{g_n\} \subset G$ is called a *minimizing sequence* for an $x \in E$, if *) $\lim_{n \to \infty} \rho(x, g_n) = \rho(x, G)$. The set G is called [1] **) *approximatively compact*, if for each $x \in E$ every minimizing sequence $\{g_n\} \subset G$ contains a subsequence converging to an element of G.

Let us also recall the following remarks of [1]: *a*) Every approximatively compact set G is an existence set (i.e. for each $x \in E$ there exists a $g_x \in G$ such that $\rho(x, g_x) = \rho(x, G)$) and hence closed. *b*) Every closed boundedly compact set is approximatively compact. Thus approximative compactness is an intermediate notion between closed boundedly compact sets and existence sets. The importance of the notion of approximative compactness is obvious from the fundamental result of [1], according to which a Chebyshev set G (i.e. a set G such that for each $x \in E$ there exists a unique $g_x \in G$ satisfying $\rho(x, g_x) = \rho(x, G)$) in a uniformly convex and smooth Banach space E is convex if and only if it is approximatively compact.

In the present paper we shall study some further properties of approximative compactness.

Recently V. Klee [5] has asked whether in a Hilbert space the metric projection onto a Chebyshev set G (i.e. the mapping which maps every $x \in E$ into the unique point $g_x \in G$ satisfying $\rho(x, g_x) = \rho(x, G)$) is continuous. The first continuity property of the metric projection in a general metric space has been given by M. Nicolescu ([8], theorem 2).

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*) We denote by ρ the distance in the metric space E.

**) Actually in [1] this definition is given only for $E =$ a real Banach space.

Various other continuity properties of the metric projection have been given by V. Klee ([5], propositions 2.3 and 2.4), Ky Fan and I. Glicksberg ([2], theorem 8). In § 1 of the present paper we shall give some semi-continuity properties of the set-valued mapping $P_G: x \rightarrow P_G(x) = \{g \in G \mid \rho(x, g) = \rho(x, G)\}$ in a general metric space. In the particular case when G is a Chebyshev set, the mapping P_G reduces to the metric projection onto G and our results will imply that the metric projection onto an approximatively compact Chebyshev set is continuous. This contains the known continuity properties mentioned above as well as some new results (which we shall give at the end of § 2).

In § 2 we shall be concerned with approximative compactness in Banach spaces. N. V. Efimov and S. B. Stechkin have given the following result ([1], lemma 1): In a uniformly convex Banach space every sequentially weakly closed set is approximatively compact. Let us call this property of uniformly convex spaces *the Efimov-Stechkin property*. In § 2 we shall give some characterizations of Banach spaces having the Efimov-Stechkin property. It will turn out that this is an intermediate class between Banach spaces satisfying the equivalent conditions given by Ky Fan and I. Glicksberg in paper [2] and reflexive Banach spaces. Finally, we shall give some characterizations of reflexivity for separable Banach spaces in terms of the Efimov-Stechkin property and of the Fan-Glicksberg conditions.

§ 1. APPROXIMATIVE COMPACTNESS AND THE MAPPING P_G IN GENERAL METRIC SPACES

For a metric space F we shall denote by 2^F the collection of all *closed nonvoid* subsets of F. Let us recall that a mapping $U: E \rightarrow 2^F$ is called (see e.g. [7]) *upper semi-continuous* respectively *lower semi-continuous*, if the set

$$\{x \in E \mid U(x) \subset M\}$$

is open for each open subset M of F, respectively if it is closed for each closed subset M of F. These conditions are obviously equivalent to the following: the set

$$\{x \in E \mid U(x) \cap N \neq \emptyset\}$$

is closed for each closed subset N of F, respectively it is open for each open subset N of F.

THEOREM 1. *Let E be a metric space and G an approximatively compact subset of E. Then P_G maps E into 2^G and it is upper semi-continuous.*

Proof. By remark *a*) of the Introduction, G is an existence set, whence P_G maps E into 2^G. Let N be an arbitrary closed subset of G. We shall prove that the set $B = \{x \in E \mid P_G(x) \cap N \neq \emptyset\}$ is closed. Let $\{x_n\}$ be a sequence in B, converging to an element $x \in E$. Since $\{x_n\} \subset B$, there exists a sequence $\{g_n\} \subset G$ such that $g_n \in P_G(x_n) \cap N$ $(n = 1, 2, \ldots)$. Then, by $g_n \in P_G(x_n)$ $(n = 1, 2, \ldots)$ we have

$$\rho(x, G) \leqslant \rho(x, g_n) \leqslant \rho(x, x_n) + \rho(x_n, g_n) =$$
$$= \rho(x, x_n) + \rho(x_n, G) \leqslant 2\rho(x, x_n) + \rho(x, G) \quad (n = 1, 2, \ldots) \tag{1}$$

whence, by $\lim_{n\to\infty} \rho(x, x_n) = 0$ we infer $\lim_{n\to\infty} \rho(x, g_n) = \rho(x, G)$, i.e. $\{g_n\}$ is a minimizing sequence for x. Hence, since G is approximatively compact, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$, converging to an element $g \in G$. By (1) we have then

$$\rho(x, G) \leqslant \rho(x, g) \leqslant \rho(x, g_{n_k}) + \rho(g_{n_k}, g) \leqslant$$
$$\leqslant 2\rho(x, x_{n_k}) + \rho(x, G) + \rho(g_{n_k}, g) \quad (k = 1, 2, \ldots),$$

whence, for $k \to \infty$,

$$\rho(x, g) = \rho(x, G),$$

i.e. $g \in P_G(x)$. On the other hand, since N is closed and since $\{g_{n_k}\} \subset N$, $\lim_{k\to\infty} g_{n_k} = g$, we also have $g \in N$. Consequently, $g \in P_G(x) \cap N$, whence $x \in B$, which completes the proof.

THEOREM 2. *Let E be a metric space and G an approximatively compact subset of E. Then*

1° $\lim_{n\to\infty} x_n = x$ *implies* $\lim_{n\to\infty} \rho(P_G(x_n), P_G(x)) = 0$.

2° $\lim_{n\to\infty} x_n = x$ *implies the existence of two sequences* $\{g_n\}, \{g'_n\} \subset G$, *with* $g_n \in P_G(x_n)$, $g'_n \in P_G(x)$ $(n = 1, 2, \ldots)$, *such that* $\lim_{n\to\infty} \rho(g_n, g'_n) = 0$.

3° $\lim_{n\to\infty} x_n = x$ *implies the existence, for each sequence* $\{g_n\} \subset G$ *with* $g_n \in P_G(x_n)$ $(n = 1, 2, \ldots)$, *of a sequence* $\{g'_n\} \subset P_G(x)$ *such that* $\lim_{n\to\infty} \rho(g_n, g'_n) = 0$.

4° $\lim_{n\to\infty} x_n = x$ *implies the existence, for each sequence* $\{g_n\} \subset G$ *with* $g_n \in P_G(x_n)$ $(n = 1, 2, \ldots)$, *of a subsequence* $\{g_{n_k}\} \subset \{g_n\}$ *and of an element* $g \in P_G(x)$ *such that* $\lim_{k\to\infty} g_{n_k} = g$.

Proof. The assertion 4° is implicitly contained in the above proof of theorem 1. On the other hand, obviously 3° ⇒ 2° ⇒ 1°. Thus it remains to prove 3°. Assume, a contrario, that for a sequence $\{g_n\} \subset G$ with $g_n \in P_G(x_n)$ $(n = 1, 2, \ldots)$ there exists no sequence $\{g'_n\} \subset P_G(x)$ satisfying $\lim_{n\to\infty} \rho(g_n, g'_n) = 0$. Then, taking $\{g'_n\} \subset P_G(x)$ such that $\rho(g_n, g'_n) \leqslant \leqslant \rho(g_n, P_G(x)) + \frac{1}{n}$ $(n = 1, 2, \ldots)$, it follows that $\overline{\lim_{n\to\infty}} \rho(g_n, P_G(x)) \neq 0$, i.e. that there exists an infinite subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and an $\varepsilon_0 > 0$ such that

$$\rho(g_{n_k}, P_G(x)) \geqslant \varepsilon_0 \quad (k = 1, 2, \ldots). \tag{2}$$

On the other hand, from 4° applied to $\{x_{n_k}\}$, $\{g_{n_k}\}$ it follows that there exists a subsequence $\{g_{n_{k_m}}\}$ of $\{g_{n_k}\}$ and an element $g \in P_G(x)$ such that $\lim_{m\to\infty} g_{n_{k_m}} = g$. Then

$$0 = \lim_{m\to\infty} \rho(g_{n_{k_m}}, g) \geqslant \overline{\lim_{m\to\infty}} \rho(g_{n_{k_m}}, P_G(x)),$$

which contradicts (2). This completes the proof.

Remark 1. One can also give the following direct proof of 3°. Assume that $\lim_{n\to\infty} x_n = x$ and that $\{g_n\} \subset G$, $g_n \in P_G(x_n)$ $(n = 1, 2, \ldots)$. Let

$$M_k(x) = \bigcup_{g \in P_G(x)} \operatorname{int} S\left(g, \frac{1}{k}\right) \qquad (k = 1, 2, \ldots), \tag{3}$$

where $\operatorname{int} S\left(g, \frac{1}{k}\right) = \left\{z \in E \mid \rho(z, g) < \frac{1}{k}\right\}$. Then $M_k(x)$ is open and we have $P_G(x) \subset M_k(x)$ $(k = 1, 2, \ldots)$. By theorem 1, the sets

$$D_k(x) = \{z \in E \mid P_G(z) \subset M_k(x)\} \qquad (k = 1, 2, \ldots)$$

are open. Consequently, since $x \in D_k(x)$ $(k = 1, 2, \ldots)$ and $\lim_{n\to\infty} x_n = x$, there exists a sequence of natural numbers $\{n_k\}$ such that

$$x_n \in D_k(x) \quad \text{for } n \geqslant n_k \quad (k = 1, 2, \ldots),$$

i.e. such that

$$P_G(x_n) \subset M_k(x) \quad \text{for } n \geqslant n_k \quad (k = 1, 2, \ldots).$$

Hence, by (3) and $g_n \in P_G(x_n)$ $(n = 1, 2, \ldots)$, there exist elements $g_n^{(k)} \in P_G(x)$ such that $g_n \in \operatorname{int} S\left(g_n^{(k)}, \frac{1}{k}\right)$ for $n \geqslant n_k$ $(k = 1, 2, \ldots)$, i.e. such that

$$\rho(g_n, g_n^{(k)}) < \frac{1}{k} \quad \text{for } n \geqslant n_k \quad (k = 1, 2, \ldots). \tag{4}$$

Now, let $g'_1, \ldots, g'_{n_1 - 1}$ be arbitrary elements of $P_G(x)$ and let

$$g'_m = g_m^{(k)} \quad (n_k \leqslant m < n_{k+1} - 1, \quad k = 1, 2, \ldots).$$

Then, by (4), we shall have $\lim_{n\to\infty} \rho(g_m, g'_m) = 0$, which completes the proof.

Remark 2. In the particular case when G is compact, 2° has been proved by M. Nicolescu ([8], theorem 2).

COROLLARY 1. *Let E be a metric space, G an approximatively compact subset of E and x an element of E such that $P_G(x)$ consists of exactly* *one element, say g. Then* $\lim_{n\to\infty} x_n = x$ *implies that* $\lim_{n\to\infty} g_n = g$ *for each sequence* $\{g_n\} \subset G$ *satisfying* $g_n \in P_G(x_n)$ $(n = 1, 2, \ldots)$.

In fact, this is a particular case of the assertion 3° of theorem 2.

COROLLARY 2. *Let E be a metric space and G an approximatively compact Chebyshev set in E. Then the metric projection onto G is continuous.*

Remark 3. a) In the particular case when G is a boundedly compact Chebyshev set, the continuity of the metric projection onto G was established by V. Klee ([5], proposition 2.3). b) In the particular case when E is a Banach space satisfying the equivalent conditions given by Ky Fan and I. Glicksberg in the paper [2] and G a closed convex subset of E, the set G is approximatively compact (this is a consequence of condition (E. 3) of [2]), whence corollary 2 above implies theorem 8 of [2]. (In its turn, this latter implies proposition 2.4 of [5]). c) We shall give some new consequences of corollary 2, concerning the continuity of metric projections onto Chebyshev sets in Banach spaces, in § 2, corollaries 4 and 5.

Remark 4. Let us mention that sometimes another notion of semi-continuity is also used. We shall call it (K)-semi-continuity, since it has been investigated by C. Kuratowski [6]. A mapping $U : E \to 2^F$ is called *upper* (K)*-semi-continuous*, respectively *lower* (K)*-semi-continuous*, if the relations $\lim_{n\to\infty} x_n = x$, $y_n \in U(x_n)$ $(n = 1, 2, \ldots)$, $\lim_{n\to\infty} y_n = y$ imply $y \in U(x)$, respectively if the relations $\lim_{n\to\infty} x_n = x$, $y \in U(x)$ imply the existence of a sequence $\{y_n\}$ with $y_n \in U(x_n)$ $(n = 1, 2, \ldots)$, such that $\lim_{n\to\infty} y_n = y$. It is known that *every upper (lower) semi-continuous mapping is upper (lower)* (K)*-semi-continuous, and the converse is true when F is compact* [6]. However, this latter is no more true if F is only approximatively compact. Now, let us consider the mapping P_G. From the proof of theorem 1 given above, it follows immediately that *if G is an arbitrary existence set in a metric space E, then the mapping* $P_G : E \to 2^G$ *is upper* (K)*-semi-continuous.* In the particular case when G is a finite-dimensional linear subspace of a Banach space E, this result has been proved by K. Tatarkiewicz [9]; actually, the method of proof of Tatarkiewicz remains also valid if G is an arbitrary existence set in a metric space E.

We conclude this paragraph with some examples showing that in certain respects the above results cannot be improved.

Example 1. If G is an existence set, but not approximatively compact, then the conclusions of theorems 1, 2 and corollary 1 are no more valid. In fact, let E be the closed half-space $\{x = \{\xi_n\} \mid \xi_1 \leqslant 1\}$ of the real Hilbert space l^2, endowed with the metric induced by l^2, and let G be the sequence $\{g_n\}_{n=0}^{\infty} \in E$ defined by

$$g_0 = 0,\ g_n = \left\{1, \frac{1}{n}, \underbrace{0, \ldots, 0}_{n-1}, 1, 0, \ldots\right\} \quad (n = 1, 2, \ldots). \tag{5}$$

Then G is an existence set (since for each $x \in E$, the sequence $\{\|x - g_n\|\}_{n=0}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} \|x - g_n\| \geqslant \| x - g_0\|$, whence $\rho(x,G) = \inf\limits_{0 \leqslant n < \infty} \| x - g_n\|$ is attained), but G is not approximatively compact (since $\{g_n\}_{n=0}^{\infty}$ is a minimizing sequence for $x_0 = \{1, 0, 0, \ldots\} \in E$, but it has no convergent subsequence, because of $\rho(g_i, g_j) \geqslant 1$ for $i \neq j$). Let N be the closed set $\left\{x \in E \,|\, \|x\| \geqslant \frac{1}{2}\right\}$. Then for the sequence $\{x_n\}_{n=0}^{\infty} \in E$ defined by

$$x_0 = \{1, 0, 0, \ldots\},\ x_n = \left\{1, \frac{1}{n}, 0, 0, \ldots\right\} \quad (n = 1, 2, \ldots) \tag{6}$$

we have $\lim\limits_{n\to\infty} x_n = x_0$, $P_G(x_n) \cap N \ni g_n$ $(n = 1, 2, \ldots)$ and $P_G(x_0) \cap N = \emptyset$, which shows that $\{x \in E \mid P_G(x) \cap N \neq \emptyset\}$ is not closed, i. e. that P_G *is not upper semi-continuous*. The same example also invalidates the conclusions of theorem 2 and corollary 1 for non-approximatively compact G.

Example 2. The continuity properties given in theorem 2 and corollaries, 1, 2, *are not uniform*. In fact, let E be the subset $\{x = \{\xi_1, \xi_2\} \mid 0 < |\xi_1| \leqslant 1,\ \xi_2 = 0\} \cup \{1, 1\} \cup \{-1, 1\}$ of the real Euclidean plane R^2, endowed with the metric induced by R^2, and let $G = \{1, 1\} \cup \{-1, 1\}$. Then G is a compact Chebyshev set, but for $x_n = \left\{\frac{1}{n}, 0\right\} \in E$, $y_n = \left\{-\frac{1}{n}, 0\right\} \in E$ $(n = 1, 2, \ldots)$ we have $\lim\limits_{n\to\infty} \rho(x_n, y_n) = 0$ and $\rho(P_G(x_n), P_G(y_n)) = 2$ $(n = 1, 2, \ldots)$.

Finally, let us mention that, in general, the mapping P_G is *not lower semi-continuous* since it need not even be lower (K)-semi-continuous (see remark 4), as shown by an example (with a compact G) of K. Tatarkiewicz [9].

§ 2. APPROXIMATIVE COMPACTNESS AND THE EFIMOV-STECHKIN PROPERTY IN BANACH SPACES

We shall say that a Banach space*) E has the *Efimov-Stechkin property* if every sequentially weakly closed set in E is approximatively compact (see the Introduction). The following theorem gives some characterizations of Banach spaces having the Efimov-Stechkin property.

THEOREM 3. *For a Banach space E the following conditions are equivalent :*

1° *E has the Efimov-Stechkin property.*

2° *Every weakly closed set in E is approximatively compact.*

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*) All Banach spaces considered in this paragraph are assumed to be real.

3° *Every closed convex set in* E *is approximatively compact.*

4° *Every closed hyperplane in* E *is approximatively compact.*

5° E *is reflexive and satisfies the following condition*: (Ω) *Whenever* $\{x_n\} \subset E$, $x \in E$, $\|x_n\| \to \|x\|$ *and* $\{x_n\}$ *is weakly convergent to* x, *then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\|x_{n_k} - x\| \to 0$.

Proof. The implications $1° \Rightarrow 2°$ and $3° \Rightarrow 4°$ are obvious, while $2° \Rightarrow 3°$ is a consequence of the fact that every closed convex set in E is weakly closed.

Assume now that we have 4°. In order to prove that E is reflexive it is sufficient, by a theorem of R. C. James [3], to prove that each $f \in E^*$ attains its supremum on the unit sphere of each separable closed linear subspace of E. Let E_0 be an arbitrary (not necessarily separable) closed linear subspace of E, and let

$$\|f\|_0 = \sup_{\substack{x \in E_0 \\ \|x\|=1}} f(x).$$

If $\|f\|_0 = 0$, then obviously f attains its supremum on the unit sphere of E_0. Assume now that $\|f\|_0 \neq 0$. Then there exists a sequence $\{x_n\} \subset E_0$ with $\|x_n\| = 1$ $(n = 1, 2, \ldots)$ such that $0 < f(x_n) \to \|f\|_0$. Take, by the Hahn-Banach theorem, a $\varphi \in E^*$ such that $\varphi(x) = f(x)$ for all $x \in E_0$ and that $\|\varphi\| = \|f\|_0$. Then for the closed hyperplane $H = \{z \in E \mid \varphi(z) = 1\}$ and for the sequence

$$y_n = \frac{1}{\varphi(x_n)} x_n = \frac{1}{f(x_n)} x_n \in E_0 \quad (n = 1, 2, \ldots)$$

we have $y_n \in H$ $(n = 1, 2, \ldots)$ and $\rho(0, y_n) = \|y_n\| = \dfrac{1}{|f(x_n)|} \to \dfrac{1}{\|f\|_0} = \dfrac{1}{\|\varphi\|} = \rho(0, H)$, i.e. $\{y_n\}$ is a minimizing sequence in H for the element $0 \in E$. Since, by 4°, H is approximatively compact, it follows that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to an element $y \in H$. Then $\|y_{n_k}\| \to \|y\|$. On the other hand, by the above, $\|y_{n_k}\| \to \dfrac{1}{\|f\|_0}$. Consequently, $\|y\| = \dfrac{1}{\|f\|_0}$, whence, since $y \in E_0 \cap H$, we obtain

$$f\left(\frac{1}{\|y\|} y\right) = \varphi\left(\frac{1}{\|y\|} y\right) = \frac{1}{\|y\|} = \|f\|_0,$$

i.e. f attains its supremum on the unit sphere of E_0. Thus E is reflexive.

Let us prove that E also satisfies condition (Ω). Assume that $\{x_n\} \subset E$, $x \in E$, $\|x_n\| \to \|x\|$ and that $\{x_n\}$ is weakly convergent to x. Choose $f \in E^*$ such that $\|f\| = 1$, $f(x) = \|x\|$. Then $f(x_n) \to f(x) = \|x\|$

and we may assume (omitting, if necessary, a finite number of the x_n) that $|f(x_n)| > 0$ $(n = 1, 2, \ldots)$. Let

$$y_n = \frac{1}{f(x_n)} x_n \quad (n = 1, 2, \ldots), \qquad y = \frac{1}{f(x)} x. \tag{8}$$

Then we have

$$f(y_n) = 1\,(n = 1, 2, \ldots),\ f(y) = 1,\ \|y_n\| \to \frac{\|x\|}{|f(x)|} = 1, \tag{9}$$

and $\{y_n\}$ is weakly convergent to y. Since for the closed hyperplane $H = \{z \in E \mid f(z) = 1\}$ we have $\rho(0, H) = \frac{1}{\|f\|} = 1$, it follows from (9) that $\{y_n\}$ is a minimizing sequence in H for 0. Hence, by 4°, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$, converging to an element $z \in H$. Since $\{y_{n_k}\}$ is also weakly convergent to y, we infer $z = y$, i.e. $y_{n_k} \to y$. Consequently,

$$x_{n_k} = f(x_{n_k})\, y_{n_k} \to f(x)\, y = x,$$

which proves that E satisfies condition (Ω). Thus $4° \Rightarrow 5°$.

Assume, finally, that we have 5°, and let G be an arbitrary sequentially weakly closed set in E. Furthermore, let x be an element of E and $\{g_n\} \subset G$ a minimizing sequence for x. Then $\lim\limits_{n\to\infty} \|x - g_n\| = \rho(x, G)$, whence $\{g_n\}$ is bounded. Since E is reflexive, there exists, by the theorem of Eberlein, a subsequence $\{g_{n_k}\}$ of $\{g_n\}$, converging weakly to an element $g \in E$. Since G is sequentially weakly closed, we have $g \in G$. Now, choose $f \in E^*$ such that $\|f\| = 1$, $f(x - g) = \|x - g\|$. Then we have

$$\|x - g\| = f(x - g) = \lim_{k\to\infty} f(x - g_{n_k}) \leqslant \lim_{k\to\infty} \|x - g_{n_k}\| =$$

$$= \rho(x, G) \leqslant \|x - g\|,$$

whence $\|x - g_{n_k}\| \to \|x - g\|$. Since $\{x - g_{n_k}\}$ is also weakly convergent to $x - g$, it follows from (Ω) that there exists a subsequence $\{x - g_{n_{k_m}}\}$ of $\{x - g_{n_k}\}$ such that $x - g_{n_{k_m}} \to x - g$, i.e. such that $g_{n_{k_m}} \to g$. This proves that G is approximatively compact. Thus $5° \Rightarrow 1°$, which completes the proof of theorem 3.

Remark 5. Every Banach space satisfying the equivalent conditions given by Ky Fan and I. Glicksberg in paper [2], has the Efimov-Stechkin property. In fact, this follows comparing theorem 3 with anyone of the following conditions of [2] *):

(E.3) *For each convex set G in E, every sequence $\{g_n\} \subset G$ satisfying $\lim\limits_{n\to\infty} \|g_n\| = \inf\limits_{g\in G} \|g\|$ is convergent.*

*) It is also interesting to transform the other conditions of [2] in order to obtain some more characterizations of Banach spaces having the Efimov-Stechkin property.

(E.4) *For each closed hyperplane H in E, every sequence $\{y_n\} \subset H$ satisfying $\lim_{n\to\infty} \|y_n\| = \inf_{y\in H} \|y\|$ is convergent.*

$(H) \wedge (R)$ *E is strictly convex and reflexive, and satisfies the following condition:*

(Ω_1) *Whenever $\{x_n\} \subset E$, $x \in E$, $\|x_n\| \to \|x\|$ and $\{x_n\}$ is weakly convergent to x, then $\|x_n - x\| \to 0$.*

Let us observe that the converse is not true, since for instance a non strictly convex finite dimensional Banach space has the Efimov-Stechkin property but does not satisfy the conditions of Ky Fan and I. Glicksberg. However, we have the following corollary of theorem 3:

COROLLARY 3. *For a Banach space E the following conditions are equivalent:*

1° *E satisfies the conditions of Ky Fan and I. Glicksberg given in* [2].

2° *E is strictly convex and has the Efimov-Stechkin property.*

Proof. The implication 1° ⇒ 2° has been remarked above. In order to prove the implication 2° ⇒ 1° it is sufficient, by 5° of theorem 3 and $(H) \wedge (R)$, to prove that *in an arbitrary Banach space E the conditions (Ω) and (Ω_1) are equivalent.* Since obviously $(\Omega_1) \Rightarrow (\Omega)$, we have to prove that $(\Omega) \Rightarrow (\Omega_1)$. Assume, a contrario, that E satisfies (Ω) but there exist $\{x_n\} \subset E$ and $x \in E$ with $\|x_n\| \to \|x\|$ and $\{x_n\}$ weakly converging to x, such that $\|x_n - x\| \not\to 0$. Then there exist $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\|x_{n_k} - x\| \geqslant \varepsilon_0 \qquad (k = 1, 2, \ldots). \tag{10}$$

However, since $\|x_{n_k}\| \to \|x\|$ and $\{x_{n_k}\}$ is weakly convergent to x, there exists, by (Ω), a subsequence $\{x_{n_{k_m}}\}$ of $\{x_{n_k}\}$ such that $\|x_{n_{k_m}} - x\| \to 0$, which contradicts (10). This completes the proof.

In order to complete the picture of the intermediate situation of Banach spaces having the Efimov-Stechkin property, between Banach spaces satisfying the conditions of Ky Fan and I. Glicksberg [2] and reflexive Banach spaces, let us also give the following example of a reflexive Banach space which does not have the Efimov-Stechkin property:

Example 3. Let

$$S' = \left\{ x = \{\xi_n\} \in l^2 \mid \|x\| \leqslant 1, |\xi_1| \leqslant \frac{1}{2} \right\}, \tag{11}$$

$$\|x\|_{S'} = \inf_{x \in \lambda S'} |\lambda| \qquad (x \in l^2). \tag{12}$$

Then $\|x\|_{S'}$ is a norm on l^2, equivalent to the l^2-norm $\|x\| = \sqrt{\sum_{i=1}^{\infty} |\xi_i|^2}$ $\left(\text{since } \|x\|_{S'} \geqslant \|x\| \geqslant \frac{1}{2}\|x\|_{S'} \text{ for all } x \in l^2\right)$. Let E be the space l^2 endowed with the norm $\|x\|_{S'}$. Then E is a reflexive Banach space (since E is isomorphic to l^2), but E does not have the Efimov-Stechkin property $\left(\text{since the closed hyperplane } H = \left\{x = \{\xi_n\} \in E \mid \xi_1 = \frac{1}{2}\right\} \text{ is not approximatively compact}\right)$.

However, we have the following characterizations of reflexivity for separable Banach spaces, in terms of the Efimov-Stechkin property and of the Fan-Glicksberg conditions :

THEOREM 4. *For a separable Banach space E the following conditions are equivalent :*

1° E is reflexive.

2° There exists an equivalent norm $|||x|||$ on E, such that E endowed with this new norm has the Efimov-Stechkin property.

3° There exists an equivalent norm $|||x|||$ on E, such that E endowed with this new norm satisfies the conditions of Ky Fan and I. Glicksberg given in [2].

Proof. The implications $3° \Rightarrow 2° \Rightarrow 1°$ are obvious consequences of corollary 3 and theorem 3. On the other hand, if E is reflexive, then there exists, by a result of M. I. Kadec ([4], theorem 2 and formula (3a)), an equivalent norm $|||x|||$ on E, such that E endowed with this new norm satisfies the condition $(H) \wedge (R)$. This completes the proof of theorem 4.

Finally, let us return to the problem of the continuity of metric projections onto Chebyshev sets (see the Introduction). From corollary 2 and the implication $5° \Rightarrow 1°$ of theorem 1 follows

COROLLARY 4. *In a reflexive Banach space satisfying condition (Ω)* *) *the metric projection onto a sequentially weakly closed Chebyshev set is continuous.*

From corollary 2 and theorem 4 follows

COROLLARY 5. *In every reflexive separable Banach space there exists an equivalent norm $|||x|||$ such that in this new norm the metric projection onto a sequentially weakly closed Chebyshev set is continuous.*

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*) Hence, in particular, in every Banach space satisfying the Fan-Glicksberg conditions.

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