ŞERBAN STRĂTILĂ • LÁSZLÓ ZSIDÓ

LECTURES ON VON NEUMANN ALGEBRAS

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Preface

This book is based on lectures delivered in July-August 1972, at the Suceava Summer School organized by the Institute of Mathematics of the Academy of the Socialist Republic of Romania, in cooperation with the Society of Mathematical Sciences.

The study of the algebras of operators in Hilbert spaces was initiated by F. J. Murray and J. von Neumann, in connection with some problems of theoretical physics. The wealth of the mathematical facts contained in their fundamental papers interested many mathematicians. This soon led to the crystallization of a new branch of mathematics: the theory of algebras of operators. The first systematic exposition of this theory appeared in the well-known monograph by J. Dixmier [26], *which was subtitled* Algèbres de von Neumann. *It expounded almost all the significant results achieved until its appearance. Afterwards, the theory continued to develop, for it had important applications in the theory of group representations, in mathematical physics and in other branches of mathematics. Of great importance were the results obtained by M. Tomita, who exhibited canonical forms for arbitrary von Neumann algebras. In recent times fine classifications and structure theorems have been obtained for von Neumann algebras especially by A. Connes.*

The present book contains what we consider to be the fundamental part of the theory of von Neumann algebras. The book also contains the essential elements of the spectral theory in Hilbert spaces. The material is divided into ten chapters; besides the basic text, each chapter has two complementary sections: exercises, comments and bibliographical comments. The book ends with a bibliography, which includes all the titles we know of, which deal with the theory of algebras of operators and some related fields.

The reader is supposed to know only some elementary facts from functional analysis.

In writing this book we made use of existing books and courses (J. Dixmier [26], [42], *I. Kaplansky* [22], *J. R. Ringrose* [3], [4], [5], *S. Sakai* [10], [32], *M. Takesaki* [17], [18], *D. M. Topping* [8]*), as well as many articles, some of them available only as preprints. Some of the exercises are borrowed from J. Dixmier's book* [26]. *For the bibliography we made much use of Israel Halperin's Operator Algebras Newsletter.*

Thanks are due to Grigore Arsene and Dan Voiculescu for the help given during the writing of this book, for the useful discussions and for the bibliographical information they gave us.

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The Authors

Contents

Introduction

In the study of operator algebras there are two main methods, the first is of an algebraic character, while the second is more analytic.

The algebraic method proceeds by a successive reduction of problems concerning the arbitrary operators to problems about positive operators and from these to problems about projections, where one can avail oneself of the lattice-theoretical geometry of projections. In this geometry the main notion is that of equivalence and the main result is the comparison theorem, an important technical device being the polar decomposition of operators. These methods are elementary, but they afford a clear classification of the von Neumann algebras into general types. The results obtained by these methods are presented in Chapter 4 and in the first sections of Chapter 7.

The analytic method, which is more complex and profound, consists of a systematic manipulation with linear forms defined on operator algebras; they may be bounded, or unbounded. Here the important facts are concentrated around certain results which extend the classical Lebesgue-Radon-Nikodym theorem, the main technical tool here being the polar decomposition of linear forms. The analytic methods permit the analysis of relations existing between the given algebra and its commutant, as well as of those which relate the predual of the given algebra to the Hilbert space in which this algebra is operating. In Chapter 6 the relations existing between the type of the given algebra and of its commutant are studied, whereas Chapters 7 and 8 exhibit the quantitative relations which measure the relative wealth of the given algebra and of its commutant. For finite von Neumann algebras the existence of a trace which measures the relative dimension of projections allows the evaluation of the quantitative relations between the given algebra and its commutant by a coupling function of a metric nature. In other, more general, cases, the coupling between the given algebra and its commutant can be measured only by projective objects, namely cardinals associated with central projections, but the information thus obtained is not always satisfactory.

The von Neumann algebras which are well equilibrated with their commutants are called standard von Neumann algebras, and the main result of Chapter 10 is that any von Neumann algebra is isomorphic to a standard von Neumann algebra in a canonical form. This has been known for a long time in the case of the semifinite von Neumann algebras; to be extended to the general case, it required a new technique namely a "polar decomposition" for the involution of the algebra. Chapter 10 is dedicated to the study of the canonical forms of the von Neumann algebras as well as to some applications to the theory of arbitrary von Neumann algebras.

The theory of operator algebras is based on two fundamental results: the density theorem of J.von Neumann and the density theorem of I. Kaplansky, both presented in Chapter 3.

The present book covers results contained in M. Takesaki's work [18], and, with the exception of the reduction theory and of the examples of factors included there, those of J. Dixmier's book [26].

The reduction theory aims at decomposing an arbitrary von Neumann algebra into a family of von Neumann algebras with trivial centers (the so-called factors), in such a manner that the algebra be obtainable from this family, whereas its properties will be derivable from those of the factors. In this manner, the reduction theory transfers to the factors the purely non-commutative part of the algebra, whereas the commutative part is reflected in the space of the indices of the family of factors; the main problem of the structure and classification of the von Neumann algebras is thus reduced to the corresponding problems for factors. For the reduction theory one can read J. Dixmier's book [26], as well as the expository article by L. Zsidó [3], based on the ideas of S. Sakai [11]. Both develop the classical reduction theory of J. von Neumann, but from seemingly different points of view, which can easily be shown to be similar. For factor theory we recommend the works of J. Dixmier [26], [52], S. Sakai [32], D. McDuff [3], H. Araki and E. J. Woods [3], A. Connes [15], [19], [21 — 24]. Important results concerning the structure of von Neumann algebras are contained in the works of A. Connes [6], [7], and M. Takesaki [29], [33].

Our exposition refers to the spatial theory of von Neumann algebras, which considers them as being subalgebras of the algebra of all bounded linear operators on a Hilbert space. S. Sakai obtained in [3] the abstract characterization of von Neumann algebras and developed the theory of von Neumann algebras by non-spatial methods. Thus, in S. Sakai's book [32] the reader will find some of the results we present here, with different proofs. Also, S. Sakai's book [32] contains some other results which are not included in the present book.

"Algebras of operators" usually designate something more general than von Neumann algebras, the so-called C^*-algebras. In our exposition we have only incidentally referred to the C^*-algebras, but this theory makes full use of the theory of von Neumann algebras. For this theory, as well as for its applications to the theory of group representations, we refer the reader to J. Dixmier's monograph [42].

Other topics connected with the theory of operator algebras, but not treated in the present book, are the following: the problem of the generation of von Neumann algebras (see D. M. Topping [8], T. Saitô [10]), non-commutative harmonic analysis and duality theory for locally compact groups (see P. Eymard [1],[1] M. Takesaki [23], M. Walter [2], [4], J. Ernest [5], [8]), non-commutative ergodic theory (see A. Guichardet [18]), applications to the theory of operators (see R. G. Douglas [3], [4], J. Ernest [9]), connections with some problems of theoretical physics (see D. Kastler [1], [3], G. E. Emch [2], D. Ruelle [4]).

Although rather a long time has elapsed since the publication of the works by F. J. Murray and J. von Neumann and their results are included in the books mentioned above, we consider that their works are still worth reading for those interested in the theory of operator algebras.

The present book is self-contained with complete proofs. The exercises contain results which enrich the text and which can be proved with the methods described in it; the more difficult exercises are marked by an asterisk, whereas some of the exercises which offer no difficulty are used in the main text and are marked by the symbol "!".

The final sections of each chapter include complements which contain bibliographical references, as well as the names of the mathematicians to whom the results contained in each chapter are to be ascribed.

The bibliography lists the works on operator algebras theory, as well as entries concerning the theory of group representations, mathematical physics and operator theory.

1

Topologies on spaces of operators

In this chapter we introduce the main topologies in the space $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a Hilbert space.

1.1. Lemma. Let $\mathscr{E}$ *be a vector space,* φ *a linear form on* $\mathscr{E}$ *and* $p_1, p_2, \ldots, p_n$ *semi-norms on* $\mathscr{E}$, *such that*

$$|\varphi(x)| \leqslant \sum_{k=1}^{n} p_k(x), \quad x \in \mathscr{E}.$$

Then there exist linear forms $\varphi_1, \ldots, \varphi_n$ *on* $\mathscr{E}$, *such that*

$$\varphi = \sum_{k=1}^{n} \varphi_k,$$

$$|\varphi_k(x)| \leqslant p_k(x), \quad x \in \mathscr{E}, \quad k = 1, \ldots, n.$$

Proof. Let $\mathscr{E}^n$ be the Cartesian product of n copies of $\mathscr{E}$, $\mathscr{D} \subset \mathscr{E}^n$ the diagonal of $\mathscr{E}^n$, p the semi-norm on $\mathscr{E}^n$ defined by

$$p(x_1, \ldots, x_n) = \sum_{k=1}^{n} p_k(x_k), \quad (x_1, \ldots, x_n) \in \mathscr{E}^n,$$

and $\tilde{\varphi}_0$ the linear form on $\mathscr{D}$ defined by

$$\tilde{\varphi}_0(x, \ldots, x) = \varphi(x), \quad x \in \mathscr{E}.$$

From the hypothesis we immediately infer that the linear form $\tilde{\varphi}_0$ on $\mathscr{D}$ is majorized on $\mathscr{D}$ by the semi-norm p. With the Hahn-Banach theorem we infer that there exists a linear form $\tilde{\varphi}$ on $\mathscr{E}^n$, having the following properties

$$\tilde{\varphi}(x, \ldots, x) = \tilde{\varphi}_0(x, \ldots, x), \; x \in \mathscr{E},$$

$$|\tilde{\varphi}(x_1, \ldots, x_n)| \leqslant p(x_1 \ldots, x_n), \quad (x_1, \ldots, x_n) \in \mathscr{E}^n.$$

We then define the forms φ_k by the relations

$$\varphi_k(x) = \varphi(0,\dots,0,x,0,\dots,0), \quad x \in \mathscr{E}, \quad k = 1,\dots,n,$$

where, in the right-hand member, x stands on the k-th place.

The linear forms thus defined satisfy the conditions of the statement.

Q.E.D.

1.2. Let $\mathscr{E}$ be a Banach space, $\mathscr{E}^*$ the dual of $\mathscr{E}$ and $\mathscr{F}$ a vector subspace of $\mathscr{E}^*$. We denote by $\sigma(\mathscr{E};\mathscr{F})$ the weak topology defined in $\mathscr{E}$ by the family $\mathscr{F}$ of linear forms; then the $\sigma(\mathscr{E};\mathscr{F})$-topology is defined by the family of semi-norms $\{p_\varphi;\ \varphi \in \mathscr{F}\}$, where

$$p_\varphi(x) = |\varphi(x)|, \quad x \in \mathscr{E}.$$

We consider the norm topology in $\mathscr{E}^*$ and we denote by $\overline{\mathscr{F}}$ the closure of $\mathscr{F}$ in this topology. We denote by $\mathscr{E}_1$ the closed unit ball in $\mathscr{E}$.

Lemma. *Let $\mathscr{E}$ be a Banach space, $\mathscr{F} \subset \mathscr{E}^*$ a vector subspace and φ a linear form on $\mathscr{E}$.*

(i) *φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous iff*) $\varphi \in \mathscr{F}$.*

(ii) *φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous on $\mathscr{E}_1$ iff $\varphi \in \overline{\mathscr{F}}$.*

(iii) *The topologies $\sigma(\mathscr{E};\mathscr{F})$ and $\sigma(\mathscr{E};\overline{\mathscr{F}})$ coincide on $\mathscr{E}_1$.*

(iv) *If $\mathscr{F}$ is closed in the norm topology and φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous on $\mathscr{E}_1$, then φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous on $\mathscr{E}$.*

Proof. (i) Obviously, if $\varphi \in \mathscr{F}$, then φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous. Conversely, if φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous, then there exist $\psi_1,\dots,\psi_n \in \mathscr{F}$, such that

$$|\varphi(x)| \leqslant \sum_{k=1}^{n} p_{\psi_k}(x), \quad x \in \mathscr{E}.$$

By virtue of Lemma 1.1, there exist linear forms $\varphi_1,\dots,\varphi_n$ on $\mathscr{E}$, such that

$$\varphi = \sum_{k=1}^{n} \varphi_k$$

$$|\varphi_k(x)| \leqslant p_{\psi_k}(x) = |\psi_k(x)|, \quad x \in \mathscr{E}, \quad k = 1,\dots,n.$$

If $\psi_k = 0$, then $\varphi_k = 0$. If $\psi_k \neq 0$, then there exists $x_k \in \mathscr{E}$, such that $\psi_k(x_k) = 1$ and, for any $x \in \mathscr{E}$, we have

$$|\varphi_k(x - \psi_k(x)x_k)| \leqslant |\psi_k(x - \psi_k(x)x_k)| = 0.$$

Consequently, we have

$$\varphi_k = \varphi_k(x_k)\psi_k \in \mathscr{F} \text{ and } \varphi = \sum_{k=1}^{n} \varphi_k \in \mathscr{F}.$$

*) 'Iff' stands for 'if and only if'.

(ii) It is easily seen that if $\varphi \in \overline{\mathscr{F}}$, then the restriction of φ to $\mathscr{E}_1$ is $\sigma(\mathscr{E};\mathscr{F})$-continuous. Conversely, let φ be a linear form on $\mathscr{E}$, whose restriction to $\mathscr{E}_1$ is $\sigma(\mathscr{E};\mathscr{F})$ continuous. Then φ is norm-continuous and, therefore $\varphi \in \mathscr{E}^*$. Let $\varepsilon > 0$ be an arbitrary positive real number. Since the restriction of φ to $\mathscr{E}_1$ is $\sigma(\mathscr{E};\mathscr{F})$-conitnuous at 0, we infer that there exist linear forms $\psi_1, \dots, \psi_n \in \mathscr{F}$ such that:

$$\|x\| \leqslant 1, \quad \sum_{k=1}^{n} p_{\psi_k}(x) < 1 \Rightarrow |\varphi(x)| < \varepsilon.$$

Hence we immediately infer that, for any $x \in \mathscr{E}$, we have

$$|\varphi(x)| \leqslant \varepsilon \|x\| + \|\varphi\| \sum_{k=1}^{n} p_{\psi_k}(x).$$

By virtue of Lemma 1.1, it follows that there exist linear forms φ_1, φ_2 on $\mathscr{E}$, such that

$$\varphi = \varphi_1 + \varphi_2,$$

$$|\varphi_1(x)| \leqslant \varepsilon\|x\|, \quad x \in \mathscr{E},$$

$$|\varphi_2(x)| \leqslant \|\varphi\| \sum_{k=1}^{n} p_{\psi_k}(x), \quad x \in \mathscr{E}.$$

Consequently, $\varphi_2 \in \mathscr{F}$ and $\|\varphi - \varphi_2\| = \|\varphi_1\| \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $\varphi \in \overline{\mathscr{F}}$.

Statements (iii), (iv) immediately follow from (i) and (ii).

Q.E.D.

1.3. Let $\mathscr{H}$ be a Hilbert space and $\mathscr{B}(\mathscr{H})$ the space of all bounded linear operators on $\mathscr{H}$. We consider $\mathscr{B}(\mathscr{H})$ as a Banach space only with respect to the usual operator norm:

$$\|x\| = \sup\{\|x\xi\|; \quad \xi \in \mathscr{H}, \quad \|\xi\| = 1\}.$$

For $\xi, \eta \in \mathscr{H}$ we define a linear form $\omega_{\xi,\eta}$ on $\mathscr{B}(\mathscr{H})$ by:

$$\omega_{\xi,\eta}(x) = (x\xi \mid \eta), \quad x \in \mathscr{B}(\mathscr{H}).$$

Obviously, $\omega_{\xi,\eta} \in \mathscr{B}(\mathscr{H})^*$ and it is easily checked that $\|\omega_{\xi,\eta}\| = \|\xi\| \cdot \|\eta\|$. The form $\omega_{\xi,\xi}$ will be simply denoted by ω_ξ.

Let $\mathscr{B}(\mathscr{H})_{\sim}$ be the vector space generated in $\mathscr{B}(\mathscr{H})^*$ by the forms $\omega_{\xi,\eta}$, $\xi, \eta \in \mathscr{H}$, whereas $\mathscr{B}(\mathscr{H})_*$ denotes the norm closure of $\mathscr{B}(\mathscr{H})_{\sim}$ in $\mathscr{B}(\mathscr{H})^*$.

Besides the *norm topology* we shall also consider the following topologies in $\mathscr{B}(\mathscr{H})$: the *weak operator topology*, or the *wo-topology*: it is the topology defined by the family of semi-norms

$$\mathscr{B}(\mathscr{H}) \ni x \mapsto |(x\xi \mid \eta)|, \quad \xi, \eta \in \mathscr{H};$$

in other words, it is just the $\sigma(\mathscr{B}(\mathscr{H}); \mathscr{B}(\mathscr{H})_{\sim})$-topology;

the *strong operator topology*, or the *so-topology*: it is the topology defined by the family of semi-norms:

$$\mathscr{B}(\mathscr{H}) \ni x \mapsto \|x\xi\|, \quad \xi \in \mathscr{H};$$

the *ultraweak operator topology*, or the *w-topology*: it is, by definition, the $\sigma(\mathscr{B}(\mathscr{H}); \mathscr{B}(\mathscr{H})_*)$-topology.

We now apply Lemma 1.2, where we make $\mathscr{E} = \mathscr{B}(\mathscr{H})$, $\mathscr{F} = \mathscr{B}(\mathscr{H})_\sim$ and $\bar{\mathscr{F}} = \mathscr{B}(\mathscr{H})_*$, and by taking into account the terminology just introduced, we get the following

Lemma. *Let $\mathscr{H}$ be a Hilbert space. Then:*

(i) *$\mathscr{B}(\mathscr{H})_\sim$ is the set of all wo-continuous linear forms on $\mathscr{B}(\mathscr{H})$.*

(ii) *$\mathscr{B}(\mathscr{H})_*$ is the set of all w-continuous linear forms on $\mathscr{B}(\mathscr{H})$.*

(iii) *A linear form φ on $\mathscr{B}(\mathscr{H})$ is w-continuous iff its restriction to $\mathscr{B}(\mathscr{H})_1$ is wo-continuous.*

(iv) *In $\mathscr{B}(\mathscr{H})_1$ the wo-topology and the w-topology coincide.*

1.4. Theorem. *A linear form φ on $\mathscr{B}(\mathscr{H})$ is wo-continuous iff it is so-continuous.*

Proof. It is easy to see that the *so*-topology is finer (stronger), than the *wo*-topology; therefore, any *wo*-continuous linear form is *so*-continuous. Conversely, if φ is *so*-continuous, then there exist non-zero vectors $\xi_1, \ldots, \xi_n \in \mathscr{H}$, such that

$$|\varphi(x)| \leqslant \sum_{k=1}^{n} \|x\xi_k\|, \quad x \in \mathscr{B}(\mathscr{H}).$$

From Lemma 1.1, there exist linear forms $\varphi_1, \ldots, \varphi_n$ on $\mathscr{B}(\mathscr{H})$, such that

$$\varphi = \sum_{k=1}^{n} \varphi_k$$

$$|\varphi_k(x)| \leqslant \|x\xi_k\|, \quad x \in \mathscr{B}(\mathscr{H}), \quad k = 1, \ldots, n.$$

Let $k \in \{1, \ldots, n\}$ be any fixed index. We obviously have $\mathscr{H} = \{x\xi_k;\, x \in \mathscr{B}(\mathscr{H})\}$. As a consequence of what we have already proved, the mapping

$$x\xi_k \mapsto \varphi_k(x)$$

is a bounded linear form on $\mathscr{H}$. With Riesz' theorem we infer that there exists $\eta_k \in \mathscr{H}$, such that $\varphi_k(x) = (x\xi_k|\eta_k)$, $x \in \mathscr{B}(\mathscr{H})$.

Consequently, for any k there exists an $\eta_k \in \mathscr{H}$, such that $\varphi_k = \omega_{\xi_k, \eta_k}$. Therefore

$$\varphi = \sum_{k=1}^{n} \varphi_k \in \mathscr{B}(\mathscr{H})_\sim,$$

i.e., φ is *wo*-continuous.

1.5. From Theorem 1.4 and from Mackey's theorem we infer the following:

Corollary. *A convex subset of $\mathscr{B}(\mathscr{H})$ is wo-closed if it is so-closed.*

1.6. From the Hahn-Banach theorem and from Theorem 1.4, we infer the following.

Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a vector subspace. A linear form on $\mathscr{M}$ is wo-continuous iff it is so-continuous.*

1.7. Lemma. *The Banach space $\mathscr{B}(\mathscr{H})$ is isomorphic to the dual of the Banach space $\mathscr{B}(\mathscr{H})_*$ by the mapping given by the canonical bilinear form*

$$\mathscr{B}(\mathscr{H}) \times \mathscr{B}(\mathscr{H})_* \ni (x, \varphi) \mapsto \varphi(x).$$

Proof. Let $x \in \mathscr{B}(\mathscr{H})$. By the formula

$$\Phi_x(\varphi) = \varphi(x), \quad \varphi \in \mathscr{B}(\mathscr{H})_*,$$

one defines a bounded linear form on $\mathscr{B}(\mathscr{H})_*$, such that $\|\Phi_x\| \leqslant \|x\|$. In fact, we have the equality $\|\Phi_x\| = \|x\|$, as can be inferred from the following computation:

$$\begin{aligned}
\|x\| &= \sup\{\|x\xi\|;\ \xi \in \mathscr{H},\ \|\xi\| = 1\} \\
&= \sup\{|(x\xi|\eta)|;\ \xi, \eta \in \mathscr{H},\ \|\xi\| = \|\eta\| = 1\} \\
&= \sup\{|\omega_{\xi,\eta}(x)|;\ \xi,\ \eta \in \mathscr{H},\ \|\xi\| = \|\eta\| = 1\} \\
&\leqslant \sup\{|\Phi_x(\omega_{\xi,\eta});\ \|\omega_{\xi,\eta}\| = 1\} \leqslant \|\Phi_x\|.
\end{aligned}$$

Conversely, let $\Phi \in (\mathscr{B}(\mathscr{H})_*)^*$. By the formula

$$\tilde{\varphi}(\xi, \eta) = \Phi(\omega_{\xi,\eta}), \quad \xi,\ \eta \in \mathscr{H},$$

we define a bounded sesquilinear form on $\mathscr{H}$. With the help of Riesz' theorem we get a uniquely determined operator $x \in \mathscr{B}(\mathscr{H})$, such that

$$\tilde{\varphi}(\xi, \eta) = (x\xi|\eta), \quad \xi, \eta \in \mathscr{H}.$$

It follows that $\Phi(\omega_{\xi,\eta}) = \Phi_x(\omega_{\xi,\eta})$, for any $\xi, \eta \in \mathscr{H}$. Consequently, Φ and Φ_x coincide on $\mathscr{B}(\mathscr{H})_\sim$. This shows that $\Phi = \Phi_x$.

Q.E.D.

1.8. Theorem. *For any Hilbert space $\mathscr{H}$ the closed unit ball $\mathscr{B}(\mathscr{H})_1$ of $\mathscr{B}(\mathscr{H})$ is wo-compact.*

Proof. According to Lemma 1.7, $\mathscr{B}(\mathscr{H}) = (\mathscr{B}(\mathscr{H})_*)^*$; from the Alaoglu theorem, it follows that $\mathscr{B}(\mathscr{H})_1$ is w-compact. By taking into account statement (iv) from Lemma 1.3, we infer that $\mathscr{B}(\mathscr{H})_1$ is *wo*-compact.

Q.E.D.

1.9. Lemma. *Let $\mathscr{B}$ be a Banach space and $\mathscr{B}_* \subset \mathscr{B}^*$ a norm closed vector subspace, such that $\mathscr{B} = (\mathscr{B}_*)^*$ through the canonical bilinear form on $\mathscr{B} \times \mathscr{B}_*$. Let*

$\mathscr{M} \subset \mathscr{B}$ be a $\sigma(\mathscr{B}; \mathscr{B}_*)$-closed vector subspace. We denote $\mathscr{M}_* = \{\varphi|_{\mathscr{M}}; \varphi \in \mathscr{B}_*\} \subset \mathscr{M}^*$. Then:

(i) *$\mathscr{M}_*$ is a norm closed vector subspace of $\mathscr{M}^*$;*

(ii) *for any $\psi \in \mathscr{M}_*$ and any $\varepsilon > 0$ there exists a $\varphi \in \mathscr{B}_*$, such that*

$$\psi = \varphi|_{\mathscr{M}}, \ \|\varphi\| \leqslant \|\psi\| + \varepsilon,$$

(iii) *$\mathscr{M} = (\mathscr{M}_*)^*$, through the canonical bilinear form on $\mathscr{M} \times \mathscr{M}_*$.*

Proof. (i) Let $\mathscr{M}^\circ = \{\varphi \in \mathscr{B}_*; \varphi|_{\mathscr{M}} = 0\}$ be the polar of $\mathscr{M}$ in $\mathscr{B}_*$. Since $\mathscr{M}$ is $\sigma(\mathscr{B}; \mathscr{B}_*)$-closed, from the bipolar theorem we infer that

$$\mathscr{M} = \mathscr{M}^{\circ\circ} = \{x \in \mathscr{B};\ \varphi(x) = 0 \text{ for any } \varphi \in \mathscr{M}^\circ\}.$$

The mapping $\mathscr{B}_* \ni \varphi \mapsto \varphi|_{\mathscr{M}} \in \mathscr{M}_*$ is linear, of norm $\leqslant 1$ and its kernel is equal to $\mathscr{M}^\circ$. Consequently, it induces a linear mapping

$$\mathscr{B}_*/\mathscr{M}^\circ \ni \varphi/\mathscr{M}^\circ \mapsto \varphi|_{\mathscr{M}} \in \mathscr{M}_*.$$

We shall show that this mapping is isometric. Let $\varphi_0 \in \mathscr{B}_*$ be such that $\|\varphi_0/\mathscr{M}^\circ\| = 1$, i.e., dist $(\varphi_0, \mathscr{M}^\circ) = 1$. From the Hahn-Banach theorem we infer that there exists a bounded linear form Φ on $\mathscr{B}_*$, such that $\|\Phi\| = \Phi(\varphi_0) = 1$ and $\Phi(\varphi) = 0$ for any $\varphi \in \mathscr{M}^\circ$. Since $\mathscr{B} = (\mathscr{B}_*)^*$, we infer that there exists an $x \in \mathscr{B}$, such that $\|x\| = \varphi_0(x) = 1$ and $\varphi(x) = 0$ for any $x \in \mathscr{M}^\circ$. It follows that $x \in \mathscr{M}^{\circ\circ} = \mathscr{M}$ and, therefore, $\|\varphi_0|_{\mathscr{M}}\| \geqslant \varphi_0(x) = 1$. Thus, the mapping $\mathscr{B}_*/\mathscr{M}^\circ \to \mathscr{M}_*$ we have just defined is an isometric isomorphism.

Since $\mathscr{B}_*/\mathscr{M}^\circ$ is complete, it follows that $\mathscr{M}_*$ is a complete subspace of $\mathscr{M}^*$ and, therefore, it is norm closed.

(ii) For any $\psi \in \mathscr{M}_*$ there exists, by virtue of what we have just proved, a $\varphi_0 \in \mathscr{B}_*$, such that $\psi = \varphi_0|_{\mathscr{M}}$ and $\|\psi\| = \|\varphi_0/\mathscr{M}^\circ\| = \text{dist}(\varphi_0, \mathscr{M}^\circ)$. Then, for any $\varepsilon > 0$ there exists a $\varphi_1 \in \mathscr{M}^\circ$ such that $\|\varphi_0 + \varphi_1\| \leqslant \|\psi\| + \varepsilon$. Let us define $\varphi = \varphi_0 + \varphi_1 \in \mathscr{B}_*$. Then $\varphi|_{\mathscr{M}} = \psi$ and $\|\varphi\| \leqslant \|\psi\| + \varepsilon$.

(iii) For any $x \in \mathscr{M}$ the mapping

$$\Psi_x\colon \mathscr{M}_* \ni \psi \mapsto \psi(x)$$

is a bounded linear form on $\mathscr{M}_*$. By taking into account (ii) and the canonical identification $\mathscr{B} = (\mathscr{B}_*)^*$, we get

$$\begin{aligned}\|\Psi_x\| &= \sup\{|\psi(x)|;\ \psi \in \mathscr{M}_*,\ \|\psi\| < 1\}\\ &= \sup\{|\varphi(x)|;\ \varphi \in \mathscr{B}_*,\ \|\varphi\| < 1\} = \|x\|.\end{aligned}$$

Conversely, let $\psi \in (\mathscr{M}_*)^*$. We define a linear form on $\mathscr{B}_*$ by the formula

$$\Phi(\varphi) = \Psi(\varphi|_{\mathscr{M}}), \quad \varphi \in \mathscr{B}_*.$$

Then $\Phi \in (\mathscr{B}_*)^*$ and $\Phi|_{\mathscr{M}^\circ} = 0$. Since $(\mathscr{B}_*)^* = \mathscr{B}$, there exists an $x \in \mathscr{B}$ such that $\Phi(\varphi) = \varphi(x)$, for any $\varphi \in \mathscr{B}_*$ and $\varphi(x) = 0$ for any $\varphi \in \mathscr{M}^\circ$. Consequently, $x \in \mathscr{M}^{\circ\circ} = \mathscr{M}$ and $\Psi(\varphi|_{\mathscr{M}}) = \varphi|_{\mathscr{M}}(x) = \Psi_x(\varphi|_{\mathscr{M}})$, for any $\varphi \in \mathscr{B}_*$, i.e., $\Psi = \Psi_x$.

Q.E.D.

1.10. Let $\mathscr{H}$ be a Hilbert space and $\mathscr{B}(\mathscr{H})$ the Banach space of all bounded linear operators on $\mathscr{H}$.

Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a w-closed vector subspace. We introduce the following notations:

$\mathscr{M}_\sim =$ the wo-dual of $\mathscr{M}$, i.e., the set of all the wo-continuous linear forms on $\mathscr{M}$. Obviously, $\mathscr{M}_\sim$ is a vector subspace of $\mathscr{M}^*$.

$\mathscr{M}_* =$ the w-dual of $\mathscr{M}$, i.e., the set of all w-continuous linear forms on $\mathscr{M}$. Obviously, $\mathscr{M}_*$ is a vector subspace of $\mathscr{M}^*$.

Theorem. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a w-closed vector subspace. Then the following statements are true:*

(i) $\mathscr{M}_\sim = \{\varphi|_{\mathscr{M}};\ \varphi \in \mathscr{B}(\mathscr{H})_\sim\}$;

(ii) $\mathscr{M}_* = \{\varphi|_{\mathscr{M}};\ \varphi \in \mathscr{B}(\mathscr{H})_*\}$;

(iii) $\mathscr{M}_* = \overline{\mathscr{M}}_\sim$, *i.e., $\mathscr{M}_*$ is equal to the norm closure of $\mathscr{M}_\sim$ in $\mathscr{M}^*$;*

(iv) $\mathscr{M} = (\mathscr{M}_*)^*$, *i.e., $\mathscr{M}$ is identified, as a normed space, to the dual of $\mathscr{M}_*$ through the canonical bilinear form on $\mathscr{M} \times \mathscr{M}_*$;*

(v) *for any $\psi \in \mathscr{M}_*$ and any $\varepsilon > 0$ there exists a $\varphi \in \mathscr{B}(\mathscr{H})_*$, such that:*

$$\varphi|_{\mathscr{M}} = \psi, \quad \|\varphi\| \leqslant \|\psi\| + \varepsilon;$$

(vi) *if ψ is a linear form on $\mathscr{M}$, then $\psi \in \mathscr{M}_*$ iff the restriction of ψ to the closed unit ball $\mathscr{M}_1$ of $\mathscr{M}$ is wo-continuous.*

Proof. The statements (i), (ii) follow from Lemma 1.3 (i), (ii), with the help of the Hahn-Banach theorem.

The statements (iv), (v), as well as the fact that $\mathscr{M}_*$ is a closed subset of $\mathscr{M}^*$, follow from Lemma 1.9, by taking into account the statement (ii) from above and Lemma 1.7.

By virtue of statements (i), (ii) from above, the bounded linear mapping $\varphi \mapsto \varphi|_{\mathscr{M}}$ maps $\mathscr{B}(\mathscr{H})_\sim$ on $\mathscr{M}_\sim$ and $\mathscr{B}(\mathscr{H})_*$ on $\mathscr{M}_*$. Since by Definition 1.3, $\mathscr{B}(\mathscr{H})_\sim$ is uniformly dense in $\mathscr{B}(\mathscr{H})_*$, it follows that $\mathscr{M}_\sim$ is uniformly dense in $\mathscr{M}_*$. Thus, statement (iii) is proved.

From statements (i), (ii) from above, it follows that the topology induced on $\mathscr{M}$ by the wo-topology (resp., the w-topology) of $\mathscr{B}(\mathscr{H})$ is the $\sigma(\mathscr{M};\mathscr{M}_\sim)$-topology (resp., the $\sigma(\mathscr{M};\mathscr{M}_*)$-topology). By virtue of statement (iii), $\mathscr{M}_* = \overline{\mathscr{M}}_\sim$. Consequently, statement (vi) follows from Lemma 1.2 (ii).

Q.E.D.

1.11. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a w-closed vector subspace. The norm-closed vector subspace $\mathscr{M}_*$ of the *dual* $\mathscr{M}^*$ of $\mathscr{M}$ is called the *predual* of $\mathscr{M}$. This term is justified by statement (iv) from Theorem 1.10.

Exercises

In the exercises that follow some elementary notions about bounded linear operators on Hilbert spaces are assumed, although these are expounded in Chapter 2.

*E.1.1. Let $\{x_i\} \subset \mathscr{B}(\mathscr{H})$ be a directed set. Then

$$x_i \xrightarrow{so} 0 \Leftrightarrow x_i^* x_i \xrightarrow{wo} 0.$$

*E.1.2. Let φ be a wo-continuous linear form on $\mathscr{B}(\mathscr{H})$. Then there exist two families of mutually orthogonal vectors $\{\xi_1, \ldots, \xi_n\}$, $\{\eta_1, \ldots, \eta_n\} \subset \mathscr{H}$, such that

$$\varphi = \sum_{k=1}^{n} \omega_{\xi_k, \eta_k},$$

$$\|\varphi\| = \sum_{k=1}^{n} \|\xi_k\| \, \|\eta_k\|.$$

E.1.3. With the help of E.1.2, show that for any w-continuous linear form φ on $\mathscr{B}(\mathscr{H})$, there exist two sequences $\{\xi_k\}$, $\{\eta_k\} \subset \mathscr{H}$, such that $\sum\limits_{k=1}^{\infty} \|\xi_k\|^2 < +\infty$, $\sum\limits_{k=1}^{\infty} \|\eta_k\|^2 < +\infty$, and:

$$\varphi = \sum_{k=1}^{\infty} \omega_{\xi_k, \eta_k}.$$

In particular, the w-topology in $\mathscr{B}(\mathscr{H})$ is defined by the system of semi-norms

$$x \mapsto \left| \sum_{k=1}^{\infty} (x\xi_k | \eta_k) \right|,$$

where $\{\xi_k\}_k, \{\eta_k\}_k \subset \mathscr{H}$, $\sum\limits_{k=1}^{\infty} \|\xi_k\|^2 < +\infty$, $\sum\limits_{k=1}^{\infty} \|\eta_k\|^2 < +\infty$.

E.1.4. The *ultrastrong* topology on $\mathscr{B}(\mathscr{H})$ is defined by the system of semi-norms

$$x \to \left(\sum_{k=1}^{\infty} \|x\xi_k\|^2 \right)^{1/2},$$

where $\{\xi_k\}_k \subset \mathscr{H}$, $\sum\limits_{k=1}^{\infty} \|\xi_k\|^2 < +\infty$.

Show that in the closed unit ball of $\mathscr{B}(\mathscr{H})$ the ultrastrong topology coincides with the strong topology.

Prove the following relations between the indicated topologies:

$$\begin{array}{ccccc}
\text{weak topology} & \leqslant & \text{strong topology} & & \\
\wedge & & \wedge & & \\
\text{ultraweak topology} & \leqslant & \text{ultrastrong topology} & \leqslant & \text{norm topology}
\end{array}$$

!E.1.5. Show that the $*$-mapping

$$\mathscr{B}(\mathscr{H}) \ni x \mapsto x^* \in \mathscr{B}(\mathscr{H})$$

Is weakly, ultraweakly and norm-continuous.

E.1.6. Let $\mathscr{H}$ be an infinitely dimensional Hilbert space and $\{\xi_n\}$ an orthonormal sequence in $\mathscr{H}$. One defines the following operators:

$$v_n: \ \xi \mapsto (\xi \mid \xi_n)\xi_1, \quad n = 1, 2, \dots$$

Show that $v_n \to 0$ in the ultrastrong topology but, for any n, one has $\|v_n^* \xi_1\| = 1$. Infer that:

(i) the $*$-operator is not strongly (resp., ultrastrongly) continuous on $\mathscr{B}(\mathscr{H})$.

(ii) the strong (resp., ultrastrong) topology is strictly finer (i.e. stronger) than the weak (resp., ultraweak) topology in $\mathscr{B}(\mathscr{H})$.

(iii) the norm topology is strictly finer than the ultrastrong topology in $\mathscr{B}(\mathscr{H})$.

!E.1.7. Show that, for any $a \in \mathscr{B}(\mathscr{H})$, the mappings

$$\mathscr{B}(\mathscr{H}) \ni x \mapsto ax \in \mathscr{B}(\mathscr{H})$$

$$\mathscr{B}(\mathscr{H}) \ni x \mapsto xa \in \mathscr{B}(\mathscr{H})$$

are weakly, strongly, ultraweakly and ultrastrongly continuous.

!E.1.8. Which of the following mappings

$$\mathscr{B}(\mathscr{H})_1 \times \mathscr{B}(\mathscr{H}) \ni (x, y) \mapsto xy \in \mathscr{B}(\mathscr{H})$$

$$\mathscr{B}(\mathscr{H}) \times \mathscr{B}(\mathscr{H})_1 \ni (x, y) \mapsto xy \in \mathscr{B}(\mathscr{H})$$

is strongly and ultrastrongly continuous ?

E.1.9. Let $\mathscr{V}$ be an ultrastrong neighbourhood of $0 \in \mathscr{B}(\mathscr{H})$. With the help of E.1.6 (iii), show that there exists a $\xi \in \mathscr{H}$ such that

$$\sup \{\|x\xi\|;\ x \in \mathscr{V}\} = +\infty.$$

Infer from this that, by endowing the space $\mathscr{B}(\mathscr{H}) \times \mathscr{B}(\mathscr{H})$ with the product of the ultrastrong topologies, and $\mathscr{B}(\mathscr{H})$ with the weak topology, the mapping

$$\mathscr{B}(\mathscr{H}) \times \mathscr{B}(\mathscr{H}) \ni (x, y) \mapsto xy \in \mathscr{B}(\mathscr{H})$$

is not continuous.

E.1.10. Let $\mathscr{H}$ be an infinitely dimensional Hilbert space and $\{\xi_n\}$ an orthonormal sequence in $\mathscr{H}$. We define the operators

$$e_n: \ \xi \to (\xi \mid \xi_n)\xi_n, \quad n = 1, 2, \dots$$

$$x_{n,m} = e_n + ne_m, \quad m, n = 1, 2, \dots$$

Show that 0 is ultrastrongly adherent to the set $\{x_{n,m}\}_{n,m=1,2,\dots}$ but no sequence in this set converges weakly to 0. Infer from this fact that $\mathscr{B}(\mathscr{H})$ is not metrizable with respect to any of the topologies—weak, strong, ultraweak, ultrastrong.

Comments

C.1.1. If $\mathscr{X}$ is a Banach space and $\mathscr{X}^*$ is its dual, then for any $\lambda > 0$ we write $\mathscr{X}^*_\lambda = \{\varphi \in \mathscr{X}^*;\ \|\varphi\| \leqslant \lambda\}$.

Theorem. (*Krein-Šmulian*). *Let $\mathscr{X}$ be a Banach space and let $\mathscr{K} \subset \mathscr{X}^*$ be a convex subset. Then $\mathscr{K}$ is $\sigma(\mathscr{X}^*; \mathscr{X})$-closed iff for any $\lambda > 0$, the set $\mathscr{K} \cap \mathscr{X}^*_\lambda$ is $\sigma(\mathscr{X}^*; \mathscr{X})$-closed.*

In $\mathscr{X}^*$ one defines the *$b\sigma(\mathscr{X}^*; \mathscr{X})$-topology* as being the finest *topology* in $\mathscr{X}^*$ which coincides with the $\sigma(\mathscr{X}^*; \mathscr{X})$-topology in $\mathscr{X}^*_\lambda$, $\lambda > 0$. One shows that the $b\sigma(\mathscr{X}^*; \mathscr{X})$-topology is a *locally convex* topology (namely, the topology of uniform convergence on the sequences from $\mathscr{X}$ which converge to 0). This is the main fact needed in the proof of the Krein-Šmulian theorem. Indeed, this fact once established, the theorem easily follows by taking into account Lemma 1.2 (iv) (one takes $\mathscr{F} = \bar{\mathscr{F}} = \mathscr{X}$ and $\mathscr{E} = \mathscr{X}^*$) and Mackey's theorem.

For the full proof of the Krein-Šmulian theorem we refer the reader to N. Dunford and J. Schwartz [1], V.5.7.

By taking into account Theorem 1.10, from the Krein-Šmulian theorem the following results can be obtained:

Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a w-closed vector subspace. A convex subset $\mathscr{K} \subset \mathscr{M}$ is w-closed iff, for any $\lambda > 0$, the set $\mathscr{K} \cap \mathscr{M}_\lambda$ is w-closed.*

Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a vector subspace. Then $\mathscr{M}$ is w-closed iff $\mathscr{M}_1$ is wo-compact.*

If $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a $*$-algebra, the result stated in this corollary will be proved by another method in 3.11.

C.1.2. With the help of the Tikhonov theorem one can easily prove the following general result of "weak" compacity, proved for the first time by R. V. Kadison:

Let $\mathscr{X}$, $\mathscr{X}'$ be vector spaces, $\mathscr{F}$ a set of linear forms on $\mathscr{X}'$, which separate the elements of $\mathscr{X}'$ and $\mathscr{L}(\mathscr{X}, \mathscr{X}')$ the vector space of all linear mappings from $\mathscr{X}$ into $\mathscr{X}'$, endowed with the topology σ, generated by the sets

$$\mathscr{Q}(x, \mathscr{V}') = \{A \in \mathscr{L}(\mathscr{X}, \mathscr{X}');\ Ax \in \mathscr{V}'\},$$

where x runs over $\mathscr{X}$, whereas $\mathscr{V}'$ runs over the set of all $\sigma(\mathscr{X}'; \mathscr{F})$-open subsets of $\mathscr{X}'$.

Theorem. *If the subset $\mathscr{S} \subset \mathscr{X}$ linearly generates $\mathscr{X}$, whereas $\mathscr{S}' \subset \mathscr{X}'$ is a $\sigma(\mathscr{X}'; \mathscr{F})$-compact subset, then the set*

$$\mathfrak{S} = \{A \in \mathscr{L}(\mathscr{X}, \mathscr{X}');\ A\mathscr{S} \subset \mathscr{S}'\}$$

is σ-compact.

Corollary. *If $(\mathscr{S}_\iota)_{\iota \in I}$ is a family of subsets of $\mathscr{S}$, whereas $(\mathscr{S}'_\iota)_{\iota \in I}$ is a family of $\sigma(\mathscr{X}', \mathscr{F})$-compact subsets of $\mathscr{S}'$, then the set*

$$\mathfrak{S}_0 = \{A \in \mathfrak{S}; A\mathscr{S}_\iota \subset \mathscr{S}'_\iota,\ \iota \in I\}$$

is σ-compact.

Alaoglu's theorem as well as Theorem 1.8 are particular cases of this result.

For the proof of the theorem and for other applications we refer the reader to R. V. Kadison [17] (see also Gr. Arsene [2]).

C.1.3. Bibliographical comments. The reader can easily find the few general facts from functional analysis needed in the treatise by N. Dunford and J. Schwartz [1], Ch. II, V. The preceding exposition follows that of J. R. Ringrose [5].

2

Bounded linear operators in Hilbert spaces

This chapter contains the basic facts about bounded linear operators in Hilbert spaces, necessary in order to develop the elementary part of the theory of von Neumann algebras.

2.1. In the first chapter we considered only the Banach space structure of $\mathscr{B}(\mathscr{H})$. By taking into consideration the multiplication of the operators, $\mathscr{B}(\mathscr{H})$ becomes a Banach algebra, i.e., for any $x, y \in \mathscr{B}(\mathscr{H})$ we have

$$\|xy\| \leqslant \|x\| \, \|y\|.$$

For any $x \in \mathscr{B}(\mathscr{H})$, the relations

$$(x\xi \mid \eta) = (\xi \mid x^*\eta), \quad \xi, \eta \in \mathscr{H},$$

determine an operator $x^* \in \mathscr{B}(\mathscr{H})$, called the *adjoint* of x. The mapping

$$\mathscr{B}(\mathscr{H}) \ni x \mapsto x^* \in \mathscr{B}(\mathscr{H})$$

is called the *canonical involution* on $\mathscr{B}(\mathscr{H})$, or the *$*$-operation* on $\mathscr{B}(\mathscr{H})$. Thus, $\mathscr{B}(\mathscr{H})$ becomes, in a canonical manner, an *involutive algebra*, or a *$*$-algebra*, i.e., for any $x, y \in \mathscr{B}(\mathscr{H})$, $\lambda \in \mathbb{C}$, we have

$$(x + y)^* = x^* + y^*,$$
$$(\lambda x)^* = \bar{\lambda} x^*,$$
$$(xy)^* = y^* x^*,$$
$$x^{**} = x.$$

We shall call a *$*$-algebra of operators* any $*$-subalgebra of $\mathscr{B}(\mathscr{H})$. The notions of $*$-homomorphism and $*$-isomorphism between $*$-algebras of operators are now obvious.

The connection existing between the norm, the multiplication and the $*$-operation in $\mathscr{B}(\mathscr{H})$ is expressed by the following

Lemma. *For any $x \in \mathscr{B}(\mathscr{H})$ we have the equality*

$$\|x^* x\| = \|x\|^2$$

Proof. We have

$$\|x^*\| = \sup_{\|\xi\|=1} \|x^*\xi\| = \sup_{\|\xi\|=1} \sup_{\|\eta\|=1} |(x^*\xi|\eta)| = \sup_{\|\xi\|=1} \sup_{\|\eta\|=1} |(\xi|x\eta)| \leqslant \|x\|,$$

and this shows that

$$\|x\|^2 = \sup_{\|\xi\|=1} \|x\xi\|^2 = \sup_{\|\xi\|=1} (x\xi|x\xi) = \sup_{\|\xi\|=1} (x^*x\xi\,|\,\xi) \leqslant \|x^*x\| \leqslant \|x\|^2.$$

Q.E.D.

In the preceding proof we have also obtained the equality $\|x^*\| = \|x\|, x \in \mathscr{B}(\mathscr{H})$.

2.2. Any Banach algebra $\mathscr{A}$, which is a $*$-algebra and in which the equality $\|x^*x\| = \|x\|^2$ holds for any $x \in \mathscr{A}$, is called a *C^*-algebra*. Lemma 2.2. shows that $\mathscr{B}(\mathscr{H})$ is a C^*-algebra in a canonical manner. Moreover, any $*$-algebra of operators, which is also closed for the norm topology, is a C^*-algebra; these algebras will be called *C^*-algebras of operators* (or *concrete C^*-algebras*).

For any subset $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$ there exists a smallest C^*-algebra of operators, which contains $\mathscr{X}$. This one will be called the *C^*-algebra generated* by $\mathscr{X}$ and it will be denoted by $\mathscr{C}^*(\mathscr{X})$.

A special class of C^*-algebras of operators, which is the subject of the present book, is the class of von Neumann algebras. Any $*$-algebra of operators $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$, which contains the identity operator $1_{\mathscr{H}}$ and which is *so*-closed (or, equivalently, *wo*-closed (see 1.5.)) is called a *von Neumann algebra*.

For any subset $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$ there exists a smallest von Neumann algebra which contains $\mathscr{X}$; it will be called *the von Neumann algebra generated by $\mathscr{X}$*, and it will be denoted by $\mathscr{R}(\mathscr{X})$.

In what follows, we shall be concerned only with C^*-algebras of operators, which will be called, simply, C^*-algebras.

2.3. For any $x \in \mathscr{B}(\mathscr{H})$ its *resolvent set* is defined by

$$\rho(x) = \{\lambda \in \mathbb{C};\ \lambda - x \text{ is invertible}\};$$

its complement $\sigma(x) = \mathbb{C} \setminus \rho(x)$ is the *spectrum* of x.

It is easy to check that if $\lambda_0 \in \rho(x)$, then

$$\{\lambda \in \mathbb{C}; |\lambda - \lambda_0| < \|(\lambda_0 - x)^{-1}\|^{-1}\} \subset \rho(x)$$

and, for any λ, such that $|\lambda - \lambda_0| < \|(\lambda_0 - x)^{-1}\|^{-1}$, we have

$$(\lambda - x)^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 - x)^{-n-1}.$$

In particular, $\rho(x)$ is an open subset of $\mathbb{C}$ and the function

$$\rho(x) \ni \lambda \mapsto (\lambda - x)^{-1} \in \mathscr{B}(\mathscr{H})$$

is analytic for the norm topology of $\mathscr{B}(\mathscr{H})$.

On the other hand, we have

$$\{\lambda \in \mathbb{C};\, |\lambda| > \|x\|\} \subset \rho(x)$$

and, for any λ, such that $|\lambda| > \|x\|$, we have

$$(\lambda - x)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} x^n.$$

In particular, we have

$$\sigma(x) \subset \{\lambda \in \mathbb{C};\, |\lambda| \leqslant \|x\|\},$$

and this shows that $\sigma(x)$ is a compact subset of $\mathbb{C}$.

Lemma. *For any $x \in \mathscr{B}(\mathscr{H})$, the spectrum $\rho(x)$ is non-empty, the sequence $(\|x^n\|^{1/n})_{n \geqslant 1}$ converges and*

$$\lim_{n\to\infty} \|x^n\|^{1/n} = \sup\{|\lambda|;\, \lambda \in \sigma(x)\}.$$

Proof. Let $\alpha(x) = \inf_{n>0} \|x^n\|^{1/n}$. It is easy to check that if $x, y \in \mathscr{B}(\mathscr{H})$ and $xy = yx$, then

$$\alpha(xy) \leqslant \alpha(x)\,\alpha(y).$$

Let $\varepsilon > 0$. There exists a natural number n_ε, such that $\|x^{n_\varepsilon}\|^{1/n_\varepsilon} \leqslant \alpha(x) + \varepsilon$. For any $n > 0$ there exist natural numbers q, r, which are uniquely determined by the conditions

$$n = n_\varepsilon q + r,\ 0 \leqslant r \leqslant n_\varepsilon - 1.$$

We have

$$\|x^n\| = \|x^{n_\varepsilon q}\, x^r\| \leqslant \|x^{n_\varepsilon}\|^q \|x\|^r \leqslant (\alpha(x) + \varepsilon)^{n_\varepsilon q}\, \|x\|^r = (\alpha(x) + \varepsilon)^{n-r}\, \|x\|^r,$$

and, therefore, we have

$$\|x^n\|^{1/n} \leqslant (\alpha(x) + \varepsilon)^{1 - \frac{r}{n}}\, \|x\|^{\frac{r}{n}};$$

consequently, we can write

$$\alpha(x) \leqslant \liminf_{n\to\infty} \|x^n\|^{1/n} \leqslant \limsup_{n\to\infty} \|x^n\|^{1/n} \leqslant \alpha(x) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that the limit $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exists and it is equal to $\alpha(x)$.

If $|\lambda| > \alpha(x)$, then $\sum_{n=0}^{\infty} \|x^n\| / \lambda^n < +\infty$ and this implies that the series $\sum_{n=0}^{\infty} x^n/\lambda^n$ converges with respect to the norm. It follows that $1 - (x/\lambda)$ is invertible and this implies that $\lambda - x$ is invertible. Consequently, we have

$$\alpha(x) \geqslant \sup \{|\lambda|;\ \lambda \in \sigma(x)\}.$$

In order to prove the reversed inequality, as well as the fact that the spectrum $\sigma(x)$ is non-empty, we distinguish two cases.

If $\alpha(x) = 0$, then $0 \in \sigma(x)$, i.e., x is not invertible. Indeed, if x is invertible, then

$$1 = \alpha(1) = \alpha(xx^{-1}) \leqslant \alpha(x)\alpha(x^{-1}) = 0,$$

and this is a contradiction.

If $\alpha(x) > 0$, let us assume that $\alpha(x) > \sup \{|\lambda|;\ \lambda \in \sigma(x)\}$, in order to get a contradiction. Since $\sigma(x)$ is a compact set, there exists an $r \in (0, \alpha(x))$, such that

$$\sigma(x) \subset \{\lambda \in \mathbb{C};\ |\lambda| \leqslant r\}.$$

Consequently, we have

$$D = \{\lambda \in \mathbb{C};\ |\lambda| > r\} \subset \rho(x).$$

It is easily checked that for any bounded linear form φ on $\mathscr{B}(\mathscr{H})$, the function $\lambda \mapsto \varphi((\lambda - x)^{-1})$ is analytic in D. Moreover, for $|\lambda| > \alpha(x)$ we have

$$\varphi((\lambda - x)^{-1}) = \sum_{n=0}^{\infty} \lambda^{-n-1}\varphi(x^n).$$

It follows that by the formula

$$f(\mu) = \begin{cases} 0 & \text{for } \mu = 0 \\ \varphi\left(\left(\dfrac{1}{\mu} - x\right)^{-1}\right) & \text{for } 0 < |\mu| < r^{-1} \end{cases}$$

one defines an analytic function in the set

$$D^{-1} = \{\mu \in \mathbb{C};\ |\mu| < r^{-1}\}.$$

Since the Taylor expansion of f at 0 is

$$f(\mu) = \sum_{n=0}^{\infty} \mu^{n+1}\, \varphi(x^n),$$

the same formula holds for any $\mu \in D^{-1}$.

Now let $\lambda_0 \in \mathbb{C}$ be such that $r < |\lambda_0| < \alpha(x)$. Then $\lambda_0^{-1} \in D^{-1}$ and, therefore, for any bounded linear form φ on $\mathscr{B}(\mathscr{H})$ we have

$$\lim_{n\to\infty} \lambda_0^{-n-1}\, \varphi(x^n) = 0.$$

From the Banach-Steinhaus theorem, we infer that

$$\sup_{n>0} |\lambda_0|^{-n-1}\, \|x^n\| = c < +\infty.$$

Consequently, for any $n > 0$, we have

$$\|x^n\| \leqslant c\,|\lambda_0|^{n+1},$$

whence

$$\alpha(x) = \lim_{n\to\infty} \|x^n\|^{1/n} \leqslant \lim_{n\to\infty} c^{1/n}\, |\lambda_0|^{1+\frac{1}{n}} = |\lambda_0| < \alpha(x)$$

and this is a contradiction.

Q.E.D.

One usually denotes

$$|\sigma(x)| = \sup\{|\lambda|;\ \lambda \in \sigma(x)\} = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

and the number $|\sigma(x)|$ is called the *spectral radius* of x. Obviously, we have

$$|\sigma(x)| \leqslant \|x\|$$

and, for any $x, y \in \mathscr{B}(\mathscr{H})$, such that $xy = yx$, we have

$$|\sigma(xy)| \leqslant |\sigma(x)|\cdot|\sigma(y)|.$$

It is easily seen that $\lambda \in \sigma(x^*) \Leftrightarrow \overline{\lambda} \in \sigma(x)$; consequently, we have

$$|\sigma(x^*)| = |\sigma(x)|.$$

2.4. Now let $x \in \mathscr{B}(\mathscr{H})$ and $p(\lambda) = \alpha_0 + \alpha_1\lambda + \ldots + \alpha_n\lambda^n$ be a polynomial. One defines

$$p(x) = \alpha_0 + \alpha_1 x + \ldots\ \alpha_n x^n.$$

Lemma. *With the above notations, we have*

$$\sigma(p(x)) = \{p(\lambda);\ \lambda \in \sigma(x)\}.$$

Proof. If p is a constant, the assertion of the lemma is obvious. Therefore, let $n \geqslant 1$, $\alpha_n \neq 0$. If $\lambda_0 \in \sigma(x)$, then $\lambda_0 - x$ is not invertible, whereas from

$$p(\lambda_0) - p(x) = \sum_{k=0}^{n} \alpha_k(\lambda_0^k - x^k) = (\lambda_0 - x) \sum_{k=1}^{n} \alpha_k \sum_{j=1}^{k-1} \lambda_0^{k-1} x^j,$$

we infer that $p(\lambda_0) - p(x)$ is not invertible, too. It follows that $p(\lambda_0) \in \sigma(p(x))$. Conversely, if $\mu \notin \{p(\lambda); \lambda \in \sigma(x)\}$, and if $\lambda_1, \ldots, \lambda_n$ are the zeros of the polynomial $p(\lambda) - \mu$, then $\lambda_1, \ldots, \lambda_n \notin \sigma(x)$. Since

$$p(x) - \mu = \alpha_n(x - \lambda_1)\ldots(x - \lambda_n),$$

it follows that $p(x) - \mu$ is invertible, and this shows that $\mu \notin \sigma(p(x))$.

Q.E.D.

2.5. An operator $x \in \mathscr{B}(\mathscr{H})$ is said to be *normal* if

$$xx^* = x^*x.$$

It is easy to see that if $x \in \mathscr{B}(\mathscr{H})$ is normal, then, for any $\xi \in \mathscr{H}$ we have

$$\|x^*\xi\| = \|x\xi\|,$$

and conversely.

The operator $x \in \mathscr{B}(\mathscr{H})$ is said to be *self-adjoint* or *hermitian* if

$$x^* = x$$

Obviously, any hermitian operator is normal.

Lemma. *Let* $x \in \mathscr{B}(\mathscr{H})$. *Then*

(i) *if* x *is normal, then* $|\sigma(x)| = \|x\|$.

(ii) *if* x *is self-adjoint, then* $\sigma(x) \subset \mathbb{R}$.

Proof. If $x \in \mathscr{B}(\mathscr{H})$ is self-adjoint, then

$$|\sigma(x)| = \lim_{n\to\infty} \|x^{2^n}\|^{2^{-n}} = \|x\|,$$

from Lemma 2.1.

If $x \in \mathscr{B}(\mathscr{H})$ is normal, then x^*x is self-adjoint, x and x^* commute and, therefore, by taking into account what we have proved in Sections 2.1, 2.3, we have

$$\|x\|^2 = \|x^*x\| = |\sigma(x^*x)| \leqslant |\sigma(x^*)|\,|\sigma(x)| \leqslant \|x^*\|\,\|x\| = \|x\|^2,$$

whence

$$|\sigma(x)| = \|x\|.$$

Now, if $x \in \mathscr{B}(\mathscr{H})$ is self-adjoint and $\lambda = \alpha + i\beta \in \sigma(x)$, $\alpha, \beta \in \mathbb{R}$, then for any $n \geqslant 1$ the operator

$$x_n = x - \alpha + in\beta$$

is normal and $i(n+1)\beta \in \sigma(x_n)$. By taking into account what we have proved in Section 2.3, we have

$$\begin{aligned}(n+1)^2\beta^2 &\leqslant |\sigma(x_n)|^2 \leqslant \|x_n\|^2 = \|x_n^* x_n\| \\ &= \|(x - \alpha - in\beta)(x - \alpha + in\beta)\| = \|(x-\alpha)^2 + n^2\beta^2\| \\ &= \sup_{\|\eta\|=1} \sup_{\|\xi\|=1} (((x-\alpha)^2\xi \mid \eta) + (n^2\beta^2\xi \mid \eta)) \\ &\leqslant \|x - \alpha\|^2 + n^2\beta^2.\end{aligned}$$

Since $n \geqslant 1$ is arbitrary, we have $\beta = 0$, and this shows that $\lambda \in \mathbb{R}$.

Q.E.D.

2.6. The following theorem will enable us to construct "convenient" elements from $\mathscr{B}(\mathscr{H})$. For any compact space Ω we shall denote by $\mathscr{C}(\Omega)$ the set of all continuous complex functions, which are defined on Ω. With the pointwise defined algebraic operations and with the $*$-operation defined by complex conjugation, the set $\mathscr{C}(\Omega)$ becomes canonically a commutative C^*-algebra, if we endow it with the uniform norm. Any element from $\mathscr{C}(\Omega)$ is "normal", whereas the "self-adjoint" elements are the real functions. The "spectrum" of an element $f \in \mathscr{C}(\Omega)$ coincides with the range $f(\Omega)$ of the function f.

Theorem *(of operational calculus with continuous functions). Let $x \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator. Then there exists a unique mapping*

$$\mathscr{C}(\sigma(x)) \ni f \mapsto f(x) \in \mathscr{B}(\mathscr{H})$$

such that

(i) *if f is a polynomial, $f(\lambda) = \alpha_0 + \alpha_1 x + \ldots + \alpha_n \lambda^n$, then*

$$f(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n,$$

(ii) $\|f(x)\| = \|f\|$, *for any $f \in \mathscr{C}(\sigma(x))$.*

Moreover, this mapping is a $*$-isomorphism of the C^*-algebra $\mathscr{C}(\sigma(x))$ onto the C^*-algebra $\mathscr{C}^*(\{x, 1\})$.

Proof. The set of all polynomials can be mapped by restriction, which is a $*$-homomorphism, onto a $*$-subalgebra $\mathscr{P}(\sigma(x)$ in $\mathscr{C}(\sigma(x))$. By taking into account the Stone-Weierstrass theorem, we infer that $\mathscr{P}(\sigma(x))$ is dense in $\mathscr{C}(\sigma(x))$ for the norm topology. For any "polynomial" $p \in \mathscr{P}(\sigma(x))$ we define $p(x)$ as we have done in Section 2.4*). The set $\{p(x);\ p \in \mathscr{P}(\sigma(x))\}$ is a $*$-subalgebra and it is dense in $\mathscr{C}^*(\{x, 1\})$ with respect to the norm topology. For any $p \in \mathscr{P}(\sigma(x))$ we have

$$\|p(x)\| = |\sigma(p(x))| = \sup\{|\mu|;\ \mu \in \sigma(p(x))\}$$
$$= \sup\{|p(\lambda)|;\ \lambda \in \sigma(x)\} = \|p\|.$$

Consequently, the mapping

$$\mathscr{P}(\sigma(x)) \ni p \mapsto p(x) \in \mathscr{B}(\mathscr{H})$$

is correctly defined and isometric. Thus, there exists a unique isometric extension of this mapping to $\mathscr{C}(\sigma(x))$, and this proves the existence and the uniqueness of the mapping having properties (i) and (ii).

The relations

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x), && f, g \in \mathscr{C}(\sigma(x)),\\
(fg)(x) &= f(x)\,g(x) &&, f, g \in \mathscr{C}(\sigma(x)),\\
(\lambda f)(x) &= \lambda f(x) &&, \lambda \in \mathbb{C},\quad f \in \mathscr{C}(\sigma(x)),\\
\bar{f}(x) &= (f(x))^* &&, f \in \mathscr{C}(\sigma(x)),
\end{aligned}$$

are easy to prove, first for polynomials, and then by tending to the limit, for arbitrary continuous functions. This shows that the mapping just defined is indeed a $*$-isomorphism of the C^*-algebra $\mathscr{C}(\sigma(x))$ onto the C^*-algebra $\mathscr{C}^*(\{x, 1\})$.

Q.E.D.

If $x, y \in \mathscr{B}(\mathscr{H})$ are commuting self-adjoint operators, $xy = yx$, and if $f \in \mathscr{C}(\sigma(x))$, $g \in \mathscr{C}(\sigma(y))$, then $f(x)$ and $g(y)$ are commuting normal operators: $f(x)g(y) = g(y)f(x)$; indeed, this fact is obvious if f and g are polynomials; for the general case one tends to the limit.

If $f(0) = 0$, then $f(x) \in \mathscr{C}^*(\{x\})$, because in this case f can be approximated by polynomials without constant terms.

2.7. By taking into account the isomorphism we have just obtained, as well as its uniqueness, one can immediately get the following:

Corollary. *Let* $x \in \mathscr{B}(\mathscr{H})$ *be a self-adjoint operator. For any* $f \in \mathscr{C}(\sigma(x))$ *we have*

$$\sigma(f(x)) = \{f(\lambda);\ \lambda \in \sigma(x)\},$$

whereas if f *is real and* $g \in \mathscr{C}(\sigma(f(x)))$, *then*

$$g(f(x)) = (g \circ f)(x).$$

---

*) By using the lemmas from 2.4 and 2.5, it is easy to show that if p is a polynomial and $p|\sigma(x) = 0$, then $p(x) = 0$; this shows that $p(x)$ is correctly defined for any $p \in \mathscr{P}(\sigma(x))$. (Translator's Note).

2.8. Corollary. *If $x \in \mathscr{B}(\mathscr{H})$ is an invertible operator, then $x^{-1} \in \mathscr{C}^*(\{x, 1\})$.*

Proof. If x is self-adjoint, the statement immediately follows from Theorem 2.6.

Now let $x \in \mathscr{B}(\mathscr{H})$ be any invertible operator, and $y = x^{-1}$. Then $y^* = (x^*)^{-1}$, $yy^* = (x^*x)^{-1}$, and, therefore, $yy^* \in \mathscr{C}^*(\{x^*x, 1\}) \subset \mathscr{C}^*(\{x, 1\})$. Since $x^* \in \mathscr{C}^*(\{x, 1\})$, we have

$$x^{-1} = y = y(y^*x^*) = (yy^*)x^* \in \mathscr{C}^*(\{x, 1\}).$$

Q.E.D.

2.9. An operator $x \in \mathscr{B}(\mathscr{H})$ is said to be *positive* if it is self-adjoint and

$$\sigma(x) \subset \mathbb{R}^+ = \{\lambda \in \mathbb{R};\ \lambda \geqslant 0\}.$$

Corollary. *If $x \in \mathscr{B}(\mathscr{H})$ is a positive operator, then there exists a unique positive operator $a \in \mathscr{B}(\mathscr{H})$, such that*

$$a^2 = x.$$

Proof. Since we have the inclusion $\sigma(x) \subset \mathbb{R}^+$, the function defined by

$$f(\lambda) = \lambda^{1/2},\ \lambda \in \sigma(x),$$

belongs to $\mathscr{C}(\sigma(x))$. From Corollary 2.7, by denoting $a = f(x)$, we have $a^2 = x$ and a is a positive operator.

Let $b \in \mathscr{B}(\mathscr{H})$ be an arbitrary positive operator, such that $b^2 = x$. Let us consider a sequence $\{p_n\}$ of polynomials which converges uniformly on $\sigma(x)$ to the function f we have just defined. Since $\sigma(b) \subset \mathbb{R}^+$, from $\sigma(x) = \{\lambda^2;\ \lambda \in \sigma(b)\}$, we infer that the polynomials $p_n(\lambda^2)$ converge uniformly on $\sigma(b)$ to the identical function. By taking into account Theorem 2.6, we have

$$\lim_{n\to\infty} \|b - p_n(x)\| = \lim_{n\to\infty} \|b - p_n(b^2)\| = 0,$$

and this shows that $b = \lim_{n\to\infty} p_n(x) = f(x) = a.$

Q.E.D.

The unique positive operator $a \in \mathscr{B}(\mathscr{H})$, such that $a^2 = x$ is denoted by $x^{1/2}$. We observe that $x^{1/} \in \mathscr{C}^*(\{x\})$.

2.10. Corollary. *Let $x \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator. Then there exist positive operators $a, b \in \mathscr{B}(\mathscr{H})$, such that*

$$x = a - b,$$

$$ab = 0,$$

and they are uniquely determined by these conditions.

Proof. Let us consider on $\sigma(x)$ the continuous functions defined by the equalities

$$f(\lambda)=\begin{cases}\lambda & \text{for} \quad \lambda\geqslant 0\\ 0 & \text{for} \quad \lambda\leqslant 0\end{cases}$$

$$g(\lambda)=\begin{cases}0 & \text{for} \quad \lambda\geqslant 0\\ -\lambda & \text{for} \quad \lambda\leqslant 0.\end{cases}$$

Then the operators $a=f(x)$, $b=g(x)$ satisfy the two conditions required in the corollary, and this proves the existence part of the corollary.

In order to prove the uniqueness, we observe that

$$(a+b)^2=x^2.$$

By taking into account a remark we made in Section 2.6, we get:

$$a+b=(a^{1/2}+b^{1/2})^2.$$

This shows that $a+b$ and x^2 are positive operators (see 2.5, 2.7) and from Corollary 2.9, we infer that

$$a+b=(x^2)^{1/2}.$$

This implies that

$$a=\frac{1}{2}((x^2)^{1/2}+x),\ b=\frac{1}{2}(x^2)^{1/2}-x).$$

Q.E.D.

The operators a and b, given by this corollary, are denoted: $a=x^+$, $b=x^-$. We observe that $x^+, x^-\in\mathscr{C}^*(\{x\})$. Thus, we have

$$x=x^+-x^-$$

$$x^+x^-=0.$$

2.11. Lemma. *Let* $x\in\mathscr{B}(\mathscr{H})$. *Then:*

(i) $x=0\Leftrightarrow(x\xi\mid\xi)=0$ *for any* $\xi\in\mathscr{H}$,

(ii) x *is self-adjoint* $\Leftrightarrow(x\xi\mid\xi)\in\mathbb{R}$ *for any* $\xi\in\mathscr{H}$.

Proof. It is easy to prove the following "polarization formula"

$$\begin{aligned}4(x\xi\mid\eta)&=(x(\xi+\eta)\mid\xi+\eta)-(x(\xi-\eta)\mid\xi-\eta)\\&\quad+i(x(\xi+i\eta)\mid\xi+i\eta)-i(x(\xi-i\eta)\mid\xi-i\eta),\end{aligned}$$

which holds for any $\xi,\ \eta\in\mathscr{H}$.

If $(x\xi \mid \xi) = 0$ for any $\xi \in \mathscr{H}$, by taking into account the polarization formula we infer that $(x\xi \mid \eta) = 0$, for any $\xi, \eta \in \mathscr{H}$; in particular, we have $\|x\xi\|^2 = (x\xi \mid x\xi) = 0$, for any $\xi \in \mathscr{H}$, and this shows that $x = 0$.

If x is self-adjoint, then for any $\xi \in \mathscr{H}$,

$$(x\xi \mid \xi) = (\xi \mid x\xi) = \overline{(x\xi \mid \xi)},$$

i.e., $(x\xi \mid \xi)$ is real. Conversely, if $(x\xi \mid \xi)$ is real, we have

$$((x - x^*)\xi \mid \xi) = (x\xi \mid \xi) - (x^*\xi \mid \xi) = (x\xi \mid \xi) - (\xi \mid x\xi)$$
$$= (x\xi \mid \xi) - (x\xi \mid \xi) = 0.$$

With the first part of the lemma, we infer that $x^* = x$.

Q.E.D.

2.12. The following proposition characterizes the positive operators:

Proposition. *For any operator $x \in \mathscr{B}(\mathscr{H})$, the following statements are equivalent:*

(i) *x is positive;*
(ii) *there exists a positive operator $a \in \mathscr{B}(\mathscr{H})$, such that $x = a^2$;*
(iii) *there exists an operator $y \in \mathscr{B}(\mathscr{H})$, such that $x = y^*y$;*
(iv) $(x\xi \mid \xi) \geqslant 0$ *for any $\xi \in \mathscr{H}$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Corollary 2.9, whereas the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.

Let us now assume that $(x\xi \mid \xi) \geqslant 0$ for any $\xi \in \mathscr{H}$. From Lemma 2.11, x is self-adjoint. From Corollary 2.10, we infer that there exist positive operators x^+, $x^- \in \mathscr{C}^*(\{x\})$, such that $x = x^+ - x^-$, $x^+x^- = 0$. Then, for any $\xi \in \mathscr{H}$, we have

$$0 \leqslant (x(x^-\xi) \mid x^-\xi) = (x^-xx^-\xi \mid \xi) = -((x^-)^3\xi \mid \xi)$$
$$= -(x^-(x^-\xi) \mid x^-\xi) \leqslant 0,$$

and this shows that

$$((x^-)^3\xi \mid \xi) = 0, \ \xi \in \mathscr{H}.$$

From Lemma 2.11, it follows that $(x^-)^3 = 0$. By taking into account Theorem 2.6, we hence infer that $x^- = 0$. Consequently, $x = x^+$ is a positive operator.

Q.E.D.

2.13. An operator $e \in \mathscr{B}(\mathscr{H})$ is said to be a *projection* if $e^* = e$ and $e^2 = e$. Any projection is a positive operator. If e is a projection, then $1 - e$ is a projection too; it is sometimes denoted by $e^\perp (= 1 - e)$.

If $e \in \mathscr{B}(\mathscr{H})$ is a projection, then $e\mathscr{H}$ is a closed subspace of $\mathscr{H}$, called the *projection subspace* of e, whereas $(1 - e)\mathscr{H}$ is the orthogonal complement $(e\mathscr{H})^\perp$ of $e\mathscr{H}$. Conversely, to any closed subspace $\mathscr{S} \subset \mathscr{H}$ there corresponds a unique projection $e \in \mathscr{B}(\mathscr{H})$, such that $e\mathscr{H} = \mathscr{S}$; this projection will be called the *projection on the subspace* $\mathscr{S}$ and it will also sometimes be denoted by $\mathscr{S}$; for example, we have $\mathscr{S}^{\perp} = 1 - \mathscr{S}$.

The set of all projection in $\mathscr{B}(\mathscr{H})$ will be denoted by $\mathscr{P}_{\mathscr{B}(\mathscr{H})}$. It is easily checked that if $e \in \mathscr{P}_{\mathscr{B}(\mathscr{H})}$, $e \neq 0$, then

$$\|e\| = 1.$$

Two projections $e_1, e_2 \in \mathscr{P}_{\mathscr{B}(\mathscr{H})}$ are said to be *orthogonal* if $e_1e_2=0$. The following statements are easily checked:

$$e_1e_2 \in \mathscr{P}_{\mathscr{B}(\mathscr{H})} \Leftrightarrow e_1e_2 = e_2e_1,$$

$$e_1 + e_2 \in \mathscr{P}_{\mathscr{B}(\mathscr{H})} \Leftrightarrow e_1e_2 = 0.$$

Let $x \in \mathscr{B}(\mathscr{H})$. We introduce the following notations:

$\mathfrak{n}(x)$ = the projection on the kernel of $x: \{\xi \in \mathscr{H};\ x\xi = 0\}$,

$\mathfrak{l}(x)$ = the projection on the closure of the range of $x: \overline{x\mathscr{H}}$,

$\mathfrak{r}(x) = 1 - \mathfrak{n}(x)$.

It is easily checked that

$$\mathfrak{r}(x) = \mathfrak{l}(x^*)$$

and $\mathfrak{l}(x)$ (resp., $\mathfrak{r}(x)$) is the smallest projection $e \in \mathscr{B}(\mathscr{H})$ such that $ex=x$ (resp., $xe=x$). One says that $\mathfrak{l}(x)$ (resp., $\mathfrak{r}(x)$) is the *left support* (resp., the *right support)* of x. If x is self-adjoint, then the following notation is used

$$\mathfrak{s}(x) = \mathfrak{l}(x) = \mathfrak{r}(x)$$

and one says that $\mathfrak{s}(x)$ is the *support* of x.

For any $x \in \mathscr{B}(\mathscr{H})$ the following relations hold

$$\mathfrak{l}(x) = \mathfrak{s}(xx^*),$$

$$\mathfrak{r}(x) = \mathfrak{s}(x^*x).$$

An operator $v \in \mathscr{B}(\mathscr{H})$ is said to be a *partial isometry* if there exists a closed subspace $\mathscr{S} \subset \mathscr{H}$, such that

$$\|v\xi\| = \|\xi\|, \quad \text{for any } \xi \in \mathscr{S},$$

$$v\xi = 0, \quad \text{for any } \xi \in \mathscr{S}^{\perp}.$$

The closed subspace $\mathscr{S}$ (resp. vS) is called the *initial subspace* (resp., the *final subspace)* of the partial isometry v. It is easy to check that

$$\mathfrak{r}(v) = v^*v, \quad \mathfrak{l}(v) = vv^*.$$

The projection v^*v (resp., vv^*) is called the *initial projection* (resp., the *final projection)* of the partial isometry v. The projection subspace of the initial (resp., of the final) projection of v is just the initial (resp., the final) subspace of v.

Conversely, if $v \in \mathscr{B}(\mathscr{H})$ and if v^*v is a projection, then vv^* is also a projection, whereas v is a partial isometry whose initial subspace is $\mathscr{S} = (v^*v)\mathscr{H}$. Indeed, by taking into account the relation $(v^*v)^2 = v^*v$, it follows that

$$\|v - vv^*v\|^2 = \|(v^* - v^*vv^*)(v - vv^*v)\| = \ldots = 0,$$

whence $v = vv^*v$ and $(vv^*)^2 = vv^*$. We then have

$$\xi \in \mathscr{S} \Rightarrow \xi = v^*v\xi \Rightarrow \|\xi\|^2 = (\xi \mid \xi) = (v^*v\,\xi \mid \xi) = \|v\xi\|^2,$$

$$\xi \in \mathscr{S}^\perp \Rightarrow 0 = v^*v\xi \Rightarrow 0 = vv^*v\xi = v\xi.$$

2.14. Theorem. *(the polar decomposition). For any operator $x \in \mathscr{B}(\mathscr{H})$ there exist a unique positive operator $a \in \mathscr{B}(\mathscr{H})$ and a unique partial isometry $v \in \mathscr{B}(\mathscr{H})$ such that*

$$x = va$$

$$v^*v = \mathrm{s}(a).$$

Proof. We define $a = (x^*x)^{1/2}$ and the operator v_0 on $a\mathscr{H}$ by the relation

$$v_0(a\xi) = x\xi, \quad \xi \in \mathscr{H}.$$

Since, for any $\xi \in \mathscr{H}$, we have

$$\|v_0(a\xi)\|^2 = \|x\xi\|^2 = (x^*x\xi \mid \xi) = (a^2\xi \mid \xi) = \|a\xi\|^2,$$

it follows that the operator v_0 can be extended, in a unique manner, to an isometric operator (i.e., one which conserves the norm), for which we shall keep the same notation, defined on the space $\overline{a\mathscr{H}} = \mathrm{s}(a)\mathscr{H}$. We then define a partial isometry $v \in \mathscr{B}(\mathscr{H})$ by the relations

$$v\xi = \begin{cases} v_0\xi & \text{for} \quad \xi \in \mathrm{s}(a)\mathscr{H}, \\ 0 & \text{for} \quad \xi \in (\mathrm{s}(a)\mathscr{H})^\perp. \end{cases}$$

It is now easy to check that

$$x = va,$$

$$v^*v = \mathrm{s}(a),$$

and this establishes the existence part of the statement.

In order to prove the uniqueness, let us remark that from the conditions of the statement it follows that

$$x^*x = av^*va = a\mathrm{s}(a)a = a^2$$

and this implies that $a = (x^*x)^{1/2}$. Then one can easily see that the partial isometry v necessarily maps according to the above definition.

Q.E.D.

The operator $a = (x^*x)^{1/2}$ is called the *absolute value* (or the *modulus*) of x and is denoted by $|x|$. We remark that $|x| \in \mathscr{C}^*(\{x\})$. The relations

$$x = v|x|, \quad v^*v = \mathrm{s}(|x|)$$

are called the *polar decomposition* of x.

2.15. Let $x \in \mathscr{B}(\mathscr{H})$ and let

$$x = v|x|, \quad v^*v = \mathrm{s}(|x|)$$

be the polar decomposition of x. It is easy to check the relations

$$x^* = v^*(v|x|v^*), \quad (v^*)^*v^* = \mathrm{s}(v|x|v^*),$$

and, therefore, according to Theorem 2.14, they yield the polar decomposition of the operator x^*. In particular, we have

$$|x^*| = v|x|v^*$$

and

$$x = |x^*|v, \quad vv^* = \mathrm{s}(|x^*|).$$

For this reason one sometimes says that (2.14) is the left polar decomposition, whereas the preceding formulas yield the right polar decomposition of the operator x.

It is easy to check the following relations

$$\mathrm{r}(x) = \mathrm{s}(|x|) = v^*v,$$

$$\mathrm{l}(x) = \mathrm{s}(|x^*|) = vv^*.$$

If $x \in \mathscr{B}(\mathscr{H})$ is a self-adjoint operator, and if

$$x = v|x|, \quad v^*v = \mathrm{s}(|x|),$$

is its polar decomposition, then the following relations are immediately obtained

$$|x| = x^+ + x^-,$$

$$v = \mathrm{s}(x^+) - \mathrm{s}(x^-);$$

in particular, we have $v = v^*$.

2.16. We shall sometimes denote by $\mathscr{B}(\mathscr{H})^h$ the set of all self-adjoint operators in $\mathscr{B}(\mathscr{H})$. The notion of positive operator allows the introduction of an *order relation* in $\mathscr{B}(\mathscr{H})^h$. Namely, for $x, y \in \mathscr{B}(\mathscr{H})$ we shall write $x \leqslant y$ if the operator $y - x$ is positive; by taking into account Proposition 2.12, it is easy to check that the relation "$\leqslant$" is indeed an order relation in $\mathscr{B}(\mathscr{H})^h$. From now on we shall use the notation $x \geqslant 0$ in order to express the fact that the operator x is positive and we shall sometimes denote $\mathscr{B}(\mathscr{H})^+ = \{x \in \mathscr{B}(\mathscr{H});\ x \geqslant 0\}$.

The set $\mathscr{B}(\mathscr{H})^h$ is a real *wo*-closed vector subspace of $\mathscr{B}(\mathscr{H})$, whereas $\mathscr{B}(\mathscr{H})^+$ is a *wo*-closed convex cone.

If $a, b \in \mathscr{B}(\mathscr{H})^+$ and $ab = ba$, then $ab \in \mathscr{B}(\mathscr{H})^+$.

If $a, b \in \mathscr{B}(\mathscr{H})^h$ and $a \leqslant b$, then $x^*ax \leqslant x^*bx$, for any $x \in \mathscr{B}(\mathscr{H})$.

Proposition. *Let $\{x_\iota\}_{\iota \in I} \subset \mathscr{B}(\mathscr{H})^h$ be an increasing net, such that there exists a $y \in \mathscr{B}(\mathscr{H})$ for which $x_\iota \leqslant y$, $\iota \in I$. Then there exists an $x \in \mathscr{B}(\mathscr{H})^h$, such that*

$$x = \sup_{\iota \in I} x_\iota.$$

Moreover, x is the limit of the net $(x_\iota)_{\iota \in I}$ for the so-topology in $\mathscr{B}(\mathscr{H})$.

Proof. We can assume, without any loss of generality, that $0 \leqslant x_\iota \leqslant 1$, $\iota \in I$.

For any $\xi \in \mathscr{H}$ we define

$$F(\xi, \xi) = \sup_{\iota \in I} (x_\iota \xi \mid \xi) = \lim_{\iota \in I} (x_\iota \xi \mid \xi)$$

and, for any $\xi, \ \eta \in \mathscr{H}$,

$$F(\xi, \eta) = \frac{1}{4}(F(\xi + \eta, \xi + \eta) - F(\xi - \eta, \xi - \eta) + \mathrm{i}\, F(\xi + \mathrm{i}\eta, \ \xi + \mathrm{i}\eta)$$
$$- \mathrm{i} F(\xi - \mathrm{i}\eta, \xi - \mathrm{i}\eta).$$

Then $F(.,.)$ is a bounded positive sesquilinear form, whose norm is $\leqslant 1$.

According to the Riesz theorem, there exists a unique operator $x \in \mathscr{B}(\mathscr{H})$, $\|x\| \leqslant 1$, such that

$$F(\xi, \eta) = (x\xi \mid \eta), \quad \xi, \eta \in \mathscr{H},$$

and, according to Proposition 2.12, we have $x \geqslant 0$.

Since, for any $\iota \in I$ and any $\xi \in \mathscr{H}$, we have

$$(x\xi \mid \xi) = F(\xi, \xi) \geqslant (x_\iota \xi \mid \xi),$$

it follows that x is an upper bound of the net $(x_\iota)_{\iota \in I}$. On the other hand, if $x_0 \in \mathscr{B}(\mathscr{H})^h$ is an upper bound of the family $(x_\iota)_{\iota \in I}$, then for any $\xi \in \mathscr{H}$, we have

$$(x\xi \mid \xi) = F(\xi, \xi) = \lim_{\iota \in I} (x_\iota \xi \mid \xi) \leqslant (x_0 \xi \mid \xi)$$

and this implies that $x \leqslant x_0$. Hence

$$x = \sup_{\iota \in I} x_\iota \text{ in } \mathscr{B}(\mathscr{H})^h.$$

Finally, for any $\xi \in \mathscr{H}$, we have

$$\|(x - x_\iota)\xi\|^2 \leqslant \|(x - x_\iota)^{1/2}\|^2 \|(x - x_\iota)^{1/2}\xi\|^2$$
$$\leqslant ((x - x_\iota)\xi \mid \xi) = F(\xi, \xi) - (x_\iota \xi \mid \xi) \to 0$$

and this implies that x is the limit of the net $(x_\iota)_{\iota \in I}$ for the *so*-topology in $\mathscr{B}(\mathscr{H})$.

Q.E.D.

If $(x_i)_{i\in I} \subset \mathscr{B}(\mathscr{H})^h$ is an increasing net and if $x \in \mathscr{B}(\mathscr{H})^h$ belongs to the *wo*-closure of the range of this family, then $x = \sup_{i\in I} x_i$ and, therefore, x belongs to the *so*-closure of the range of the same family. In this case we shall write

$$x_i \uparrow x.$$

This notation means that the net $(x_i)_{i\in I}$ is increasing, $x = \sup x_i$ in $\mathscr{B}(\mathscr{H})^h$ and that x belongs to the *so*-closure of the range of the family $(x_i)_{i\in I}$. The same notation will be used for the "increasing convergence" of real numbers.

2.17. Let $x \in \mathscr{B}(\mathscr{H})$, $0 \leqslant x \leqslant 1$, and $e \in \mathscr{P}_{\mathscr{B}(\mathscr{H})}$. Then

$$x \leqslant e \Leftrightarrow x = xe.$$

Indeed, from the relation $x \leqslant 1$ we infer that $exe \leqslant e$, whence if $x = xe$, we deduce that $x \leqslant e$. Conversely, if $0 \leqslant x \leqslant e$, then we get successively:

$$0 = (1-e)0(1-e) \leqslant (1-e)x(1-e) \leqslant (1-e)e(1-e) = 0,$$

$$(1-e)x(1-e) = 0,$$

$$((1-e)x^{1/2})((1-e)x^{1/2})^* = 0,$$

$$(1-e)x^{1/2} = 0,$$

$$(1-e)x = 0,$$

and this implies that $x = ex = xe$.

In particular, if $e_1, e_2 \in \mathscr{P}_{\mathscr{B}(\mathscr{H})}$, then $e_1 \leqslant e_2$ iff $e_1 = e_1e_2$. It is easy to check that we have $e_1 \leqslant e_2$ iff $e_1\mathscr{H} \subset e_2\mathscr{H}$.

Let $(e_i)_{i\in I} \subset \mathscr{P}_{\mathscr{B}(\mathscr{H})}$ be any family of projections. One can define the following projections

$$\bigvee_{i\in I} e_i = \text{the projection on the subspace } \overline{\sum_{i\in I} e_i\mathscr{H}}.$$

$$\bigwedge_{i\in I} e_i = \text{the projection on the subspace } \bigcap_{i\in I} e_i\mathscr{H}.$$

One can then immediately check that $\bigvee_{i\in I} e_i$ (resp., $\bigwedge_{i\in I} e_i$) is the least upper bound (resp., the greatest lower bound) of the family $(e_i)_{i\in I}$ with respect to the order relation induced on $\mathscr{P}_{\mathscr{B}(\mathscr{H})}$ by the order relation just defined in $\mathscr{B}(\mathscr{H})^h$. If the set of indices is finite, $I = \{1,\dots,n\}$, then one also uses the following notations

$$e_1 \vee \dots \vee e_n = \bigvee_{i\in I}^{n} e_i, \text{ for } \bigvee_{i\in I} e_i,$$

$$e_1 \wedge \dots \wedge e_n = \bigwedge_{i=1}^{n} e_i, \text{ for } \bigwedge_{i\in I} e_i.$$

From the preceding results and from Proposition 2.16, we get the following

Corollary. (i) *$\mathscr{P}_{\mathscr{B}(\mathscr{H})}$ is a complete lattice.*

(ii) *If $(e_\iota)_{\iota\in I}$ is an increasing net, then*

$$e_\iota \uparrow \bigvee_{\iota\in I} e_\iota.$$

(iii) *If $(e_\iota)_{\iota\in I}$ is a family of mutually orthogonal projections, then the family $(e_\iota)_{\iota\in I}$ is summable for the so-topology, and*

$$\sum_{\iota\in I} e_\iota = \bigvee_{\iota\in I} e_\iota.$$

We observe that for any $e_1,\dots, e_n \in \mathscr{P}_{\mathscr{B}(\mathscr{H})}$, we have

$$s(e_1 + \dots + e_n) = e_1 \vee \dots \vee e_n.$$

2.18. We shall now extend the operational calculus given by Theorem 2.6 to a larger class than that of the continuous functions. In order to do this we shall need the following.

Lemma. *Let $x \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator and let $\{f_n\}$ and $\{g_n\}$ be two bounded increasing sequences of positive functions from $\mathscr{C}(\sigma(x))$, such that*

$$\sup_n f_n(\lambda) \leqslant \sup_n g_n(\lambda), \quad \lambda \in \sigma(x).$$

Then

$$\sup_n f_n(x) \leqslant \sup_n g_n(x).$$

Proof. By taking into account Theorem 2.6 and Proposition 2.16, we infer the existence of the elements $\sup_n f_n(x)$, $\sup_n g_n(x)$, and the relations

$$f_n(x) \uparrow \sup_n f_n(x), \quad g_n(x) \uparrow \sup_n g_n(x).$$

Let n be a natural number and $\varepsilon > 0$. For any $\lambda \in \sigma(x)$ we have

$$f_n(\lambda) - \varepsilon < f_n(\lambda) \leqslant \sup_m f_m(\lambda) \leqslant \sup_m g_m(\lambda);$$

consequently, there exists a neighbourhood V_λ of λ and a natural number m_λ, such that

$$f_n(\mu) - \varepsilon < g_{m_\lambda}(\mu), \quad \mu \in V_\lambda.$$

Since $\sigma(x)$ is compact, it follows that there exists a natural number m_n, such that

$$f_n - \varepsilon \leqslant g_{m_n} \quad \text{in} \quad \mathscr{C}(\sigma(x)).$$

Consequently, by taking into account Theorem 2.6, we have

$$f_n(x) - \varepsilon \leqslant g_{m_n}(x) \leqslant \sup_m g_m(x),$$

whence, by taking into account the fact that $\varepsilon > 0$ is arbitrary, we get

$$f_n(x) \leqslant \sup_m g_m(x);$$

since n is arbitrary, we have

$$\sup_n f_n(x) \leqslant \sup_m g_m(x)$$

Q.E.D.

2.19. Let $x \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator. We shall write

$$m(x) = \inf\{\lambda; \lambda \in \sigma(x)\}, \quad M(x) = \sup\{\lambda; \lambda \in \sigma(x)\}.$$

Since $\sigma(x)$ is compact, we have

$$m(x),\ M(x) \in \sigma(x).$$

For any $\lambda \in \mathbb{R}$ we shall consider the continuous functions

$$f_n^\lambda(t) = \begin{cases} 1 & , \text{ for } t \in \left(-\infty, \lambda - \frac{1}{n}\right], \\ n(\lambda - t) & , \text{ for } t \in \left[\lambda - \frac{1}{n}, \lambda\right], \\ 0 & , \text{ for } t \in [\lambda, +\infty). \end{cases}$$

Then we have

$$f_n^\lambda(t) \uparrow \chi_{(-\infty,\lambda)}(t), \quad t \in \mathbb{R},$$

$$(f_n^\lambda)^2(t) \uparrow \chi_{(-\infty,\lambda)}(t), \quad t \in \mathbb{R},$$

where by χ_D we denote the characteristic function of the set $D \subset \mathbb{R}$.

According to Proposition 2.16 and Lemma 2.18, there exists a projection $e_\lambda \in \mathscr{B}(\mathscr{H})$, such that

$$f_n^\lambda(x) \uparrow e_\lambda.$$

In what follows we shall prove some properties of the projections e_λ.

(i) $e_\lambda \in \mathscr{R}(\{x\})$; in particular, e_λ commutes with any operator commuting with x.

This fact follows from the definition of the projections e_λ, from Theorem 2.6 and from the obvious equality $\mathscr{R}(\{x\}) = so$-closure of $\mathscr{C}^*(\{x, 1\})$.

(ii)

$$\lambda_1 \leqslant \lambda_2 \Rightarrow e_1 \leqslant e_2.$$

Indeed, for any n we have

$$f_n^{\lambda_1} \leqslant f_n^{\lambda_2} \text{ in } \mathscr{C}(\sigma(x)),$$

and this implies that

$$f_n^{\lambda_1}(x) \leqslant f_n^{\lambda_2}(x);$$

the required property now follows by tending to the limit

(iii) $$\lambda_n \uparrow \lambda \Rightarrow e_{\lambda_n} \uparrow e_\lambda.$$

Indeed, we then have

$$f_n^{\lambda_n} \uparrow \chi_{(-\infty, \lambda)}, \text{ pointwise;}$$

by taking into account Lemma 2.18, the definition of the projection e_λ and (ii), we get

$$e_\lambda \geqslant e_{\lambda_n} \geqslant f_n^{\lambda_n}(x) \uparrow e_\lambda,$$

and, therefore, $e_{\lambda_n} \uparrow e_\lambda$.

(iv) $$\lambda \leqslant m(x) \Rightarrow e_\lambda = 0; \quad \lambda > M(x) \Rightarrow e_\lambda = 1.$$

Indeed, if $\lambda \leqslant m(x)$, then $f_n^\lambda = 0$ in $\mathscr{C}(\sigma(x))$, for any n, whereas for $\lambda > M(x)$ we have $f_n^\lambda = 1$ in $\mathscr{C}(\sigma(x))$, if n is sufficiently great.

(v) $$xe_\lambda \leqslant \lambda e_\lambda, \quad x(1 - e_\lambda) \geqslant \lambda(1 - e_\lambda).$$

Indeed, we have the following relations

$$tf_n^\lambda(t) \leqslant \lambda f_n^\lambda(t), \quad t \in \mathbb{R},$$

$$t(1 - f_n^\lambda(t)) \geqslant \left(\lambda - \frac{1}{n}\right)(1 - f_n^\lambda(t)), \quad t \in \mathbb{R},$$

whence

$$xf_n^\lambda(x) \leqslant \lambda f_n^\lambda(x),$$

$$x(1 - f_n^\lambda(x)) \geqslant \left(\lambda - \frac{1}{n}\right)(1 - f_n^\lambda(x)),$$

and the stated inequalities can be obtained by tending to the limit.

From property (v) we infer that if $\mu \leqslant \lambda$, then

$$\mu(e_\lambda - e_\mu) \leqslant x(e_\lambda - e_\mu) \leqslant \lambda(e_\lambda - e_\mu).$$

Let now, $\delta > 0$, $\varepsilon > 0$, and let

$$\Delta = \{m(x) = \lambda_0 < \lambda_1 < \ldots < \lambda_n = M(x) + \delta\}$$

be a partition of the interval $[m(x), M(x) + \delta]$, whose norm is $\|\Delta\| = \sup\{\lambda_i - \lambda_{i-1};\ i = 1, 2, \ldots, n\} < \varepsilon$. We shall now consider the "Darboux sums":

$$s(\Delta) = \sum_{i=1}^{n} \lambda_{i-1}(e_{\lambda_i} - e_{\lambda_{i-1}}),$$

$$S(\Delta) = \sum_{i=1}^{n} \lambda_i(e_{\lambda_i} - e_{\lambda_{i-1}}).$$

By taking into account the preceding results, we can easily prove the following relations

$$s(\Delta) \leqslant x \leqslant S(\Delta),$$

$$\|S(\Delta) - s(\Delta)\| < \varepsilon,$$

and these enable us to write

(vi) $$x = \int_{-\infty}^{+\infty} \lambda \, de_\lambda = \int_{m(x)}^{M(x)+0} \lambda de_\lambda,$$

where the integral is to be considered as a vector Stieltjes integral, which converges with respect to the norm.

Assertions (i)—(vi) make up what is usually called the *spectral theorem* for the self-adjoint operator x, whereas the family of projections $(e_\lambda)_\lambda$ is called the *spectral scale* of the self-adjoint operator x.

For any ξ, $\eta \in \mathscr{H}$ we shall consider the function $e_{\xi,\eta}$ defined by the relation

$$e_{\xi,\eta}(\lambda) = (e_\lambda \xi \,|\, \eta), \quad \lambda \in \mathbb{R}.$$

Then the functions $e_{\xi,\xi}$ are positive, increasing, and

$$e_{\xi,\xi} \leqslant \|\xi\|^2,$$

whereas the functions $e_{\xi,\eta}$ are of bounded variation, and their total variation can be majorized with the help of the Cauchy-Buniakovsky inequality

$$V(e_{\xi,\eta}) \leqslant \|\xi\| \, \|\eta\|.$$

We recall that any function of bounded variation on $\mathbb{R}$ determines a bounded, Borel measure, whose norm is equal to the total variation of the function; the integral corresponding to such a measure is usually called the Lebesgue-Stieltjes integral. In particular, the functions $e_{\xi,\eta}$ determine bounded Borel measures; by the same method as in the proof of statement (iv), one can show that the support of the measures defined in this manner is contained in the spectrum $\sigma(x)$ of x.

From property (vi), or by direct verification, it follows that

(vii) $$(x\xi \,|\, \eta) = \int_{-\infty}^{+\infty} \lambda de_{\xi,\eta}(\lambda), \quad \xi, \ \eta \in \mathscr{H}.$$

2.20. For any topological space Ω we shall denote by $\mathscr{B}(\Omega)$ the set of all bounded complex Borel functions, defined on Ω. The set $\mathscr{B}(\Omega)$ can be endowed canonically with a structure of a C^*-algebra, with the usual algebraic operations and with the uniform norm (i.e., the sup-norm). If the space Ω is metrizable, then, by virtue of a theorem of Baire, $\mathscr{B}(\Omega)$ *is the smallest class of functions, closed with respect to the pointwise convergence of the bounded sequences and which contains the bounded continuous functions.*

Theorem *(of operational calculus with Borel functions). Let $x \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator. There exists a unique mapping*

$$\mathscr{B}(\sigma(x)) \ni f \mapsto f(x) \in \mathscr{B}(\mathscr{H})$$

such that

(i) *if f is a polynomial, $f(\lambda) = \alpha_0 + \alpha_1\lambda + \ldots + \alpha_n\lambda^n$, then $f(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n$,*

(ii) *if $f, f_n \in \mathscr{B}(\sigma(x))$, $\sup_n \|f_n\| < +\infty$ and $f_n \mapsto f$ pointwise, then $f_n(x) \to f(x)$ for the so-topology in $\mathscr{B}(\mathscr{H})$.*

Moreover, this mapping is a $*$-homomorphism of the C^*-algebra $\mathscr{B}(\sigma(x))$ into the von Neumann algebra $\mathscr{R}(\{x\})$, and it is an extension of the $*$-isomorphism given by Theorem 2.6.

Proof. Any mapping which satisfies conditions (i) and (ii) obviously coincides with the mapping given by Theorem 2.6, when restricted to $\mathscr{C}(\sigma(x))$. In this way the uniqueness is an immediate consequence of the theorem of Baire, already mentioned above.

In order to prove the existence, as well as the other properties of the mapping, described in the statement, we shall define, for any $f \in \mathscr{B}(\sigma(x))$:

$$F_f(\xi, \eta) = \int_{-\infty}^{+\infty} f(\lambda)\, \mathrm{d}e_{\xi,\eta}(\lambda), \quad \xi, \eta \in \mathscr{H},$$

where we used the Lebesgue-Stieltjes integral; more precisely, the function f can be extended to a Borel function on $\mathbb{R}$, whereas the integral does not depend on this extension, since the support of the measure is included in $\sigma(x)$ (see Section 2.19). Then $F_f(.,.)$ is a bounded sesquilinear form, defined on $\mathscr{H} \times \mathscr{H}$:

$$|F_f(\xi, \eta)| \leqslant \|f\| V(e_{\xi,\eta}) \leqslant \|f\|\, \|\xi\|\, \|\eta\|, \quad \xi, \eta \in \mathscr{H}.$$

From the theorem of Riesz it follows that there exists a unique operator $f(x) \in \mathscr{B}(\mathscr{H})$, such that

$$(f(x)\xi | \eta) = \int_{-\infty}^{+\infty} f(\lambda)\, \mathrm{d}e_{\xi,\eta}(\lambda), \quad \xi, \eta \in \mathscr{H}.$$

In this manner we have defined the mapping:

$$\mathscr{B}(\sigma(x)) \ni f \mapsto f(x) \in \mathscr{B}(\mathscr{H}).$$

It is easy to show that this mapping is linear. By taking into account the relation $\bar{e}_{\xi,\eta} = e_{\eta,\xi}$, $\xi, \eta \in \mathscr{H}$, it is easy to show that $\bar{f}(x) = (f(x))^*$.

Let $f, g \in \mathscr{B}(\sigma(x))$. For any $\xi, \eta \in \mathscr{H}$ we have

$$(f(x)\, g(x)\xi | \eta) = (g(x)\xi | (f(x))^*\eta) = \int_{-\infty}^{+\infty} g(\lambda)\, \mathrm{d}e_{\xi,(f(x))^*\eta}(\lambda).$$

But we have

$$e_{\xi,(f(x))^*\eta}(\lambda) = (e_\lambda \xi \,|\, (f(x))^* \eta) = (f(x) e_\lambda \xi \,|\, \eta)$$

$$= \int_{-\infty}^{+\infty} f(\mu)\, \mathrm{d}e_{e_\lambda \xi, \eta}(\mu) = \int_{-\infty}^{\lambda} f(\mu)\, \mathrm{d}e_{\xi,\eta}(\mu),$$

where the last equality follows from 2.19 (ii). Consequently, we have

$$(f(x)\, g(x) \xi \,|\, \eta) = \int_{-\infty}^{+\infty} g(\lambda)\, \mathrm{d}\left(\int_{-\infty}^{\lambda} f(\mu)\, \mathrm{d}e_{\xi,\eta}(\mu)\right)$$

$$= \int_{-\infty}^{+\infty} g(\lambda) f(\lambda)\, \mathrm{d}e_{\xi,\eta}(\lambda)$$

$$= \int_{-\infty}^{+\infty} (fg)(\lambda)\, \mathrm{d}e_{\xi,\eta}(\lambda) = ((fg)(x)\xi \,|\, \eta);$$

the second equality above is obvious if g is the characteristic function of an interval; this fact already implies that the measures $\mathrm{d}\left(\int_{-\infty}^{\lambda} f(\mu)\, \mathrm{d}e_{\xi,\eta}(\mu)\right)$ and $f(\lambda)\, \mathrm{d}e_{\xi,\eta}(\lambda)$ are equal, and, therefore, the same equality is true for any $g \in \mathscr{B}(\sigma(x))$. We have thus shown that $(fg)(x) = f(x)\, g(x)$. Consequently, the mapping $f \mapsto f(x)$ is a $*$-homomorphism of the C^*-algebra $\mathscr{B}(\sigma(x)$ into $\mathscr{B}(\mathscr{H})$.

If $f_0(\lambda) = 1$, $\lambda \in \sigma(x)$, then, obviously, $f_0(x) = 1$. If $f_1(\lambda) = \lambda$, $\lambda \in \sigma(x)$, then, by taking into account 2.19, (vii), we get $f_1(x) = x$. Since the mapping $f \mapsto f(x)$ is multiplicative, we now immediately get property (i).

For any $f \in \mathscr{B}(\sigma(x))$ and any $\xi \in \mathscr{H}$, we have the relation

$$(\bullet) \qquad \|f(x)\xi\|^2 = \int_{-\infty}^{+\infty} |f(\lambda)|^2\, \mathrm{d}e_{\xi,\xi}(\lambda).$$

Indeed, we have

$$\|f(x)\xi\|^2 = (f(x)\xi \,|\, f(x)\xi) = (f(x)^* f(x)\xi \,|\, \xi)$$

$$= (\bar{f}(x) f(x) \xi \,|\, \xi) = (|f|^2(x)\xi \,|\, \xi)$$

$$= \int_{-\infty}^{+\infty} |f(\lambda)|^2\, \mathrm{d}e_{\xi,\xi}(\lambda).$$

Property (ii) now easily follows since

$$\|(f_n(x) - f(x))\xi\|^2 = \int_{-\infty}^{+\infty} |f_n(\lambda) - f(\lambda)|^2\, \mathrm{d}e_{\xi,\xi}(\lambda),$$

whereas the integral converges to 0, by virtue of the dominated convergence theorem of Lebesgue.

Finally, the set $\{f \in \mathscr{B}(\sigma(x));\ f(x) \in \mathscr{R}(\{x\})\}$ contains the polynomials and it is closed with respect to the pointwise convergence of bounded sequences; therefore, from the above mentioned theorem of Baire, it equals $\mathscr{B}(\sigma(x))$.

Q.E.D.

If $x, y \in \mathscr{B}(\mathscr{H})$ are commuting self-adjoint operators (i.e., $xy = yx$), and if $f \in \mathscr{B}(\sigma(x))$, $g \in \mathscr{B}(\sigma(y))$, then $f(x)$ and $g(y)$ are commuting normal operators.

If $x \in \mathscr{B}(\mathscr{H})$ is a self-adjoint operator, and if $e \in \mathscr{B}(\mathscr{H})$ is a projection which commutes with x, then for any function $f \in \mathscr{B}(\sigma(x))$, $f(0) = 0$, we have

$$f(ex) = ef(x).$$

Indeed, this equality is easily checked for f, a polynomial without the constant term, and then, by tending to the limit, it obtains for any $f \in \mathscr{B}(\sigma(x))$, $f(0) = 0$.

Relation (*) from the proof of the theorem is useful in other situations, too. For example, with its help one can easily prove that if $f_n, f \in \mathscr{B}(\sigma(x))$, and $f_n \to f$ uniformly, then $f_n(x) \to f(x)$ for the norm topology.

We also observe that, since it is a *-homomorphism, the mapping $f \mapsto f(x)$ is positive.

2.21. The following fact has already been established in Section 2.19, but we mention it again due to its special usefulness:

Corollary. *Let $x \in \mathscr{B}(\mathscr{H})$ be a positive operator and $\alpha > 0$ a positive number. Then there exists a projection $e \in \mathscr{R}(\{x\})$, such that*

$$xe \geqslant \alpha e,$$

$$x(1-e) \leqslant \alpha(1-e).$$

We observe that one can take $e = \chi_{(\alpha, +\infty)}(x)$ or $e = \chi_{[\alpha, +\infty)}(x)$, and this fact shows that the projection e is not uniquely determined by the preceding conditions.

2.22. If $x \in \mathscr{B}(\mathscr{H})$ is a self-adjoint operator, then:

$$s(x) = \chi_{\mathbf{R}\setminus\{0\}}(x).$$

Indeed, from the obvious equality $\lambda \cdot \chi_{\mathbf{R}\setminus\{0\}}(\lambda) = \lambda$, $\lambda \in \mathbf{R}$, it follows that $x\chi_{\mathbf{R}\setminus\{0\}}(x) = x$, and this implies that $s(x) \leqslant \chi_{\mathbf{R}\setminus\{0\}}(x)$. On the other hand, from the relation $xs(x) = x$, it follows that $f(x)\,s(x) = f(x)$, first for f a polynomial without constant term, and then, by tending to the limit, for any $f \in \mathscr{B}(\sigma(x))$, $f(0) = 0$. In particular, we have $\chi_{\mathbf{R}\setminus\{0\}}(x)s(x) = \chi_{\mathbf{R}\setminus\{0\}}(x)$, and, therefore, $\chi_{\mathbf{R}\setminus\{0\}}(x) \leqslant s(x)$.

From the proof we also inferred that for any $f \in \mathscr{B}(\sigma(x))$, such that $f(0) = 0$, we have

$$s(f(x)) \leqslant s(x).$$

If $f_n \in \mathscr{B}(\sigma(x))$, $\sup_n \|f_n\| < +\infty$, and $f_n \to \chi_{\mathbf{R}\setminus\{0\}}$, then

$$f_n(x) \xrightarrow{so} s(x).$$

For example, if $x \geqslant 0$, we have

$$nx(1+nx)^{-1} \xrightarrow{so} \mathrm{s}(x),$$

$$x^{1/n} \xrightarrow{so} \mathrm{s}(x).$$

Also, it is easy to show that there exists a sequence of polynomials without constant terms, in x, which is *so*-convergent to $\mathrm{s}(x)$.

Corollary. *Let $x \in \mathscr{B}(\mathscr{H})$ be a positive operator. Then there exists a sequence of projections $\{e_n\} \subset \mathscr{R}(\{x\})$, such that*

$$xe_n \geqslant \frac{1}{n} e_n,$$

$$e_n \uparrow \mathrm{s}(x).$$

One can take

$$e_n = \chi_{\left(\frac{1}{n}, +\infty\right)}(x).$$

2.23. Already the spectral theorem (2.19, (vi)) implied that any self-adjoint operator $x \in \mathscr{B}(\mathscr{H})$ is the limit, for the norm topology, of linear combinations of projections from $\mathscr{R}(\{x\})$. In particular, any von Neumann algebra coincides with the norm-closed linear span of its projections. These results can be further strengthened by the following.

Corollary. *Let $x \in \mathscr{B}(\mathscr{H})$, $0 \leqslant x \leqslant 1$. Then there exists a sequence of projections $\{e_n\} \subset \mathscr{R}(\{x\})$, such that*

$$x = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n;$$

the series converges in the norm topology.

Proof. The sequence $\{e_n\}$ can be defined inductively in the following manner: According to Corollary 2.21, there exists a projection $e_1 \in \mathscr{R}(\{x\})$, such that

$$xe_1 \geqslant \frac{1}{2} e_1,$$

$$x(1-e_1) \leqslant \frac{1}{2}(1-e_1).$$

From Corollary 2.21, there exists a projection $e_n \in \mathscr{R}(\{x\})$, such that

$$\left(x - \sum_{k=1}^{n-1} \frac{1}{2^k} e_k\right) e_n \geqslant \frac{1}{2^n} e_n,$$

$$\left(x - \sum_{k=1}^{n-1} \frac{1}{2^k} e_k\right)(1-e_n) \leqslant \frac{1}{2^n}(1-e_n).$$

One can easily prove, by induction, and by using the hypothesis that $0 \leqslant x \leqslant 1$, that we have

$$0 \leqslant x - \sum_{k=1}^{n} \frac{1}{2^k} e_k \leqslant \frac{1}{2^n},$$

whence the desired assertion immediately follows.

Q.E.D.

The preceding corollary corresponds to the dyadic decomposition of the eal numbers be tween 0 and 1.

2.24. If $x \in \mathscr{B}(\mathscr{H})$ is an arbitrary operator, then the operators

$$x_1 = \frac{1}{2}(x + x^*),\ x_2 = \frac{i}{2}(x^* - x) \in \mathscr{C}^*(\{x\})$$

are self-adjoint, and

$$x = x_1 + ix_2.$$

An operator $u \in \mathscr{B}(\mathscr{H})$ is said to be *unitary* if it maps isometrically $\mathscr{H}$ *onto* $\mathscr{H}$. It is easily checked that $u \in \mathscr{B}(\mathscr{H})$ is unitary iff

$$u^*u = uu^* = 1.$$

Proposition. *Let $x \in \mathscr{B}(\mathscr{H})$ be an arbitrary operator. Then x is a linear combination of unitary operators from $\mathscr{C}^*(\{x, 1\})$.*

Proof. Because of the preceding remark, we can assume, without any loss of generality, that

$$x^* = x, \quad \|x\| \leqslant 1.$$

In this case, we define

$$u = x + i(1 - x^2)^{1/2},$$

and it is easy to check that u is unitary and

$$x = \frac{1}{2}(u + u^*).$$

Q.E.D.

In fact, we proved that any operator (resp., any self-adjoint, positive operator) is a linear combination of 4 (resp., a linear combination with positive coefficients of 2) unitary operators from the C^*-algebra with identity, generated by it We observe, therefore, that any C^*-algebra of operators (resp., anyC^*-algebra of operators with identity) is the vector space generated by the self-adjoin (resp., the unitary) operators it contains.

2.25. For arbitrary bounded linear operators on $\mathscr{H}$ one can define an operational calculus with functions analytic on a neighbourhood of the spectrum.

Let $x \in \mathscr{B}(\mathscr{H})$. We shall denote by $\mathscr{A}(\sigma(x))$ the set of all analytic functions defined on a neighbourhood of the spectrum $\sigma(x)$ of x (neighbourhood which can depend on the considered function). By identifying two functions from $\mathscr{A}(\sigma(x))$, if they coincide on a neighbourhood of the spectrum $\sigma(x)$, we can define canonically, in $\mathscr{A}(\sigma(x))$, an algebra structure. For any $f \in \mathscr{A}(\sigma(x))$ we shall consider closed rectifiable Jordan curves, with the positive orientation, $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, such that the interiors of these curves be mutually disjoint, the union of the interiors of these curves contain $\sigma(x)$, whereas the closure of this region be included in the domain of f. We denote $\Gamma = \{\Gamma_1, \ldots, \Gamma_k\}$ and define

$$f_{\mathscr{A}}(x) = (2\pi i)^{-1} \int_\Gamma f(\lambda)(\lambda - x)^{-1} d\lambda,$$

as a Cauchy integral, which converges in norm. From the well-known theorem of Cauchy, $f_{\mathscr{A}}(x)$ does not depend on the choice of Γ.

Theorem *(of operational calculus with analytic functions). Let $x \in \mathscr{B}(\mathscr{H})$ be an arbitrary operator.*

(i) *If f is a polynomial, $f(\lambda) = \alpha_0 + \alpha_1\lambda + \ldots + \alpha_n\lambda^n$, then $f \in \mathscr{A}(\sigma(x))$ and $f_{\mathscr{A}}(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n$.*

ii) *The mapping*

$$\mathscr{A}(\sigma(x)) \ni f \mapsto f_{\mathscr{A}}(x) \in \mathscr{B}(\mathscr{H})$$

is an algebra homomorphism.

Moreover, $f_{\mathscr{A}}(x) \in \mathscr{C}^*(\{x, 1\})$, for any $f \in \mathscr{A}(\sigma(x))$.

Proof. The mapping $f \mapsto f_{\mathscr{A}}(x)$ is obviously linear. Consequently, in order to prove (i), it suffices to consider the case $f(\lambda) = \lambda^n$, $n \geqslant 0$. Let Γ be a circle centered at 0 and of radius $> \|x\|$, positively oriented. For any $\lambda \in \Gamma$ we have

$$(\lambda - x)^{-1} = \sum_{k=0}^{\infty} \lambda^{-k-1} x^k,$$

he series being convergent in norm. Hence

$$f_{\mathscr{A}}(x) = (2\pi i)^{-1} \int_\Gamma \lambda^n(\lambda - x)^{-1} d\lambda = \sum_{k=0}^{\infty} ((2\pi i)^{-1} \int_\Gamma \lambda^{n-k-1} d\lambda) x^k = x^n.$$

We still have to prove the multiplicativity of the mapping $f \mapsto f_{\mathscr{A}}(x)$. Let f, $g \in \mathscr{A}(\sigma(x))$ and $\Gamma_1, \ldots, \Gamma_k, \Gamma'_1, \ldots, \Gamma'_j$ be closed rectifiable Jordan curves, positively oriented, such that the interiors of the curves $\Gamma_1, \ldots, \Gamma_k$ be mutually disjoint and their union include $\sigma(x)$, the closure of this union be included in the union of the mutually disjoint interiors of the curves $\Gamma'_1, \ldots, \Gamma'_j$ whereas the closure of this last union be included in the intersection of the domains of f and g. We denote $\Gamma = \{\Gamma_1, \ldots, \Gamma_k\}$ and $\Gamma' = \{\Gamma'_1, \ldots, \Gamma'_j\}$. From the identity

$$(\lambda - x)^{-1} - (\mu - x)^{-1} = (\mu - \lambda)(\lambda - x)^{-1}(\mu - x)^{-1},$$

we infer that

$$f_{\mathscr{A}}(x)\, g_{\mathscr{A}}(x) = -(4\pi^2)^{-1}\left(\int_\Gamma f(\lambda)(\lambda - x)^{-1}\mathrm{d}\lambda\right)\left(\int_{\Gamma'} g(\mu)(\mu - x)^{-1}\mathrm{d}\mu\right)$$

$$= -(4\pi^2)^{-1}\int_\Gamma\int_{\Gamma'} f(\lambda)g(\mu)(\lambda - x)^{-1}(\mu - x)^{-1}\mathrm{d}\lambda\mathrm{d}\mu$$

$$= -(4\pi^2)^{-1}\int_\Gamma\int_{\Gamma'} \frac{f(\lambda)\, g(\mu)}{\mu - \lambda}((\lambda - x)^{-1} - (\mu - x)^{-1})\mathrm{d}\lambda\mathrm{d}\mu$$

$$= -(4\pi^2)^{-1}\int_\Gamma f(\lambda)(\lambda - x)^{-1}\left(\int_{\Gamma'} \frac{g(\mu)}{\mu - \lambda}\mathrm{d}\mu\right)\mathrm{d}\lambda$$

$$+ (4\pi^2)^{-1}\int_{\Gamma'} g(\mu)(\mu - x)^{-1}\left(\int_\Gamma \frac{f(\lambda)}{\mu - \lambda}\mathrm{d}\lambda\right)\mathrm{d}\mu$$

$$= (2\pi\mathrm{i})^{-1}\int_\Gamma f(\lambda)g(\lambda)(\lambda - x)^{-1}\mathrm{d}\lambda = (fg)_{\mathscr{A}}(x).$$

Q.E.D.

The mapping $\mathscr{A}(\sigma(x)) \ni f \mapsto f_{\mathscr{A}}(x) \in \mathscr{B}(\mathscr{H})$ is also called the Dunford operational calculus for the operator x.

2.26. Corollary. *Let $x \in \mathscr{B}(\mathscr{H})$ and $f \in \mathscr{A}(\sigma(x))$. Then*

$$\sigma(f_{\mathscr{A}}(x)) = \{f(\lambda);\ \lambda \in \sigma(x)\}.$$

Proof. Let $\lambda_0 \in \sigma(x)$. We shall consider, on the domain of f, the analytic function g defined by the formula

$$g(\lambda) = \begin{cases} \dfrac{f(\lambda_0) - f(\lambda)}{\lambda_0 - \lambda} & \text{for } \lambda \neq \lambda_0 \\ f'(\lambda_0) & \text{for } \lambda = \lambda_0. \end{cases}$$

Then, from Theorem 2.25 (ii), we have

$$f(\lambda_0) - f_{\mathscr{A}}(x) = (\lambda_0 - x)\, g_{\mathscr{A}}(x);$$

hence, the invertibility of $f(\lambda_0) - f_{\mathscr{A}}(x)$ implies the invertibility of $\lambda_0 - x$, a contradiction. Consequently, $f(\lambda_0) - f_{\mathscr{A}}(x)$ is not invertible and $f(\lambda_0) \in \sigma(f_{\mathscr{A}}(x))$.

Conversely, let $\mu_0 \in \sigma(f_{\mathscr{A}}(x))$. If $\mu_0 \notin \{f(\lambda);\ \lambda \in \sigma(x)\}$, then the formula

$$h(\lambda) = \frac{1}{\mu_0 - f(\lambda)}$$

defines a function from $\mathscr{A}(\sigma(x))$ and, from Theorem 2.25 (ii), we have

$$h_{\mathscr{A}}(x)\,(\mu_0 - f_{\mathscr{A}}(x)) = 1,$$

contrary to the assumption that $\mu_0 \in \sigma(f_{\mathscr{A}}(x))$. Therefore, we have $\mu_0 \in \{f(\lambda); \lambda \in \sigma(x)\}$.

Q.E.D.

2.27. Corollary. *Let $x \in \mathscr{B}(\mathscr{H})$, $f \in \mathscr{A}(\sigma(x))$ and $g \in \mathscr{A}(\sigma(f_{\mathscr{A}}(x))$. Then we have*

$$g_{\mathscr{A}}(f_{\mathscr{A}}(x)) = (g \circ f)_{\mathscr{A}}(x).$$

Proof. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_k, \Gamma'_1, \ldots, \Gamma'_j$ be positively oriented, closed, rectifiable, Jordan curves, such that the interiors of the curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ be mutually disjoint and their union include $\sigma(x)$, the closure of this union be included in the domain of f and its image by f be included in the mutually disjoint union of the interiors of the curves $\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_j$, whereas the closure of this union be included in the domain of g. We denote $\Gamma = \{\Gamma_1, \ldots, \Gamma_k\}$ and $\Gamma' = \{\Gamma'_1, \ldots, \Gamma'_j\}$. For any $\mu \in \Gamma'$, the formula

$$h(\lambda) = \frac{1}{\mu - f(\lambda)}$$

determines a function $h \in \mathscr{A}(\sigma(x))$. From Theorem 2.25 (ii), we have

$$(\mu - f_{\mathscr{A}}(x))\, h_{\mathscr{A}}(x) = 1,$$

and, therefore,

$$(\mu - f_{\mathscr{A}}(x))^{-1} = h_{\mathscr{A}}(x).$$

Therefore, we have

$$\begin{aligned} g_{\mathscr{A}}(f_{\mathscr{A}}(x)) &= (2\pi i)^{-1} \int_{\Gamma'} g(\mu)(\mu - f_{\mathscr{A}}(x))^{-1}\, d\mu \\ &= -(4\pi^2)^{-1} \int_{\Gamma'} g(\mu) \left(\int_{\Gamma} \frac{1}{\mu - f(\lambda)} (\lambda - x)^{-1} d\lambda \right) \\ &= (2\pi i)^{-1} \int_{\Gamma} (g \circ f)(\lambda)\, (\lambda - x)^{-1}\, d\lambda = (g \circ f)_{\mathscr{A}}(x). \end{aligned}$$

Q.E.D.

2.28. In the set $\mathbb{C} \setminus \{\lambda; \operatorname{Re}\lambda \leqslant 0, \operatorname{Im}\lambda = 0\}$ we define the function ln by the formula

$$\ln \lambda = \ln |\lambda| + \arg \lambda, \quad -\pi < \arg \lambda < \pi.$$

In the same set, and for any $\alpha \in \mathbb{C}$, we define the function

$$\lambda \mapsto \lambda^{\alpha} = \exp(\alpha \ln \lambda).$$

The functions $\lambda \mapsto \ln \lambda$ and $\lambda \mapsto \lambda^{\alpha}$ are analytic in their domain of definition. Therefore, for any operator $x \in \mathscr{B}(\mathscr{H})$, such that $\sigma(x) \subset \mathbb{C} \setminus \{\lambda;\ \operatorname{Re}\lambda \leqslant 0,\ \operatorname{Im}\lambda = 0\}$, the operators $\ln x$ and x^{α}, $\alpha \in \mathbb{C}$, are well defined.

Corollary. *Let* $x \in \mathscr{B}(\mathscr{H})$ *be such that* $\sigma(x) \subset \mathbb{C} \setminus \{\lambda;\ \mathrm{Re}\,\lambda \leqslant 0,\ \mathrm{Im}\,\lambda = 0\}$. *Then the mapping*

$$\mathbb{C} \ni \alpha \mapsto x^\alpha \in \mathscr{B}(\mathscr{H})$$

is an entire function (with respect to the norm topology in $\mathscr{B}(\mathscr{H})$*).*

Proof. Let $y = \ln x$. From Corollary 2.27, for any $\alpha \in \mathbb{C}$ we have

$$x^\alpha = \exp(\alpha y).$$

By taking into account Theorem 2.25, it is easy to verify the relation

$$\exp(\alpha y) = \sum_{n=0}^{\infty} \frac{1}{n} y^n \alpha^n,$$

whence it immediately follows that the mapping $\alpha \mapsto \exp(\alpha y)$ is an entire function.

Q.E.D.

2.29. The following proposition yields a natural connection between the operational calculus with continuous functions (2.6) and the operational calculus with analytic functions (2.25). We obviously have $\mathscr{A}(\sigma(x)) \subset \mathscr{C}(\sigma(x))$, $x \in \mathscr{B}(\mathscr{H})$.

Proposition. *Let* $x \in \mathscr{B}(\mathscr{H})$ *be a self-adjoint operator and* $f \in \mathscr{A}(\sigma(x))$. *Then we have*

$$f_{\mathscr{A}}(x) = f(x).$$

Proof. For any $g \in \mathscr{A}(\sigma(x))$, the operator $g_{\mathscr{A}}(x)$ is normal. By virtue of Corollary 2.26, we have

$$\sigma(g_{\mathscr{A}}(x)) = \{g(\lambda);\ \lambda \in \sigma(x)\},$$

and, therefore, from Lemmas 2.5, 2.3, we have

$$\|g_{\mathscr{A}}(x)\| = |\sigma(g_{\mathscr{A}}(x))| = \sup\{|g(\lambda)|;\ \lambda \in \sigma(x)\}.$$

Therefore, the mapping

$$g|_{\sigma(x)} \mapsto g_{\mathscr{A}}(x),\ g \in \mathscr{A}(\sigma(x)),$$

is isometric. Since $\{g|_{\sigma(x)};\ g \in \mathscr{A}(\sigma(x))\}$ is a dense subset of $\mathscr{C}(\sigma(x))$, the preceding mapping can be uniquely extended to an isometric mapping of $\mathscr{C}(\sigma(x))$ into $\mathscr{B}(\mathscr{H})$. By taking into account Theorem 2.25 (i), and also the uniqueness part of Theorem 2.6, we infer that this extension coincides with the mapping

$$\mathscr{C}(\sigma(x)) \ni f \mapsto f(x) \in \mathscr{B}(\mathscr{H}),$$

which was defined in Theorem 2.6.

Q.E.D.

We observe that from the preceding proposition it follows, in particular, that for any self-adjoint operator $x \in \mathscr{B}(\mathscr{H})$ and any $f \in \mathscr{A}(\sigma(x))$ the operator $f_{\mathscr{A}}(x)$ depends only on $f|_{\sigma(x)}$.

2.30. Let $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$. We consider the mapping

$$[0, +\infty) \ni \lambda \mapsto \lambda^\alpha \in \mathbb{C},$$

defined on $(0, +\infty)$ as in 2.28, and equal to zero at 0. This mapping is Borel measurable and bounded on compact sets. Thus, for any positive operator $x \in \mathscr{B}(\mathscr{H})$ the (normal) operator x^α makes sense. From Proposition 2.29, this definition is compatible with that given in 2.28.

Corollary. *Let $x \in \mathscr{B}(\mathscr{H})$ be a positive operator and $\xi \in \mathscr{H}$; then the mapping*

$$\alpha \mapsto x^\alpha \xi \in \mathscr{H}$$

is continuous on $\{\alpha;\ \operatorname{Re}\alpha \geqslant 0\}$ and analytic in $\{\alpha;\ \operatorname{Re}\alpha > 0\}$, with respect to the norm topology in $\mathscr{H}$.

Proof. From Corollary 2.22, there exists a sequence of operators $\{e_n\} \subset \mathscr{R}(\{x\})$, such that

$$xe_n \geqslant \frac{1}{n} e_n, \quad e_n \uparrow s(x).$$

We write

$$x_n = \frac{1}{n}(1 - e_n) + xe_n.$$

Since $\sigma(x_n) \subset \left\{\lambda;\ \operatorname{Re}\lambda \geqslant \frac{1}{n}\right\}$, from Corollary 2.28 it follows that, for any $\xi \in e_n\mathscr{H}$, the mapping

$$\alpha \mapsto x^\alpha \xi = (x_n)^\alpha \xi$$

is continuous on $\{\alpha;\ \operatorname{Re}\alpha \geqslant 0\}$ and analytic in $\{\alpha;\ \operatorname{Re}\alpha > 0\}$.

Obviously, if $\xi \in \mathfrak{n}(x)\mathscr{H}$ (with the notation from 2.13), the mapping

$$\alpha \mapsto x^\alpha \xi = 0$$

is continuous on $\{\alpha;\ \operatorname{Re}\alpha \geqslant 0\}$ and analytic in $\{\alpha;\ \operatorname{Re}\alpha > 0\}$.

Let now $\xi \in \mathscr{H}$ be arbitrary. Since the set

$$\mathfrak{n}(x)\mathscr{H} \cup \bigcup_{n=1}^{\infty} e_n \mathscr{H}$$

is total in $\mathscr{H}$, there exists a sequence $\{\xi_k\}$, which converges to ξ, and is such that the mappings

$$\alpha \mapsto x^\alpha \xi_k;\ k = 1, 2, \ldots$$

be continuous on $\{\alpha;\ \operatorname{Re}\alpha \geqslant 0\}$ and analytic in $\{\alpha;\ \operatorname{Re}\alpha > 0\}$. Since the mappings $\alpha \mapsto x^\alpha \xi_k$ converge uniformly, on any compact subset of $\{\alpha;\ \operatorname{Re}\alpha \geqslant 0\}$, to the mapping $\alpha \mapsto x^\alpha \xi$, it follows that the mapping $\alpha \mapsto x^\alpha \xi$ is continuous on $\{\alpha;\ \operatorname{Re}\alpha \geqslant 0\}$ and analytic in $\{\alpha;\ \operatorname{Re}\alpha > 0\}$.

Q.E.D.

2.31. In the preceding sections we already used the fact that, since the function exp is an entire function, for any operator $x \in \mathscr{B}(\mathscr{H})$ the operator $\exp(x)$ makes sense and the relation

$$\exp(x) = \sum_{n=1}^{\infty} \frac{1}{n!} x^n$$

holds.

We hence infer that, for any $x \in \mathscr{B}(\mathscr{H})$, we have

$$(\exp(x))^* = \exp(x^*);$$

also, if $x, y \in \mathscr{B}(\mathscr{H})$, $xy = yx$, we have

$$\exp(x)\exp(y) = \exp(x + y).$$

In particular, if $x \in \mathscr{B}(\mathscr{H})$ is self-adjoint, then the operator $\exp(\mathrm{i}x)$ is unitary.

Proposition. *Let* x_1, x_2, $y \in \mathscr{B}(\mathscr{H})$. *If* x_1, x_2 *are normal and if* $x_1 y = y x_2$, *then* $x_1^* y = y x_2^*$.

Proof. The function

$$f\colon \mathbb{C} \ni \lambda \mapsto \exp(-\lambda x_1^*)\, y \exp(\lambda x_2^*) \in \mathscr{B}(\mathscr{H})$$

is an entire analytic function with respect to the norm topology of $\mathscr{B}(\mathscr{H})$.

From the relation $yx_2 = x_1 y$ we infer that, for any $\lambda \in \mathbb{C}$, we have

$$y = \exp \bar{\lambda} x_1)\, y \exp(-\bar{\lambda} x_2),$$

and, therefore, we have

$$f(\lambda) = \exp(-\lambda x_1^*) \exp(\bar{\lambda} x_1)\, y \exp(-\bar{\lambda} x_2) \exp(\lambda x_2^*)$$

$$= \exp(\mathrm{i}(\mathrm{i}\lambda x_1^* - \mathrm{i}\bar{\lambda} x_1))\, y \exp(\mathrm{i}(\mathrm{i}\bar{\lambda} x_2 - \mathrm{i}\lambda x_2^*)).$$

Since the operators $\mathrm{i}\lambda x_1^* - \mathrm{i}\bar{\lambda} x_1$ and $\mathrm{i}\bar{\lambda} x_2 - \mathrm{i}\lambda x_2^*$ are self-adjoint their exponentials are unitary operators; therefore, the function f is also bounded.

From the Liouville theorem it follows that f is constant. Consequently, its derivative is equal to zero:

$$0 = f'(\lambda) = -x_1^* \exp(-\lambda x_1^*)\, y \exp(\lambda x_2^*) + \exp(-\lambda x_1^*)\, y \exp(\lambda x_2^*)\, x_2^*.$$

In particular, we have $f'(0) = 0$, and this implies that

$$x_1^* y = y x_2^*.$$

Q.E.D.

From the preceding proposition we infer, in particular, that if an operator y commutes with a normal operator x, then it commutes with its adjoint x^*, too.

2.32. In this section we recall the structure of operators in Hilbert direct sums of Hilbert spaces and we introduce some notations.

Let $\mathscr{H}$ be a Hilbert space, γ an arbitrary cardinal number, I a set of indices, whose cardinal is γ, and $(\mathscr{H}_i)_{i\in I}$ a family of Hilbert spaces, such that $\mathscr{H}_i = \mathscr{H}$ for any $i \in I$. We consider the Hilbert direct sum

$$\widetilde{\mathscr{H}}_\gamma = \bigoplus_{i\in I} \mathscr{H}_i.$$

The elements of the Hilbert space $\widetilde{\mathscr{H}}_\gamma$ are the families $\tilde{\xi}=(\xi_i)_{i\in I} \subset \mathscr{H}$, such that $\sum_{i\in I} \|\xi_i\|^2 < +\infty$, whereas for any two elements $\tilde{\xi}=(\xi_i)_{i\in I}, \tilde{\eta} = (\eta_i)_{i\in I} \in \widetilde{\mathscr{H}}_\gamma$ we have, by definition

$$(\tilde{\xi} \mid \tilde{\eta}) = \sum_{i\in I} (\xi_i \mid \eta_i).$$

For any $i_0 \in I$ we consider the operator

$$u_{i_0} : \mathscr{H} \ni \xi \mapsto u_{i_0}(\xi) \in \widetilde{\mathscr{H}}_\gamma,$$

where, for any $u \in \mathscr{H}$, we define

$$u_{i_0}(\xi) = (\xi_i)_{i\in I}, \quad \xi_i = \begin{cases} 0 \text{ for } i \neq i_0 \\ \xi \text{ for } i = i_0. \end{cases}$$

The adjoint of this operator is

$$u_{i_0}^* : \widetilde{\mathscr{H}}_\gamma \ni \tilde{\xi} \mapsto u_{i_0}^*(\xi) \in \mathscr{H}$$

where, for any $\tilde{\xi} = (\xi_i)_{i\in I} \in \widetilde{\mathscr{H}}_\gamma$, we have

$$u_{i_0}^*(\tilde{\xi}) = \xi_{i_0}.$$

It is easily checked that for any $i \in I$ the operator u_i is a linear isometric operator, and

$$u_i^* u_i = \text{the identity on } \mathscr{H},$$

$$u_i u_i^* = \text{the projection of } \widetilde{\mathscr{H}}_\gamma \text{ onto } \mathscr{H}_i.$$

To any operator $x \in \mathscr{B}(\widetilde{\mathscr{H}}_\gamma)$ one can associate a "matrix" (x_{ik}) of operators from $\mathscr{B}(\mathscr{H})$, by the relation

$$x_{ik} = u_i^* \circ x \circ u_k, \quad i, k \in I;$$

with its help the operator x can be recovered by the formula

(*)
$$x = \sum_{i,k\in I} u_i \circ x_{ik} \circ u_k^*,$$

where the series is *so*-convergent.

Conversely, if γ is a finite cardinal number, then to any "matrix" of elements from $\mathscr{B}(\mathscr{H})$ there corresponds an operator from $\mathscr{B}(\widetilde{\mathscr{H}}_\gamma)$ by the formula (*).

If γ is an infinite cardinal, then, of course, only those "matrices" which satisfy the convergence condition from (*) can yield operators from $\mathscr{B}(\widetilde{\mathscr{H}}_\gamma)$. For example, for any family $(x_i)_{i\in I} \subset \mathscr{B}(\mathscr{H})$, such that $\sup_{i\in I} \|x_i\| < +\infty$, the "matrix" $(\delta_{ik}x_i)$ yields an operator from $\mathscr{B}(\widetilde{\mathscr{H}}_\gamma)$ which leaves invariant all the subspaces $\mathscr{H}_i$; obviously, any operator from $\mathscr{B}(\widetilde{\mathscr{H}}_\gamma)$, having this property, is of this form.

In particular, for any $x \in \mathscr{B}(\mathscr{H})$ we can consider the operator

$$\tilde{x} = (\delta_{ik}x) \in \mathscr{B}(\widetilde{\mathscr{H}}_\gamma),$$

which commutes with all operators $u_i u_k^* \in \mathscr{B}(\widetilde{\mathscr{H}}_\gamma)$. It is easy to see that, if an operator $(x_{ik}) \in \mathscr{B}(\widetilde{\mathscr{H}}_\gamma)$ commutes with all the operators $u_i u_k^* \in \mathscr{B}(\widetilde{\mathscr{H}}_\gamma)$, then $x_{ik} = 0$ for $i \neq k$ and $x_{ii} = x_{kk}$ for any $i, k \in I$; consequently, there exists an $x \in \mathscr{B}(\mathscr{H})$, such that $(x_{ik}) = \tilde{x}$.

Let $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$. We shall use the notations:

$$\mathrm{Mat}_\gamma(\mathscr{X}) = \{(x_{ik}) \in \mathscr{B}(\widetilde{\mathscr{H}}_\gamma);\, x_{ik} \in \mathscr{X} \text{ for any } i, k \in I\}$$

$$\widetilde{\mathscr{X}}_\gamma = \{\tilde{x} \in \mathscr{B}(\widetilde{\mathscr{H}}_\gamma);\, x \in \mathscr{X}\}.$$

Therefore, we have $\mathscr{B}(\widetilde{\mathscr{H}}_\gamma) = \mathrm{Mat}_\gamma(\mathscr{B}(\mathscr{H}))$.

If $\lambda \in \mathbb{C}$ and $(x_{ik}), (y_{ik}) \in \mathscr{B}(\widetilde{\mathscr{H}}_\gamma)$, it is easy to check that we have

$$(x_{ik}) + (y_{ik}) = (x_{ik} + y_{ik}),$$

$$\lambda(x_{ik}) = (\lambda x_{ik}),$$

$$(x_{ik})^* = (x_{ki}^*),$$

$$(x_{ik})(y_{ik}) = \Big(\sum_{j\in I} x_{ij} y_{jk}\Big),$$

the series in the right-hand member of the last equality being *so*-convergent. In particular, for $\lambda \in \mathbb{C}, x, y \in \mathscr{B}(\mathscr{H})$, we have

$$\tilde{x} + \tilde{y} = (x + y)^\sim,$$

$$\lambda\tilde{x} = (\lambda x)^\sim,$$

$$(x)^* = (x^*)^\sim,$$

$$\tilde{x}\tilde{y} = (xy)^\sim.$$

In what follows, if γ is a finite cardinal number, i.e., a natural number n, we shall write n instead of γ, whereas if γ is an infinite cardinal number, known from the context, then we shall omit it.

2.33. Let $\mathscr{H}$, $\mathscr{K}$ be two Hilbert spaces and $\mathscr{H} \otimes \mathscr{K}$ their (algebraic) tensor product as vector spaces. In $\mathscr{H} \otimes \mathscr{K}$ one can define a unique pre-Hilbert structure, such that

$$(\xi_1 \otimes \eta_1 \mid \xi_2 \otimes \eta_2) = (\xi_1 \mid \xi_2)(\eta_1 \mid \eta_2),$$

for any $\xi_1, \xi_2 \in \mathscr{H}$, $\eta_1, \eta_2 \in \mathscr{K}$. This pre-Hilbert structure is separated. The Hilbert space obtained by the completion of the pre-Hilbert space $\mathscr{H} \otimes \mathscr{K}$ is called the *Hilbert tensor product* of the spaces $\mathscr{H}$ and $\mathscr{K}$ and it is denoted by $\mathscr{H} \overline{\otimes} \mathscr{K}$.

Let $x \in \mathscr{B}(\mathscr{H})$, $y \in \mathscr{B}(\mathscr{K})$. The tensor product, $x \otimes y$ of the linear operators x, y is a continuous linear operator on $\mathscr{H} \otimes \mathscr{K}$. Indeed, since $x \otimes y = (x \otimes 1)(1 \otimes y)$, we can assume, for example, that $y = 1$. Let

$$\sum_{k=1}^{n} \xi_k \otimes \eta_k \in \mathscr{H} \otimes \mathscr{K};$$

we can assume that the vectors η_k are mutually orthogonal. We then have

$$\left\| (x \otimes 1) \sum_{k=1}^{n} \xi_k \otimes \eta_k \right\|^2 = \left\| \sum_{k=1}^{n} x\xi_k \otimes \eta_k \right\|^2$$

$$= \sum_{k=1}^{n} \|x\xi_k\|^2 \|\eta_k\|^2 \leqslant \|x\|^2 \sum_{k=1}^{n} \|\xi_k\|^2 \|\eta_k\|^2 = \|x\|^2 \left\| \sum_{k=1}^{n} \xi_k \otimes \eta_k \right\|^2,$$

and the assertion is proved. Consequently, $x \otimes y$ can be extended in a unique manner to a continuous operator $x \overline{\otimes} y \in \mathscr{B}(\mathscr{H} \overline{\otimes} \mathscr{K})$.

It is easily verified that the mapping

$$\mathscr{B}(\mathscr{H}) \times \mathscr{B}(\mathscr{K}) \ni (x, y) \mapsto (x \overline{\otimes} y) \in \mathscr{B}(\mathscr{H} \overline{\otimes} \mathscr{K})$$

is bilinear*); also, for any $x_1, x_2 \in \mathscr{B}(\mathscr{H})$, $y_1, y_2 \in \mathscr{B}(\mathscr{K})$, we have

$$(x_1 \overline{\otimes} y_1)(x_2 \overline{\otimes} y_2) = x_1x_2 \overline{\otimes} y_1y_2,$$

and, for any $x \in \mathscr{B}(\mathscr{H})$, $y \in \mathscr{B}(\mathscr{K})$, we have

$$(x \overline{\otimes} y)^* = x^* \overline{\otimes} y^*.$$

In particular, if $x \in \mathscr{B}(\mathscr{H})$, $y \in \mathscr{B}(\mathscr{K})$ are self-adjoint (resp., normal, unitary or projection) operators, then $x \overline{\otimes} y \in \mathscr{B}(\mathscr{H} \overline{\otimes} \mathscr{K})$ is a self-adjoint (resp., normal, unitary, projection) operator. Also if $x \in \mathscr{B}(\mathscr{H}), y \in \mathscr{B}(\mathscr{K})$ are positive operators, then $x \overline{\otimes} y \in \mathscr{B}(\mathscr{H} \overline{\otimes} \mathscr{K})$ is a positive operator. If $x \in \mathscr{B}(\mathscr{H})$, $y \in \mathscr{B}(\mathscr{K})$ and $x = u|x|$, $y = v|y|$ are the polar decompositions of these operators, then $x \overline{\otimes} y = (u \overline{\otimes} v)(|x| \overline{\otimes} |y|)$ is the polar decomposition of $x \overline{\otimes} y \in \mathscr{B}(\mathscr{H} \overline{\otimes} \mathscr{K})$.

2.34. Let $\mathscr{H}, \mathscr{K}$ be Hilbert spaces, $(\eta_i)_{i \in I}$ an orthonormal basis in $\mathscr{K}$ and $\gamma = \dim \mathscr{K} = \operatorname{card} I$. In what follows we shall show that the orthonormal basis we have chosen allows a canonical identification of the Hilbert spaces $\mathscr{H} \overline{\otimes} \mathscr{K}$ and $\tilde{\mathscr{H}}_\gamma$.

Indeed, for any $i \in I$, the linear, isometric mapping

$$\mathscr{H} \ni \xi \mapsto \xi \otimes \eta_i \in \mathscr{H} \overline{\otimes} \mathscr{K}$$

determines a canonical identification of the Hilbert space $\mathscr{H}$ with a closed subspace $\mathscr{H}_i$ of $\mathscr{H} \overline{\otimes} \mathscr{K}$. The spaces $\mathscr{H}_i$ are mutually orthogonal, whereas their

*) One can show that the corresponding linear mapping from the algebraic tensor product is injective.

union $\bigcup_{\iota \in I} \mathscr{H}_\iota$ is total in $\mathscr{H} \overline{\otimes} \mathscr{K}$. Consequently, $\mathscr{H} \overline{\otimes} \mathscr{K}$ is the Hilbert direct sum of the spaces $\mathscr{H}_\iota$. Therefore, the mapping

$$(\xi_\iota)_{\iota \in I} \mapsto \sum_{\iota \in I} \xi_\iota \otimes \eta_\iota$$

establishes a canonical identification

$$\tilde{\mathscr{H}}_\gamma = \mathscr{H} \overline{\otimes} \mathscr{K}.$$

Once this identification has been done, the operators from $\mathscr{B}(\mathscr{H} \overline{\otimes} \mathscr{K})$ can be represented by "matrices" of operators from $\mathscr{B}(\mathscr{H})$. For example, it is easily checked that, for any $x \in \mathscr{B}(\mathscr{H})$, we have

$$\tilde{x} = x \overline{\otimes} 1.$$

Exercises

E.2.1. Let $x, y \in \mathscr{B}(\mathscr{H})$. If $1 - xy$ is invertible, then $1 - yx$ is also invertible. Infer that

$$\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\}.$$

E.2.2. Let $x \in \mathscr{B}(\mathscr{H})$ be self-adjoint. Show that

$$\|x\| = \sup\{|(x\xi \mid \xi)|;\ \xi \in \mathscr{H}, \|\xi\| = 1\}.$$

!E.2.3. Let $e, f \in \mathscr{B}(\mathscr{H})$ be projections. Show that

$$\mathbf{l}(ef) = e - e \wedge (1-f), \qquad \mathbf{r}(ef) = f - (1-e) \wedge f;$$

$$\mathbf{l}(e(1-f)) = e - e \wedge f, \qquad \mathbf{r}(e(1-f)) = e \vee f - f.$$

E.2.4. Let $e, f \in \mathscr{B}(\mathscr{H})$ be projections. Then the sequence $\{(ef)^n\}$ *so*-converges to $e \wedge f$.

E.2.5. For any projection $e \in \mathscr{B}(\mathscr{H})$, the operator $s = 1 - 2e$ is a *symmetry* (i.e., self-adjoint and unitary). Conversely, any symmetry is of this form.

!E.2.6. Let $a, b \in \mathscr{B}(\mathscr{H})$, $0 \leqslant a \leqslant b$. Show that there exists an $x \in \mathscr{B}(\mathscr{H})$, $\|x\| \leqslant 1$, such that $a^{1/2} = xb^{1/2}$.

!E.2.7. Let $a, b \in \mathscr{B}(\mathscr{H})$, $0 \leqslant a \leqslant b$, $ab = ba$. Show that $0 \leqslant a^2 \leqslant b^2$. Infer that $a\mathscr{H} \subset b\mathscr{H}$.

E.2.8. Let $a, b \in \mathscr{B}(\mathscr{H})$, $0 \leqslant a \leqslant b$, a invertible. Then b is invertible and $0 \leqslant b^{-1} \leqslant a^{-1}$.

E.2.9. Let $\{x_\iota\}_{\iota \in I} \subset \mathscr{B}(\mathscr{H})$ be a net of normal operators, which *so*-converges to the normal operator $x \in \mathscr{B}(\mathscr{H})$. Then the net $\{x_\iota^*\}_{\iota \in I}$ is *so*-convergent to x^*. In other words, the restriction of the $*$-operation to the set of normal operators is *so*-continuous.

E.2.10. Show that if $x \in \mathscr{B}(\mathscr{H})$ is normal, then $\|x\|^2 = \|x^2\|$.

E.2.11. Two operators $x, y \in \mathscr{B}(\mathscr{H})$ are said to be *similar* (resp., *unitarily equivalent)* if there exists an invertible (resp., unitary) operator $s \in \mathscr{B}(\mathscr{H})$, such that $y = sxs^{-1}$. Show that two normal similar operators are unitarily equivalent.

E.2.12. Let $\mathscr{A}$ be a commutative Banach algebra with unit $1 \in \mathscr{A}$. Show that any element $x \in A$, such that $\|1 - x\| < 1$ is invertible. Infer that any maximal ideal $\mathfrak{M}$ of $\mathscr{A}$ is closed and any non-zero element from the Banach algebra $\mathscr{A}/\mathfrak{M}$ is invertible. Then, with the help of the Liouville theorem infer that $\mathscr{A}/\mathfrak{M}$ consists of the scalar multiples of the unit element.

E.2.13. Let $x \in \mathscr{B}(\mathscr{H})$ be normal and $\lambda \in \mathbb{C}$. Then $\lambda \in \sigma(x)$ iff $\lambda - x$ belongs to a maximal ideal of $\mathscr{C}^*(\{x, 1\})$.

E.2.14. Let $x \in \mathscr{B}(\mathscr{H})$ be normal and $p\,(.,.)$ be a complex polynomial in two variables. Then:

$$\sigma(p(x, x^*)) = \{p(\lambda, \bar{\lambda});\ \lambda \in \sigma(x)\}.$$

E.2.15. Extend Theorem 2.6 and Corollary 2.7 to the case of normal operators $x \in \mathscr{B}(\mathscr{H})$.

E.2.16. Let $x \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator, $\{e_\lambda\}$ its spectral scale and $f \in \mathscr{C}(\sigma(x))$. Then

$$f(x) = \int_{-\infty}^{+\infty} f(\lambda)\, de_\lambda,$$

where the integral is a norm-convergent vector Stieltjes integral.

E.2.17. Let $x \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator. For any Borel subset D of the spectrum $\sigma(x)$ of x we define the *spectral projection* of x, which corresponds to the D, by the formula

$$e(D) = \chi_D(x).$$

Then, for any $f \in \mathscr{B}(\sigma(x))$, we have

$$\|f(x)\| = \inf_{e(D)=1} \sup_{\lambda \in D} |f(\lambda)|$$

and

$$\sigma(f(x)) = \bigcap_{e(D)=1} \overline{\{f(\lambda);\ \lambda \in D\}} \subset \overline{\{f(\lambda);\ \lambda \in \sigma(x)\}}.$$

E.2.18. Let $x \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator and $f \in \mathscr{B}(\sigma(x))$, real. Then, for any $g \in \mathscr{B}(\overline{\{f(\lambda);\ \lambda \in \sigma(x)\}})$, we have

$$g(f(x)) = (g \circ f)(x).$$

***E.2.19.** An operator $x \in \mathscr{B}(\mathscr{H})$ is said to be *compact* if for any bounded subset $\mathscr{S} \subset \mathscr{H}$ the set $x(\mathscr{S})$ is relatively compact.

One usually denotes by $\mathscr{K}($ the set of all com $\mathscr{H}$pact operators from $\mathscr{B}(\mathscr{H})$.

Show that $\mathcal{K}(\mathcal{H})$ is the smallest non-zero, norm-closed two-sided ideal of $\mathcal{B}(\mathcal{H})$. If $\mathcal{H}$ is infinitely dimensional and separable, then $\mathcal{K}(\mathcal{H})$ is the only proper, norm closed, two-sided ideal of $\mathcal{B}(\mathcal{H})$.

***E.2.20.** Let $x \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. One says that λ is an *eigenvalue* of x if $\mathcal{E}_\lambda = \ker(x - \lambda) \neq 0$; in this case the non-zero vectors from $\mathcal{E}_\lambda$ are called *eigenvectors*, and the dimension of $\mathcal{E}_\lambda$ is called the *multiplicity* of the eigenvalue λ. If λ is an eigenvalue of x, then $\lambda \in \sigma(x)$. The eigenvectors which correspond to different eigenvalues of a self-adjoint operator are orthogonal.

Show that if $x \in \mathcal{K}(\mathcal{H})$ and $0 \neq \lambda \in \sigma(x)$, then λ is an eigenvalue of finite multiplicity of x. Thus, the spectrum of a compact operator is either a finite set, or forms a sequence converging to zero.

E.2.21. Let $x \in \mathcal{K}(\mathcal{H})$, $x \geq 0$, $\sigma(x) \setminus \{0\} = \{\lambda_1, \lambda_2, \ldots\}$ and let e_k be the orthogonal projection onto $\mathcal{E}_{\lambda_k}$. Then the projections e_k are mutually orthogonal and

$$x = \sum_k \lambda_k e_k,$$

the series being norm-convergent.

***E.2.22.** Let $x \in \mathcal{B}(\mathcal{H})$, $\|x\| \leq 1$. Show that there exists an isometry v of $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ and a unitary operator $u \in \mathcal{B}(\mathcal{K})$, such that

(i) $x^n = v^* u^n v$; $n = 1, 2, \ldots$

(ii) the set $\sum_{n \in \mathbb{Z}} u^n v(\mathcal{H})$ is total in $\mathcal{K}$.

The pair $(v: \mathcal{H} \to \mathcal{K}, u)$ is called the *minimal unitary dilation* of the "contraction" x and it is unique, in an obvious sense.

E.2.23. Let $x \in \mathcal{B}(\mathcal{H})$, $\|x\| \leq 1$, and $p(\cdot)$ be a complex polynomial. With the help of the unitary dilation of x, prove the following *von Neumann inequality*

$$\|p(x)\| \leq \sup\{|p(\lambda)|; \lambda \in \mathbb{C}, |\lambda| = 1\}.$$

E.2.24. Let $x \in \mathcal{B}(\mathcal{H})$, $\|x\| \leq 1$. Show that for any $\xi \in \mathcal{H}$, we have

$$x\xi = \xi \Leftrightarrow x^*\xi = \xi,$$

whence infer that

$$\mathbf{n}(1 - x) = \mathbf{n}(1 - x^*) \leq \mathbf{l}(x) \wedge \mathbf{r}(x).$$

By denoting $y = x - \mathbf{n}(1 - x)$, show that

$$\mathbf{n}(1 - y) = \mathbf{n}(1 - y^*) = 0.$$

E.2.25. Let $x \in \mathcal{B}(\mathcal{H})$, $\|x\| \leq 1$. Show that for any $\xi \in \mathcal{H}$ we have the following convergence

$$\frac{1}{n}(\xi + x\xi + \ldots + x^{n-1}\xi) \to p\xi,$$

where p is the orthogonal projection onto $\{\zeta \in \mathcal{H}; x\zeta = \zeta\}$.

This result is the *mean ergodic theorem of von Neumann*. (Hint: Let $\mathscr{H}_1 = \{\zeta \in \mathscr{H};\ x\zeta = \zeta\}$ and $\mathscr{H}_2 = \{\eta - x\eta;\ \eta \in \mathscr{H}\}$; in accordance with E.2.24, we have $\mathscr{H}_1 = \mathscr{H}_2^{\perp}$, and, therefore, $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$; the convergence can be checked, separately, for $\xi \in \mathscr{H}_1$ and $\xi \in \mathscr{H}_2$. The proof we have sketched here is by B. Sz.-Nagy. For other information we refer the reader to N. Dunford and J. Schwartz [1], ch. VIII).

Comments

C.2.1. In this section we will state briefly some results concerning the theory of abstract C^*-algebras. In doing so we will repeat the operational calculus for normal operators.

If $\mathscr{A}$ is a C^*-algebra, in the algebra $\tilde{\mathscr{A}}$ obtained by the adjunction of the unit element to $\mathscr{A}$ one can introduce canonically a structure of a C^*-algebra. For simplicity's sake, we shall assume in what follows that all C^*-algebras encountered have a unit. The ideas and results from Sections 2.3, 2.4, 2.5 also extend to the case of the abstract C^*-algebras, and with the same proofs.

Let $\mathscr{A}$ be a commutative C^*-algebra. A *character* of $\mathscr{A}$ is any non-zero homomorphism $\omega\colon \mathscr{A} \to \mathbb{C}$. For any $x \in \mathscr{A}$ the element $\omega(x) - x$ belongs to the kernel $\ker\omega$, which is a two-sided ideal of $\mathscr{A}$, and, therefore, $\omega(x) \in \sigma(x)$. It follows that $|\omega(x)| \leqslant |\sigma(x)| \leqslant \|x\|$ and, if x is self-adjoint, then $\omega(x)$ is real. Consequently, $\|\omega\| = 1$ and $\omega(x^*) = \overline{\omega(x)}$, $x \in \mathscr{A}$. The set $\Omega_{\mathscr{A}}$ of all characters of $\mathscr{A}$, endowed with the topology induced by the $\sigma(\mathscr{A}^*, \mathscr{A})$-topology, is a compact space, called the *spectrum* of $\mathscr{A}$. Any element $x \in \mathscr{A}$ determines a function $\hat{x} \in \mathscr{C}(\Omega_{\mathscr{A}})$, given by

$$\hat{x}(\omega) = \omega(x), \quad \omega \in \Omega_{\mathscr{A}}.$$

The mapping $\omega \mapsto \ker\omega$ establishes a bijection between the set of all characters of $\mathscr{A}$ and the maximal (two-sided) ideals of $\mathscr{A}$ (see E.2.12). If $x \in \mathscr{A}$ and $\lambda \in \sigma(x)$, then $\lambda - x$ belongs to a maximal ideal of $\mathscr{A}$ (E.2.13); therefore, there exists an $\omega \in \Omega_{\mathscr{A}}$, such that $\omega(x) = \lambda$. Consequently, for any $x \in \mathscr{A}$, we have

$$\|x\| = |\sigma(x)| = \sup\{|\omega(x)|;\ \omega \in \Omega_{\mathscr{A}}\} = \|\hat{x}\|.$$

From the preceding results and, by taking into account the Stone-Weierstrass theorem, we infer the following

Theorem *(the Gelfand representation). Let $\mathscr{A}$ be a commutative C^*-algebra. The mapping*

$$x \mapsto \hat{x}$$

establishes an isometric $*$-isomorphism of the C^*-algebra $\mathscr{A}$ onto the C^*-algebra $\mathscr{C}(\Omega_{\mathscr{A}})$.

Let x be a normal element of an arbitrary C^*-algebra, e.g., a normal operator from $\mathscr{B}(\mathscr{H})$. If $\mathscr{A} = \mathscr{C}^*(\{x, 1\})$, then the mapping

$$\omega \mapsto \omega(x)$$

establishes a homeomorphism of $\Omega_{\mathscr{A}}$ onto $\sigma(x)$. Consequently, there exists an isometric $*$-isomorphism

$$\mathscr{C}(\sigma(x)) \ni f \mapsto f(x) \in \mathscr{C}^*(\{x, 1\}),$$

which is called the *operational calculus* for the normal element x. In particular, Theorem 2.6 also extends to normal operators (see E.2.15).

The notion of positive operator extends, with the same definition, to the notion of a positive element, whereas the results from 2.7—2.10 extend to arbitrary C*-algebras, with the same proof. Also, the equivalence of statements (i), (ii), (iii) from Proposition 2.12 remains true, but, in order to prove this, the following remarks are necessary. Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{A}^+ = \{x \in \mathscr{A};\ x \geqslant 0\}$; a self-adjoint element $x \in \mathscr{A}$, $\|x\| \leqslant 1$, is positive iff $\|1 - x\| \leqslant 1$; this can easily be proved by using the Gelfand representation; with the help of this result one can easily prove that $\mathscr{A}^+$ is a closed convex cone and $\mathscr{A}^+ \cap (-\mathscr{A}^+) = \{0\}$. If $x \in \mathscr{A}$ and $x^*x \in (-\mathscr{A}^+)$, then $x = 0$. Indeed, let $x = h + ik$, where $h, k \in \mathscr{A}$ are self-adjoint. From the hypothesis $-x^*x \in \mathscr{A}^+$ and from E.2.1 it follows that $-xx^* \in \mathscr{A}^+$. Then one can immediately check that

$$x^*x = 2h^2 + 2k^2 + (-xx^*) \in \mathscr{A}^+,$$

hence $x^*x \in \mathscr{A}^+ \cap (-\mathscr{A}^+) = \{0\}$, and this implies that $x = 0$. With the help of these hints, the implication (iii) $\Rightarrow$ (i) from 2.12 follows with a slight modification of the argument just used in the implication (iv) $\Rightarrow$ (i) from 2.12.

By using the Gelfand representation and the first remark from Section 3.12, one can show that any injective $*$-homomorphism of C^*-algebras is isometric.

Let $\mathscr{A}$ be an abstract C^*-algebra. For any element $x \in \mathscr{A}$, $x \neq 0$, we have $-x^*x \notin \mathscr{A}^+$. From the Krein-Rutman theorem, there exists a positive form (see 5.1) φ_x on $\mathscr{A}$, such that $\varphi_x(-x^*x) < 0$. With the help of the form φ_x one gets, as in Section 5.18, a $*$-homomorphism $\pi_{\varphi_x}\colon \mathscr{A} \to \mathscr{B}(\mathscr{H}_{\varphi_x})$, $\pi_{\varphi_x}(x) \neq 0$. By denoting

$$\pi\colon \mathscr{A} \to \mathscr{B}(\mathscr{H})$$

the *direct sum* of the mappings π_{φ_x}, $x \in \mathscr{A}$, it follows that π is an injective $*$-homomorphism of the C^*-algebra $\mathscr{A}$ into the C^*-algebra $\mathscr{B}(\mathscr{H})$. We can thus obtain the following:

Theorem. *Any C^*-algebra is isometrically-$*$-isomorphic with a C^*-algebra of operators on a Hilbert space.*

I. M. Gelfand and M. A. Naimark defined the notion of a C^*-algebra by the following axioms:

(I) $\mathscr{A}$ is $*$-algebra;

(II) $\mathscr{A}$ is a Banach space, with the vector structure of (I);

(III) $\|xy\| \leqslant \|x\| \, \|y\|$, for any $x, y \in \mathscr{A}$;
(IV) $\|x^*x\| = \|x^*\| \, \|x\|$, for any $x \in \mathscr{A}$;
(V) $\|x^*\| = \|x\|$, for any $x \in \mathscr{A}$;
(VI) $1 + x^*x$ is invertible in $\mathscr{A}$, for any $x \in \mathscr{A}$.

With this definition they proved the preceding theorem and they made the conjecture that axiom (VI) and axiom (V) follow from the other axioms.

M. Fukamiya [2]*) and J. L. Kelley and R. L. Vaught [1] proved that, indeed axiom (VI) follows from the other axioms, that is, they proved the equivalence of statements (i), (ii), (iii) from 2.12 for the abstract case, by using the arguments we have briefly mentioned above.

J. Glimm and R. V. Kadison [1] and T. Ono [3] proved that axiom (V) also follows from axioms (I)–(IV), if $\mathscr{A}$ has the unit element; J. Vowden [1] proved the same for the general case, thus solving positively and completely the Gelfand-Naimark conjecture.

The conjunction of axioms (IV) and (V) is obviously equivalent to
(IV′) $\|x^*x\| = \|x\|^2$, for any $x \in \mathscr{A}$,
this being the axiom we have adopted here (2.2). H. Araki and G. A. Elliott [1] have shown that from axioms (I), (II) and (IV′), axiom (III) follows; they have also shown that axiom (III) follows from axioms (I), (II), (IV), by assuming that the $*$-operation is continuous (see also Z. Sebestyén [1], [2]).

For the theory of C^*-algebras the reader is also referred to the books of J. Dixmier [42] and M. A. Naimark [6], where he can find a detailed exposition of the arguments presented in this section (see also R.S. Doran and J. Wichmann [1]).

C.2.2. Many results from the theory of C^*-algebras extend in a natural, but not trivial manner, to more general Banach algebras with involution. The older results in this direction can be found in the classical books of M. A. Naimark [6] and C. Rickart [6]. An elegant exposition of the new results can be found in V. Pták [2] and F. Bonsall and J. Duncan [3].

C.2.3. To any normal operator x in a separable Hilbert space one can associate canonically a class of absolute continuity of finite Borel measures on $\sigma(x)$, called the *spectral type* of x, and a function, defined on $\sigma(x)$ and taking values in $\mathbb{N} \cup \{+\infty\}$, measurable with respect to the spectral type, and called the *spectral multiplicity function* of x. One can prove that two normal operators in separable Hilbert spaces are unitarily equivalent iff they have the same spectral type and the same spectral multiplicity function. The spectral type and the spectral multiplicity function allow the construction of a canonical form of the normal operator, called the *spectral representation*. This theory is presented, in detail, in the books of P. R. Halmos [1] and N. Dunford and J. Schwartz [1], Ch. X.

C.2.4. Bibliographical comments. There exist many treatises and monographs containing the spectral theory of bounded linear operators in Hilbert spaces. We mention especially the books by R. Fiesz and B. Sz.-Nagy [1], N. Dunford and

*) Fukamiya considered only the commutative case, but his arguments were general, as noticed by I. Kaplansky and recorded by J. A. Schatz in his review of the paper of M. Fukamiya [2].

J. Schwartz [1], Ch. VI, VII, IX, X, P. R. Halmos [5] and C. T. Ionescu-Tulcea [2]. In our exposition we used these sources, as well as a course by J. R. Ringrose [4]. See also C. Foiaş [1].

For the analytic operational calculus we refer the reader to N. Dunford and J. Schwartz [1], Ch. VII. The case of the finite dimensional Hilbert spaces is masterfully treated in the book by P. R. Halmos [2]. One was able to develop an analytic operational calculus of several commuting operators; this culminates with a deep theorem due to G. Shilov, R. Arens and A. Calderón (see N. Bourbaki [3]). New investigations in this direction have been initiated by J. L. Taylor [4], [5], [9].

Proposition 2.31 is known as the Fuglede-Putnam theorem, whereas the proof given here is due to M. Rosenblum [1] (see also C. R. Putnam [1]).

A systematic approach to the theory of unitary dilation can be found in the book by B. Sz.-Nagy and C. Foiaş [1]. The unitary dilation theorems are due to M. A. Naimark and B. Sz.-Nagy, whereas extensions of such theorems to C^*-algebras were made by W. F. Stinespring [1] and W. Arveson [7] (See also I.Suciu [1]).

3

Von Neumann algebras

In this chapter we present the density theorems of J. von Neumann and I. Kaplansky and we introduce the elementary operations on von Neumann algebras.

3.1. Let $\mathscr{H}$ be a Hilbert space. For any subsets $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$ and $\mathscr{S} \subset \mathscr{H}$ we write

$$\mathscr{X}\mathscr{S} = \{x\xi;\ x \in \mathscr{X},\ \xi \in \mathscr{S}\},$$

$$[\mathscr{X}\mathscr{S}] = \text{the closed vector subspace generated by } \mathscr{X}\mathscr{S}.$$

The projection in $\mathscr{B}(\mathscr{H})$ which corresponds to the closed vector subspace $[\mathscr{X}\mathscr{S}]$ will be denoted by $[\mathscr{X}\mathscr{S}]$ too. If $\mathscr{S} = \{\xi\}$, $\xi \in \mathscr{H}$, we shall simply write

$$\mathscr{X}\mathscr{S} = \mathscr{X}\xi,$$

$$[\mathscr{X}\mathscr{S}] = [\mathscr{X}\xi].$$

For any subset $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$ we shall denote by $\mathscr{X}'$ the *commutant* of $\mathscr{X}$:

$$\mathscr{X}' = \{x' \in \mathscr{B}(\mathscr{H});\ x'x = xx', \text{ for any } x \in \mathscr{X}\},$$

and by X'' the *bicommutant* of $\mathscr{X}$:

$$\mathscr{X}'' = (\mathscr{X}')';$$

by induction we can define the $(n + 1)$-th commutant of $\mathscr{X}$ to be the commutant of the n-th commutant of $\mathscr{X}$. It is now easy to see that, whereas the obvious inclusion

$$\mathscr{X} \subset \mathscr{X}''$$

may be strict, for any $k \geqslant 1$ we have:

$$\text{the } (2k - 1)\text{-th commutant of } \mathscr{X} = \mathscr{X}',$$

$$\text{the } (2k)\text{-th commutant of } \mathscr{X} = \mathscr{X}''.$$

For any subset $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$, $\mathscr{X}'$ is an algebra which contains the identity operator $1 \in \mathscr{B}(\mathscr{H})$; moreover, it is easy to check that $\mathscr{X}'$ is *so*-closed (equivalently, it is *wo*-closed (1.4)). If $\mathscr{X} = \mathscr{X}^*$, then $\mathscr{X}'$ is a von Neumann algebra (see 2.2). In particular, if $\mathscr{M}$ is a von Neumann algebra, then the commutant $\mathscr{M}'$ is a von Neumann algebra. The *passage to the commutant* is the first elementary operation on von Neumann algebras.

If $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$, $\mathscr{X} = \mathscr{X}^*$ and $e \in \mathscr{B}(\mathscr{H})$ is a projection, then $e \in \mathscr{X}'$ iff $xe = exe$ for any $x \in \mathscr{X}$. In particular, if $\mathscr{X} = \mathscr{X}^*$, then $[\mathscr{X}\xi] \in \mathscr{X}'$ for any $\xi \in \mathscr{H}$.

3.2. The fundamental result of the theory of von Neumann algebras is the following

Theorem *(von Neumann's density theorem). Let $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$, $1 \in \mathscr{A}$, be a $*$-algebra of operators. Then the so-closure of $\mathscr{A}$ coincides with the bicommutant of $\mathscr{A}$.*

Proof. It is sufficient to show that $\mathscr{A}$ is *so*-dense in $\mathscr{A}''$. To this end, let us choose an element $x'' \in \mathscr{A}''$.

Let $\xi \in \mathscr{H}$. Then $[\mathscr{A}\xi] \in \mathscr{A}'$ and this implies, in particular, that the subspace $[\mathscr{A}\xi]$ is invariant for any operator from $\mathscr{A}''$. Since $1 \in \mathscr{A}$, it follows that $\xi \in [\mathscr{A}\xi]$ and, therefore,

$$x''\xi \in [\mathscr{A}\xi].$$

Let now $\xi_1, \ldots, \xi_n \in \mathscr{H}$. We introduce the notations (see 2.32):

$$\tilde{\xi} = (\xi_1, \ldots, \xi_n) \in \tilde{\mathscr{H}}_n,$$

$$\tilde{\mathscr{A}}_n = \{\tilde{x}\,;\, x \in \mathscr{A}\} \subset \mathscr{B}(\tilde{\mathscr{H}}_n).$$

Then $\tilde{\mathscr{A}}_n \subset \mathscr{B}(\tilde{\mathscr{H}}_n)$ is a $*$-algebra, $1 \in \tilde{\mathscr{A}}_n$ and

$$(\tilde{\mathscr{A}}_n)' = \{(x'_{ij});\, x'_{ij} \in \mathscr{A}',\ i, j = 1, \ldots, n\} = \mathrm{Mat}_n(\mathscr{A}').$$

Indeed, the relation

$$0 = \tilde{x}(x'_{ij}) - (x'_{ij})\tilde{x} = (xx'_{ij} - x'_{ij}x), \text{ in } \mathscr{B}(\tilde{\mathscr{H}}_n)$$

is satisfied for any $\tilde{x} \in \tilde{\mathscr{A}}_n$, iff the relations

$$xx'_{ij} = x'_{ij}x, \text{ in } \mathscr{B}(\mathscr{H}),\ i, j = 1, \ldots, n$$

are satisfied for any $x \in \mathscr{A}$.

Consequently, for any $(x'_{ij}) \in (\tilde{\mathscr{A}}_n)'$, we have

$$\tilde{x}''(x'_{ij}) - (x'_{ij})\tilde{x}'' = (x''x'_{ij} - x'_{ij}x'') = 0,$$

i.e., $\tilde{x}'' \in (\tilde{\mathscr{A}}_n)''$.

According to the first part of the proof, it follows that

$$\tilde{x}''\tilde{\xi} \in [\tilde{\mathscr{A}}_n\tilde{\xi}].$$

Hence, there exists a sequence $\{x_m\} \subset \mathscr{A}$, such that

$$\lim_{m \to \infty} \|(x'' - x_m)\xi_k\| = 0,\ k = 1, \ldots, n.$$

It follows that x'' is *so*-adherent to $\mathscr{A}$.

Q.E.D.

3.3. Corollary. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, $1 \in \mathcal{M}$, be a $*$-algebra of operators. Then the following statements are equivalent:*

(i) *$\mathcal{M}$ is a von Neumann algebra;*
(ii) *$\mathcal{M} = \mathcal{M}''$.*

3.4. Corollary. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $x \in \mathcal{B}(\mathcal{H})$. Then the following properties are equivalent*

(i) *$x \in \mathcal{M}$*
(ii) *$xe' = e'x$, for any projection $e' \in \mathcal{M}'$,*
(iii) *$u'^*xu' = x$, for any unitary operator $u' \in \mathcal{M}'$.*

Proof. By taking into account 2.23 (resp., 2.24), from property (ii) (resp., (iii)) we infer that $x \in \mathcal{M}''$ and, therefore, with Corollary 3.3, we have $x \in \mathcal{M}$.

Q.E.D.

3.5. Corollary. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $x \in \mathcal{M}$. Then $\mathbf{l}(x)$, $\mathbf{r}(x) \in \mathcal{M}$.*

Proof. We recall (see 2.13) that $\mathbf{l}(x)$ is the smallest projection $e \in \mathcal{B}(\mathcal{H})$, such that $ex = x$.

Let $u' \in \mathcal{M}'$ be a unitary operator and $e \in \mathcal{B}(\mathcal{H})$ a projection. Then $u'^*xu' = x$ and, hence, $ex = x$ iff $(u'^*eu')x = x$. Consequently, we have $u'^*\mathbf{l}(x)u' = \mathbf{l}(x)$.

From Corollary 3.4, it follows that $\mathbf{l}(x) \in \mathcal{M}$. One can similarly prove that $\mathbf{r}(x) \in \mathcal{M}$.

Q.E.D.

3.6. Corollary. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $x \in \mathcal{M}$. If $x = v|x|$ is the polar decomposition of x, then $|x|$, $v \in \mathcal{M}$.*

Proof. Since $\mathcal{M}$ is closed in the norm topology, we have $|x| = (x^*x)^{1/2} \in \mathcal{M}$. Let $u' \in \mathcal{M}'$ be a unitary operator. Then $u'^*xu' = x$, $u'^*|x|u' = |x|$, hence $x = (u'^*vu')|x|$, and u'^*vu' is a partial isometry whose initial projection equals the support of $|x|$. From the uniqueness of the polar decomposition (see Theorem 2.14) it follows that $u'^*vu' = v$. With Corollary 3.4, it follows that $v \in \mathcal{M}$.

Q.E.D.

3.7. We recall (see 2.17) that the set of all projections in $\mathcal{B}(\mathcal{H})$ is a complete lattice in a canonical manner.

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. We shall denote by $\mathcal{P}_{\mathcal{M}}$ the set of all projections in $\mathcal{M}$. We shall consider on $\mathcal{P}_{\mathcal{M}}$ the order relation which is induced by the order relation already defined in the set of all projections in $\mathcal{B}(\mathcal{H})$.

Corollary. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $\mathcal{P}_{\mathcal{M}}$ is a complete lattice.*

Proof. Let $\{e_\iota\}_{\iota \in I} \subset \mathcal{P}_{\mathcal{M}}$ and $e = \bigvee_{\iota \in I} e_\iota \in \mathcal{B}(\mathcal{H})$. For any unitary operator $u' \in \mathcal{M}'$ we have:

$$u'^*eu' = \bigvee_{\iota \in I} (u'^*e_\iota u') = \bigvee_{\iota \in I} e_\iota = e,$$

and hence, with Corollary 3.4, we have $e \in \mathcal{M}$. It follows that $e \in \mathcal{P}_{\mathcal{M}}$ is the l.u.b. of the family $\{e_\iota\}_{\iota \in I}$ in $\mathcal{P}_{\mathcal{M}}$.

Q.E.D.

3.8. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\xi \in \mathscr{H}$. We shall write

$$p_\xi = [\mathscr{M}'\xi], \quad p'_\xi = [\mathscr{M}\xi].$$

By taking into account the last remark from Section 3.1, and also Corollary 3.3, we get the following:

Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\xi \in \mathscr{H}$. Then*

$$p_\xi \in \mathscr{M}, \quad p'_\xi \in \mathscr{M}'.$$

The projections of the form p_ξ (resp., p'_ξ), $\xi \in \mathscr{H}$, are called the *cyclic projections* in $\mathscr{M}$ (resp., $\mathscr{M}'$).

If $\{\xi_n\}$ is a sequence of vectors in $\mathscr{H}$, and if the projections p'_{ξ_n} are mutually orthogonal then $\bigvee_{n=1}^{\infty} p_{\xi_n}$ is a cyclic projection in $\mathscr{M}$. Indeed, we can assume that $\|\xi_n\| \leqslant 2^{-n}$, and we then define $\xi = \sum_{n=1}^{\infty} \xi_n \in \mathscr{H}$. Since the projections p'_{ξ_n} are mutually orthogonal, and $p'_{\xi_n}\xi_n = \xi_n$, we have $p'_{\xi_n}\xi = \xi_n$. Then we have

$$p_\xi = [\mathscr{M}'\xi] \geqslant [\mathscr{M}'p'_{\xi_n}\xi] = [\mathscr{M}'\xi_n] = p_{\xi_n},$$

and this implies that $p_\xi \geqslant \bigvee_{n=1}^{\infty} p_{\xi_n}$; since the reversed inequality is obvious, the proof is complete.

A set of vectors $\mathscr{S} \subset \mathscr{H}$ is said to be *totalizing* for $\mathscr{M}$ if $[\mathscr{M}\mathscr{S}] = \mathscr{H}$; it is said to be *separating* for $\mathscr{M}$, if

$$x \in \mathscr{M}, \; x\xi = 0 \text{ for any } \xi \in \mathscr{S} \Rightarrow x = 0.$$

It is easy to prove that $\mathscr{S}$ is totalizing (resp., separating) for $\mathscr{M}$ iff $\mathscr{S}$ is separating (resp., totalizing) for $\mathscr{M}'$.

A vector $\xi \in \mathscr{H}$ is said to be *cyclic* (or *totalizing*) for $\mathscr{M}$ (resp., *separating* for $\mathscr{M}$) iff the set $\{\xi\}$ is totalizing (resp., separating) for $\mathscr{M}$.

The vector $\xi \in \mathscr{M}$ is cyclic (resp., separating) for $\mathscr{M}$ iff $p'_\xi = 1$ (resp., $p_\xi = 1$).

3.9. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\mathscr{M}' \subset \mathscr{B}(\mathscr{H})$ its commutant Then

$$\mathscr{Z} = \mathscr{M} \cap \mathscr{M}'$$

is the common *center* of the algebras $\mathscr{M}$ and $\mathscr{M}'$.

It is obvious that $\mathscr{Z} \subset \mathscr{B}(\mathscr{H})$ is a (commutative) von Neumann algebra. We shall denote by $\mathscr{R}(\mathscr{M}, \mathscr{M}') \subset \mathscr{B}(\mathscr{H})$ the von Neumann algebra generated by $\mathscr{M} \cup \mathscr{M}'$. It is easy to check that

$$\mathscr{R}(\mathscr{M}, \mathscr{M}') = \mathscr{Z}'.$$

A von Neumann algebra is said to be a *factor* if its center is equal to the set of all scalar multiples of the unit operator.

A projection in $\mathscr{M}$ will be called a *central projection* if it belongs to the center of $\mathscr{M}$. A factor is characterized by the fact that its only central projections are 0 and 1.

Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $x \in \mathscr{M}$. The set of all central projections p such that $px = x$ has a smallest element, denoted by $\mathrm{z}(x)$, which can be calculated by the formula.*

$$\mathrm{z}(x) = [(\mathscr{M}x)\mathscr{H}].$$

Proof. By taking into account Corollary 3.7 (applied to $\mathscr{Z}$), we define $\mathrm{z}(x)$ by the formula

$$\mathrm{z}(x) = \wedge \{p \in \mathscr{P}_{\mathscr{Z}};\ px = x\} \in \mathscr{P}_{\mathscr{Z}}.$$

If $p \in \mathscr{P}_{\mathscr{Z}}$ and $px = x$, then $p\mathscr{H} \supset x\mathscr{H}$. It follows that $\mathrm{z}(x)\mathscr{H} \supset x\mathscr{H}$ and, therefore, $\mathrm{z}(x)x = x$.

Let $p = [(\mathscr{M}x)\mathscr{H}]$. Since $[(\mathscr{M}x)\mathscr{H}]$ is a subspace invariant with respect to the operators in $\mathscr{M}$ and $\mathscr{M}'$, it follows that $p \in \mathscr{M}' \cap \mathscr{M} = \mathscr{Z}$. Since $[(\mathscr{M}x)\mathscr{H}] \supset x\mathscr{H}$, it follows that $px = x$. Hence $p \geqslant \mathrm{z}(x)$. On the other hand, it is obvious that $\mathrm{z}(x)\mathscr{H}$ is invariant with respect to the operators in $\mathscr{M}$ and $\mathrm{z}(x)\mathscr{H} \supset x\mathscr{H}$; hence, we have $\mathrm{z}(x)\mathscr{H} \supset [(\mathscr{M}x)\mathscr{H}]$, i.e., $\mathrm{z}(x) \geqslant p$. Consequently, $\mathrm{z}(x) = [(\mathscr{M}x)\mathscr{H}]$.

Q.E.D.

The projection $\mathrm{z}(x)$ will be called the *central support* of x. We shall consider, in particular, the central support of the projections in $\mathscr{M}$. Obviously, $\mathrm{z}(x) = \mathrm{z}(\mathrm{l}(x)) = \mathrm{z}(\mathrm{r}(x))$. It is easy to check that for any $e \in \mathscr{P}_{\mathscr{M}}$, we have $\mathrm{z}(e) = \vee \{u^*eu;\ u \in \mathscr{M}$, unitary$\}$.

3.10. A very important result in the theory of von Neumann algebras is contained in the following

Theorem (*I. Kaplansky's density theorem*). *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\mathscr{A} \subset \mathscr{M}$ a so-dense $*$-subalgebra of $\mathscr{M}$. Then the unit ball (resp., the self-adjoint part of the unit ball, the positive part of the unit ball) of $\mathscr{A}$ is so-dense in the unit ball (resp., in the self-adjoint part of the unit ball, in the positive part of the unit ball) of $\mathscr{M}$.*

Proof. For any subset $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$ we shall denote, for this proof, by $\mathscr{X}_1$ (resp., $\mathscr{X}^h$, resp., $\mathscr{X}^+$) the unit ball (resp., the self-adjoint, resp., the positive part) of $\mathscr{X}$. In accordance with Corollary 1.5, the statement is equivalent to the following assertions:

$$\mathscr{A}_1 \text{ (resp. } \mathscr{A}_1^h, \text{ resp. } \mathscr{A}_1^+) \text{ is } wo\text{-dense in } \mathscr{M}_1 \text{ (resp. } \mathscr{M}_1^h, \text{ resp. } \mathscr{M}_1^+).$$

In order to carry out the proof, we shall assume, without any loss of generality, that $\mathscr{A}$ is closed for the norm topology. Without any further comment, we shall use the theorem (2.6) on the operational calculus.

(I) The mapping

$$\mathcal{M} \ni x \mapsto \frac{1}{2}(x + x^*) \in \mathcal{M}^h$$

is *wo*-continuous and the image of $\mathcal{A}$ by this mapping is $\mathcal{A}^h$. Since $\mathcal{A}$ is *wo*-dense in $\mathcal{M}$, it follows that $\mathcal{A}^h$ is *wo*-dense in $\mathcal{M}^h$.

(II) Let now $x \in \mathcal{M}_1^h$. The function

$$[-1, 1] \ni t \mapsto 2t(1 + t^2)^{-1} \in [-1, 1]$$

is continuous, strictly increasing and onto; therefore, it has a continuous inverse. It follows that there exists an element $y \in \mathcal{M}^h$, such that

$$x = 2y(1 + y^2)^{-1}.$$

In accordance with (I) there exists a net $\{b_i\} \subset \mathcal{A}^h$ which is *so*-convergent to y. For any i we define

$$a_i = 2b_i(1 + b_i^2)^{-1}.$$

Then $\{a_i\} \subset \mathcal{A}_1^h$ and for any i we have:

$$\begin{aligned} a_i - x &= (1 + b_i^2)^{-1}(2b_i(1 + y^2) - (1 + b_i^2)2y)(1 + y^2)^{-1} \\ &= 2(1 + b_i^2)^{-1}(b_i - y)(1 + y^2)^{-1} + 2(1 + b_i^2)^{-1}b_i(y - b_i)y(1 + y^2)^{-1}, \end{aligned}$$

and this equality obviously shows that the net $\{a_i\}$ is *so*-convergent to x.

Consequently, $\mathcal{A}_1^h$ is *so*-dense in $\mathcal{M}_1^h$.

(III) Let $x \in \mathcal{M}_1^+$ and $y = x^{1/2}$. In accordance with (II), there exists a net $\{b_i\} \subset \mathcal{A}_1^h$, which is *so*-convergent to y. We denote $a_i = b_i^* b_i \in \mathcal{A}_1^+$. Then the net $\{a_i\}$ is *wo*-convergent to x, as one can see from the following formula

$$\begin{aligned} |((x - a_i)\xi \mid \eta)| &= |((y^*y - b_i^*b_i)\xi \mid \eta)| \\ &= |(y^*(y - b_i)\xi \mid \eta) + ((y^* - b_i^*)b_i\xi \mid \eta)| \\ &= |((y - b_i)\xi \mid y\eta) + (b_i\xi \mid (y - b_i)\eta)| \\ &\leqslant \|(y - b_i)\xi\| \, \|\eta\| + \|(y - b_i)\eta\| \, \|\xi\|, \quad \xi, \eta \in \mathcal{H}. \end{aligned}$$

Consequently, $\mathcal{A}_1^+$ is *wo*-dense in $\mathcal{M}_1^+$.

(IV) Let us now consider the Hilbert space $\tilde{\mathcal{H}}_2$ and the $*$-algebra $\mathrm{Mat}_2(\mathcal{A}) \subset \mathcal{B}(\tilde{\mathcal{H}}_2)$, as well as the von Neumann algebra $\mathrm{Mat}_2(\mathcal{M}) \subset \mathcal{B}(\tilde{\mathcal{H}}_2)$. It is easily verified that $\mathrm{Mat}_2(\mathcal{A})$ is *so*-dense in $\mathrm{Mat}_2(\mathcal{M})$.

Let $x \in \mathcal{M}_1$. Then the element

$$\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{M})$$

is self-adjoint and its norm is less than 1. From (II) we infer that there exists a net

$$\left\{\begin{pmatrix} x_{11}^{\iota} & x_{12}^{\iota} \\ x_{21}^{\iota} & x_{22}^{\iota} \end{pmatrix}\right\} \subset (\mathrm{Mat}_2(\mathscr{A}))_1^h,$$

which is *so*-convergent to the element given by the preceding matrix. It follows that $\|x_{21}^{\iota}\| \leqslant 1$, and the net $\{x_{21}^{\iota}\}$ is *so*-convergent to x. Consequently, $\mathscr{A}_1$ is *so*-dense in $\mathscr{M}_1$.

Q.E.D.

3.11. Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$, $\mathscr{M} \ni 1$, be a $*$-algebra. Then the following statements are equivalent*

(i) *$\mathscr{M}$ is a von Neumann algebra,*

(ii) *$\mathscr{M}$ is w-closed,*

(iii) *$\mathscr{M}_1$ is w-compact,*

where by $\mathscr{M}_1$ we here denoted the closed unit ball of $\mathscr{M}$.

Proof. If $\mathscr{M}$ is a von Neumann algebra, then $\mathscr{M}$ is *wo*-closed, and, therefore, *w*-closed. If $\mathscr{M}$ is *w*-closed, then, from Theorem 1.10, we have $\mathscr{M} = (\mathscr{M}_*)^*$, and this implies that $\mathscr{M}_1$ is *w*-compact, in accordance with Alaoglu's theorem.

We still have to prove the implication (iii) $\Rightarrow$ (i); in other words, we must prove that $\mathscr{M}$ is *wo*-closed if we know that $\mathscr{M}_1$ is *wo*-closed (see 1.2 (iii) and 1.10). If $x \in \mathscr{B}(\mathscr{H})$ is a *wo*-adherent point to $\mathscr{M}_1$, then, in accordance with Kaplansky's density theorem, there exists a net $\{x_\iota\} \subset \mathscr{M}$, $\|x_\iota\| \leqslant \|x\|$, which is *wo*-convergent to x. Since $\mathscr{M}_1$ is *wo*-closed, it follows that $x \in \mathscr{M}$.

Q.E.D.

3.12. Let $\mathscr{A}_j \subset \mathscr{B}(\mathscr{H}_j)$, $\mathscr{A}_j \ni 1$, $j = 1, 2$, be two C^*-algebras of operators and let $\pi : \mathscr{A}_1 \to \mathscr{A}_2$ be a $*$-homomorphism, such that $\pi(1) = 1$. Then $\|\pi\| = 1$.

Indeed, it is easily verified that for any element $x_1 \in \mathscr{A}_1$ we have $\sigma(\pi(x_1^* x_1)) \subset$ $\subset \sigma(x_1^* x_1)$ and, therefore, by taking into account Lemma 2.5, we get:

$$\|\pi(x_1)\|^2 = \|\pi(x_1)^*\pi(x_1)\| = \|\pi(x_1^* x_1)\| \leqslant \|x_1^* x_1\| = \|x_1\|^2.$$

Let $\mathscr{M}_j \subset \mathscr{B}(\mathscr{H}_j)$, $j = 1, 2$, be two von Neumann algebras and let $\pi : \mathscr{M}_1 \to \mathscr{M}_2$ be a *wo*-continuous $*$-homomorphism, such that $\pi(1) = 1$. Then π is *w*-continuous.

Indeed, it is sufficient to show that, for any *w*-continuous linear form φ_2 on $\mathscr{M}_2$, the restriction of the linear form $\varphi_2 \circ \pi$ to the unit ball of $\mathscr{M}_1$ is *w*-continuous. But this fact is obvious since the *w*-topology coincides with the *wo*-topology on the unit ball of a von Neumann algebra, whereas π is *wo*-continuous and $\|\pi\| = 1$.

Corollary. *Let $\pi : \mathscr{M}_1 \to \mathscr{M}_2 \subset \mathscr{B}(\mathscr{H}_2)$ be a w-continuous $*$-homomorphism between two von Neumann algebras, such that $\pi(1) = 1$. Then $\pi(\mathscr{M}_1) \subset \mathscr{B}(\mathscr{H}_2)$ is a von Neumann algebra.*

Proof. In accordance with Corollary 3.11, it is sufficient to show that the closed unit ball of the $*$-algebra $\pi(\mathscr{M}_1)$ is *w*-compact.

Let $x_2 \in \pi(\mathscr{M}_1)$, such that $\|x_2\| < \alpha < 1$. Then there exists an $x_1 \in \mathscr{M}_1$, such that $x_2 = \pi(x_1)$. Let $x_1 = v|x_1|$, v, $|x_1| \in \mathscr{M}_1$, be the polar decomposition of x_1 (see Corollary 3.6), and let $e \in \mathscr{P}_{\mathscr{M}}$, in accordance with Corollary 2.21, be such that:

$$|x_1|e \geqslant \alpha e, \quad |x_1|(1-e) \leqslant \alpha(1-e).$$

We have

$$\pi(|x_1|)\pi(e) \geqslant \alpha\pi(e),$$

whence

$$\alpha\|\pi(e)\| \leqslant \|\pi(|x_1|)\| = \|\pi(v^*v|x_1|)\| \leqslant \|\pi(v|x_1|)\| = \|x_2\|,$$

and, therefore,

$$\|\pi(e\| \leqslant (\|x_2\|/\alpha) < 1;$$

since $\pi(e)$ is a projection, we have

$$\pi(e) = 0.$$

It follows that $v|x_1|(1-e) \in \mathscr{M}_1$, $\|v|x_1|(1-e)\| < 1$ and

$$x_2 = \pi(v|x_1|(1-e)).$$

Consequently, we have

$$\{x_2 \in \pi(\mathscr{M}_1);\ \|x_2\| < 1\} = \pi(\{x_1 \in \mathscr{M}_1;\ \|x_1\| < 1\}).$$

Since the closed unit ball of $\mathscr{M}_1$ is w-compact, and since π is w-continuous, it follows that the closed unit ball of $\pi(\mathscr{M}_1)$, i.e.

$$\{x_2 \in \pi(\mathscr{M}_1);\ \|x_2\| \leqslant 1\} = \pi(\{x_1 \in \mathscr{M}_1;\ \|x_1\| \leqslant 1\}),$$

is w-compact.

Q.E.D.

3.13. Let $x \in \mathscr{X} \subset \mathscr{B}(\mathscr{H})$ and $e \in \mathscr{P}_{\mathscr{B}(\mathscr{H})}$. We shall write

$$x_e = ex|_{e\mathscr{H}} \in \mathscr{B}(e\mathscr{H}),$$
$$\mathscr{X}_e = \{x_e;\ x \in \mathscr{X}\} \subset \mathscr{B}(e\mathscr{H}).$$

As another consequence of Kaplansky's density theorem we shall prove a theorem which will enable us to introduce other elementary operations on von Neumann algebras.

Theorem. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $e \in \mathscr{P}_{\mathscr{M}}$. Then we have that*

(i) *$\mathscr{M}_e \subset \mathscr{B}(e\mathscr{H})$ and $(\mathscr{M}')_e \subset \mathscr{B}(e\mathscr{H})$ are von Neumann algebras,*

(ii) *$(\mathscr{M}_e)' = (\mathscr{M}')_e$.*

Proof. The mapping

$$\mathscr{M}' \ni x' \mapsto x'_e \in (\mathscr{M}')_e \subset \mathscr{B}(e\mathscr{H})$$

is a *wo*-continuous $*$-homomorphism, which is onto; from Section 3.12, we infer that $(\mathscr{M}')_e$ is a von Neumann algebra.

It is obvious that $\mathscr{M}_e \subset ((\mathscr{M}')_e)'$. Conversely, any element in $((\mathscr{M}')_e)'$ is of the form x_e, where $x \in \mathscr{B}(\mathscr{H})$, $x = exe$. For any $x' \in \mathscr{M}'$ we have

$$x_e x'_e = x'_e x_e \qquad \text{in } \mathscr{B}(e\mathscr{H}),$$

whence

$$xx' = x'x \qquad \text{in } \mathscr{B}(\mathscr{H}),$$

and this implies that $x \in \mathscr{M}'' = \mathscr{M}$ (see 3.3).

Thus, we proved that $\mathscr{M}_e = ((\mathscr{M}')_e)'$; in particular, $\mathscr{M}_e$ is a von Neumann algebra. By passing to the commutant in this equality, we get

$$(\mathscr{M}_e)' = (\mathscr{M}')_e,$$

because $(\mathscr{M}')_e$ is a von Neumann algebra (see Corollary 3.3).

Q.E.D.

Henceforth we shall write

$$\mathscr{M}'_e = (\mathscr{M}_e)' = (\mathscr{M}')_e.$$

3.14. We now introduce other elementary operations on von Neumann algebras: *the reduction* and *the induction.*

Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $e \in \mathscr{P}_{\mathscr{M}}$. The von Neumann algebra $\mathscr{M}_e$ is called *the reduced von Neumann algebra* of $\mathscr{M}$ with respect to e.

It is easy to check that the mapping

$$e\mathscr{M}e \ni x \mapsto x_e \in \mathscr{M}_e$$

is a $*$-isomorphism of $*$-algebras.

Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $e' \in \mathscr{P}_{\mathscr{M}'}$. The von Neumann algebra $\mathscr{M}_{e'}$ is called *the induced von Neumann algebra* of $\mathscr{M}$ with respect to e'.

In the proof of Theorem 3.13, we observed and used the fact that the mapping

$$\mathscr{M} \ni x \mapsto x_{e'} \in \mathscr{M}_{e'}$$

is a *wo*-continuous $*$-homomorphism of von Neumann algebras. This $*$-homomorphism is called the *canonical induction* determined by $e' \in \mathscr{P}_{\mathscr{M}'}$.

Proposition. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $e' \in \mathscr{P}_{\mathscr{M}'}$. Then the canonical induction $\mathscr{M} \to \mathscr{M}_{e'}$ is a $*$-isomorphism iff $z(e') = 1$.*

Proof. We have $(1 - z(e'))_{e'} = 0$ and, therefore, if the canonical induction is a $*$-isomorphism, we have $z(e') = 1$.

Conversely, if $z(e') = 1$, then, in accordance with Corollary 3.9, we have $[(\mathscr{M}'e')\mathscr{H}] = \mathscr{H}$. If $x \in \mathscr{M}$ and $x_{e'} = 0$, then $xe'\mathscr{H} = 0$ and, therefore, we have $x[(\mathscr{M}'e')\mathscr{H}] = 0$, hence $x = 0$. Consequently, the canonical induction is a $*$-isomorphism.

Q.E.D.

3.15. If p is a central projection in the von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$, then the mappings

$$\mathscr{M}p \ni x \mapsto x_p \in \mathscr{M}_p,$$

$$\mathscr{M}'p \ni x' \mapsto x'_p \in \mathscr{M}'_p$$

establish the canonical identifications

$$\mathscr{M}p = \mathscr{M}_p \subset \mathscr{B}(p\mathscr{H}),$$

$$\mathscr{M}'p = \mathscr{M}'_p \subset \mathscr{B}(p\mathscr{H}).$$

Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra with the center $\mathscr{Z}$, and let $e \in \mathscr{P}_{\mathscr{M}}$. Then the common center of the algebras $\mathscr{M}_e, \mathscr{M}'_e$ is equal to $\mathscr{Z}_e$.*

Proof. If e is a central projection, the assertion is obvious. Consequently, we can assume, without any loss of generality, that $z(e) = 1$. Then, in accordance with Proposition 3.14, the canonical induction $\mathscr{M}' \to \mathscr{M}'_e$ is a $*$-isomorphism and, therefore, the center of $\mathscr{M}'_e$ is the image of the center $\mathscr{Z}$ of $\mathscr{M}'$ by this $*$-isomorphism.

Q.E.D.

In particular, if $\mathscr{M}$ is a factor, then $\mathscr{M}_e$ and $\mathscr{M}'_e$ are factors.

3.16. Before introducing a new elementary operation on von Neumann algebra, namely the tensor product, we prove some commutation relations for "matrices" of operators; such relations have already been considered in the proofs of Theorems 3.2 and 3.10.

Let $\mathscr{H}$ be a Hilbert space, γ any cardinal number and I a set of indices, such that card $I = \gamma$. We use the notations already introduced in Section 2.32.

Lemma. *Let $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$ be any subset. Then for the sets $\tilde{\mathscr{X}}_\gamma$,* $\mathrm{Mat}_\gamma(\mathscr{X}) \subset \mathscr{B}(\tilde{\mathscr{H}}_\gamma)$ *the following relations*

$$(\tilde{\mathscr{X}}_\gamma)' = \mathrm{Mat}_\gamma(\mathscr{X}'), \quad (\tilde{\mathscr{X}}_\gamma)'' = (\tilde{\mathscr{X}}'')_\gamma.$$

hold; if $\mathscr{X} \ni 0, 1$, then we also have the following relations

$$(\mathrm{Mat}_\gamma(\mathscr{X}))' = (\tilde{\mathscr{X}}')_\gamma, \quad (\mathrm{Mat}_\gamma(\mathscr{X}))'' = \mathrm{Mat}_\gamma(\mathscr{X}'').$$

Proof. The relation $(\tilde{\mathscr{X}}_\gamma)' = \mathrm{Mat}_\gamma(\mathscr{X}')$ can be proved in the same manner as in the proof of Theorem 3.2.

We observe that, for any $i_0, k_0 \in I$ we have

$$u_{i_0}u^*_{k_0} = (\delta_{i_0 i}\, \delta_{k_0 k}).$$

Consequently, if $\mathscr{X} \ni 0,1$, then all the operators $u_i u^*_k$ belong to $\mathrm{Mat}_\gamma(\mathscr{X})$ and, therefore, in accordance with a remark from Section 2.32, it follows that

$$(\mathrm{Mat}_\gamma(\mathscr{X}))' \subset \widetilde{\mathscr{B}(\mathscr{H})}_\gamma.$$

On the other hand, we have $\tilde{\mathscr{X}}_\gamma \subset \mathrm{Mat}_\gamma(\mathscr{X})$, and, therefore,

$$(\mathrm{Mat}_\gamma(\mathscr{X}))' \subset (\tilde{\mathscr{X}}_\gamma)' = \mathrm{Mat}_\gamma(\mathscr{X}'),$$

Consequently, we have

$$(\mathrm{Mat}_\gamma(\mathscr{X}))' \subset \widetilde{\mathscr{B}(\mathscr{H})}_\gamma \cap \mathrm{Mat}_\gamma(\mathscr{X}') = (\tilde{\mathscr{X}}')_\gamma,$$

Since the reversed inclusion is obvious, the relation $(\mathrm{Mat}_\gamma(\mathscr{X}))' = (\tilde{\mathscr{X}}')_\gamma$ is proved.

The other relations are immediate consequences of the already proved ones.

Q.E.D.

3.17. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$, $\mathscr{N} \subset \mathscr{B}(\mathscr{K})$ be von Neumann algebras. Then:

$$\mathscr{M} \otimes \mathscr{N} = \left\{ \sum_{k=1}^{n} x_k \overline{\otimes} y_k;\ x_k \in \mathscr{M},\ y_k \in \mathscr{N},\ n = 1, 2, \ldots \right\}$$

is a $*$-algebra of operators on $\mathscr{H} \overline{\otimes} \mathscr{K}$. The von Neumann algebra generated in $\mathscr{B}(\mathscr{H} \overline{\otimes} \mathscr{K})$ by $\mathscr{M} \otimes \mathscr{N}$ is denoted by

$$\mathscr{M} \overline{\otimes} \mathscr{N}$$

and it is called *the tensor product* of the von Neumann algebras $\mathscr{M}$ and $\mathscr{N}$.

We recall that for any Hilbert space $\mathscr{H}$ one denotes by $\mathscr{C}(\mathscr{H})$ the set of all the scalar multiples of the identity operator on $\mathscr{H}$; obviously, $\mathscr{C}(\mathscr{H}) \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra. It is easy to check that

$$\mathscr{B}(\mathscr{H})' = \mathscr{C}(\mathscr{H}), \quad \mathscr{C}(\mathscr{H})' = \mathscr{B}(\mathscr{H}).$$

In the following proposition we use the identifications we have agreed upon in Section 2.34.

Proposition. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\mathscr{K}$ a Hilbert space, $\gamma = \dim \mathscr{K}$. Then we have*

(i) $\mathscr{M} \overline{\otimes} \mathscr{C}(\mathscr{K}) = \mathscr{M} \otimes \mathscr{C}(\mathscr{K}) = \tilde{\mathscr{M}}_\gamma$,

(ii) $\mathscr{M} \overline{\otimes} \mathscr{B}(\mathscr{K}) = \mathrm{Mat}_\gamma(\mathscr{M})$,

(iii) $(\mathscr{M} \overline{\otimes} \mathscr{C}(\mathscr{K}))' = \mathscr{M}' \overline{\otimes} \mathscr{B}(\mathscr{K})$, $(\mathscr{M} \overline{\otimes} \mathscr{B}(\mathscr{K}))' = \mathscr{M}' \overline{\otimes} \mathscr{C}(\mathscr{K})$.

Proof. The relation $\mathscr{M} \otimes \mathscr{C}(\mathscr{K}) = \tilde{\mathscr{M}}_\gamma$ is an immediate consequence of Section 2.34, whereas from Lemma 3.16 it follows that $(\tilde{\mathscr{M}}_\gamma)'' = (\tilde{\mathscr{M}}'')_\gamma = \tilde{\mathscr{M}}_\gamma$, and assertion (i) is proved.

For any von Neumann algebra $\mathscr{N} \subset \mathscr{B}(\mathscr{K})$, it is easy to prove that

$$\mathscr{M} \overline{\otimes} \mathscr{N} = \mathscr{R}(\mathscr{M} \overline{\otimes} \mathscr{C}(\mathscr{K}) \cup \mathscr{C}(\mathscr{H}) \overline{\otimes} \mathscr{N}),$$

whence we get

$$(\mathscr{M} \overline{\otimes} \mathscr{N})' = (\mathscr{M} \overline{\otimes} \mathscr{C}(\mathscr{K}))' \cap (\mathscr{C}(\mathscr{H}) \overline{\otimes} \mathscr{N})'.$$

In particular, we have

$$(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{K}))' = (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K}))' \cap (\mathscr{C}(\mathscr{H}) \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{K}))'$$

$$= (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K}))' \cap \mathscr{B}(\mathscr{H}) \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K}) = \mathscr{M}' \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K}),$$

where the two last equalities are easily checked by direct verification. We then have

$$(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K}))' = (\mathscr{M}'' \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K}))' = (\mathscr{M}' \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{K}))'' = \mathscr{M}' \otimes \mathscr{B}(\mathscr{K}),$$

and this proves assertion (iii).

Finally, by taking into account properties (i), (iii) and Lemma 3.16, we get

$$\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{K}) = (\mathscr{M}' \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K}))' = (\tilde{\mathscr{M}}'_\gamma)'$$

$$= (\mathrm{Mat}_\gamma(\mathscr{M}))'' = \mathrm{Mat}_\gamma(\mathscr{M}'') = \mathrm{Mat}_\gamma(\mathscr{M}),$$

and assertion (ii) is proved.

Q.E.D.

3.18. If $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra and $\mathscr{K}$ a Hilbert space, then the mapping

$$\mathscr{M} \ni x \mapsto x \mathbin{\overline{\otimes}} 1 \in \mathscr{M} \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K})$$

is a $*$-isomorphism, called *amplification*. With the usual identifications, this isomorphism can also be written in the following form

$$\mathscr{M} \ni x \mapsto \tilde{x} \in \tilde{\mathscr{M}}_\gamma, \quad \gamma = \dim \mathscr{K}.$$

With the notations from Section 2.32, let $e_\iota = u_\iota u_\iota^*$ be the projection of $\tilde{\mathscr{H}}_\gamma$ onto $\mathscr{H}_\iota$. Then $e_\iota \in \mathrm{Mat}_\gamma(\mathscr{C}(\mathscr{H})) = \mathscr{C}(\mathscr{H}) \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{K})$, for any $\iota \in I$; it follows that

$$e_\iota \in \mathscr{M} \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{K}), \quad \iota \in I,$$

$$e_\iota \in \mathscr{M}' \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{K}) = (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K}))', \quad \iota \in I.$$

We can, therefore, consider the reduced algebra

$$(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{K}))_{e_\iota} \subset \mathscr{B}(\mathscr{H}_\iota)$$

and the induced algebra

$$(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K}))_{e_\iota} \subset \mathscr{B}(\mathscr{H}_\iota).$$

It is easily seen that the mapping

$$x \mapsto u_\iota x u_\iota^* |_{\mathscr{H}_\iota},$$

is a $*$-isomorphism of $\mathscr{M}$ onto $(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{K}))_{e_\iota} = (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K}))_{e_\iota}$.

Thus, the passage from $\mathscr{M}$ to $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{K})$ (resp., to $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{C}(\mathscr{K})$) and the passage from a von Neumann algebra to a reduced algebra (resp., to an induced algebra) are reciprocal operations (see also Section 4.22), whereas the $*$-isomorphism reciprocal to an amplification is an induction.

3.19. Since the closed unit ball of a von Neumann algebra is w-compact (see 3.11), it has extreme points, by virtue of the Krein-Milman theorem. The following lemma, which describes the nature of these extreme points, will be used in the proof of Theorem 5.16.

Lemma. *Let $\mathscr{M}$ be a von Neumann algebra and v an extreme point of the closed unit ball of $\mathscr{M}$. Then v is a partial isometry.*

Proof. We must show that v^*v is a projection and, in order to achieve this, we must show that $\sigma(v^*v) \subset \{0, 1\}$. If this be not true, let $\lambda \in \sigma(v^*v), 0 < \lambda < < \alpha < 1$, and $\varepsilon > 0$ be such that $\varepsilon(1 + \varepsilon\alpha)^2 < 1$. Let us define

$$a = \varepsilon\chi_{[0,\alpha]}(v^*v).$$

It is easily verified, by taking into account the operational calculus, that the following relations hold

$$\|(1 + a)v^*v\,(1 + a)\| \leqslant 1,$$

$$\|(1 - a)v^*v\,(1 - a)\| \leqslant 1.$$

Consequently, the elements

$$v_1 = v(1 + a), \quad v_2 = v(1 - a)$$

belong to the closed unit ball of $\mathscr{M}$ and

$$v = \frac{1}{2}\, v_1 + \frac{1}{2}\, v_2.$$

Since v is an extreme point, it follows that $v_1 = v_2 = v$, whence $va = 0$. Consequently, we have $v^*va = 0$, and, therefore, $v^*v\chi_{[0,\alpha]}(v^*v) = 0$, a contradiction.

Q.E.D.

3.20. In the last sections of this chapter, we consider some general properties of the ideals of the algebras of operators.

Proposition. *Let $\mathscr{A}$ be a C^*-algebra and $\mathfrak{N} \subset \mathscr{A}$ a left ideal of $\mathscr{A}$. Then there exists a net $\{u_\alpha\}_{\alpha\in\Gamma} \subset \mathfrak{N}$, such that*

(i) $0 \leqslant u_\alpha \leqslant 1, \quad \alpha \in \Gamma,$

(ii) $\alpha \leqslant \beta \Rightarrow u_\alpha \leqslant u_\beta;$

(iii) $\|x - xu_\alpha\| \xrightarrow[\alpha]{} 0$, *for any* $x \in \mathfrak{N}$.

Proof. We denote by Γ the set of all pairs (n, F), where n is a natural integer and F is a finite subset of $\mathfrak{N}$. Endowed with the order relation

$$(n, F) \leqslant (m, G) \Leftrightarrow n \leqslant m \text{ and } F \subset G,$$

Γ becomes a directed set. For any $\alpha = (n, F) \in \Gamma$, we define

$$v_\alpha = \sum_{x \in F} x^* x \in \mathfrak{N}, \quad u_\alpha = (n^{-1} + v_\alpha)^{-1} v_\alpha.$$

We observe that $u_\alpha = f_n(v_\alpha)$, where $f_n(t) = (n^{-1} + t)^{-1} t$. Since $0 \leqslant f_n(t) \leqslant 1$, for any $t \geqslant 0$, from Theorem 2.6, it follows that

$$0 \leqslant u_\alpha \leqslant 1.$$

If $\alpha = (n, F) \leqslant (m, G) = \beta$, then $v_\alpha \leqslant v_\beta$ and, therefore, with E.2.8, we have

$$(n^{-1} + v_\alpha)^{-1} \geqslant (n^{-1} + v_\beta)^{-1}.$$

Since $n^{-1}(n^{-1} + t)^{-1} \geqslant m^{-1}(m^{-1} + t)^{-1}$, for any $t \geqslant 0$, with Theorem 2.6, we infer that

$$n^{-1}(n^{-1} + v_\beta)^{-1} \geqslant m^{-1}(m^{-1} + v_\beta)^{-1}.$$

Consequently, we have

$$1 - n^{-1}(n^{-1} + v_\alpha)^{-1} \leqslant 1 - n^{-1}(n^{-1} + v_\beta)^{-1} \leqslant 1 - m^{-1}(m^{-1} + v_\beta)^{-1},$$

i.e.,

$$u_\alpha \leqslant u_\beta.$$

Finally, for any $\alpha = (n, F) \in \Gamma$, we have

$$\sum_{x \in F} [x(1 - u_\alpha)]^* [x(1 - u_\alpha)] = (1 - u_\alpha) v_\alpha (1 - u_\alpha)$$

$$= n^{-2}(n^{-1} + v_\alpha)^{-2} v_\alpha \leqslant 4^{-1} n^{-1},$$

because we have $(n^{-1} + t)^{-2} t \leqslant 4^{-1} n$, for any $t \in \mathbb{R}$. We, therefore, have

$$[x(1 - u_\alpha)]^* [x(1 - u_\alpha)] \leqslant 4^{-1} n^{-1}, \quad x \in F,$$

whence

$$\| x(1 - u_\alpha) \|^2 \leqslant 4^{-1} n^{-1}, \quad x \in F.$$

It follows that

$$\| x - x u_\alpha \| \xrightarrow[\alpha]{} 0, \text{ for any } x \in \mathfrak{N}.$$

Q.E.D.

The net $\{u_\alpha\}_{\alpha \in \Gamma}$ is called an *approximate unit* for the left ideal $\mathfrak{N}$. We observe that property (iii) remains in force for any x belonging to the norm closure $\bar{\mathfrak{N}}$ of $\mathfrak{N}$.

In the preceding proposition we did not assume the existence of a unit element in the C^*-algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$, all inverses being considered in $\mathscr{B}(\mathscr{H})$. In particular, it follows that any two-sided ideal of $\mathscr{A}$, which is also dense in $\mathscr{A}$ for the norm topology, contains an approximate unit $\{u_\alpha\}_{\alpha \in \Gamma}$ for $\mathscr{A}$; in this case, by replacing, in property (iii), the element x by x^*, it follows that we also have

$$\|x - u_\alpha x\| \to 0, \text{ for any } x \in \mathscr{A}.$$

Corollary. *Let $\mathscr{M}$ be a von Neumann algebra and $\mathfrak{N} \subset \mathscr{M}$ a left ideal. Then there exists a unique projection $e \in \mathscr{M}$, such that*

$$\overline{\mathfrak{N}}^w = \mathscr{M}e.$$

Any approximate unit of $\mathfrak{N}$ is so-convergent to e.

Proof. Let $\{u_\alpha\}$ be an approximate unit of $\mathfrak{N}$, in accordance with the preceding proposition, and let $e = \sup_\alpha u_\alpha \in \overline{\mathfrak{N}}^w$ (see 2.16). For any $x \in \mathfrak{N}$ we have $x - xu_\alpha \to 0$, hence $x = xe$. Consequently, we have

$$x = xe,$$

for any $x \in \overline{\mathfrak{N}}^w$. In particular, we have $e^2 = e$, i.e., e is a projection. We have $\mathscr{M}e \subset \overline{\mathfrak{N}}^w$, since $\overline{\mathfrak{N}}^w$ is a left ideal, and also $\overline{\mathfrak{N}}^w \subset \mathscr{M}e$, as we have already proved; it follows that

$$\overline{\mathfrak{N}}^w = \mathscr{M}e.$$

The uniqueness of the projection e is immediate.

Q.E.D.

We observe that the adjoint of a left ideal of $\mathscr{M}$ is a right ideal. Consequently, the w-closed left (resp. right) ideals of $\mathscr{M}$ are of the form $\mathscr{M}e$ (resp., $e\mathscr{M}$), where e is a projection in $\mathscr{M}$. If $\mathfrak{N} = \mathscr{M}e$ is a w-closed two-sided ideal of $\mathscr{M}$, then, for any unitary $u \in \mathscr{M}$, we have

$$\mathscr{M}e = \mathfrak{N} = u\mathfrak{N}u^* = \mathscr{M}(ueu^*),$$

and this implies that $e = ueu^*$. From Corollary 3.4 we infer that $e \in \mathscr{M}'$, and, therefore, e is central. Consequently, the w-closed two-sided ideals of $\mathscr{M}$ are of the form $\mathscr{M}p$, where p is a central projection, and conversely.

3.21. Let $\mathscr{A}$ be a C^*-algebra. A subset $\mathfrak{F} \subset \mathscr{A}^+$ is said to be a *face* if it has the property

$$a, b \in \mathfrak{F}, \quad c \in \mathscr{A}^+, \quad c \leqslant a + b \Rightarrow c \in \mathfrak{F}.$$

It is easy to prove that if $\mathfrak{F} \subset \mathscr{A}^+$ is a face, then $\mathfrak{F} + \mathfrak{F} \subset \mathfrak{F}$, $\mathbb{R}^+\mathfrak{F} \subset \mathfrak{F}$.

For any face $\mathfrak{F} \subset \mathscr{A}^+$ one defines

$$\mathfrak{N} = \{x \in \mathscr{A};\ x^*x \in \mathfrak{F}\},$$

$$\mathfrak{M} = \mathfrak{N}^*\mathfrak{N} = \left\{\sum_{j=1}^n y_j^* x_j;\ x_j, y_j \in \mathfrak{N},\ n \in \mathbb{N}\right\}.$$

Proposition. *Let $\mathfrak{F} \subset \mathscr{A}^+$ be a face. Then*

(i) *$\mathfrak{N}$ is a left ideal and $\mathfrak{M}$ is a $*$-subalgebra of $\mathscr{A}$;*

(ii) *$\mathfrak{M}^+ = \mathfrak{F}$ and $\mathfrak{M}$ is the linear hull of $\mathfrak{M}^+$;*

(iii) *there exists an approximate unit of $\mathfrak{N}$, which is contained in $\mathfrak{M}^+$.*

Proof. (i) Let $x, y \in \mathfrak{N}$ and $a \in \mathscr{A}$. We have

$$(x + y)^*(x + y) \leqslant 2(x^*x + y^*y) \in \mathfrak{F}$$

and, therefore, we have $x + y \in \mathfrak{N}$. We then have

$$(ax)^*(ax) = x^*a^*ax \leqslant \|a\|^2x^*x \in \mathfrak{F},$$

and this implies that $ax \in \mathfrak{N}$. Consequently, $\mathfrak{N}$ is a left ideal of $\mathscr{A}$.

(ii) Let $a \in \mathfrak{F}$. From Proposition 2.12, we infer that there exists an $x \in \mathscr{A}$, such that $a = x^*x$. Then we have $x \in \mathfrak{N}$ and, therefore, $x^*x \in \mathfrak{M} \cap \mathscr{A}^+ = \mathfrak{M}^+$.

It is easy to prove the following polarization relation

$$y^*x = 4^{-1} \sum_{k=0}^{3} i^k(x + i^ky)^*(x + i^ky), \quad x, y \in \mathscr{A}.$$

If $a = \sum_{j=1}^{n} y_j^*x_j \in \mathfrak{M}^+$, $y_i, x_j \in \mathfrak{N}$, then, by taking into account the polarization relation, we get

$$\begin{aligned} a &= 4^{-1} \sum_{j=1}^{n} (x_j + y_j)^* (x_j + y_j) - (x_j - y_j)^* (x_j - y_j) \\ &\leqslant 4^{-1} \sum_{j=1}^{n} (x_j + y_j)^* (x_j + y_j) \in \mathfrak{F}, \end{aligned}$$

and, therefore, we have $a \in \mathfrak{F}$.

We have thus shown that $\mathfrak{M}^+ = \mathfrak{F}$.

It is obvious that $\mathfrak{M}$ contains the linear hull of $\mathfrak{M}^+$, whereas the reversed inclusion easily follows from the polarization relation.

(iii) We shall use the notations introduced in the proof of Proposition 3.20. For any $\alpha = (a, F) \in \Gamma$, we have $v_\alpha \in \mathfrak{F} = \mathfrak{M}^+$. Since $(n^{-1} + t)^{-1}t \leqslant nt$, for any $t \geqslant 0$, by taking into account Theorem 2.6, we get

$$u_\alpha = (n^{-1} + v_\alpha)^{-1}v_\alpha \leqslant nv_\alpha \in \mathfrak{F},$$

and, since $\mathfrak{M}^+$ is a face, we get

$$u_\alpha \in \mathfrak{M}^+.$$

Q.E.D.

A subalgebra $\mathfrak{M}$ of the C^*-algebra $\mathscr{A}$, such that $\mathfrak{M}^+$ is a face and $\mathfrak{M}$, itself, is the linear hull of $\mathfrak{M}^+$, is called a *facial subalgebra*. Any facial subalgebra is self-adjoint.

The following corollary characterizes the reduced algebras of a von Neumann algebra.

Corollary 1. *Let $\mathcal{M}$ be a von Neumann algebra and $\mathfrak{M} \subset \mathcal{M}$ a w-closed subset. The following statements are equivalent:*

(i) *$\mathfrak{M}$ is a facial subalgebra of $\mathcal{M}$;*

(ii) *there exists a projection $e \in \mathcal{M}$, such that $\mathfrak{M} = e\mathcal{M}e$.*

Proof. We assume that $\mathfrak{M}$ is a facial subalgebra. Since $\mathfrak{M}$ is w-closed, the face $\mathfrak{F} = \mathfrak{M}^+$ is w-closed, and, therefore, the left ideal $\mathfrak{N} = \{x \in \mathcal{M};\ x^*x \in \mathfrak{F}\}$ is w-closed. In accordance with Corollary 3.20, there exists a projection $e \in \mathcal{M}$, such that $\mathfrak{N} = \mathcal{M}e$. In accordance with the preceding proposition, we have

$$\mathfrak{M} = \mathfrak{N}^*\mathfrak{N} = (\mathcal{M}e)^*(\mathcal{M}e) = e\mathcal{M}e.$$

We have thus proved that (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (i) offers no difficulties.

Q.E.D.

Let $\mathfrak{M}$ be a two-sided ideal of the von Neumann algebra $\mathcal{M}$. If $x \in \mathcal{M}$ and $x = v|x|$ is the polar decomposition of x (see 3.6) then $|x| = v^*x \in \mathfrak{M}$, and $x^* = |x|\, v^* \in \mathfrak{M}$ (see 2.15). Consequently, *any two-sided ideal of a von Neumann algebra is self-adjoint.*

A face $\mathfrak{F} \subset \mathcal{M}^+$ is said to be *invariant* if

$$a \in \mathfrak{F}, \quad u \in \mathcal{M}, \quad \text{unitary} \Rightarrow uau^* \in \mathfrak{F}.$$

By using the polar decomposition it is easy to show that a face $\mathfrak{F} \subset \mathcal{M}^+$ is invariant iff

$$x \in \mathcal{M}, \quad x^*x \in \mathfrak{F} \Rightarrow xx^* \in \mathfrak{F}.$$

The following corollary characterizes the positive parts of the two-sided ideals of von Neumann algebras.

Corollary 2. *Let $\mathcal{M}$ be a von Neumann algebra. Then mappings*

$$\mathfrak{M} \mapsto \mathfrak{M}^+,$$

$$\mathfrak{F} \mapsto \textit{the linear hull of } \mathfrak{F},$$

are reciprocal bijections between the set of all two-sided ideals $\mathfrak{M} \subset \mathcal{M}$ and the set of all invariant faces $\mathfrak{F} \subset \mathcal{M}^+$.

Proof. Let $\mathfrak{M} \subset \mathcal{M}$ be a two-sided ideal and $b \in \mathfrak{M}^+$. If $a \in \mathcal{M}^+$ and $a \leqslant b$, then, in accordance with exercise E.2.6, there exists an $x \in \mathcal{M}$ such that $a^{1/2} = xb^{1/2}$. It follows that $a = xbx^* \in \mathfrak{M}^+$. We infer that $\mathfrak{M}^+$ is a face, obviously invariant, since $\mathfrak{M}$ is a two-sided ideal.

The remaining assertions in the corollary are easily verified, by taking into account the preceding proposition.

Q.E.D.

Exercises

E.3.1. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra, $\{x_n\} \subset \mathscr{M}$ and $x \in \mathscr{B}(\mathscr{H})$. If we have

$$x_n\xi \to x\xi, \text{ weakly, and } x_n^*\xi \to x^*\xi, \text{ weakly,}$$

for any ξ belonging to a total subset of $\mathscr{H}$, then $x \in \mathscr{M}$.

Let $\mathscr{H}$ be a separable Hilbert space and $\{\xi_k\}$ an orthonormal basis of $\mathscr{H}$. We define the operators x_n, x, $y \in \mathscr{B}(\mathscr{H})$ by the formulas

$$x_n\xi_k = \begin{cases} 2\xi_2 - 3\xi_3 & \text{for } k = 2, \\ -n\xi_2 + \dfrac{3}{2}n\xi_3 & \text{for } k = n, \\ 0 & \text{for } k \neq 2, n. \end{cases}$$

$$x\xi_k = \begin{cases} 2\xi_2 \quad - 3\xi_3 & \text{for } k = 2, \\ 0 & \text{for } k \neq 2, \end{cases}$$

$$x\xi_k = \begin{cases} \displaystyle\sum_{j=1}^{\infty} \frac{1}{j}\,\xi_j & \text{for } k = 1, \\ \dfrac{1}{k}\,\xi_1 & \text{for } k \neq 1. \end{cases}$$

Show that $x_n\xi_k \to x\xi_k$ for any k, but, although we have $x_n y = yx_n$, for any n, we have $(xy\xi_1 \mid \xi_2) \neq (yx\xi_1 \mid \xi_2)$; it follows that $x \notin \mathscr{R}(\{x_n\})$.

E.3.2. A von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a factor iff for any $x, y \in \mathscr{M}$ there exists an $a \in \mathscr{M}$, such that $xay \neq 0$.

E.3.3. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\{e_i\}_{i \in I} \subset \mathscr{P}_{\mathscr{M}}$. Then we have

$$\mathrm{z}(\bigvee_{i \in I} e_i) = \bigvee_{i \in I} \mathrm{z}(e_i).$$

E.3.4. Let $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ be a $*$-algebra and $x \in \mathscr{B}(\mathscr{H})$ an invertible operator, such that the mapping $a \mapsto x^{-1}ax$ be a $*$-automorphism of $\mathscr{A}$. Then there exists a unitary $u \in \mathscr{B}(\mathscr{H})$, such that $x^{-1}ax = u^*au$, for any $a \in \mathscr{A}$. If, moreover, $\mathscr{A}$ is a von Neumann algebra and $x \in \mathscr{A}$, then one can find such a u, having the above properties and, moreover, belonging to $\mathscr{A}$.

E.3.5. One says that a von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is of *countable type* if any family of mutually orthogonal non-zero projections in $\mathscr{M}$ is at most countable. Any von Neumann algebra in a separable Hilbert space is of countable type.

Show that the closed unit ball of a von Neumann algebra of countable type is *so*-metrizable, whereas the closed unit ball of a von Neumann algebra in a separable Hilbert space is *wo*-metrizable.

E.3.6. Let $\mathscr{A} \subset \mathscr{M} \subset \mathscr{N} \subset \mathscr{B}(\mathscr{H})$ be von Neumann algebras, such that $\mathscr{A}$ be included in the center of $\mathscr{N}$.

For any $x \in \mathscr{N}$, $x \notin \mathscr{M}$, there exists a $p \in \mathscr{P}_{\mathscr{A}}$, $p \neq 0$, such that

$$q \in \mathscr{P}_{\mathscr{A}},\ 0 \neq q \leqslant p \Rightarrow xq \notin \mathscr{M}.$$

If $\mathscr{M}q \neq \mathscr{N}q$, for any $q \in \mathscr{P}_{\mathscr{A}}$, $q \neq 0$, then there exists a projection $e \in \mathscr{N}$, such that $eq \notin \mathscr{M}$, for any $q \in \mathscr{P}_{\mathscr{A}}$, $q \neq 0$.

!E.3.7. A von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is of countable type iff there exists an $\mathscr{M}$-separating orthonormal sequence in $\mathscr{H}$.

!E.3.8. A commutative von Neumann algebra is of countable type iff it has a separating vector.

!E.3.9. A vector $\xi \in \mathscr{H}$ is called a *trace vector* for the von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ if

$$(xy\xi \mid \xi) = (yx\xi \mid \xi),\ x, y \in \mathscr{M}.$$

Obviously, if $\mathscr{M}$ is commutative, then any $\xi \in \mathscr{H}$ is a trace vector for $\mathscr{M}$.

Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra which has a cyclic trace vector $\xi \in \mathscr{H}$. Then ξ is separating for $\mathscr{M}$ and the mapping

$$\mathscr{M}\xi \ni x\xi \mapsto x^*\xi \in \mathscr{M}\xi$$

is isometric; hence, it extends in a unique manner to a *conjugation* J on $\mathscr{H}$; i.e., an antilinear, involutive ($J^2 = 1$) and isometric mapping $J: \mathscr{H} \to \mathscr{H}$.

Show that

$$J(x'\xi) = (x')^*\xi, \quad x' \in \mathscr{M}'.$$

(Hint: $x'\xi = \lim_{n\to\infty} x_n\xi$, $x_n \in \mathscr{M}$).

Infer from the preceding properties that the mapping

$$x \mapsto Jx^*J$$

is a $*$-antiisomorphism of $\mathscr{M}$ onto $\mathscr{M}'$, which acts identically on the center of $\mathscr{M}$.

!E.3.10. A von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is said to be *maximal Abelian* if it is commutative and maximal (with respect to the inclusion) with this property, in $\mathscr{B}(\mathscr{H})$. The von Neumann algebra $\mathscr{M}$ is maximal Abelian iff $\mathscr{M} = \mathscr{M}'$. Any commutative von Neumann algebra in $\mathscr{B}(\mathscr{H})$ is included in a maximal Abelian von Neumann algebra in $\mathscr{B}(\mathscr{H})$.

Show that a commutative von Neumann algebra, of countable type, is maximal Abelian iff it has a cyclic vector.

E.3.11. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with the center $\mathcal{Z} = \mathcal{M} \cap \mathcal{M}'$. Show that for subsets $\{x_{ij};\ 1 \leqslant i, j \leqslant n\} \subset \mathcal{M}$ and $\{x'_{ij};\ 1 \leqslant i, j \leqslant n\} \subset \mathcal{M}'$, the following assertions are equivalent:

(i) $\sum_{k=1}^{n} x_{ik} x'_{kj} = 0, \quad 1 \leqslant i, j \leqslant n,$

(ii) there exists a subset $\{z_{ij}; 1 \leqslant i, j \leqslant n\} \subset \mathcal{Z}$, such that

$$\sum_{k=1}^{n} x_{ik} z_{kj} = 0, \quad \sum_{k=1}^{n} z_{ik} x'_{kj} = x'_{ij}, \quad 1 \leqslant i, j \leqslant n.$$

(Hint: $(x_{ij}) \in \mathcal{M} \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H}_n)$, $(x'_{ij}) \in \mathcal{M}' \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H}_n)$; (i) $\Leftrightarrow (x_{ij})\,(x'_{ij}) = 0$).

E.3.12. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a factor, $\{x_1, \ldots, x_n\} \subset \mathcal{M}$ a linearly independent subset of $\mathcal{M}$, and $\{x'_1, \ldots, x'_n\} \subset \mathcal{M}'$. Then

$$\sum_{k=1}^{n} x_k x'_k = 0 \Leftrightarrow x'_k = 0, \quad 1 \leqslant k \leqslant n.$$

Infer that the mapping

$$\sum_{k=1}^{n} x_k \mathbin{\overline{\otimes}} x'_k \mapsto \sum_{k=1}^{n} x_k x'_k$$

is a $*$-isomorphism of the $*$-algebra $\mathcal{M} \otimes \mathcal{M}'$ onto the $*$-algebra generated in $\mathcal{B}(\mathcal{H})$ by $\mathcal{M}$ and $\mathcal{M}'$.

E.3.13. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $\{x_1, \ldots, x_n\} \subset \mathcal{M}$ and $\{\xi_1, \ldots, \xi_n\} \subset \mathcal{H}$ be such that $\sum_{k=1}^{n} x_k \xi_k = 0$. Then there exists a subset $\{x_{ij}; 1 \leqslant i, j \leqslant n\} \subset \mathcal{M}$, such that

$$\sum_{k=1}^{n} x_k x_{kj} = 0, \quad 1 \leqslant j \leqslant n,$$

$$\sum_{k=1}^{n} x_{ik} \xi_k = \xi_i, \quad 1 \leqslant i \leqslant n.$$

In other words, $\mathcal{H}$ is a *flat* $\mathcal{M}$-module.

E.3.14. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $e \in \mathcal{P}_{\mathcal{M}}$. Show that:
(i) if $\mathcal{X} \subset \mathcal{M}$ is a $*$-subalgebra and $\mathcal{M} = \mathcal{R}(\mathcal{X})$, then $\mathcal{M}_e = \mathcal{R}(\mathcal{X}_e)$
(ii) if $\mathcal{X}' \subset \mathcal{M}'$ and $\mathcal{M}' = \mathcal{R}(\mathcal{X}')$, then $\mathcal{M}'_e = \mathcal{R}(\mathcal{X}'_e)$.

!**E.3.15.** Let $\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1)$, $\mathcal{M}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be von Neumann algebras and $e_1 \in \mathcal{P}_{\mathcal{M}_1}$, $e_2 \in \mathcal{P}_{\mathcal{M}_2}$. Then $e_1 \mathbin{\overline{\otimes}} e_2 \in \mathcal{P}_{\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2}$ and the following relations hold:

$$(\mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2)_{e_1 \overline{\otimes} e_2} = (\mathcal{M}_1)_{e_1} \mathbin{\overline{\otimes}} (\mathcal{M}_2)_{e_2},$$

$$(\mathcal{M}'_1 \mathbin{\overline{\otimes}} \mathcal{M}'_2)_{e_1 \overline{\otimes} e_2} = (\mathcal{M}'_1)_{e_1} \mathbin{\overline{\otimes}} (\mathcal{M}'_2)_{e_2}.$$

E.3.16. Let $\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1)$, $\mathcal{M}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be von Neumann algebras. Show that if $\mathcal{M}_1 = \mathcal{R}(\mathcal{X}_1)$, $\mathcal{M}_2 = \mathcal{R}(\mathcal{X}_2)$, then $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 = \mathcal{R}(\mathcal{X})$, where we have denoted $\mathcal{X} = \{x_1 \overline{\otimes} 1;\ x_1 \in \mathcal{X}_1\} \cup \{1 \overline{\otimes} x_2;\ x_2 \in \mathcal{X}_2\}$.

!E.3.17. State and prove an associativity property for the tensor product of von Neumann algebras.

E.3.18. Let $\mathcal{A}$ be a C^*-algebra, $\mathfrak{N}$ a left ideal of $\mathcal{A}$ and $\{u_\alpha\}_{\alpha \in \Gamma}$ an approximate unit for $\mathfrak{N}$.

Show that

$$\overline{\mathfrak{N}} = \{x \in \mathcal{A};\ \|x - xu_\alpha\| \to 0\}.$$

E.3.19. Let $\mathcal{A}$ be a C^*-algebra and $\mathfrak{N} \subset \mathcal{A}$ a closed left ideal of $\mathcal{A}$. With the help of an approximate unit of $\mathfrak{N}$, show that

$$a \in \mathfrak{N}^+,\ \lambda > 0 \Rightarrow a^\lambda \in \mathfrak{N}^+.$$

(Hint: it is sufficient to prove that $a^{1/2} \in \mathfrak{N}^+$).

E.3.20. Let $\mathcal{A}$ be a C^*-algebra, $a \in \mathcal{A}^+$ and $\mathfrak{N}_a$ the closed left ideal of $\mathcal{A}$ generated by a in $\mathcal{A}$. Show that

(i) $u_n = (n^{-1} + a)^{-1} a$ is an approximate unit of $\mathfrak{N}_a$;

(ii) if $x \in \mathcal{A}$ and $x^*x \leqslant a$, then $x \in \mathfrak{N}_a$.

Comments

C.3.1. A *wo*-closed subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, which contains the identity operator (but which is not assumed to be self-adjoint) is said to be *transitive* if 0 and $\mathcal{H}$ are the only closed linear subspaces $\mathcal{K} \subset \mathcal{H}$, which are $\mathcal{A}$-invariant, i.e., such that

$$x \in \mathcal{A} \Rightarrow x(\mathcal{K}) \subset \mathcal{K}.$$

Obviously, $\mathcal{B}(\mathcal{H})$ is transitive.

From the von Neumann density theorem (3.2) it easily follows that any self-adjoint transitive subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ coincides with $\mathcal{B}(\mathcal{H})$.

If $\mathcal{H}$ is finitely dimensional, then a classical theorem of Burnside (see H. Weyl [2]) states that *any* transitive subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ coincides with $\mathcal{B}(\mathcal{H})$.

The problem of establishing whether this statement is, or is not, true in the general case, is still unresolved; nor is the weaker problem of the existence of an invariant closed non-trivial vector subspace, for any operator in $\mathcal{B}(\mathcal{H})$.

The first problem is known as the "problem of the transitive algebras", whereas the second is known as the "problem of the invariant subspaces".

An important contribution towards the solution of this problem, which has a surprisingly simple proof, was obtained by V. I. Lomonosov [1]. We state it in the form given by H. Radjavi and P. Rosenthal [1]:

Theorem. *Any transitive subalgebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$, which contains a non-zero compact operator, coincides with $\mathscr{B}(\mathscr{H})$.*

Corollary. *Any operator $x \in \mathscr{B}(\mathscr{H})$ which commutes with a non-zero compact operator, has a non-trivial closed invariant vector subspace.*

The result contained in the preceding corollary was obtained independently by D. Voiculescu [3], who used methods completely different from those of V. I. Lomonosov.

A vector subspace $\mathscr{X} \subset \mathscr{H}$ is said to be *para-closed* if it is the range of an operator in $\mathscr{B}(\mathscr{H})$. C. Foiaş [2] has shown that any subalgebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$, whose only invariant para-closed vector subspaces are 0 and $\mathscr{H}$, coincides with $\mathscr{B}(\mathscr{H})$; this is another extension of the Burnside theorem.

Extensions to von Neumann algebras have been obtained by D. Voiculescu [1] and C. Peligrad [2].

For other information we refer the reader to H. Radjavi and P. Rosenthal [1].

C.3.2. The essential fact in the proof of the density theorem of I. Kaplansky (3.10) is the *so*-continuity on $\mathscr{B}(\mathscr{H})^h$ of the mapping $x \mapsto f(x)$, where $f(t) = 2t(1+t^2)^{-1}$. The continuity in the *so*-topology of the functions of normal operators has been studied by I. Kaplansky [11] and, more recently, by R. V. Kadison [26]. We mention the following result of R. V. Kadison:

Theorem. *Let $\Omega \subset \mathbb{C}$ be a subset such that:*

$$(\overline{\overline{\Omega} \setminus \Omega}) \cap \Omega = \emptyset$$

and let $f: \Omega \to \mathbb{C}$ be a complex function. Then the following properties are equivalent:

(i) *the mapping $x \mapsto f(x)$ is so-continuous on the set of all normal operators, whose spectrum is contained in Ω;*

(ii) *the function f is continuous, bounded on bounded sets and such that $f(z)/z$ is bounded at infinity.*

Therefore, the theorem is valid for open or closed sets Ω. In particular, by taking $\Omega = \mathbb{R}$ in the preceding theorem, it follows that any continuous and bounded function $f: \mathbb{R} \to \mathbb{C}$ is *so*-continuous on the set of all self-adjoint operators, a result which extends the fact that was used in the proof of the density theorem of I. Kaplansky.

C.3.3. By using the density theorem of I. Kaplansky, R. V. Kadison [15] proved the following:

Theorem. *Let $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ be a wo-dense C^*-algebra. Then for any $x \in \mathscr{B}(\mathscr{H})$ and any finite dimensional vector subspace $\mathscr{X} \subset \mathscr{H}$ there exists an $a \in \mathscr{A}$, such that*

$$x\xi = a\xi, \quad \xi \in \mathscr{X}.$$

Moreover, if x is self-adjoint, then a can be chosen so as to be self-adjoint too, whereas if x is unitary and $\mathscr{A}$ contains the identity operator, then a can be chosen to be unitary.

A C^*-algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ is said to be *irreducible* if the only closed vector subspaces of $\mathscr{X}$, which are invariant for all operators in $\mathscr{A}$, are 0 and $\mathscr{H}$; the

C^*-algebra $\mathscr{A}$ is said to be *strictly-irreducible* if the only invariant (not necessarily closed) vector subspace of $\mathscr{H}$ are 0 and $\mathscr{H}$. From the preceding theorem one can infer the following

Corollary. *A C^*-algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ is irreducible iff it is stricly irreducible.*

For the proof of these results, with more precise formulations, we refer to the books of J. Dixmier [42] and S. Sakai [32].

M. Tomita [8] (see also K. Saitô [2]) has obtained extensions of these results. For an exposition of these extensions we refer to L. Zsidó [4].

A study of the strict irreducibility for the representations of the Banach algebras with involution has been carried out by B. A. Barnes [8].

C.3.4. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. For any set $\mathscr{X} \subset \mathscr{M}^+$ we denote by $\mathscr{X}^\sigma$ (resp., $\mathscr{X}_\delta$) the set of all elements of $\mathscr{M}^+$ which are suprema (resp., infima) of increasing sequences (resp., decreasing sequences) of elements in $\mathscr{X}$; we also denote by $\mathscr{X}^m$ (resp., $\mathscr{X}_m$) the set of all elements of $\mathscr{M}^+$ which are suprema (resp., infima) of bounded increasingly (resp., decreasingly) directed subsets of $\mathscr{X}$. Recently, G. K. Pedersen [5], [7], proved the following remarkable result:

Theorem 1. *If $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra of countable type, and if $\mathscr{A} \subset \mathscr{M}$ is a C^*-algebra, which is wo-dense in $\mathscr{M}$, then*

$$((\mathscr{A}_1^+)^\sigma)_\delta = \mathscr{M}_1^+.$$

If $\mathscr{M}$ is not assumed to be of countable type, then this theorem fails to be true even in the commutative case. Nevertheless, we have the following result, due to G. K. Pedersen *(loc. cit.)*:

Theorem 2. *If $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra and if $\mathscr{A} \subset \mathscr{M}$ is a wo-dense C^*-subalgebra, then*

$$(((\mathscr{A}_1^+)^m)_m)^m = \mathscr{M}_1^+.$$

From these theorems one can get the older results of R. V. Kadison [13], [14], which we shall now state:

Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a C^*-algebra. If $(\mathscr{M}_1^+)^m \subset \mathscr{M}$, or if $\mathscr{H}$ is separable and $(\mathscr{M}_1^+)^\sigma \subset \mathscr{M}$, then $\mathscr{M}$ is a von Neumann algebra.*

A proof of the results of G. K. Pedersen can be found in L. Zsidó [4]. The results of R. V. Kadison are also presented in Ş. Strătilă [1].

C.3.5. J. Dixmier and O. Maréchal [1] proved the following

Theorem. *The set of all cyclic vectors of a von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a G_δ-set, which is either empty, or dense in $\mathscr{H}$.*

We observe that for the proof of this result they also showed that *any element x in a von Neumann algebra $\mathscr{M}$ is the so-limit of a sequence $\{x_n\}$ of invertible elements in $\mathscr{M}$, such that $\|x_n\| \leqslant \|x\|$.*

We also mention the following result of M. Broise [5]:

Theorem. *Let $\mathscr{M}, \mathscr{N} \subset \mathscr{B}(\mathscr{H})$ be von Neumann algebras, such that $\mathscr{N}'$ is abelian and of countable type. If $\mathscr{M}$ and $\mathscr{N}$ have the same cyclic vectors, then $\mathscr{M} = \mathscr{N}$.*

C.3.6. Let $(\Omega, \mathfrak{B}, \mu)$ be a finite measure space. We shall now consider the Hilbert space $\mathscr{H}_\mu = \mathscr{L}^2(\mu)$ and for any $f \in \mathscr{L}^\infty(\mu)$ we shall denote by $x_f \in \mathscr{B}(\mathscr{H}_\mu)$ the operator given by the multiplication by f. Then

$$\mathscr{M}_\mu = \{x_f;\ f \in \mathscr{L}^\infty(\mu)\} \subset \mathscr{B}(\mathscr{H})$$

is a maximal abelian von Neumann algebra of countable type.

On the other hand, if $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a maximal abelian von Neumann algebra of countable type, then $\mathscr{M}$ has a cyclic and separating vector (see E.3.10), which determines in a natural manner a measure μ on the spectrum of $\mathscr{M}$ (see C.2.1). It is easily shown that $\mathscr{M}$ is $*$-isomorphic with $\mathscr{M}_\mu$ and, therefore, $\mathscr{M}$ and $\mathscr{M}_\mu$ are spatially isomorphic (E.5.22).

Consequently, the maximal abelian von Neumann algebras of countable type can be described in a simple manner, modulo the spatial isomorphism. This description can be easily extended to arbitrary maximal abelian von Neumann algebras.

C.3.7. In this chapter we introduced the following elementary operations on von Neumann algebras: the passing to the commutant, the reduction, the induction and the tensor product; the amplification, which is a particular case of the latter.

Another important operation, which did not explicitly appear in our presentation is, of course, the *direct sum:* if $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ and $\mathscr{N} \subset \mathscr{B}(\mathscr{H})$ are von Neumann algebras, then

$$\mathscr{M} \oplus \mathscr{N} = \{x \oplus y \in \mathscr{B}(\mathscr{H} \oplus \mathscr{K});\ x \in \mathscr{M},\quad y \in N\}$$

is a von Neumann algebra.

Another operation is the *cross-product*, which we shall describe in what follows.

Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra, Aut $(\mathscr{M})$ the group of all $*$-automorphisms of $\mathscr{M}$, G a locally compact group, dg its left Haar measure, and

$$G \ni g \mapsto \pi_g \in \text{Aut}\ (\mathscr{M})$$

a homomorphism of groups, such that for any $x \in \mathscr{M}$, the mapping

$$G \ni g \mapsto \pi_g(x) \in \mathscr{M}$$

is continuous for the *wo*-topology in $\mathscr{M}$.

One considers the space $\mathscr{K}(G, \mathscr{H})$ of all functions defined on G and taking values in $\mathscr{H}$, which have compact supports and are continuous for the norm topology; we endow it with the scalar product

$$(f_1 \mid f_2) = \int_G (f_1(g) \mid f_2(g))\, dg$$

and we denote by $\mathscr{L}^2(G; \mathscr{H})$ the Hilbert space obtained by completion.

For any $x \in \mathscr{M}$ the operator $t_x \in \mathscr{B}(\mathscr{L}^2(G, \mathscr{H}))$ is defined by the relations

$$(t_x(f))(g') = \pi_{g'}^{-1}(x)\,(f(g')),\ f \in \mathscr{K}(G; \mathscr{H}),\ g' \in G,$$

whereas for any $g \in G$ one defines the (unitary) operator $u_g \in \mathscr{B}(\mathscr{L}^2(G;\mathscr{H})$ by the relations:

$$(u_g(f))(g') = f(g^{-1}g'), \quad f \in \mathscr{K}(G;\mathscr{H}), \quad g' \in G.$$

The von Neumann algebra generated in $\mathscr{B}(\mathscr{L}^2(G;\mathscr{H}))$ by the operators t_x, $x \in \mathscr{M}$, and u_g, $g \in G$, is called the *cross-product of $\mathscr{M}$ by the action π of G* and it is denoted by $\mathscr{R}(\mathscr{M}, \pi)$ or, simply, by $\mathscr{M} \times G$.

If G is discrete, the preceding construction appears in the work of F. J. Murray and J. von Neumann [1] in connection with the construction of different types of factors (see C.4.3), whereas systematic expositions of this construction appear in J. Dixmier [26] (Ch. I, § 9.2), T. Turumaru [3], N. Suzuki [4], V. I. Golodets [22], etc.

In the general case, this construction first appears in the paper of S. Doplicher, D. Kastler and D. W. Robinson [1]; it is systematically studied by M. Takesaki [33]. If G is a separable *abelian* locally compact group, which acts by $*$-automorphisms of the von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is assumed to be separable, then the group $\hat{G}$ of the characters of G acts in a natural manner by $*$-automorphisms of $\mathscr{M} \times G$; M. Takesaki [33] has shown that the von Neumann algebra

$$(\mathscr{M} \times G) \times \hat{G}$$

is $*$-isomorphic to the von Neumann algebra

$$\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{L}^2(G)).$$

In particular, if $\mathscr{M}$ is properly infinite, then the following isomorphism

$$(\mathscr{M} \times G) \times \hat{G} \cong \mathscr{M}$$

holds, thus yielding a duality theorem. For extensions to the non-abelian case we refer to Ş. Strătilă, D. Voiculescu, L. Zsidó [1], [2], [3], M. Landstad [1], [3] and Y. Nakagami [5], [6].

Another definition of a cross-product, which is better adapted to the construction of factors, has been given by W. Krieger [3].

Finally, let us mention the fact that one can define a notion of infinite tensor product for von Neumann algebras, for which we refer to J. von Neumann [12], D. Bures [1] and A. Guichardet [15].

C.3.8. The extreme points of the closed unit ball of a C^*-algebra have been thoroughly studied, the fundamental results being the following two theorems of R. V. Kadison [2], which we state in the improved version given by S. Sakai [32]:

Theorem 1. *The closed unit ball of a C^*-algebra $\mathscr{A}$ has extreme points iff $\mathscr{A}$ has the unit element. In this case the unit element is an extreme point.*

Theorem 2. *An element x of the closed unit ball of a C^*-algebra $\mathscr{A}$ is extreme iff*

$$(1 - xx^*)\,\mathscr{A}(1 - x^*x) = \{0\}.$$

Lemma 3.19 is an easy consequence of the last theorem.

R. V. Kadison used these results in order to study the isometries between algebras of operators, in generalizing the classical theorem of S. Banach and M. H. Stone (see N. Dunford and J. Schwartz [1], V.8.8).

We also mention the following strong result due to B. Russo and H. A. Dye [1]:

Theorem 3. *In any C^*-algebra $\mathscr{A}$ with a unit element, the uniformly closed convex hull of the set $\{u \in \mathscr{A}; u \text{ unitary}\}$ coincides with the closed unit ball of $\mathscr{A}$.*

For other related results we refer to: P. E. Miles [2], B. Yood [3], J. B. Conway and J. Szücs [1], F. Bonsall and J. Ducan [3].

C.3.9. Bibliographical comments. Theorem 3.2 is due to J. von Neumann [2], whereas Theorem 3.10 to I. Kaplansky [10], [11]. Elementary operations on von Neumann algebras have been considered by many authors, among whom we mention: F. J. Murray and J. von Neumann [1], J. von Neumann [12], J. Dixmier [15], I. E. Segal [9], Y. Misonou [4], M. Tomita [2], [4]. For a detailed exposition of the properties of ideals of algebra of operators, as well as for the corresponding references, we refer to Ş. Strătilă [1].

The term "von Neumann algebra" was introduced by J. Dixmier [26]; F. J. Murray and J. von Neumann called these algebras "rings of operators".

I. M. Gelfand and M. A. Naimark called the Banach algebras "normed rings"; the term "Banach algebra" was introduced by E. Hille. The terms "C^*-algebra" and "W^*-algebra" (see C.5.3) have been introduced by I. E. Segal. Sometimes, for C^*-algebras one uses the equivalent term "B^*-algebras"; this double terminology is related to the problems discussed at the end of Section C.2.1.

In writing this chapter, we used the books by J. Dixmier [26], [42], and also the course by D. M. Topping [8].

4

The geometry of projections and the classification of von Neumann algebras

In this chapter we study the relations existing between the lattice operations and the equivalence of projections and we also introduce the classification of von Neumann algebras according to *types*.

4.1. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. We shall denote by $\mathcal{P}_{\mathcal{M}}$ the set of the projections in $\mathcal{M}$. Then $\mathcal{P}_{\mathcal{M}}$ is a complete lattice (see Corollary 3.7).

Two projections $e, f \in \mathcal{P}_{\mathcal{M}}$ are said to be *equivalent*, and this relation is denoted by $e \sim f$, if there exists a partial isometry $u \in \mathcal{M}$, such that $e = u^*u$ and $f = uu^*$; then $ue = u = fu$. We say that e is *dominated* by f, and we denote by $e \prec f$ this relation, if e is equivalent to a subprojection of f. The relation "$\sim$" is an equivalence relation in $\mathcal{P}_{\mathcal{M}}$, whereas the relation "$\sim$" is a preorder relation in $\mathcal{P}_{\mathcal{M}}$.

If $e \prec p_{\xi}$, $\xi \in \mathcal{H}$, then there exists an $\eta \in \mathcal{H}$, such that $e = p_{\eta}$.

If $e \sim f$, then $\mathrm{z}(e) = \mathrm{z}(f)$; in particular, if $e \sim 0$, then $e = 0$.

If $e \sim f$ and $p \in \mathcal{P}_{\mathcal{Z}}$, then $ep \sim fp$.

If $e \sim f$ *via* the partial isometry u, then the mapping

$$e\mathcal{M}e \ni x \mapsto uxu^* \in f\mathcal{M}f$$

is an isomorphism of $*$-algebras. The image ugu^* of the projection $g \in e\mathcal{M}e$ is a projection equivalent to g.

If $e = \bigvee_{i \in I} e_i$, where e_i are mutually orthogonal projections, and if $f \sim e$, then there exists a family $\{f_i\}_{i \in I} \subset \mathcal{P}_{\mathcal{M}}$, where the f_i are mutually orthogonal projections, such that $f = \bigvee_{i \in I} f_i$ and $f_i \sim e_i$, for any $i \in I$.

4.2. Proposition. *Let $\{e_i\}_{i \in I}$, $\{f_i\}_{i \in I} \subset \mathcal{P}_{\mathcal{M}}$, where the e_i are mutually orthogonal projections, and the f_i are also mutually orthogonal projections. If $e_i \sim f_i$, for any $i \in I$, then $\bigvee_{i \in I} e_i \sim \bigvee_{i \in I} f_i$.*

Proof. Let $u_i \in \mathcal{M}$ be such that

$$e_i = u_i^* u_i, \quad f_i = u_i u_i^*, \qquad i \in I,$$

and, for any finite subset $J \subset I$, let

$$u_J = \sum_{i \in J} u_i.$$

We then have a net $\{u_J\}_J$ of operators, about which we shall prove that it is *so*-convergent. In order to do this, we have only to show that for any $\xi \in \mathscr{H}$, the net $\{u_J\xi\}_J$ is convergent (in accordance with the Banach-Steinhaus theorem). Since the vectors $u_i\xi$ are mutually orthogonal, this is equivalent to the following condition

$$\sum_{i \in I} \|u_i\xi\|^2 = \lim_J \sum_{i \in J} \|u_i\xi\|^2 < +\infty;$$

indeed, we have

$$\sum_{i \in I} \|u_i\xi\|^2 = \sum_{i \in I} (u_i^* u_i \xi \mid \xi) = \sum_{i \in I} (e_i\xi \mid \xi) = ((\bigvee_{i \in I} e_i)\xi \mid \xi) < +\infty.$$

Consequently, the operators

$$u = \sum_{i \in I} u_i, \qquad v = \sum_{i \in I} u_i^*$$

exist by *so*-convergence and they belong to $\mathscr{M}$. From $u = so\text{-}\sum_{i \in I} u_i$, it follows that $u^* = wo\text{-}\sum_{i \in I} u_i^*$ and, therefore, $v = u^*$. From the equalities $u_i^* u_k = \delta_{ik} e_i$, it follows that $u^*u = \bigvee_{i \in I} e_i$, $uu^* = \bigvee_{i \in I} f_i$.

Q.E.D

4.3. Theorem. *For any $x \in \mathscr{M}$ one has* $\mathrm{l}(x) \sim \mathrm{r}(x)$.

Proof. Let $x = u|x|$ be the polar decomposition of x in $\mathscr{M}$. Then the partial isometry u implements the equivalence $\mathrm{l}(x) \sim \mathrm{s}(|x|)$. From $\mathrm{n}(|x|) = \mathrm{n}(x)$ we also infer that $\mathrm{s}(|x|) = \mathrm{l}(|x|) = \mathrm{l}(x^*) = \mathrm{r}(x)$.

Q.E.D.

4.4. Corollary. *(The parallelogram rule). For any $e, f \in \mathscr{P}_{\mathscr{M}}$ one has the relations*

(i) $e \vee f - f \sim e - e \wedge f$,

(ii) $e - e \wedge (1 - f) \sim f - (1 - e) \wedge f$.

Proof. It follows from Theorem 4.3 and exercise E.2.3.

Q.E.D.

4.5. Corollary. *Let $e, f \in \mathscr{P}_{\mathscr{M}}$. The following assertions are equivalent:*

(i) $e\mathscr{M}f \neq \{0\}$,

(ii) *there exist* $e_1, f_1 \in \mathscr{P}_{\mathscr{M}}$, $0 \neq e_1 \leqslant e$, $0 \neq f_1 \leqslant f$, *such that* $e_1 \sim f_1$,

(iii) $\mathrm{z}(e)\,\mathrm{z}(f) \neq 0$.

Proof. (i) $\Rightarrow$ (ii): if $x \in \mathscr{M}$ and $exf \neq 0$, then $e_1 = \mathrm{l}(exf)$ and $f_1 = \mathrm{r}(exf)$ satisfy condition (ii);

(ii) $\Rightarrow$ (i): if $e \geqslant e_1 = u^*u$, $f \geqslant f_1 = uu^* \neq 0$, then we have $eu^*f = u^* \neq 0$;

(i) $\Rightarrow$ (iii): if $\mathrm{z}(e)\,\mathrm{z}(f) = 0$, then $exf = \mathrm{z}(e)\,exf\,\mathrm{z}(f) = 0$, for any $x \in \mathscr{M}$;

(iii) $\Rightarrow$ (i): by taking into account Corollary 3.9, from $e\mathscr{M}f = \{0\}$, it follows that $e\mathrm{z}(f) = 0$, and therefore $\mathrm{z}(e)\mathrm{z}(f) = 0$.

Q.E.D.

4.6. Theorem *(the comparison theorem). For any $e, f \in \mathscr{P}_{\mathscr{M}}$ there exists a $p \in \mathscr{P}_{\mathscr{Z}}$, such that*

$$ep \prec fp,$$
$$e(1-p) \succ f(1-p).$$

Proof. Let $(\{e_i\}_{i\in I}, \{f_i\}_{i\in I})$ be a maximal pair of families of mutually orthogonal projections, such that

$$e_i \leqslant e, \quad f_i \leqslant f, \quad e_i \sim f_i, \quad i \in I.$$

In accordance with Proposition 4.2, it follows that

$$e_1 = \bigvee_{i\in I} e_i \sim \bigvee_{i\in I} f_i = f_1.$$

If $e_2 = e - e_1$ and $f_2 = f - f_1$, then, due to the maximality of the chosen pair and to Corollary 4.5, it follows that

$$\mathrm{z}(e_2)\,\mathrm{z}(f_2) = 0.$$

Let us define $p = \mathrm{z}(f_2)$. Then we have

$$ep = e_1p + e_2p = e_1p + e_2\mathrm{z}(e_2)\mathrm{z}(f_2) = e_1p \sim f_1p \leqslant fp,$$
$$f(1-p) = f_1(1-p) + f_2(1-p) = f_1(1-p) \sim e_1(1-p) \leqslant e(1-p).$$

Q.E.D.

4.7. Theorem *(von Neumann's Schröder-Bernstein type theorem). Let $e, f \in \mathscr{P}_{\mathscr{M}}$. If $e \prec f$ and $f \prec e$, then $e \sim f$.*

Proof. Let $w, v \in \mathscr{M}$ be such that

$$ww^* = e, \quad w^*w \leqslant f, \quad vv^* = f, \quad v^*v \leqslant e.$$

For $g \in \mathscr{P}_{\mathscr{M}}$, $g \leqslant f$, we define

$$\varphi(g) = f - w^*(e - v^*gv)w.$$

It is easy to see that φ is an increasing function on the complete lattice $\{g \in \mathscr{P}_{\mathscr{M}};\ g \leqslant f\}$. Let $\mathscr{X} = \{g;\ g \leqslant \varphi(g)\}$ and $h = \bigvee_{g\in\mathscr{X}} g$. If $g \in \mathscr{X}$, then $g \leqslant h$ and, therefore, $g \leqslant \varphi(g) \leqslant \varphi(h)$; hence $h \leqslant \varphi(h)$. Consequently, we have $\varphi(h) \leqslant \varphi(\varphi(h))$ and, therefore, $\varphi(h) \in \mathscr{X}$; it follows that $\varphi(h) \leqslant h$. Consequently, we have

$$h = f - w^*(e - v^*hv)\,w.$$

The partial isometries hv and $(e - v^*hv)\,w$ yield the equivalences

$$h \sim v^*hv \text{ and } f - h \sim e - v^*hv.$$

Consequently, we have $e \sim f$.

Q.E.D.

4.8. A projection $e \in \mathcal{P}_{\mathcal{M}}$ is said to be *abelian* if the reduced algebra $e\mathcal{M}e$ is commutative.

A projection $e \in \mathcal{P}_{\mathcal{M}}$ is said to be *finite* if

$$f \in \mathcal{P}_{\mathcal{M}}, f \leqslant e, f \sim e \Rightarrow f = e.$$

One says that a projection $e \in \mathcal{P}_{\mathcal{M}}$ is *properly infinite* if

$$p \in \mathcal{P}_{\mathcal{Z}}, pe \text{ finite} \Rightarrow pe = 0.$$

Any abelian projection is finite.

If e is finite and $f \prec e$, then f is finite.

If e is abelian and $f \prec e$, then f is abelian.

4.9. Proposition. *Let $e, f \in \mathcal{P}_{\mathcal{M}}$. If e is abelian and $f \leqslant e$, then $f = ez(f)$.*

Proof. Since $e\mathcal{M}e$ is commutative, for any $x \in \mathcal{M}$ we have

$$fx(e-f) = f(exe)(e-f) = f(e-f)(exe) = 0,$$

hence $f\mathcal{M}(e-f) = \{0\}$. In accordance with Corollary 4.5, it follows that $z(f)z(e-f) = 0$. In particular, we have $f = ez(f)$.

Q.E.D.

4.10. Proposition. *Let $e, f \in \mathcal{P}_{\mathcal{M}}$ be abelian projections. If $z(e) \leqslant z(f)$, then $e \prec f$. If $z(e) = z(f)$, then $e \sim f$.*

Proof. In accordance with Theorem 4.7, it is sufficient to prove only the first assertion. For this, in accordance with the comparison theorem (4.6), we can suppose that $f \leqslant e$. But then $f = ez(f)$, in accordance with Proposition 4.9. Therefore, we have

$$e = ez(e) = ez(f) = f.$$

Q.E.D.

4.11. Proposition. *Let $e \in \mathcal{P}_{\mathcal{M}}$ be such that it does not contain any non-zero abelian subprojection. Then there exist $e_1, e_2 \in \mathcal{P}_{\mathcal{M}}$, such that*

$$e = e_1 + e_2, \qquad e_1 e_2 = 0, \qquad e_1 \sim e_2.$$

Proof. Let $(\{e_{1,i}\}_{i \in I}, \{e_{2,i}\}_{i \in I})$ be a maximal pair of families of mutually orthogonal projections, such that

$$e_{1,i} \leqslant e, \quad e_{2,i} \leqslant e, \quad e_{1,i}e_{2,i} = 0, \; e_{1,i} \sim e_{2,i}, \quad i \in I.$$

In accordance with Proposition 4.2, it follows that

$$e_1 = \bigvee_{i \in I} e_{1,i} \sim \bigvee_{i \in I} e_{2,i} = e_2, \qquad e_1 e_2 = 0.$$

If

$$h = e - e_1 - e_2 \neq 0,$$

then h is not abelian; consequently, h contains a non-zero subprojection g, which is not central in $h\mathcal{M}h$. It follows that

$$g\mathcal{M}(h-g) \neq \{0\}.$$

In accordance with Corollary 4.5, this result contradicts the maximality of the chosen pair; consequently, $h = 0$, i.e.,

$$e = e_1 + e_2.$$

Q.E.D.

4.12. Proposition. *Let $e \in \mathcal{P}_{\mathcal{M}}$. Then e is properly infinite iff there exists a countable family $\{e_n\} \subset \mathcal{P}_{\mathcal{M}}$ of mutually orthogonal projections, which are bounded from above by e, and such that*

$$e = \bigvee_n e_n$$

$$e_n \sim e, \quad \textit{for any } n.$$

Corollary. *A projection $e \in \mathcal{P}_{\mathcal{M}}$ is properly infinite, iff there exist $e_1, e_2 \in \mathcal{P}_{\mathcal{M}}$, such that*

$$e = e_1 + e_2, \quad e_1 e_2 = 0, \quad e_1 \sim e_2 \sim e.$$

Proof. We proceed by steps:

(I) If e is not finite, then e contains an infinite set of mutually orthogonal subprojections, which are equivalent and different from zero.

Indeed, let $e_1 \leqslant e$, $e_1 \neq e$, $e_1 \sim e$ and $f_1 = e - e_1$. There exist $e_2, f_2 \in \mathcal{P}_{\mathcal{M}}$, such that

$$e_1 = e_2 + f_2, \quad e_2 f_2 = 0, \quad e_1 \sim e_2, \quad f_1 \sim f_2.$$

Then there exist $e_3, f_3 \in \mathcal{P}_{\mathcal{M}}$, such that

$$e_2 = e_3 + f_3, \quad e_3 f_3 = 0, \quad e_2 \sim e_3, \quad f_2 \sim f_3.$$

Now we proceed by induction. The required set is

$$\{f_1, f_2, f_3, \ldots\}.$$

(II) If $\{e_i\}_{i \in I}$ is an infinite family of mutually orthogonal, equivalent projections, $e = \bigvee_{i \in I} e_i$ and if $f \in \mathcal{P}_{\mathcal{M}}$, $fe = 0$, $f \prec e_i$, then there exist $\{h_i\}_{i \in I} \subset \mathcal{P}_{\mathcal{M}}$, such that the h_i be mutually orthogonal and $e + f = \bigvee_{i \in I} f_i$, $e_i \sim h_i$ for any $i \in I$.

Indeed, let $i_0 \in I$ and let $f_{i_0}, g_{i_0} \in \mathcal{P}_{\mathcal{M}}$ be such that

$$e_{i_0} = f_{i_0} + g_{i_0}, \qquad f_{i_0} g_{i_0} = 0, \qquad f_{i_0} \sim f.$$

For any $i \in I$, there exist $f_i, g_i \in \mathcal{P}_{\mathcal{M}}$, such that

$$e_i = f_i + g_i, \qquad f_i g_i = 0, \qquad f_i \sim f_{i_0}, \qquad g_i \sim g_{i_0}.$$

Let φ be a bijection between the sets $\{g_i\}_{i \in I}$ and $\{f, f_i\}_{i \in I}$, and let

$$h_i = g_i + \varphi(g_i), \qquad i \in I.$$

Now the assertion is obvious.

(III) If $\{e_i\}_{i \in I}$ is a maximal infinite family of mutually orthogonal, equivalent projections, then there exist $p \in \mathscr{P}_{\mathscr{Z}}$, $p \neq 0$, and $\{g_i\}_{i \in I} \subset \mathscr{P}_{\mathscr{M}}$, where the g_i's are mutually orthogonal, such that $p = \bigvee_{i \in I} g_i$ and $g_i \sim pe_i$, for any $i \in I$.

Indeed, let $e = \bigvee_{i \in I} e_i$ and $i_0 \in I$. Since the relation $e_{i_0} \prec 1 - e$ is not possible (due to the maximality of the family), by applying the comparison theorem to e_{i_0} and $1 - e$, it follows that there exists a $p \in \mathscr{P}_{\mathscr{Z}}$, $p \neq 0$, such that $p(1 - e) \prec pe_{i_0}$. But $p = p(1 - e) \vee (\bigvee_{i \in I} pe_i)$, and the assertion follows from (II).

(IV) If $1 \in \mathscr{M}$ is properly infinite, then 1 is the least upper bound of a countable family of equivalent, mutually orthogonal projections.

Indeed, with (I), there exists a maximal infinite family of equivalent, mutually orthogonal projections, whereas from (III) we infer that there exists a $p \in \mathscr{P}_{\mathscr{Z}}$, $p \neq 0$, which is the l.u.b. of a countable family of equivalent, mutually orthogonal projections.

Since $1 - p$ is also properly infinite, the proof proceeds by transfinite induction.

(V) If $e \in \mathscr{P}_{\mathscr{M}}$ is properly infinite, then, in accordance with (IV), there exists a countable family $\{f_n\}$ of equivalent, mutually orthogonal projections, such that $\bigvee_n f_n = e$. Let us define

$$e_n = \vee \; \{f_m;\; n \text{ divides } m \text{ and } f_m e_k = 0 \text{ for } k < n\}.$$

Then $e = \bigvee_n e_n$ and $e_n \sim e$, for any n; the first part of the proposition is proved.

The other assertions follow immediately.

Q.E.D.

4.13. One says that a projection $e \in \mathscr{P}_{\mathscr{M}}$ is of *countable type* if any family of mutually orthogonal non-zero projections, majorized by e, is at most countable.

Proposition. *Let $e, f \in \mathscr{P}_{\mathscr{M}}$, with e of countable type and f properly infinite. If $z(e) \leqslant z(f)$, then $e \prec f$.*

Proof. Let $\{e_i\}_{i \in I}$ be a maximal family of mutually orthogonal non-zero projections, such that

$$e_i \leqslant e, \quad e_i \prec f, \quad i \in I.$$

By virtue of Corollary 4.5 and of the maximality of the chosen family, we infer that

$$e = \bigvee_{i \in I} e_i.$$

Since e is of countable type, the set I is at most countable. With Proposition 4.12, it follows that there exists a family $\{f_i\}_{i\in I}$ of mutually orthogonal projections, such that

$$f = \bigvee_{i\in I} f_i, \quad f_i \sim f, \qquad i \in I.$$

It follows that $e_i \prec f_i$, $i \in I$, hence, $e < f$.

Q.E.D.

4.14. Lemma. *Let $\{e_i\}_{i\in I}$ be a family of finite (resp., abelian) projections, whose central supports are mutually orthogonal. Then $e = \bigvee_{i\in I} e_i$ is a finite (resp., abelian) projection.*

Proof. Let $f \in \mathscr{P}_{\mathscr{A}}$, $f \leqslant e$, $f \sim e$. Since $e - e_i \leqslant \bigvee_{n\neq i} z(e_n)$, it follows that $e_i = ez(e_i)$. Since e_i is finite, it follows that

$$e_i = fz(e_i) \leqslant f, \qquad i \in I.$$

Consequently, $e \doteq f$.

For the case of abelian projections, the proof is similar.

Q.E.D.

4.15. Proposition. *If $e, f \in \mathscr{P}_{\mathscr{A}}$ are finite projections, then $e \vee f$ is a finite projection.*

Proof. We can assume that $e \vee f = 1$. We shall also assume that 1 is not finite and we shall get a contradiction. From Lemma 4.14, there exists a finite central projection $p \in \mathscr{P}_{\mathscr{Z}}$, such that $1 - p$ be properly infinite. Therefore, we can assume that 1 is properly infinite. From Corollary 4.12, there exists a $g \in \mathscr{P}_{\mathscr{A}}$, such that

$$g \sim 1 \sim 1 - g.$$

From the parallelogram rule (4.4), we get

$$1 - e = e \vee f - e \sim f - e \wedge f \leqslant f,$$

hence $1 - e$ is a finite projection.

We now apply the comparison theorem (4.6) to the projections $g \wedge (1 - e)$ and $(1 - g) \wedge e$. It follows that we can consider that one of the following relations holds, without any loss of generality

(1) $$g \wedge (1 - e) \prec (1 - g) \wedge e,$$

(2) $$(1 - g) \wedge e \prec g \wedge (1 - e).$$

In the first case, by taking into account the parallelogram rule, we get

$$\begin{aligned} g &= g \wedge (1 - e) + (g - g \wedge (1 - e)) \\ &\prec (1 - g) \wedge e + (g \vee (1 - e) - (1 - e)) \leqslant e, \end{aligned}$$

and this contradicts the finiteness of e. In the second case, a similar argument leads to $1-g \prec 1-e$, thus contradicting the finiteness of $1-e$. The proposition is proved.

Q.E.D.

4.16. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra.

$\mathscr{M}$ is said to be *finite* if 1 is a finite projection.

$\mathscr{M}$ is said to be *semifinite* if any non-zero central projection contains a non-zero finite projection.

$\mathscr{M}$ is said to be of *type I* if any non-zero central projection contains a non-zero abelian projection.

$\mathscr{M}$ is said to be of *type II* if it is semifinite and it does not contain any non-zero abelian projection.

$\mathscr{M}$ is said to be of *type III* if it does not contain any non-zero finite projection.

$\mathscr{M}$ is said to be of *type* I_{fin} if it is finite and of type I.

$\mathscr{M}$ is said to be of *type* I_∞ if it is not finite and it is of type I.

$\mathscr{M}$ is said to be of *type* II_1 if it is finite and of type II.

$\mathscr{M}$ is said to be of *type* II_∞ if it is not finite, but it is of type II.

Other terms used in this connection are introduced by the following definitions:

$\mathscr{M}$ is said to be *discrete* if it is of type I.

$\mathscr{M}$ is said to be *continuous* if it is of type II or III.

$\mathscr{M}$ is said to be *properly infinite* if 1 is a properly infinite projection.

$\mathscr{M}$ is said to be *purely infinite* if it is of type III.

4.17. Theorem *(of classification). Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. Then there exist unique projections $p_i \in \mathscr{P}_{\mathscr{Z}}$, $i = 1, 2, 3, 4, 5$, such that $\sum_{i=1}^{5} p_i = 1$ and*

$$\begin{gathered}\mathscr{M}p_1 \textit{ is of type } \mathrm{I}_{\mathrm{fin}},\\ \mathscr{M}p_2 \textit{ is of type } \mathrm{I}_\infty,\\ \mathscr{M}p_3 \textit{ is of type } \mathrm{II}_1,\\ \mathscr{M}p_4 \textit{ is of type } \mathrm{II}_\infty,\\ \mathscr{M}p_5 \textit{ is of type } \mathrm{III}.\end{gathered}$$

Proof. The theorem follows from the superposition of the following three decompositions:

(i) There exist unique projections $p_0, q_0 \in \mathscr{P}_{\mathscr{Z}}$, such that $p_0q_0 = 0, p_0 + q_0 = 1$, $\mathscr{M}p_0$ is semifinite and $\mathscr{M}q_0$ is purely infinite.

Indeed, let us define $p_0 = \vee\{p \in \mathscr{P}_{\mathscr{Z}};\ \mathscr{M}p$ is semifinite$\}$ and $q_0 = 1 - p_0$.

(ii) There exist unique $p_0, q_0 \in \mathscr{P}_{\mathscr{Z}}$, such that $p_0q_0 = 0$, $p_0 + q_0 = 1$, $\mathscr{M}p_0$ is finite and $\mathscr{M}q_0$ is properly infinite.

Indeed, let us define $p_0 = \vee\{p \in \mathscr{P}_{\mathscr{Z}};\ p$ is finite$\}$ and $q_0 = 1 - p_0$. The fact that p_0 is finite follows from Lemma 4.14.

(iii) There exist unique $p_0, q_0 \in \mathscr{P}_{\mathscr{Z}}$, such that $p_0q_0 = 0$, $p_0 + q_0 = 1$, $\mathscr{M}p_0$ is discrete and $\mathscr{M}q_0$ is continuous.

Indeed, let us define $p_0 = \vee\{p \in \mathscr{P}_{\mathscr{Z}};\ \mathscr{M}p \text{ is discrete}\}$ and $q_0 = 1 - p_0$.

Q.E.D.

4.18. Corollary. *A factor $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is of one and only one of the types* $\mathrm{I}_{\mathrm{fin}}$, I_∞, II_1, II_∞, III.

4.19. Proposition. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. Then $\mathscr{M}$ is discrete (resp., semifinite)) iff $\mathscr{M}$ contains an abelian (resp., finite) projection, whose central support is equal to* 1.

Proof. Let $\{e_i\}_{i\in I}$ be a maximal family of abelian (resp., finite) non-zero projections, whose central supports be mutually orthogonal. With Lemma 4.14, we infer that the projection $e = \bigvee_{i\in I} e_i$ is abelian (resp., finite), whereas from the maximality of the chosen family we infer that $z(e) = 1$.

Q.E.D.

4.20. Corollary. *$\mathscr{M}$ is discrete (resp., semifinite) iff any non-zero projection in $\mathscr{M}$ contains an abelian (resp., finite) non-zero projection.*

Proof. Assuming that $\mathscr{M}$ is discrete (resp., semifinite), let $e \in \mathscr{P}_{\mathscr{M}}$ be an abelian (resp., finite) projection, such that $z(e) = 1$ and let $f \in \mathscr{P}_{\mathscr{M}}, f \neq 0$. With the comparison theorem we infer that there exists a $p \in \mathscr{P}_{\mathscr{Z}}$, such that $ep \prec fp$ and $e(1-p) \succ \succ f(1-p)$. It follows that $f(1-p)$ is abelian (resp., finite), whereas fp contains an abelian (resp., finite) projection, which is equivalent to ep. If $p \neq 0$, then $ep \neq 0$, since $z(e) = 1$.

Q.E.D.

4.21. Table about the classification of von Neumann algebras

Type I		Type II		Type III
Type $\mathrm{I}_{\mathrm{fin}}$	Type I_∞	Type II_1	Type II_∞	
discrete	discrete	continuous	continuous	continuous
semifinite	semifinite	semifinite	semifinite	purely infinite
finite	properly infinite	finite	properly infinite	properly infinite

4.22. Two von Neumann algebras $\mathscr{M}_1 \subset \mathscr{B}(\mathscr{H}_1)$, $\mathscr{M}_2 \subset \mathscr{B}(\mathscr{H}_2)$ are said to be *spatially isomorphic* if there exists a unitary operator $u : \mathscr{H}_1 \to \mathscr{H}_2$ (i.e. u is isometric and onto), such that $u\mathscr{M}_1u^{-1} = \mathscr{M}_2$. In this case $\mathscr{M}_1$ and $\mathscr{M}_2$ are $*$-isomorphic.

The following theorem provides a link between the geometry of projections and the tensor product (see also Section 3.18).

Theorem. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\{e_i\}_{i\in I} \subset \mathscr{M}$ a family of equivalent, mutually orthogonal projections, such that $\sum_{i\in I} e_i = 1$. We denote by $\mathscr{H}_I$ a Hilbert space whose dimension is equal to card (I). Then $\mathscr{M}$ is spatially isomorphic to $\mathscr{M}_{e_i} \overline{\otimes} \mathscr{B}(\mathscr{H}_I)$, for any $i \in I$.*

Proof. Let us fix an index $i_0 \in I$. For any $i \in I$, let $v_i \in \mathscr{M}$ be a partial isometry, such that

$$v_i^* v_i = e_{i_0} \quad , \quad v_i v_i^* = e_i.$$

Let then $\{\eta_i\}_{i \in I}$ be an orthonormal basis in $\mathscr{H}_I$. We define the linear operator

$$u : \mathscr{H} \ni \xi \mapsto \sum_{i \in I} v_i^*(\xi) \otimes \eta_i \in e_{i_0}(\mathscr{H}) \overline{\otimes} \mathscr{H}_I.$$

It is easy to see that it is a unitary operator.

Let $x \in \mathscr{B}(\mathscr{H})$, $\zeta \in e_{i_0}(\mathscr{H})$ and $i \in I$. We then have

$$uxu^{-1}(\zeta \otimes \eta_i) = uxv_i(\zeta) = \sum_{k \in I} v_k^* x v_i(\zeta) \otimes \eta_k.$$

If we denote by $w_{k,i}$ the partial isometry in $\mathscr{B}(\mathscr{H}_I)$, such that $w_{k,i}^* w_{k,i} = [\mathbf{C}\eta_i]$, whereas $w_{k,i} w_{k,i}^* = [\mathbf{C}\eta_k]$, we have

$$uxu^{-1} = \sum_{i, k \in I} (v_k^* x v_i) \overline{\otimes} w_{k,i}.$$

Hence we immediately infer that

$$u\mathscr{M}u^{-1} \subset \mathscr{M}_{e_{i_0}} \overline{\otimes} \mathscr{B}(\mathscr{H}_I)$$

and it is easy to see that

$$u\mathscr{M}'u^{-1} \subset (\mathscr{M}_{e_{i_0}})' \overline{\otimes} \mathscr{C}(\mathscr{H}_I) \subset (\mathscr{M}_{e_{i_0}} \overline{\otimes} \mathscr{B}(\mathscr{H}_I))'.$$

By taking into account Corollary 3.3, we deduce that

$$u\mathscr{M}u^{-1} = \mathscr{M}_{e_{i_0}} \overline{\otimes} \mathscr{B}(\mathscr{H}_I).$$

Q.E.D.

4.23. In this final section we exhibit the relations existing between the two-sided ideals of a von Neumann algebra and the ideals of its center.

Let $\mathscr{M}$ be a von Neumann algebra, whose center is $\mathscr{Z}$.

Lemma 1. *For any ideal* $\mathfrak{J}$ *of* $\mathscr{Z}$ *we have* $(\mathscr{M}\mathfrak{J}) \cap \mathscr{Z} = \mathfrak{J}$.

Proof. Let $a \in (\mathscr{M}\mathfrak{J}) \cap \mathscr{Z}$, $a \geqslant 0$. With the help of the polar decomposition (see 3.6), we can write

$$a = \sum_{j=1}^{n} x_j y_j, \; x_j = x_j^* \in \mathscr{M}, \; 0 \leqslant y_j \in \mathfrak{J}.$$

Then $0 \leqslant a \leqslant \sum_{j=1}^{n} \|x_j\| y_j$. With the help of exercise E.2.6, we find an element $z \in \mathscr{Z}$, such that $a = z \sum_{j=1}^{n} \|x_j\| y_j$. Hence $a \in \mathfrak{J}$.

From the foregoing result and with the help of the polar decomposition we get the inclusion $(\mathscr{M}\mathfrak{J}) \cap \mathscr{Z} \subset \mathfrak{J}$. Since the reversed inclusion is obvious, the lemma is proved.

Q.E.D.

Lemma 2. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two-sided ideals in $\mathscr{M}$. Then we have*

$$(\mathfrak{M} + \mathfrak{N}) \cap \mathscr{Z} = \mathfrak{M} \cap \mathscr{Z} + \mathfrak{N} \cap \mathscr{Z}.$$

Proof. Let $a \in (\mathfrak{M} + \mathfrak{N}) \cap \mathscr{Z}$, $a \geqslant 0$ and let $x \in \mathfrak{M}$, $x = x^*$, $y \in \mathfrak{N}$, $y = y^*$ be such that

$$a = x + y.$$

Since $a \in \mathscr{Z}$, it follows that x and y commute. Let $e = s((2x - a)^+)$ (see 2.10, 2.15); it follows that $e \in \mathscr{M}$ and

$$0 \leqslant ae \leqslant 2xe, \qquad 0 \leqslant a(1 - e) \leqslant 2y(1 - e).$$

From exercise E.2.6, we infer that there exist $s, t \in \mathscr{M}$, such that

$$ae = sxe, \qquad a(1 - e) = ty(1 - e),$$

hence $ae \in \mathfrak{M}$ and $a(1 - e) \in \mathfrak{N}$.

With the comparison theorem (4.6) we infer that there exists a projection $p \in \mathscr{Z}$, such that

$$ep \prec (1 - e)p, \quad (1 - e)(1 - p) \prec e(1 - p).$$

Consequently, there exist two partial isometries $u, v \in \mathscr{M}$, such that

$$u^*u = ep, uu^* \leqslant (1 - e)p \text{ and } v^*v = (1 - e)(1 - p), vv^* \leqslant e(1 - p).$$

Hence we infer that

$$u^*(1 - e)pu = ep, \qquad v^*e(1 - p)v = (1 - e)(1 - p).$$

Thus, by taking into account the fact that $a \in \mathscr{Z}$, from what we have already proved we infer that

$$u^*a(1 - e)pu = aep \in \mathfrak{M} \cap \mathfrak{N},$$

$$v^*ae(1 - p)v = a(1 - e)(1 - p) \in \mathfrak{M} \cap \mathfrak{N}.$$

Consequently, we have

$$ap = a(1 - e)p + aep \in \mathfrak{N}$$

$$a(1 - p) = a(1 - e)(1 - p) + ae(1 - p) \in \mathfrak{M}$$

and

$$a = a(1 - p) + ap \in \mathfrak{M} \cap \mathscr{Z} + \mathfrak{N} \cap \mathscr{Z}.$$

We have thus proved the inclusion $(\mathfrak{M} + \mathfrak{N}) \cap \mathscr{Z} \subset \mathfrak{M} \cap \mathscr{Z} + \mathfrak{N} \cap \mathscr{Z}$. The reversed inclusion is obvious.

Q.E.D.

Theorem. *Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{Z}$ its center. For any ideal $\mathfrak{J}$ of $\mathcal{Z}$, the set of all two-sided ideals $\mathfrak{M}$ of $\mathcal{M}$, such that $\mathfrak{M} \cap \mathcal{Z} = \mathfrak{J}$ has a smallest element, denoted by $\mathfrak{M}_0(\mathfrak{J})$, and a greatest element, denoted by $\mathfrak{M}_\infty(\mathfrak{J})$.*

Proof. With Lemma 1, the ideal $\mathfrak{M}_0(\mathfrak{J}) = \mathcal{M}\mathfrak{J}$ obviously is the smallest element of the considered set. This set is inductively ordered by inclusion and, therefore, it has a maximal element $\mathfrak{M}_\infty(\mathfrak{J})$. From Lemma 2 it follows that the considered set is increasingly directed and, therefore, $\mathfrak{M}_\infty(\mathfrak{J})$ is its greatest element.

Q.E.D.

From the preceding theorem one can easily infer the following

Corollary. *Let $\mathcal{M}$ be a von Neumann algebra with the center $\mathcal{Z}$. Then the mapping*

$$\mathfrak{M} \mapsto \mathfrak{M} \cap \mathcal{Z} = \mathfrak{J}$$

is a bijection between the set of all maximal two-sided ideals $\mathfrak{M}$ of $\mathcal{M}$ and the set of all maximal ideals $\mathfrak{J}$ of $\mathcal{Z}$.

Exercises

In the exercises in which the symbols $\mathcal{M}$ and $\mathcal{Z}$ are not otherwise explained, they will denote a von Neumann algebra and its center.

E.4.1. Let $e, f \in \mathcal{P}_\mathcal{M}$. Then

$$e \prec f \Leftrightarrow \text{there exists a } g \in \mathcal{P}_\mathcal{M}, \text{ such that } e \leqslant g \sim f.$$

E.4.2. Let $e, f \in \mathcal{P}_\mathcal{M}$. Then there exists a $p \in \mathcal{P}_\mathcal{Z}$, such that

$$ep \prec fp,$$
$$(1-e)(1-p) \prec (1-f)(1-p).$$

E.4.3. Let $e_1, e_2, f_1, f_2 \in \mathcal{P}_\mathcal{M}$, $e_1 e_2 = f_1 f_2 = 0$. If

$$e_1 + e_2 = f_1 + f_2,$$
$$e_1 \sim e_2,\ f_1 \sim f_2,$$

then $e_1 \sim f_1$.

E.4.4. Let $e, f \in \mathcal{P}_\mathcal{M}$, $a = e + f - 1$ and $s = 1 - 2s(a^-)$. Then s is a symmetry and $sefs = fe$.

E.4.5. Let $e, f \in \mathcal{P}_\mathcal{M}$, $e \wedge (1-f) = (1-e) \wedge f = 0$. Then there exists a symmetry $s \in \mathcal{M}$, such that $ses = f$.

E.4.6. For any $e, f \in \mathcal{P}_\mathcal{M}$ there exists a symmetry $s \in \mathcal{M}$, such that $s(e \vee f - e)s = f - e \wedge f$. In particular, the parallelogram rule now easily follows (see 4.4).

E.4.7. For any pair of equivalent projections $e, f \in \mathcal{M}$, there exist projections $e_1, e_2, f_1, f_2 \in \mathcal{P}_\mathcal{M}$, $e_1 e_2 = f_1 f_2 = 0$ and unitary elements $u_1, u_2 \in \mathcal{M}$, such that

$$e = e_1 + e_2, \qquad f = f_1 + f_2,$$
$$u_1^* e_1 u_1 = f_1, \qquad u_2^* e_2 u_2 = f_2.$$

!E.4.8. Let $e, f \in \mathscr{P}_{\mathscr{M}}$, with e finite, f properly infinite. Then

$$e \prec f \Leftrightarrow \mathrm{z}(e) \leqslant \mathrm{z}(f).$$

!E.4.9. Let $e, f \in \mathscr{P}_{\mathscr{M}}$ be equivalent finite projections. Then

$$1 - e \sim 1 - f.$$

Consequently, there exists a unitary element $u \in \mathscr{M}$, such that $u^*eu = f$.

E.4.10. Let $e \in \mathscr{P}_{\mathscr{M}}$ be a *continuous projection* (i.e., the reduced algebra $\mathscr{M}_e$ is continuous). For any natural number n there exists a finite set $\{e_1, \ldots, e_n\} \subset \mathscr{P}_{\mathscr{M}}$ of equivalent, mutually orthogonal projections, such that $e = \sum_{i=1}^{n} e_i$.

E.4.11. Let $\mathscr{M}$ be a (properly) infinite factor and $e, f \in \mathscr{P}_{\mathscr{M}}$. Then

$$e \vee f \sim 1 \Leftrightarrow e \sim 1 \text{ or } f \sim 1.$$

E.4.12. A projection $e \in \mathscr{P}_{\mathscr{M}}$ is said to be *minimal relatively to the central support* if $e \neq 0$ and

$$f \in \mathscr{P}_{\mathscr{M}},\ f \leqslant e,\ \mathrm{z}(f) = \mathrm{z}(e) \Rightarrow f = e.$$

Show that a non-zero projection is minimal relatively to the central support iff it is abelian.

A factor is of type I iff it contains a *minimal projection* and, in this case, it is $*$-isomorphic to a $\mathscr{B}(\mathscr{H})$, the dimension of $\mathscr{H}$ being uniquely determined.

E.4.13. Any finitely dimensional von Neumann algebra is of type $\mathrm{I}_{\mathrm{fin}}$. Infer from this result that if $\mathscr{M}$ is continuous and $e \in \mathscr{P}_{\mathscr{M}}$, $e \neq 0$, then $\dim(e\mathscr{H}) = +\infty$.

!E.4.14. One says that $\mathscr{M}$ is *homogeneous of type* I_γ (resp., *uniform of type* S_γ). where γ is any cardinal, if there exists a family $\{e_\iota\}_{\iota \in I}$ of abelian (resp., finite) equivalent, mutually orthogonal projections in $\mathscr{M}$, such that $\sum_{\iota \in I} e_\iota = 1$, and card $I = \gamma$. In this case, $\mathscr{M}$ is of type I (resp., semifinite).

Show that for any von Neumann algebra $\mathscr{M}$ of type I (resp., semifinite and properly infinite) there exists a family Γ of distinct cardinals (resp., distinct infinite cardinals) and a family $\{p_\gamma\}_{\gamma \in \Gamma}$ of mutually orthogonal central projections, such that $\sum_{\gamma \in \Gamma} p_\gamma = 1$, and, moreover, $\mathscr{M}p_\gamma$ be homogeneous of type I_γ (resp., uniform of type S_γ).

E.4.15. If n is a natural number, and $\mathscr{M}$ is homogeneous of type I_n, then any family of non-zero, equivalent, mutually orthogonal projections in $\mathscr{M}$ has at most n elements. In particular, if m, n are natural numbers, whereas $\mathscr{M}$ is homogeneous of type I_m and of type I_n, then $m = n$ (see also Section 8.4).

E.4.16. Let $\mathscr{M}$ be a von Neumann algebra, which is homogeneous, of type I_γ Show that there exists a commutative von Neumann algebra $\mathscr{Z}$, which is $*$-isomorphic to the center of $\mathscr{M}$, and a Hilbert space $\mathscr{H}$, such that $\mathscr{M}$ be spatially isomorphic to $\mathscr{Z} \overline{\otimes} \mathscr{B}(\mathscr{H})$.

In particular, any factor of type I is $*$-isomorphic to $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a suitable Hilbert space.

E.4.17. Let $e, f_1, \ldots, f_n \in \mathscr{P}_{\mathscr{M}}$, with abelian $f_1, \ldots, f_n$. If $e \leqslant \sum_{k=1}^{n} f_k$, then there exist mutually orthogonal, abelian $e_1, \ldots, e_m \in \mathscr{P}_{\mathscr{M}}$, such that $e = \sum_{j=1}^{m} e_j$.

E.4.18. Let $\mathscr{M}$ be a von Neumann algebra of type I (resp., of type II; resp., of type III), and $e \in \mathscr{P}_{\mathscr{M}}$. Then $\mathscr{M}_e$ is of type I (resp., of type II; resp., of type III).

E.4.19. Show that any factor of type II is spatially isomorphic to the tensor product of a factor of type II_1 by a $\mathscr{B}(\mathscr{H})$.

!**E.4.20.** Let $\mathscr{M}$ be a finite von Neumann algebra and $\mathscr{H}$ a Hilbert space. Show that the von Neumann algebra $\mathscr{M} \overline{\otimes} \mathscr{B}(\mathscr{H})$ is finite if $\dim(\mathscr{H}) < +\infty$, and properly infinite, if $\dim(\mathscr{H}) = +\infty$.

!**E.4.21.** Let $e \in \mathscr{P}_{\mathscr{M}}$ be such that

$$p \in \mathscr{P}_{\mathscr{Z}},\ ep \text{ abelian} \Rightarrow ep = 0.$$

Show that there exist $e_1, e_2 \in \mathscr{P}_{\mathscr{M}}$, $e_1 e_2 = 0$, such that

$$e = e_1 + e_2$$

$$z(e_1) = z(e_2) = z(e).$$

E.4.22. Show that any abelian von Neumann algebra is $*$-isomorphic to a maximal abelian von Neumann algebra.

E.4.23. Let $\mathscr{M}$ be a von Neumann algebra and "$\approx$" an equivalence relation in $\mathscr{P}_{\mathscr{M}}$, such that

(i) If $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$ are families of mutually orthogonal projections in $\mathscr{M}$, such that

$$e_i \approx f_i, \qquad i \in I,$$

then

$$\sum_{i \in I} e_i \approx \sum_{i \in I} f_i;$$

(ii) If $e, f \in \mathscr{P}_{\mathscr{M}}$ and if there exists a unitary element $u \in \mathscr{M}$, such that

$$e = u^* f u,$$

then

$$e \approx f.$$

With the help of Proposition 4.12 and of E.4.9, show that if $e, f \in \mathscr{P}_{\mathscr{M}}$, and

$$e \sim f$$

then

$$e \approx f.$$

Consequently, the relation "$\sim$" is the minimal completely additive extension of the relation of "unitary equivalence".

E.4.24. Let $\mathscr{M}$ be a factor. Show that:

(1) any non-zero two-sided ideal of $\mathscr{M}$ is w-dense in $\mathscr{M}$.

(2) if $\mathscr{M}$ is finite, or of type III and of countable type, then $\mathscr{M}$ has no non-trivial two-sided ideals.

(3) if $\mathscr{M}$ is semifinite and properly infinite, then the linear hull of the set of all finite projections in $\mathscr{M}$ is equal to the smallest non-trivial two-sided ideal of $\mathscr{M}$.

(4) if $\mathscr{M}$ is of type III, but it is not of countable type, then the linear hull of the set of all projections in $\mathscr{M}$, which are of countable type, is the smallest non-trivial two-sided ideal of $\mathscr{M}$.

Comments

C.4.1. The geometry of projections developed in an algebraic frame, started by C. E. Rickart [1] and by I. Kaplansky [13], [17], [22], for the so-called $*$-Baer rings. A complete exposition of the results obtained in this direction can be found in the book by S. K. Berberian [11].

A C^*-algebra, which is, at the same time, a $*$-Baer ring, is called an *AW^*-algebra* (algebraic W^*-algebra; for the notion of a W^*-algebra, which is, essentially, the same as that of von Neumann algebra, see C.5.3). For AW^*-algebras almost all results in this chapter are true, with essentially the same proofs. Any commutative AW^*-algebra is $*$-isomorphic to $\mathscr{C}(\Omega)$, where Ω is a *stonean space* (a Hausdorff compact space, in which the closure of any open set is open), whereas any commutative von Neumann algebra (commutative W^*-algebra) is $*$-isomorphic to $\mathscr{C}(\Omega)$, where Ω is a *hyperstonean* space (cf. J. Dixmier [17]). For expositions of these results see W. G. Bade [2] and L. Zsidó [4].

C.4.2. R. V. Kadison and G. K. Pedersen [1] defined an equivalence relation for the positive elements of a von Neumann algebra $\mathscr{M}$; namely, two elements $a, b \in \mathscr{M}$, $a, b \geq 0$, are said to be equivalent if there exists a family $\{x_\iota\}_{\iota \in I} \subset \mathscr{M}$, such that

$$a = \sum_{\iota \in I} x_\iota^* x_\iota, \qquad b = \sum_{\iota \in I} x_\iota x_\iota^*.$$

They developed a theory for this equivalence relation, which is similar to the geometry of projections, and they showed that, for projections, the equivalence they introduced coincides with the usual equivalence of projections (4.1).

C.4.3. From exercises E.4.14 and E.4.16 there follows a complete description of the structure of von Neumann algebras of type I.

The structure of the continuous von Neumann algebras is still far from being well understood, even for the case of separable Hilbert spaces. Since, in this case, a "reduction theory" exists, which reduces the study of general von Neumann algebras to that of the factors — theory which was developed by J. von Neumann [15] (see also J. Dixmier [26], L. Zsidó [3]), the difficulty remains the classification of the continuous factors.

F. J. Murray and J. von Neumann [3] constructed two non-isomorphic factors of type II_1. As late as 1963, J. T. Schwartz [1] constructed a third factor of type II_1. S. Sakai [25] and Wai-Mee Ching [1] added two more examples of non-isomorphic factors of type II_1. J. Dixmier and E. C. Lance [1] constructed two other factors of type II_1, whereas G. Zeller-Meier [5] succeeded in constructing another two new factors. Thus, in 1969, only nine non-isomorphic factors of type II_1 were known. In the same year, D. McDuff [1] constructed a countable family of mutually non-isomorphic factors of type II_1; afterwards, D. McDuff [2] and S. Sakai [28] have shown that there exists a family of mutually non-isomorphic factors of type II_1, having the power of the continuum. In the same article, S. Sakai has shown the existence of a family, having the power of the continuum, of mutually non-isomorphic factors of type II_∞. For an exposition of the present state of the theory of factors of type II, we refer to D. McDuff [3], S. Sakai [32], and to the recent papers of A. Connes [14 — 19], [21], [22].

As far as the factors of type III are concerned, F. J. Murray and J. von Neumann have shown the existence of a factor of type III. Only in 1956, L. Pukánszky [4] found a second example of a factor of type III and, afterwards, in 1963, J. T. Schwartz [2] found a third example. Thus, in 1967, only three mutually non-isomorphic factors of type III were known. In the same year, R. T. Powers [1] proved the existence of a family of "hyperfinite" factors of type III, having the power of the continuum; in 1968, H. Araki and E. J. Woods [3] found another family of mutually non-isomorphic hyperfinite factors of type III, having the power of the continuum. In 1970, S. Sakai [27] showed that there also exists a family of mutually non-isomorphic non-hyperfinite factors of type III, having the power of the continuum. Expositions on this state of the theory of factors of type III can be found in J. T. Schwartz [5] and S. Sakai [32].

Recently, the theory of factors of type III greatly expanded, overcoming the stage of the "fight with cardinals". These investigations started with the paper of A. Connes [6] and were developed by A. Connes [8 — 11], [22 — 24] and M. Takesaki [29], [33]. The main technical instruments used in these investigations are the cross-products, infinite tensor products and, especially, the theory of Tomita, which we shall present in Chapter 10.

The structure and the classification of factors is, at present, one of the main fields of research in the theory of operator algebras.

C.4.4. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\mathscr{Z}$ be the center of $\mathscr{M}$. For any $x \in \mathscr{M}$ we denote by $\mathscr{U}(x)$ the convex hull of the set $\{u^*xu;\ u \in \mathscr{M}, \text{unitary}\}$ and by $\overline{\mathscr{U}(x)}^{u}$ (resp., $\overline{\mathscr{U}(x)}^{w}$) the closure of $\mathscr{U}(x)$ in the uniform topology (resp., the w-topology). Following an idea of J. von Neumann, J. Dixmier [12] introduced the sets

$$\mathscr{K}(x) = \mathscr{Z} \cap \overline{\mathscr{U}(x)}^{n},$$
$$\mathscr{C}(x) = \mathscr{Z} \cap \overline{\mathscr{U}(x)}^{w}.$$

The fundamental result obtained by J. Dixmier is the following

Theorem 1. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. For any $x \in \mathscr{M}$ one has that*

$$\mathscr{K}(x) \neq \emptyset.$$

With the help of the spectral theorem, the proof reduces to the case in which x is a projection, whereas, in this case, the proof uses the geometry of projections (see J. Dixmier [26], Ch. III, § 5). The significance of this theorem is discussed in C.7.1.

As far as the inclusion $\mathscr{K}(x) \subset \mathscr{C}(x)$ is concerned, H. Halpern [13] and Ş. Strătilă and L. Zsidó [2] have proved the following

Theorem 2. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a properly infinite von Neumann algebra. Then the following properties are equivalent*

(i) *for any $x \in \mathscr{M}$ the equality $\mathscr{K}(x) = \mathscr{C}(x)$ holds.*

(ii) *$\mathscr{M}$ is of countable type.*

In the case of finite von Neumann algebras the equality $\mathscr{K}(x) = \mathscr{C}(x)$ always holds; more precisely, these sets reduce to a single element (see C.7.1).

By studying the derivations of von Neumann algebras, S. Sakai [17] proved the following

Theorem 3. *If $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra of type III and of countable type, then $\mathscr{K}(x) \setminus \{0\} \neq \emptyset$, for any $x \in \mathscr{M}$, $x \neq 0$.*

On the other hand, L. Zsidó [5] obtained the following result

Theorem 4. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a properly infinite von Neumann algebra. Then, for any $x \in M$, the set $\mathscr{C}(x)$ is the w-closed convex hull of the set $\mathscr{Z} \cap \overline{\{u^*xu; u \in \mathscr{M}, u \text{ unitary}\}}^{w}$.*

C.4.5. Bibliographical comments. The results in this chapter were obtained by F. J. Murray and J. von Neumann for the case of factors. The reduction theory of J. von Neumann [15] provided the possibility of extending these results to von Neumann algebras with a separable predual. In general these results have been obtained by global methods by J. Dixmier [18] and I. Kaplansky [13], [17], [22]. In the book by J. Dixmier [26] the classification of von Neumann algebras is performed on the basis of apparently different criteria, from those we have given here (*loc. cit.*, Ch. I, §§ 6, 8), but one can prove that it is equivalent to what we have given here (*loc. cit.*, Ch. III, §§ 1, 2, 8). The proof of Theorem 4.7 is due to A. Lebow [1].

The result in Theorem 4.23, for closed ideals, as well as Corollary 4.23, are due to Y. Misonou [1]. In the general form given here, the result belongs to D. Voiculescu [2]. We note that if the ideal $\mathfrak{J}$ of $\mathscr{Z}$ is closed, then $\mathfrak{M}_{\infty}(\mathfrak{J})$ is closed, too (see J. Dixmier [26], Ch. III, § 5).

In our exposition we used J. Dixmier [26] and I. Kaplansky [22]. The results E.4.4, E.4.5, E.4.6 appeared in the course by D. Topping [8].

5

Linear forms on algebras of operators

This chapter is dedicated to the study of the predual of a von Neumann algebra, i.e., to the study of the w-topology in the algebra. In this manner, the algebra appears to be, and it is studied as, the dual Banach space of its predual.

5.1. Let $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ be a $*$-algebra of operators. By a *form* on $\mathscr{A}$ we shall mean any linear functional on $\mathscr{A}$. To any form φ on $\mathscr{A}$ one can associate its *adjoint form* φ^*, defined by

$$\varphi^*(a) = \overline{\varphi(a^*)}, \quad a \in \mathscr{A}.$$

A form φ is said to be self-adjoint (or *hermitean*) if $\varphi = \varphi^*$. The form φ is self-adjoint iff it takes real values at the self-adjoint elements of the $*$-algebra $\mathscr{A}$. Any form φ has a unique decomposition

$$\varphi = \varphi_1 + i\varphi_2,$$

where φ_1 and φ_2 are self-adjoint forms. The form φ is bounded iff φ_1 and φ_2 are bounded.

A form φ on $\mathscr{A}$ is said to be *positive* if it takes positive values at positive elements of $\mathscr{A}$. If $\mathscr{A}$ is a C^*-algebra of operators, then any positive form on $\mathscr{A}$ is self-adjoint.

5.2. Proposition. *Any positive form φ on a C^*-algebra $\mathscr{A}$ is bounded.*

Proof. Let

$$\alpha = \sup \{\varphi(a);\ a \in \mathscr{A},\ a \geq 0,\ \|a\| \leq 1\}.$$

If $\alpha = +\infty$, then there exists a sequence $\{a_n\} \subset \mathscr{A}$, $a_n \geq 0$, $\|a_n\| \leq 1$, such that $\varphi(a_n) \geq n$. On the other hand, for any sequence $(\lambda_n)_n$ of positive numbers, such that $\sum_{n=1}^{\infty} \lambda_n < +\infty$, the series $\sum_{n=1}^{\infty} \lambda_n a_n$ converges to an element $a \in \mathscr{A}$, and, since φ is positive, it follows that

$$\sum_{n=1}^{m} \lambda_n \varphi(a_n) = \varphi\left(\sum_{n=1}^{m} \lambda_n a_n\right) \leq \varphi(a), \quad m = 1, 2, \ldots$$

It follows that the series $\sum_{n=1}^{\infty} \lambda_n \varphi(a_n)$ converges. Since the sequence $\{\lambda_n\}$, $\lambda_n \geqslant 0$, such that $\sum_{n=1}^{\infty} \lambda_n < +\infty$ was arbitrary, this fact contradicts the relation $\varphi(a_n) \geqslant n$, $n \geqslant 1$. Consequently, we have $\alpha < +\infty$.

Now, for any $x = x^* \in \mathscr{A}$, $\|x\| \leqslant 1$, we have

$$|\varphi(x)| \leqslant \varphi(x^+) + \varphi(x^-) \leqslant 2\alpha,$$

whence $\|\varphi\| \leqslant 4\alpha$.

Q.E.D.

5.3. Proposition. *(the Schwarz inequality). Let φ be a positive form on the $*$-algebra $\mathscr{A}$. Then, for any $a, b \in \mathscr{A}$ we have*

$$|\varphi(ab)|^2 \leqslant \varphi(aa^*)\,\varphi(b^*b).$$

Proof. If $\varphi(ab) = 0$, the inequality is obvious and so we can assume that $\varphi(ab) \neq 0$. For any $\lambda \in \mathbb{C}$ we have

$$\varphi((\lambda a + b^*)(\lambda a + b^*)^*) \geqslant 0,$$

i.e.,

$$|\lambda|^2 \varphi(aa^*) + \lambda\varphi(ab) + \overline{\lambda}\,\overline{\varphi(ab)} + \varphi(b^*b) \geqslant 0.$$

If we take in this inequality $\lambda = t(|\varphi(ab)|/\varphi(ab))$, $t \in \mathbb{R}$, it follows that

$$t^2\varphi(aa^*) + 2t|\varphi(ab)| + \varphi(b^*b) \geqslant 0.$$

If we now write that the discriminant of this real polynomial is negative, we get

$$|\varphi(ab)|^2 - \varphi(aa^*)\,\varphi(b^*b) \leqslant 0.$$

Q.E.D.

5.4. Proposition. *Let φ be a bounded form on the C^*-algebra $\mathscr{A}$, assumed to have the unit element. Then φ is positive iff $\varphi(1) = \|\varphi\|$.*

Proof. If φ is positive, with the help of the Schwarz inequality we easily get $\varphi(1) = \|\varphi\|$.

Conversely, assume that $\varphi(1) = \|\varphi\| = 1$ and that there exists an $a \in \mathscr{A}$, $a \geqslant 0$, such that $\varphi(a)$ is not positive. Then there exists a disk $\{\lambda;\ |\lambda - \lambda_0| \leqslant r\}$ in the complex plane, which contains the spectrum of a, but does not contain $\varphi(a)$. Since the spectrum of the normal operator $a - \lambda_0$ is included in the disk $\{\lambda;\ |\lambda| \leqslant r\}$, we have $\|a - \lambda_0\| \leqslant r$. It follows that

$$|\varphi(a) - \lambda_0| = |\varphi(a) - \lambda_0\varphi(1)| = |\varphi(a - \lambda_0)| \leqslant \|a - \lambda_0\| \leqslant r,$$

a contradiction.

Q.E.D.

5.5. Proposition. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and φ a bounded form on $\mathscr{M}$. Then φ is positive iff it takes positive values at all the projections in $\mathscr{M}$.*

Proof. The proposition is an immediate consequence of Corollary 2.23.
Q.E.D.

5.6. Let φ be a bounded form on a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$. One says that φ is *completely additive* if, for any family $\{e_i\}_{i\in I}$ of mutually orthogonal projections in $\mathcal{M}$, one has

$$\varphi\Big(\sum_{i\in I} e_i\Big) = \sum_{i\in I} \varphi(e_i).$$

Obviously, any w-continuous form on $\mathcal{M}$ is completely additive. As we shall see later (Theorem 5.11), the converse is also true.

In order to prove this, we need some preparation.

Let φ be a bounded self-adjoint form on $\mathcal{M}$ and α a real number. By taking into account Corollary 2.23, it easily follows that, if there exists an $a\in\mathcal{M}$, $a \geqslant 0$, $\|a\| \leqslant 1$, such that $\varphi(a) > \alpha$, then there exists an $e\in\mathcal{P}_{\mathcal{M}}$, such that $\varphi(e) > \alpha$. Consequently, if $\varphi(e) \leqslant \alpha$ for any $e\in\mathcal{P}_{\mathcal{M}}$, then $\|\varphi\| \leqslant 4\alpha$.

5.7. Lemma. *Let φ be a bounded, self-adjoint, completely additive form on $\mathcal{M}$. For any $e\in\mathcal{P}_{\mathcal{M}}$ there exists an $f\in\mathcal{P}_{\mathcal{M}}$, $f\leqslant e$, such that $\varphi(f)\geqslant\varphi(e)$, whereas the restriction of φ to $f\mathcal{M}f$ be positive.*

Proof. Let $\{e_i\}_{i\in I}$ be a maximal family of mutually orthogonal projections, majorized by e and such that $\varphi(e_i) < 0$, $i\in I$; let $f = e - \sum_{i\in I} e_i$. Since the considered family is maximal, φ is positive at any projection which is majorized by f and therefore, according to Proposition 5.5, it is positive on $f\mathcal{M}f$. On the other hand since $\varphi(e_i) < 0$, $i\in I$, and since φ is completely additive, we have $\varphi(f) \geqslant \varphi(e)$
Q.E.D.

5.8. Lemma. *Let φ be a completely additive positive form on $\mathcal{M}$ and let $e\in\mathcal{P}_{\mathcal{M}}$, $e\neq 0$. Then there exists an $f\in\mathcal{P}_{\mathcal{M}}$, $f\neq 0$, $f\leqslant e$, and a $\xi\in\mathcal{H}$, such that*

$$|\varphi(xf)| \leqslant \|xf\xi\|, \quad x\in\mathcal{M}.$$

Proof. There exists an $\eta\in\mathcal{H}$, such that

$$(\omega_\eta - \varphi)(e) = \|e\eta\|^2 - \varphi(e) > 0.$$

From Lemma 5.7, we infer that there exists an $f\in\mathcal{P}_{\mathcal{M}}$, $f\leqslant e$, such that the restriction of $\omega_\eta - \varphi$ to $f\mathcal{M}f$ be positive and

$$(\omega_\eta - \varphi)(f) \geqslant (\omega_\eta - \varphi)(e).$$

It follows that $f\neq 0$ and, for any $x\in\mathcal{M}$,

$$\|xf\eta\|^2 - \varphi(fx^*xf) = (\omega_\eta - \varphi)(fx^*xf) \geqslant 0.$$

By using the Schwarz inequality (5.3), we get

$$|\varphi(xf)|^2 \leqslant \varphi(1)\,\varphi(fx^*xf) \leqslant \varphi(1)\,\|xf\eta\|^2,$$

and, therefore, we can choose ξ to be equal to a suitable scalar multiple of η.
Q.E.D.

5.9. Lemma. *Any positive form φ, which is completely additive on $\mathscr{M}$, is w-continuous.*

Proof. Let $\{e_i\}_{i \in I}$ be a maximal family of mutually orthogonal non-zero projections in $\mathscr{M}$, such that for any $i \in I$ there exists a $\xi_i \in \mathscr{H}$ with the property that

$$|\varphi(xe_i)| \leqslant \|xe_i\xi_i\|, \quad x \in \mathscr{M}.$$

From Lemma 5.8, we infer that

$$\sum_{i \in I} e_i = 1.$$

Since φ is completely additive, we get

$$\sum_{i \in I} \varphi(e_i) = \varphi(1).$$

Let $\varepsilon > 0$. Then there exists a finite subset $J \subset I$ such that, if we denote

$$e = \sum_{i \in J} e_i, \quad f = \sum_{i \in I \setminus J} e_i,$$

we have

$$\varphi(f) \leqslant \varepsilon^2 \|\varphi\|^{-1}.$$

We now define the bounded forms φ_1, φ_2 on $\mathscr{M}$ by

$$\varphi_1(x) = \varphi(xe), \quad x \in \mathscr{M},$$

$$\varphi_2(x) = \varphi(xf), \quad x \in \mathscr{M}.$$

Then $\varphi = \varphi_1 + \varphi_2$, φ_1 is w-continuous, because

$$|\varphi_1(x)| \leqslant \sum_{i \in J} \|xe_i\xi_i\|, \quad x \in \mathscr{M},$$

whereas $\|\varphi_2\| \leqslant \varepsilon$, as a consequence of the following computations

$$|\varphi_2(x)|^2 = |\varphi(xf)|^2 \leqslant \varphi(1)\varphi(fx^*xf) \leqslant \|\varphi\| \, \|x\|^2\varphi(f) \leqslant \varepsilon^2\|x\|^2, \quad x \in \mathscr{M}.$$

Hence we obtained a $\varphi_1 \in \mathscr{M}_*$, such that $\|\varphi - \varphi_1\| \leqslant \varepsilon$. Since $\mathscr{M}_*$ is uniformly closed in $\mathscr{M}^*$ (cf. Theorem 1.10), we infer that $\varphi \in \mathscr{M}_*$, and this shows that φ is w-continuous.

Q.E.D.

5.10. Lemma. *Let $e \in \mathscr{B}(\mathscr{H})$ be a projection, $x \in \mathscr{B}(\mathscr{H})$, $\|x\| \leqslant 1$ and α, β, γ be real numbers, such that $\alpha, \beta, \alpha\beta - \gamma^2 \geqslant 0$. Then*

$$\alpha e + \beta(1 - e) + \gamma(ex(1 - e) + (1 - e)x^*e) \geqslant 0.$$

Proof. For any real numbers s,t we have

$$\alpha s^2 + \beta t^2 - 2|\gamma|st \geqslant 0.$$

Hence, for any $\xi \in \mathscr{H}$ we have

$$\begin{aligned}\alpha(e\xi|\xi) &+ \beta((1-e)\xi|\xi) + \gamma(ex(1-e)\xi|\xi) + \gamma((1-e)x^*e\xi|\xi)\\ &= \alpha\|e\xi\|^2 + \beta\|(1-e)\xi\|^2 + 2\gamma\,\mathrm{Re}(x(1-e)\xi|e\xi)\\ &\geqslant \alpha\|e\xi\|^2 + \beta\|(1-e)\xi\|^2 - 2|\gamma|\,\|(1-e)\xi\|\,\|e\xi\| \geqslant 0.\end{aligned}$$

Q.E.D.

5.11. Theorem. *A bounded form on a von Neumann algebra is w-continuous iff it is completely additive.*

Proof. Let φ be a completely additive bounded form on the von Neumann algebra $\mathscr{M}$. We can assume that φ is self-adjoint and $\|\varphi\| \leqslant 1$. We write

$$\mu = \sup\{\varphi(a);\quad a \in \mathscr{M},\quad 0 \leqslant a \leqslant 1\}.$$

Then $0 \leqslant \mu \leqslant 1$. Let $\varepsilon > 0$, $\varepsilon \leqslant 3/4$. There exists an $a \in \mathscr{M}$, $0 \leqslant a \leqslant 1$, such that $\varphi(a) > \mu - \varepsilon$. From Corollary 5.6 and Lemma 5.7, we infer that there exists a projection $e_1 \in \mathscr{M}$, such that $\varphi(e_1) > \mu - \varepsilon$, whereas the restriction of φ to $e_1\mathscr{M}e_1$ be positive. From Lemma 5.9, we infer that the restriction of φ to $e_1\mathscr{M}e_1$ is w-continuous.

Let $e_2 = 1 - e_1$. We define the forms $\varphi_{ij} \in \mathscr{M}^*$ by

$$\varphi_{ij}(x) = \varphi(e_i x e_j),\ x \in \mathscr{M};\ i,j = 1,2.$$

Then φ_{11} is w-continuous and $\varphi = \varphi_{11} + \varphi_{12} + \varphi_{21} + \varphi_{22}$. If $f \in \mathscr{P}_{\mathscr{M}}$, $f \leqslant e_2$, then

$$\mu \geqslant \varphi(e_1 + f) = \varphi(e_1) + \varphi(f) > \mu - \varepsilon + \varphi(f),$$

and, therefore, $\varphi(f) < \varepsilon$.

We now investigate the norms $\|\varphi_{12}\|$ and $\|\varphi_{21}\|$. Let $x \in \mathscr{M}$, $\|x\| \leqslant 1$. We denote

$$y = (1-\varepsilon)e_1 + \varepsilon e_2 + \varepsilon^{1/2}(1-\varepsilon)^{1/2}(e_1 x e_2 + e_2 x^* e_1).$$

Then

$$1 - y = \varepsilon e_1 + (1-\varepsilon)e_2 - \varepsilon^{1/2}(1-\varepsilon)^{1/2}(e_1 x e_2 + e_2 x^* e_1).$$

With Lemma 5.10 we infer that $0 \leqslant y \leqslant 1$; hence:

$$\begin{aligned}\mu \geqslant \varphi(y) &= (1-\varepsilon)\varphi(e_1) + \varepsilon\varphi(e_2) + \varepsilon^{1/2}(1-\varepsilon)^{1/2}\varphi(e_1 x e_2 + e_2 x^* e_1)\\ &> (1-\varepsilon)(\mu-\varepsilon) - \varepsilon + 2\varepsilon^{1/2}(1-\varepsilon)^{1/2}\,\mathrm{Re}\varphi(e_1 x e_2).\end{aligned}$$

We hence obtain

$$\mathrm{Re}\,\varphi_{12}(x) = \mathrm{Re}\,\varphi(e_1 x e_2) \leqslant \frac{1}{2}\varepsilon^{1/2}\frac{2+\mu-\varepsilon}{(1-\varepsilon)^{1/2}} \leqslant \frac{1}{2}\varepsilon^{1/2}\frac{3}{(1/4)^{1/2}} = 3\varepsilon^{1/2}.$$

Therefore, we have $\|\varphi_{12}\| \leqslant 3\varepsilon^{1/2}$ and, analogously, $\|\varphi_{21}\| \leqslant 3\varepsilon^{1/2}$.

We denote by ψ the restriction of $(-\varphi)$ to $e_2\mathcal{M}e_2$. Then $\|\psi\| \leqslant 1$ and, for any projection $f \leqslant e_2$, we have $\psi(f) > -\varepsilon$. By repeating for ψ the preceding argument, we get the projections f_1, f_2, such that $f_1 + f_2 = e_2$ and, by denoting

$$\psi_{ij}(x) = \psi(f_i x f_j), \quad x \in e_2\mathcal{M}e_2; \quad i,j = 1, 2,$$

the following properties should hold: ψ_{11} is w-continuous, $\|\psi_{12}\| \leqslant 3\varepsilon^{1/2}, \|\psi_{21}\| \leqslant 3\varepsilon^{1/2}$, and, for any projection $f \leqslant f_2$, $\psi_{22}(f) < \varepsilon$. It follows that $|\psi_{22}(f)| < \varepsilon$, for any projection $f \leqslant f_2$; hence $\|\psi_{22}\| \leqslant 4\varepsilon$ (see 5.6). Consequently, we have

$$\|\psi - \psi_{11}\| \leqslant 4\varepsilon + 6\varepsilon^{1/2}.$$

We define the form ψ_0 on $\mathcal{M}$ by

$$\psi_0(x) = \psi_{11}(e_2 x e_2), \quad x \in \mathcal{M}.$$

Then ψ_0 is w-continuous and

$$\|\varphi - \varphi_{11} - \psi_0\| \leqslant 4\varepsilon + 12\varepsilon^{1/2}.$$

Since $\mathcal{M}_*$ is closed in $\mathcal{M}^*$ and $0 < \varepsilon \leqslant 3/4$ is arbitrary, it follows that $\varphi \in \mathcal{M}_*$, i.e., φ is w-continuous.

Q.E.D.

5.12. Corollary. *A bounded form on a von Neumann algebra is w-continuous iff its restrictions to maximal commutative von Neumann subalgebras of $\mathcal{M}$ are w-continuous.*

5.13. Corollary. *Any $*$-isomorphism between two von Neumann algebras is w-continuous.*

Proof. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be two von Neumann algebras and let $\pi : \mathcal{M} \to \mathcal{N}$ be a $*$-isomorphism between them. Then for any element $x \in \mathcal{M}$ we have $\sigma(x^*x) = \sigma(\pi(x^*x))$ (see Corollary 2.8) and, hence, by taking into account Lemma 2.5, we get

$$\|x\|^2 = \|x^*x\| = \|\pi(x^*x)\| = \|\pi(x)^*\pi(x)\| = \|\pi(x)\|^2,$$

i.e., π is an isometry.

Let ψ be a w-continuous form on $\mathcal{N}$. Then $\varphi = \psi \circ \pi$ is a bounded form on $\mathcal{M}$. If $\{e_i\}_{i \in I}$ is a family of mutually orthogonal projections in $\mathcal{M}$, then $\{\pi(e_i)\}_{i \in I}$ is a family of mutually orthogonal projections in $\mathcal{N}$ and

$$\pi\Big(\sum_{i \in I} e_i\Big) = \pi\Big(\bigvee_{i \in I} e_i\Big) = \bigvee_{i \in I} \pi(e_i) = \sum_{i \in I} \pi(e_i).$$

Consequently, we have

$$\varphi\Big(\sum_{i \in I} e_i\Big) = \psi\Big(\pi\Big(\sum_{i \in I} e_i\Big)\Big) = \psi\Big(\sum_{i \in I} \pi(e_i)\Big) = \sum_{i \in I} \psi(\pi(e_i)) = \sum_{i \in I} \varphi(e_i).$$

Thus, φ is completely additive and, by virtue of Theorem 5.11, φ is w-continuous. It follows that π is w-continuous.

Q.E.D.

5.14. Another consequence of Theorem 5.11 is the characterization of the weakly relatively compact subsets of the predual of a von Neumann algebra.

Theorem. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra, $\mathscr{M}_*$ the predual of $\mathscr{M}$ and $\mathscr{F} \subset \mathscr{M}_*$ a norm-bounded subset. The following assertions are equivalent*

(i) *$\mathscr{F}$ is $\sigma(\mathscr{M}_*; \mathscr{M})$-relatively compact;*

(ii) *for any countable family $\{e_n\}$ of mutually orthogonal projections in $\mathscr{M}$ one has $\varphi(e_n) \to 0$, uniformly for $\varphi \in \mathscr{F}$.*

Proof. (ii) $\Rightarrow$ (i). Since $\mathscr{F}$ is a bounded subset of $\mathscr{M}_* \subset \mathscr{M}^*$, it follows that its $\sigma(\mathscr{M}^*;\mathscr{M})$-closure $\overline{\mathscr{F}}$ in $\mathscr{M}^*$ is $\sigma(\mathscr{M}^*;\mathscr{M})$-compact. It is, therefore, sufficient to show that $\overline{\mathscr{F}} \subset \mathscr{M}_*$; indeed, we then have $\mathscr{F} \subset \overline{\mathscr{F}} \subset \mathscr{M}_*$ and $\overline{\mathscr{F}}$ is $\sigma(\mathscr{M}_*;\mathscr{M})$-compact, since the $\sigma(\mathscr{M}_*;\mathscr{M})$-topology is just the restriction of the $\sigma(\mathscr{M}^*;\mathscr{M})$-topology to $\mathscr{M}_*$.

Let $\varphi \in \overline{\mathscr{F}}$. There then exists a net $\{\varphi_k\}_{k\in K} \subset \mathscr{F}$, which is $\sigma(\mathscr{M}^*;\mathscr{M})$-convergent to φ.

Let $\{e_\iota\}_{\iota\in I}$ be a family of mutually orthogonal projections in $\mathscr{M}$ and $e = \sum_{\iota\in I} e_\iota$. We then have

$$\varphi(e) = \lim_{k\in K} \varphi_k(e),$$

$$\varphi(e_\iota) = \lim_{k\in K} \varphi_k(e_\iota), \text{ for any } \iota \in I,$$

$$\varphi_k(e) = \sum_{\iota\in I} \varphi_k(e_\iota), \text{ uniformly for } k \in \mathrm{K}.$$

In fact, we have $\psi(e) = \sum_{\iota\in I} \psi(e_\iota)$, uniformly for $\psi \in \mathscr{F}$. Indeed, if this be not true, then there exists a sequence $\{\psi_n\}_n \subset \mathscr{F}$, a sequence $(J_n)_n$ of finite mutually disjoint subsets $J_n \subset I$, and a $\delta > 0$, such that for any n we have $\left|\sum_{\iota\in J_n} \psi_n(e_\iota)\right| \geqslant \delta$. We define $f_n = \sum_{\iota\in J_n} e_\iota$. Then $\{f_n\}_n$ is a countable family of mutually orthogonal projections in $\mathscr{M}$ and, for any n, we have $|\psi_n(f_n)| \geqslant \delta$, a contradiction if (ii) is taken into account.

It follows that

$$\varphi(e) = \sum_{\iota\in I} \varphi(e_\iota),$$

hence φ is completely additive and, therefore, by virtue of Theorem 5.11, φ is w-continuous, i.e., $\varphi \in \mathscr{M}_*$.

(i) $\Rightarrow$ (ii). We shall proceed by contradiction.

Hence, there exists a sequence $\{e_n\}_n$ of mutually orthogonal projections in $\mathscr{M}$, a sequence $\{\varphi_n\} \subset \mathscr{F}$ and a $\delta > 0$, such that, for any n, we have

$$|\varphi_n(e_n)| > 4\delta.$$

Since $\mathscr{F}$ is $\sigma(\mathscr{M}_*;\mathscr{M})$-relatively compact, we can assume that the sequence $\{\varphi_n\}$ is $\sigma(\mathscr{M}_*;\mathscr{M})$-convergent to a form $\varphi \in \mathscr{M}_*$. Since the sequence $\{e_n\}$ is

w-convergent to 0, we have $\lim_{n\to\infty} \varphi(e_n) = 0$ and, therefore, we can assume that, for any n we have

$$|\varphi(e_n)| \leqslant \delta.$$

The sequence of forms $\psi_n = \varphi_n - \varphi \in \mathcal{M}_*$ is $\sigma(\mathcal{M}_*;\mathcal{M})$-convergent to 0 and, for any n, we have

$$|\psi_n(e_n)| \geqslant 3\delta. \tag{1}$$

We shall now construct an increasing sequence $\{n(1), n(2),\ldots\}$ of natural numbers, with the following properties

$$\left|\sum_{j=1}^{k-1} \psi_{n(k)}(e_{n(j)})\right| < \delta, \quad \text{for any } k = 2, 3,\ldots \tag{2}$$

$$\sum_{j=n(k+1)}^{\infty} |\psi_{n(k)}(e_j)| < \delta, \quad \text{for any } k = 1, 2,\ldots \tag{3}$$

In order to do this, let us first observe that for any $\psi \in \mathcal{M}_*$ we have

$$\sum_{n=1}^{\infty} |\psi(e_n)| < +\infty,$$

because for any bounded sequence $\{\lambda_n\}$ of scalars, the series $\sum_{n=1}^{\infty} \lambda_n e_n$ is w-convergent, hence the series $\sum_{n=1}^{\infty} \lambda_n \psi(e_n)$ is convergent.

We begin the construction by taking $n(1) = 1$ and we assume that $n(1),\ldots, n(p-1)$ have already been constructed, such that condition (2) be satisfied for $k = 2,\ldots,p-1$, whereas condition (3) be satisfied for $k = 1,\ldots,p-2$. Since $\{\psi_n\}$ is $\sigma(\mathcal{M}_*;\mathcal{M})$-convergent to 0 and since $\sum_{j=1}^{\infty} |\psi_{n(p-1)}(e_j)| < +\infty$, for a sufficiently great n the following inequalities are satisfied

$$\left|\sum_{j=1}^{p-1} \psi_n(e_{n(j)})\right| < \delta,$$

$$\sum_{j=1}^{\infty} |\psi_{n(p-1)}(e_j)| < \delta.$$

Consequently, relation (2) is satisfied for $k = p$, whereas relation (3) is satisfied for $k = p - 1$, if we choose $n(p) > n(p-1)$ to be sufficiently great. The required construction is thus possible by induction.

From relation (3) it follows that

$$\sum_{j=k+1}^{\infty} |\psi_{n(k)}(e_{n(j)})| < \delta, \quad k = 1, 2, \ldots \tag{4}$$

We now consider the projection $f = \sum_{j=1}^{\infty} e_{n(j)} \in \mathcal{M}$. We then have

$$\psi_{n(k)}(f) = \sum_{j=1}^{\infty} \psi_{n(k)}(e_{n(j)}), \quad k = 1, 2, \ldots$$

From relations (1), (2) and (4) we infer that

$$|\psi_{n(k)}(f)| > \delta, \quad k = 1, 2, \ldots,$$

thus contradicting the fact that the sequence $\{\psi_n\}_n$ is $\sigma(\mathcal{M}_*; \mathcal{M})$-convergent to 0.
Q.E.D.

5.15. The w-continuous positive forms on a von Neumann algebra are also called *normal forms*. Let φ be a normal form on $\mathcal{M}$ and let $a \in \mathcal{M}$, $a \geqslant 0$, be such that $\varphi(a) = 0$. Then $\varphi(s(a)) = 0$. Indeed, in accordance with Corollary 2.22, there exists an increasing sequence $\{e_n\} \subset \mathcal{P}_{\mathcal{M}}$, such that $\bigvee_n e_n = s(a)$, and $ae_n \geqslant (1/n)e_n$, for any n; it follows that $\varphi(e_n) = 0$, for any n, and, therefore, $\varphi(s(a)) = 0$.

If $e, f \in \mathcal{P}_{\mathcal{M}}$, $\varphi(e) = 0$ and $\varphi(f) = 0$, then $\varphi(e \vee f) = 0$, because $e \vee f = s(e+f)$.

Consequently, if φ is a normal form on $\mathcal{M}$, then the family $\{e \in \mathcal{P}_{\mathcal{M}};\ \varphi(e) = 0\}$ is increasingly directed and, therefore, by denoting by $1 - s(\varphi)$ the l.u.b. of this family, we infer that $\varphi(1 - s(\varphi)) = 0$. The projection $s(\varphi)$ is called the *support* of φ. One says that φ is *faithful* if $s(\varphi) = 1$.

With the help of the Schwarz inequality, one can easily prove that

$$\varphi(x) = \varphi(xs(\varphi)) = \varphi(s(\varphi)x), \quad x \in \mathcal{M}.$$

From the definition of the support it follows that

$$x \in \mathcal{M},\ x \geqslant 0, \quad \varphi(x) = 0 \Rightarrow s(\varphi)xs(\varphi) = 0;$$

in particular, the form φ is faithful iff the implication

$$x \in \mathcal{M}, \quad x \geqslant 0, \quad \varphi(x) = 0 \Rightarrow x = 0.$$

holds.

Let φ be a form on $\mathcal{M}$ and $a \in \mathcal{M}$. We then define the forms

$$(L_a\varphi)(x) = \varphi(ax), \quad x \in \mathcal{M},$$

$$(R_a\varphi)(x) = \varphi(xa), \quad x \in \mathcal{M}$$

$$(T_a\varphi)(x) = \varphi(a^*xa), \quad x \in \mathcal{M}.$$

If φ is bounded (resp., w-continuous), then $L_a\varphi$, $R_a\varphi$, $T_a\varphi$ are bounded (resp., w-continuous).

5.16. The following result often allows the reduction of problems on w-continuous forms to problems on normal forms.

Theorem *(of polar decomposition for forms). Let φ be a w-continuous form on the von Neumann algebra $\mathscr{M}$. Then there exists a normal form $|\varphi|$ and a partial isometry $v \in \mathscr{M}$, uniquely determined by the conditions*

$$\varphi = R_v|\varphi|,$$

$$v^*v = s(|\varphi|).$$

Proof. The set $\{x \in \mathscr{M};\ \|x\| \leqslant 1,\ \varphi(x) = \|\varphi\|\}$ is a non-empty, w-compact, convex part of $\mathscr{M}$. Let u be an extreme point of this set. Then u is an extreme point of the unit ball of $\mathscr{M}$; consequently, by virtue of Proposition 3.19, it is a partial isometry.

We define $\psi = R_u\varphi$. Since

$$\psi(1) = \varphi(u) = \|\varphi\| \geqslant \|\psi\| \geqslant \psi(1),$$

it follows that $\psi(1) = \|\psi\|$. From Proposition 5.4, we infer that ψ is positive and, therefore, it is normal.

We define $v = u^*s(\psi)$. Since u is a partial isometry, we have $u = uu^*u$, whence $\psi(1 - uu^*) = \varphi(u - uu^*u) = 0$ and, therefore, $uu^* \geqslant s(\psi)$. It follows that

$$v^*v = s(\psi),$$

and, for any $x \in \mathscr{M}$, we have

$$\psi(x) = \psi(x\,s(\psi)) = \varphi(x\,s(\psi)u) = \varphi(xv^*).$$

We shall now prove that $\varphi = R_v\psi$, i.e., $\varphi(x) = \psi(xv)$, for any $x \in \mathscr{M}$. Indeed, if this is not true, then there exists an $x \in \mathscr{M}$, $\|x\| \leqslant 1$, such that

$$\varphi(x(1 - vv^*)) = \alpha > 0.$$

For any natural number n we have

$$\|nv^* + x(1 - vv^*)\|^2 = \|(nv^* + x(1 - vv^*))\,(nv + (1 - vv^*)x^*)\|$$

$$= \|n^2v^*v + x(1 - vv^*)x^*\| \leqslant n^2 + 1,$$

hence

$$|\varphi(nv^* + x(1 - vv^*))| \leqslant \|\varphi\|(n^2 + 1)^{1/2}.$$

On the other hand, we have

$$\varphi(nv^* + x(1 - vv^*)) = \psi(n) + \varphi(x(1 - vv^*)) = \|\varphi\|n + \alpha.$$

Thus, we have

$$\|\varphi\|n + \alpha \leqslant \|\varphi\|(n^2 + 1)^{1/2};$$

an impossible inequality if n is sufficiently great.

It follows that, if we denote $|\varphi| = \psi$, we have $\varphi = R_v|\varphi|$ and $v^*v = s(|\varphi|)$, the existence part of the theorem being thus established.

Let ψ and ψ' be two normal forms on $\mathscr{M}$, and v, v' partial isometries in $\mathscr{M}$, such that $\varphi = R_v\psi = R_{v'}\psi'$ and $v^*v = s(\psi)$, $v'^*v' = s(\psi')$.

We have

$$\psi(1) = \psi(v^*v) = \varphi(v^*) = \psi'(v^*v') = \psi'(v'^*v'v^*v') = \varphi(v'^*v'v^*)$$
$$= \psi(v'^*v'v^*v) = \psi(v'^*v'),$$

hence $v'^*v' \geqslant s(\psi) = v^*v$. Analogously, we have $v^*v \geqslant v'^*v'$, hence $v^*v = v'^*v' = e$.

Since $v'^*v = v'^*v'v'^*vv^*v = ev'^*ve \in e\mathscr{M}e$, we can write

$$v'^*v = a + ib,$$

where $a, b \in e\mathscr{M}e$ are self-adjoint. We have

$$\psi(a) + i\psi(b) = \psi(v'^*v) = \varphi(v'^*) = \psi'(v'^*v') = \|\psi'\| = \|\psi\|,$$

hence $\psi(a) = \|\psi\|$. Hence, $\psi(e - a) = 0$. Since $e - a \geqslant 0$, we find that $a = e$. It follows that $\|e + ib\| \leqslant 1$, whence $b = 0$. Consequently, we have

$$v'^*v = e.$$

Since $v'^*v = e$, we also have $v^*v' = e$, and we can write

$$v'v'^* = v'v'^*v'v'^* = v'ev'^* = v'v'^*vv^*v'v'^*,$$

$$v'v'^*(1 - vv^*)v'v'^* = 0,$$

$$(1 - vv^*)v'v'^* = 0,$$

$$v'v'^* = vv^*v'v'^* \leqslant vv^*.$$

Analogously, we find that $vv^* \leqslant v'v'^*$, and, therefore,

$$vv^* = v'v'^* = f.$$

We finally have

$$v = vv^*v = fv = v'v'^*v = v'e = v'v'^*v' = v',$$

and, for any $x \in \mathscr{M}$,

$$\psi(x) = \psi(xv^*v) = \varphi(xv^*) = \psi'(xv^*v') = \psi'(xe) = \psi'(e).$$

The uniqueness part of the theorem is thus proved.

Q.E.D.

5.17. Theorem 5.16 then gives us:

Theorem *(the Jordan decomposition). Let φ be a w-continuous self-adjoint form on the von Neumann algebra $\mathcal{M}$. Then there exist normal forms φ_1 and φ_2, uniquely determined by the conditions*

$$\varphi = \varphi_1 - \varphi_2,$$
$$s(\varphi_1)\, s(\varphi_2) = 0.$$

Proof. Let ψ be the normal form on $\mathcal{M}$ and $v \in \mathcal{M}$ the partial isometry, which, in accordance with Theorem 5.16, satisfy the properties

$$\varphi = R_v \psi,$$
$$v^* v = s(\psi).$$

We now define the normal form $\psi_0 = T_v \psi$. Then we have $s(\psi_0) = vv^*$. Since φ is self-adjoint, for any $x \in \mathcal{M}$ we have

$$\varphi(x) = \overline{\varphi(x^*)} = \overline{\psi(x^* v)} = \psi(v^* x) = \psi(v^* x v^* v) = \psi_0(x v^*).$$

From the uniqueness part of Theorem 5.16, we get

$$v = v^* \text{ and } \psi = \psi_0.$$

It follows that $v = e_1 - e_2$, where e_1 and e_2 are orthogonal projections. We define $\varphi_1 = R_{e_1}\varphi$ and $\varphi_2 = -R_{e_2}\varphi$. Since $\psi = \psi_0$ and since $s(\psi) = e_1 + e_2$, we infer that, for any $x \in \mathcal{M}$, we have

$$\psi(x) = \psi(e_1 x e_1 + e_2 x e_2).$$

We hence infer that φ_1 and φ_2 are positive. The other conditions in the statement of the theorem are now easily verified and, thus, the existence part of the theorem is proved.

Let now φ_1' and φ_2' be normal forms whose supports e_1', e_2' be orthogonal, and such that $\varphi = \varphi_1' + \varphi_2'$. We denote $\psi' = \varphi_1' + \varphi_2'$ and $v' = e_1' - e_2'$. It is easily seen that $\varphi = R_{v'}\psi'$ and $v'^* v' = s(\psi')$; hence, from the uniqueness part of Theorem 5.16, we have $v' = v$ and $\psi' = \psi$. It follows that

$$e_1' - e_2' = e_1 - e_2,$$
$$e_1' + e_2' = s(\psi') = s(\psi) = e_1 + e_2,$$

hence $e_1' = e_1$, $e_2' = e_2$. We now immediately get the equalities

$$\varphi_1' = \varphi_1, \quad \varphi_2' = \varphi_2.$$

Q.E.D.

5.18. Let $\mathcal{A}$ be a C^*-algebra with the unit element, $\mathcal{H}$ a Hilbert space and $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\pi(1) = 1$, a $*$-homomorphism. Then any vector $\xi \in \mathcal{H}$ determines a positive form $\varphi = \omega_\xi \circ \pi$ on $\mathcal{A}$.

Conversely, let φ be a positive form on a C*-algebra $\mathscr{A}$, assumed to have the unit element. With the help of the Schwarz inequality, it is easy to show that the set

$$\mathfrak{N}_\varphi = \{a \in \mathscr{A};\ \varphi(a^*a) = 0\}$$

is a left ideal of $\mathscr{A}$. For any $a \in \mathscr{A}$ we shall denote by a_φ the canonical image of a in $\mathscr{A}_\varphi = \mathscr{A}/\mathfrak{N}_\varphi$. We define on $\mathscr{A}_\varphi$ a scalar product by the relation

$$(a_\varphi | b_\varphi)_\varphi = \varphi(b^*a), \quad a_\varphi, b_\varphi \in \mathscr{A}_\varphi.$$

Then $\mathscr{A}_\varphi$ becomes a separated pre-Hilbert space. We denote by $\mathscr{H}_\varphi$ the Hilbert space obtained by the completion of $\mathscr{A}_\varphi$. For any $x \in \mathscr{A}$ we define

$$\pi^0_\varphi(x)a_\varphi = (xa)_\varphi, \quad a_\varphi \in \mathscr{A}_\varphi.$$

Since

$$\|\pi^0_\varphi(x)a_\varphi\|_\varphi \leqslant \|x\|\ \|a_\varphi\|_\varphi,$$

it follows that $\pi^0_\varphi(x)$ can be uniquely extended by continuity to an operator $\pi_\varphi(x) \in \in \mathscr{B}(\mathscr{H}_\varphi)$.

It is easily seen that the mapping

$$\pi_\varphi : \mathscr{A} \ni x \mapsto \pi_\varphi(x) \in \mathscr{B}(\mathscr{H}_\varphi)$$

is a $*$-homomorphism, $\pi_\varphi(1) = 1$ and:

$$\varphi = \omega_{1_\varphi} \circ \pi_\varphi.$$

We observe that the vector $1_\varphi \in \mathscr{H}_\varphi$ is cyclic for $\pi_\varphi(\mathscr{A})$, i.e.,

$$[\pi_\varphi(\mathscr{A})1_\varphi] = \mathscr{H}_\varphi.$$

Proposition. *Let $\mathscr{M}$ be a von Neumann algebra and φ a normal form on $\mathscr{M}$. Then π_φ is a w-continuous $*$-homomorphism, $\pi_\varphi(1) = 1$, $\pi_\varphi(\mathscr{M}) \subset \mathscr{B}(\mathscr{H}_\varphi)$ is a von Neumann algebra and*

$$\varphi = \omega_{1_\varphi} \circ \pi_\varphi,$$

$$1_\varphi \in \mathscr{H}_\varphi \text{ is a cyclic vector for } \pi_\varphi(\mathscr{M}).$$

Moreover, if $s(\varphi) = 1$, then π_φ is a $*$-isomorphism of $\mathscr{M}$ onto $\pi_\varphi(\mathscr{M})$ and 1_φ is a separating vector for $\pi_\varphi(\mathscr{M})$.

Proof. Let $\{x_i\}_{i \in I} \subset \mathscr{M}$, $x \in \mathscr{M}$, $x_i \uparrow x$. Then $\pi_\varphi(x)$ is an upper bound for the increasing net $\{\pi(x_i)\}_{i \in I}$ and, since φ is normal, for any $a \in \mathscr{M}$ we have

$$\lim_{i \in I} (\pi_\varphi(x_i)a_\varphi | a_\varphi) = \lim_{i \in I} \varphi(a^*x_ia) = \varphi(a^*xa) = (\pi_\varphi(x)a_\varphi | a_\varphi).$$

Consequently, we have $\pi_\varphi(x_i) \uparrow \pi_\varphi(x)$.

In particular, π_φ is a completely additive mapping. Since $\|\pi_\varphi\| \leqslant 1$, by virtue of Theorem 5.11 (see also E.5.17), it follows that π_φ is a w-continuous $*$-homomorphism. The fact that $\pi_\varphi(\mathscr{M}) \subset \mathscr{B}(\mathscr{H}_\varphi)$ is a von Neumann algebra is a consequence of Corollary 3.12.

Let us now assume that $s(\varphi) = 1$. If $x \in \mathscr{M}$ and $\pi_\varphi(x)1_\varphi = 0$, then $x_\varphi = 0$, i.e., $\varphi(x^*x) = 0$, whence $x = 0$.

Q.E.D.

By taking into account E.5.6, from the preceding proposition it follows that any von Neumann algebra of countable type is $*$-isomorphic to a von Neumann algebra which has a separating cyclic vector.

5.19. If φ and ψ are linear forms on a $*$-algebra of operators, we shall write $\varphi \leqslant \psi$ if $\psi - \varphi$ is a positive form.

Lemma. *Let $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$, $\mathscr{A} \ni 1$, be a $*$-algebra, $\xi \in \mathscr{H}$ and φ a positive form on $\mathscr{A}$, such that $\varphi \leqslant \omega_\xi$. Then there exists an $a' \in \mathscr{A}'$, $0 \leqslant a' \leqslant 1$, such that*

$$\varphi = \omega_{a'\xi}.$$

Proof. For any $x, y \in \mathscr{A}$, we have

$$|\varphi(y^*x)| \leqslant \varphi(y^*y)^{1/2}\varphi(x^*x)^{1/2} \leqslant \|y\xi\| \, \|x\xi\|.$$

If we write

$$(x\xi \mid y\xi)_0 = \varphi(y^*x),$$

we define a positive sesquilinear form of norm $\leqslant 1$ on $[\mathscr{A}\xi]$. Hence there exists a linear operator a_0 on $[\mathscr{A}\xi]$, $0 \leqslant a_0 \leqslant 1$, such that

$$\varphi(y^*x) = (x\xi \mid y\xi)_0 = (a_0x\xi \mid y\xi).$$

For any $x, y, z \in \mathscr{A}$ we have

$$(a_0xy\xi \mid z\xi) = \varphi(z^*xy) = (a_0y\xi \mid x^*z\xi) = (xa_0y\xi \mid z\xi),$$

hence

$$a_0(x \mid [\mathscr{A}\xi]) = (x \mid [\mathscr{A}\xi])a_0.$$

Thus, if we denote $e_0 = [\mathscr{A}\xi] \in \mathscr{A}'$, it follows that

$$e_0 \circ a_0 \circ e_0 \in \mathscr{A}'.$$

Let $a' = (e_0 \circ a_0 \circ e_0)^{1/2}$. Then $a' \in \mathscr{A}'$, $0 \leqslant a' \leqslant 1$, and, for any $x \in \mathscr{A}$, we have

$$\varphi(x) = (a_0x\xi \mid \xi) = (a'^2x\xi \mid \xi) = \omega_{a'\xi}(x).$$

Q.E.D.

5.20. Lemma. *Let φ be a positive form on the C^*-algebra $\mathscr{A}$ and $a \in \mathscr{A}$. If $L_a\varphi > 0$, then*

$$L_a\varphi \leqslant \|a\|\varphi.$$

Proof. Let $x \in \mathscr{A}$, $x \geqslant 0$. Then

$$(L_a\varphi)(x) = \varphi(ax) = \varphi(ax^{1/2}x^{1/2}) \leqslant \varphi(axa^*)^{1/2}\varphi(x)^{1/2}.$$

Since $L_a\varphi \geqslant 0$, it follows that

$$\varphi(axa^*) = (L_a\varphi)(xa^*) = \overline{(L_a\varphi)(ax^*)} = (L_a\varphi)(ax^*) = \varphi(a^2x),$$

hence

$$L_{a^2}\varphi \geqslant 0,$$

$$(L_a\varphi)(x) \leqslant (L_{a^2}\varphi)(x)^{1/2}\varphi(x)^{1/2}.$$

We proceed analogously with the forms

$$L_{a^2}\varphi, \ldots, L_{a^{2^{n-1}}}\varphi,$$

and, by induction, we get

$$\begin{aligned}(L_a\varphi)(x) &\leqslant (L_{a^{2^n}}\varphi)(x)^{1/2^n}\varphi(x)^{1/2+\ldots+1/2^n}\\ &\leqslant \|\varphi\|^{1/2^n}\|a\|\,\|x\|^{1/2^n}\varphi(x)^{1/2+\ldots+1/2^n}.\end{aligned}$$

By tending to the limit for $n \to \infty$, it follows that

$$(L_a\varphi)(x) \leqslant \|a\|\varphi(x).$$

Q.E.D.

5.21. The following result, due to S. Sakai, is an extension of the Radon-Nikodym theorem from measure theory; it is very important in itself and also because of its applications.

Theorem *(of Radon-Nikodym type). Let φ and ψ be two normal forms on the von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$, such that $\varphi \leqslant \psi$. Then there exists an $a \in \mathscr{M}$ uniquely determined by the properties*

$$0 \leqslant a \leqslant 1,$$

$$\mathrm{s}(a) \leqslant \mathrm{s}(\psi),$$

$$\varphi = L_aR_a\psi.$$

Proof. Without any loss of generality, we can assume that $\mathrm{s}(\psi) = 1$. Then, because of Proposition 5.18, we can assume that $\psi = \omega_\xi$, with a suitable vector $\xi \in \mathscr{H}$. Since $\mathrm{s}(\psi) = 1$, we have

$$x \in \mathscr{M}, \quad x\xi = 0 \Rightarrow x = 0,$$

$$[\mathscr{M}'\xi] = \mathscr{H}.$$

From Lemma 5.19, there exists an $a' \in \mathscr{M}'$, $0 \leqslant a' \leqslant 1$, such that

$$\varphi = \omega_{a'\xi}.$$

We now consider on $\mathcal{M}'$ the forms φ' and f', determined by the relations

$$\varphi'(x') = (x'\xi \mid \xi),$$
$$f'(x') = (x'\xi \mid a'\xi).$$

From the polar decomposition theorem (5.16; see also exercise E.5.10), it follows that there exists a normal form g' on $\mathcal{M}'$, and a partial isometry $v' \in \mathcal{M}'$, such that

$$f' = L_{v'}g',$$
$$g' = L_{v'^*}f'.$$

Thus, we have $g' = L_{v'^*}L_{a'}\varphi' = L_{a'v'^*}\varphi'$, and, with Lemma 5.20, we infer that

$$g' \leqslant \varphi' = \omega_\xi.$$

If we apply again Lemma 5.19, we get a $b \in \mathcal{M}$, $0 \leqslant b \leqslant 1$, such that

$$g'(x') = \omega_{b\xi}(x') = (x'\xi \mid b^2\xi), \quad x' \in \mathcal{M}'.$$

Let us denote $a = b^2$. For any $x' \in \mathcal{M}'$ we have

$$(x'\xi \mid a\,\xi) = (L_{a'v'^*}\varphi')\,(x') = (a'v'^*x'\xi \mid \xi) = (x'\xi \mid v'a'\,\xi),$$
$$(x'\xi \mid a'\xi) = f'(x') = g'(v'x') = \varphi'(a'v'^*v'x') = (x'\xi \mid v'^*v'a'\xi).$$

Since $[\mathcal{M}'\xi] = \mathcal{H}$, it follows that

$$a\xi = v'a'\xi,$$
$$a'\xi = v'^*v'a'\xi.$$

Hence, for any $x \in \mathcal{M}$, we have

$$\varphi(x) = (xa'\xi \mid a'\xi) = (xv'^*v'a'\xi \mid a'\xi) = (xv'a'\xi \mid v'a'\xi) = (xa\xi \mid a\xi)$$
$$= \omega_\xi(axa) = \psi(axa),$$

and the existence part of the theorem is proved.

In order to prove the uniqueness, let $a, b \in \mathcal{M}$, $0 \leqslant a, b \leqslant 1$, be such that for any $x \in \mathcal{M}$ we have

$$\omega_\xi(axa) = \omega_\xi(bxb).$$

Since

$$\|xa\xi\|^2 = \omega_\xi(ax^*xa) = \omega_\xi(bx^*xb) = \|xb\xi\|^2,$$

we can define a partial isometry $u' \in \mathcal{M}'$ by the relation

$$u'(xa\xi) = xb\xi.$$

We now consider the following normal forms on $\mathscr{M}'$

$$g'_a(x') = (x'a\xi \mid \xi), \quad x' \in \mathscr{M}',$$

$$g'_b(x') = (x'b\xi \mid \xi), \quad x' \in \mathscr{M}'.$$

Then we have

$$g'_b(x') = (x'u'a\xi \mid \xi) = (R_{u'}g'_a)(x'),$$

$$u'^*u' = s(g'_a).$$

From the uniqueness of the polar decomposition of g'_a it follows that the partial isometry u' maps identically $[\mathscr{M}a\xi]$ onto $[\mathscr{M}a\xi]$. In particular, we have

$$a\xi = u'a\xi = b\xi,$$

whence

$$a = b.$$

Q.E.D.

5.22. As applications of the Radon-Nikodym type theorem, in the following sections we present some fundamental results which are essentially due to J. von Neumann. These results are at the basis of the subsequent theory and are themselves Radon-Nikodym type theorems.

Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a fixed von Neumann algebra and $\xi \in \mathscr{H}$. The restriction of the form ω_ξ (1.3) to $\mathscr{M}$ will be denoted also by ω_ξ, whereas the restriction of the form ω_ξ to $\mathscr{M}'$ will be denoted by ω'_ξ. With the notations already introduced in Section 3.8, we have the following relations, which can be easily verified

$$z(p_\xi) = z(p'_\xi) = [\mathscr{M}\mathscr{M}'\xi],$$

$$s(\omega_\xi) = s(\omega_\xi \mid \mathscr{M}) = p_\xi,$$

$$s(\omega'_\xi) = s(\omega_\xi \mid \mathscr{M}') = p'_\xi.$$

Lemma. *Let φ be a normal form on the von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$, and $\xi \in \mathscr{H}$. If*

$$\varphi \geqslant \omega_\xi \text{ and } s(\varphi) = p_\xi,$$

then there exists an $\eta \in \mathscr{H}$, such that

$$\varphi = \omega_\eta \text{ and } p'_\eta = p'_\xi.$$

Proof. Since $\varphi \geqslant \omega_\xi$, and from the Radon-Nikodym type Theorem 5.21, there exists an $a \in \mathscr{M}$, $0 \leqslant a \leqslant 1$, such that

$$\omega_\xi = L_a R_a \varphi,$$

$$s(a) = s(\varphi).$$

Since $s(\omega_\xi) = s(\varphi)$, it follows that $s(a) = s(\varphi) = p_\xi$.

From Corollary 2.22, there exists a sequence $\{e_n\} \subset \mathscr{P}_{\mathscr{M}}$, such that e_n and a commute and

$$ae_n \geqslant \frac{1}{n} e_n,$$

$$e_n \uparrow s(a).$$

Then ae_n is invertible in $e_n\mathscr{M}e_n$, hence there exists an $x_n \in e_n\mathscr{M}e_n$, $x_n \geqslant 0$, such that $ax_n = e_n$.

We define $\eta_n = x_n\xi$. Then, for any $n \geqslant m$, we have

$$\|\eta_n - \eta_m\|^2 = ((x_n - x_m)\xi \mid (x_n - x_m)\xi)$$
$$= \omega_\xi((x_n - x_m)^2) = \varphi(a(x_n - x_m)^2 a) = \varphi(e_n - e_m),$$

and, since $\varphi(e_n)$ converges to $\varphi(s(a))$, it follows that $\{\eta_n\}$ is a Cauchy sequence. Let $\eta = \lim_{n\to\infty} \eta_n$. For any $x \in \mathscr{M}$ we have

$$\varphi(x) = \varphi(s(a)xs(a)) = \lim_{n\to\infty} \varphi(e_n x e_n) = \lim_{n\to\infty} \varphi(ax_n x x_n a)$$
$$= \lim_{n\to\infty} (xx_n\xi \mid x_n\xi) = (x\eta \mid \eta) = \omega_\eta(x),$$

hence $\varphi = \omega_\eta$.

On the other hand, we have $\eta_n \in \mathscr{M}\xi$ for any n, hence $\eta \in [\mathscr{M}\xi]$ and, therefore, $p'_\eta \leqslant p'_\xi$. Conversely, we have

$$a\eta = \lim_{n\to\infty} ax_n\xi = \lim_{n\to\infty} e_n\xi = s(a)\xi = p_\xi(\xi) = \xi;$$

i.e., $\xi \in \mathscr{M}\eta$, whence $p'_\xi \leqslant p'_\eta$.

Q.E.D.

5.23. Theorem. *Let ψ be a normal form on the von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ and $\xi \in \mathscr{H}$. If*

$$s(\psi) \leqslant p_\xi,$$

then there exists an $\eta \in [\mathscr{M}\xi] \cap [\mathscr{M}'\xi]$, such that

$$\psi = \omega_\eta.$$

Moreover, if $s(\psi) = p_\xi$, then there exists an $\eta \in \mathscr{H}$, such that

$$\psi = \omega_\eta \text{ and } p'_\eta = p'_\xi.$$

Proof. Let $\varphi = \psi + \omega_\xi$. By virtue of Lemma 5.22, there exists an $\eta_0 \in \mathscr{H}$, such that

$$\varphi = \omega_{\eta_0} \text{ and } p'_{\eta_0} = p'_\xi.$$

Since $\psi \leqslant \varphi$, from the Radon-Nikodym type theorem, there exists an $a \in \mathscr{M}$, $0 \leqslant a \leqslant 1$, such that

$$\psi = L_a R_a \varphi,$$

$$s(a) \leqslant s(\varphi) = p_\xi.$$

Let us define $\eta = a\eta_0$. Then, for any $x \in \mathscr{M}$, we have

$$\psi(x) = \omega_{\eta_0}(axa) = \omega_\eta(x),$$

i.e., $\psi = \omega_\eta$. Since $\eta = a\eta_0 \in \mathscr{M}\eta_0$, it follows that $\eta \in [\mathscr{M}\xi]$. On the other hand, we have

$$\eta = a\eta_0 \in s(a)\mathscr{H} \subset p_\xi \mathscr{H} = [\mathscr{M}'\xi].$$

Let us now assume that $s(\psi) = p_\xi$. Then, with the preceding notations, it follows that $s(a) = p_{\eta_0}$. By taking into account Corollary 2.22, we find two sequences $\{e_n\} \subset \mathscr{P}_\mathscr{M}$ and $\{x_n\} \subset e_n \mathscr{M} e_n$, $x_n \geqslant 0$, such that

$$ax_n = x_n a = e_n,$$

$$e_n \uparrow s(a).$$

We denote $\eta_n = x_n \eta = x_n a \eta_0 = e_n \eta_0$. Since $\eta_n = x_n \eta \in \mathscr{M}\eta$, we infer that

$$\eta_0 = p_{\eta_0}(\eta_0) = s(a)(\eta_0) = \lim_{n \to \infty} e_n \eta_0 \in [\mathscr{M}\eta].$$

Thus, we have $p'_{\eta_0} \leqslant p'_\eta$. But $p'_{\eta_0} = p'_\xi$, hence $p'_\xi \leqslant p'_\eta$. On the other hand, it is obvious that $p'_\eta \leqslant p'_\xi$. Consequently, we have $p'_\eta = p'_\xi$.

Q.E.D.

5.24. Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra with the property that there exists a vector $\xi \in \mathscr{H}$, which is separating for $\mathscr{M}$. Then, for any normal (resp., normal and faithful) form φ on $\mathscr{M}$, there exists a vector $\eta \in \mathscr{H}$, such that $\varphi = \omega_\eta$ (resp., $\varphi = \omega_\eta$ and $p'_\eta = p'_\xi$).*

5.25. Corollary. *Let $\mathscr{M}_1 \subset \mathscr{B}(\mathscr{H}_1)$, $\mathscr{M}_2 \subset \mathscr{B}(\mathscr{H}_2)$ be von Neumann algebras, such that there exist vectors $\xi_1 \in \mathscr{H}_1$, $\xi_2 \in \mathscr{H}_2$ which are cyclic and separating for $\mathscr{M}_1$ (resp., $\mathscr{M}_2$).*

For any $*$-isomorphism $\pi : \mathscr{M}_1 \to \mathscr{M}_2$ there exists a unitary operator $u : \mathscr{H}_1 \to \mathscr{H}_2$ such that, for any $x_1 \in \mathscr{M}_1$ we have

$$\pi(x_1) = u \circ x_1 \circ u^*.$$

In particular, π is wo-continuous.

Proof. We define a normal form φ on $\mathscr{M}_1$ by

$$\varphi(x_1) = \omega_{\xi_2}(\pi(x_1)), \ x_1 \in \mathscr{M}_1.$$

Then φ is faithful, because ξ_2 is separating for $\mathscr{M}_2$, whereas π is a $*$-isomorphism. From Corollary 5.24 it follows that there exists a vector η_1, which is cyclic for $\mathscr{M}_1$, such that

$$\varphi = \omega_{\eta_1}.$$

But in this case η_1 is also separating for $\mathscr{M}_1$, because φ is faithful.

We now define a linear mapping $u_0 : \mathscr{M}_1\eta_1 \to \mathscr{M}_2\eta_2$ by

$$u_0(x_1\eta_1) = \pi(x_1)\eta_2.$$

This mapping is isometric:

$$\|x_1\eta_1\|^2 = \omega_{\eta_1}(x_1^*x_1) = \varphi(x_1^*x_1) = \omega_{\xi_2}(\pi(x_1^*x_1)) = \|\pi(x_1)\xi_2\|^2.$$

Since the vectors η_1, ξ_2 are cyclic respectively for $\mathscr{M}_1$, $\mathscr{M}_2$, it follows that the mapping u_0 can be uniquely extended, by continuity, to a unitary operator $u : \mathscr{H}_1 \to \mathscr{H}_2$. For any $x_1, y_1 \in \mathscr{M}_1$ we have

$$\pi(x_1)\pi(y_1)\xi_2 = \pi(x_1y_1)\xi_2 = u_0x_1y_1\eta_1 = (u_0x_1)(y_1\eta_1) = u_0x_1u_0^{-1}\pi(y_1)\xi_2,$$

hence $\pi(x_1) = u \circ x_1 \circ u^{-1}$.

Q.E.D.

Exercises

In the exercises in which the symbols $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ are not explained, they will denote a von Neumann algebra $\mathscr{M}$ which acts on the Hilbert space $\mathscr{H}$.

E.5.1. Let $\mathscr{A}$ be a C^*-algebra and φ a bounded form on $\mathscr{A}$. If there exists an $a \in \mathscr{A}$, $0 \neq a \geq 0$, such that $\varphi(a) = \|\varphi\| \|a\|$, then φ is positive.

!E.5.2. Let $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$, $\mathscr{A} \ni 1$, be a $*$-algebra. If $\xi, \eta \in \mathscr{H}$ and if $\omega_{\xi,\eta} | \mathscr{A}$ is positive, then there exists a $\zeta \in \mathscr{H}$, such that $\omega_{\xi,\eta} | \mathscr{A} = \omega_\zeta | \mathscr{A}$. Infer that if φ is a positive wo-continuous form, then there exist $\xi_1, \dots, \xi_n \in \mathscr{H}$, such that $\varphi = \sum_{k=1}^{n} \omega_{\xi_k} | \mathscr{A}$.

E.5.3. Let φ be a normal form on $\mathscr{M}$ and $\{x_i\}$ a net in the closed unit ball of $\mathscr{M}$. Then we have

$$\varphi(x_i^*x_i) \to 0 \Leftrightarrow x_i s(\varphi) \xrightarrow{so} 0.$$

E.5.4. Let φ, ψ be normal on $\mathscr{M}$. Then $s(\psi) \leq s(\varphi)$ iff on the closed unit ball of $\mathscr{M}$ the topology determined by the seminorm $x \mapsto \varphi(x^*x)^{1/2}$ is stronger than the topology determined by the seminorm $x \mapsto \psi(x^*x)^{1/2}$.

!E.5.5. On a von Neumann algebra one considers the *s-topology* given by the semi-norms

$$s_\varphi(x) = \varphi(x^*x)^{1/2}, \quad x \in \mathscr{M},$$

where φ runs over all normal forms on $\mathscr{M}$.

Show that if φ_0 is a normal form on $\mathscr{M}$, such that $s(\varphi_0) = 1$, then, on the closed unit ball of $\mathscr{M}$, the s-topology is determined by the norm s_{φ_0}.

!E.5.6. A projection $e \in \mathscr{M}$ is of countable type iff it is the support of a normal form on $\mathscr{M}$. In particular, $\mathscr{M}$ is of countable type if there exists a normal form φ_0 on $\mathscr{M}$, such that $s(\varphi_0) = 1$.

E.5.7. Prove the following implications:
$\mathscr{H}$ is separable $\Rightarrow$ the predual $\mathscr{M}_*$ of $\mathscr{M}$ is separable $\Rightarrow \mathscr{M}$ is of countable type $\Rightarrow$ the closed unit ball of $\mathscr{M}$ is s-metrizable.

E.5.8. Show that, for any net $\{x_i\} \subset \mathscr{M}$, one has

$$x_i \xrightarrow{s} 0 \Rightarrow x_i^* x_i \xrightarrow{w} 0.$$

Infer that on the closed unit ball of $\mathscr{M}$ the s-topology coincides with so-topology.
Show that for any form φ on $\mathscr{M}$, φ is s-continuous $\Leftrightarrow$ φ is w-continuous,

E.5.9. Let φ be a normal form on $\mathscr{M}$. For any $e \in \mathscr{P}_{\mathscr{M}}$, $e \neq 0$, there exist $f \in \mathscr{P}_{\mathscr{M}}$, $0 \neq f \leqslant e$ and $\xi \in \mathscr{H}$, such that

$$\varphi(fxf) = \omega_\xi(x), \quad x \in \mathscr{M}.$$

!E.5.10. Let φ be a w-continuous form on $\mathscr{M}$ and $\varphi = R_v|\varphi|$ its polar decomposition. Then the polar decomposition of the form φ^* is

$$\varphi^* = R_{v^*}|\varphi^*|, \quad |\varphi^*| = L_{v^*}R_v|\varphi|.$$

In particular,

$$\varphi = L_v|\varphi^*|, \quad vv^* = s(|\varphi^*|).$$

E.5.11. Let φ be a w-continuous form on $\mathscr{M}$. Then $|\varphi|$ is the unique normal form ψ on $\mathscr{M}$ with the properties

$$\|\psi\| = \|\varphi\|,$$

$$|\varphi(x)|^2 \leqslant \|\varphi\|\psi(xx^*), \ x \in \mathscr{M}.$$

E.5.12. Let φ, ψ be w-continuous forms on $\mathscr{M}$. Then, for any $x \in \mathscr{M}$, we have

$$|\,|\varphi + \psi|(x)|^2 \leqslant (\|\varphi\| + \|\psi\|)(|\varphi|(xx^*) + |\psi|(xx^*)).$$

E.5.13. Let φ be a normal form on $\mathscr{M}$ and $a \in \mathscr{M}$. Then

$$|L_a\varphi| \leqslant \|a\|\varphi.$$

E.5.14. Let φ be a w-continuous form on $\mathscr{M}$ and $e \in \mathscr{P}_{\mathscr{M}}$. Show that

$$L_e\varphi = \varphi \Leftrightarrow \|L_e\varphi\| = \|\varphi\|.$$

!E.5.15. Let φ, ψ be normal forms on $\mathscr{M}$. Show that

$$s(\varphi)s(\psi) = 0 \Leftrightarrow \|\varphi - \psi\| = \|\varphi\| + \|\psi\|.$$

E.5.16. Any normal form on the von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ extends to a normal form on $\mathscr{B}(\mathscr{H})$. Infer that any w-continuous form φ on $\mathscr{M}$ extends to a w-continuous form $\tilde{\varphi}$ on $\mathscr{B}(\mathscr{H})$, such that $\|\tilde{\varphi}\| = \|\varphi\|$.

E.5.17. Let $\mathscr{M}_1, \mathscr{M}_2$ be von Neumann algebras and $\Phi : \mathscr{M}_1 \to \mathscr{M}_2$ a bounded linear mapping. Then Φ is w-continuous iff Φ is completely additive.

***E.5.18.** Show that the assertions (i), (ii) from Theorem 5.14 are equivalent to the following assertion

(iii) there exists a normal form φ_0 on $\mathscr{M}$, such that for any $\varepsilon > 0$ there exists a $\delta > 0$, with the property:

$$x \in \mathscr{M}, \quad \|x\| \leqslant 1, \quad \varphi_0(x^*x + xx^*) \leqslant \delta \Rightarrow |\varphi(x)| \leqslant \varepsilon, \text{ for any } \varphi \in \mathscr{F}.$$

E.5.19. Let $\mathscr{M}_*$ be the predual of the von Neumann algebra $\mathscr{M}$. If $\mathscr{F}$ is a $\sigma(\mathscr{M}_*, \mathscr{M})$-relatively compact part of $\mathscr{M}_*^+ = \{\varphi \in \mathscr{M}_*;\ \varphi \geqslant 0\}$, then the set $\{L_x\varphi;\ \varphi \in \mathscr{F}, x \in \mathscr{M}, \|x\| \leqslant 1\}$ is also $\sigma(\mathscr{M}_*, \mathscr{M})$-relatively compact.

E.5.20. Produce an example in order to show that there exist $\sigma(\mathscr{M}_*, \mathscr{M})$-relatively compact parts $\mathscr{F} \subset \mathscr{M}_*$, such that the set $\{|\varphi|;\ \varphi \in \mathscr{F}\}$ is not $\sigma(\mathscr{M}_*, \mathscr{M})$-relatively compact.

E.5.21. Show that the following assertions are equivalent:

(i) any normal form on $\mathscr{M}$ is of the type ω_ξ, $\xi \in \mathscr{H}$,

(ii) for any $e \in \mathscr{P}_\mathscr{M}$ of countable type, $\mathscr{M}_e \subset \mathscr{B}(e\mathscr{H})$ has a separating vector.

E.5.22. Show that two $*$-isomorphic maximal abelian von Neumann algebras are spatially isomorphic.

Comments

C.5.1. Besides the uniform topology (i.e., the norm topology) and the topologies wo, w, so, on a von Neumann algebras $\mathscr{M}$ one also considers the following topologies:

the s-topology, given by the seminorms

$$s_\varphi(x) = \varphi(x^*x)^{1/2}, \quad x \in \mathscr{M},$$

where φ runs over the set of all normal forms on $\mathscr{M}$ (see E.5.5);

the s-topology*, given by the seminorms

$$s_\varphi(x) = \varphi(x^*x)^{1/2}, \quad x \in \mathscr{M},$$

$$s_\varphi^*(x) = \varphi(xx^*)^{1/2}, \quad x \in \mathscr{M},$$

where φ runs over the set of all normal forms on $\mathscr{M}$;

the τ-topology $= \tau(\mathscr{M};\mathscr{M}_*)$, i.e., the Mackey topology associated with the topology $w = \sigma(\mathscr{M};\mathscr{M}_*)$; this topology is given by the seminorms

$$p_{\mathscr{K}}(x) = \sup_{\varphi \in \mathscr{K}} |\varphi(x)|, \quad x \in \mathscr{M},$$

where $\mathscr{K}$ runs over the set of all $\sigma(\mathscr{M}_*;\mathscr{M})$-compact, convex, equilibrated subsets of $\mathscr{M}_*$, and it is the finest locally convex topology on $\mathscr{M}$ which determines the same set of linear,continuous forms on $\mathscr{M}$, as the w-topology (see Bourbaki [1]).

The general relations existing between these topologies are represented in the following diagram

$$\begin{array}{ccccccc} & (1) & & & & & \\ wo & \leqslant & so & & & & \\ (2)\ \wedge & & \wedge\ (3) & & & & \\ w & \leqslant & s & \leqslant & s^* & \leqslant & \tau. \\ & (4) & & (5) & & (6) & \end{array}$$

Relations (1) and (2) are obvious from the definitions of the corresponding topologies (see 1.3, 1.10); relations (3) and (4) easily follow from Proposition 5.3 and E.5.8; relation (5) is trivial, whereas relation (6) follows from E.5.8, if we observe that the $*$-operation is τ-continuous.

As far as the restrictions of these topologies to the closed unit ball of $\mathscr{M}$ are concerned (denoted below by the subscript 1 to the corresponding symbol of the topology), we have the following relations:

$$\begin{array}{ccccccc} wo_1 & \leqslant & so_1 & & & & \\ (a)\ \| & & \|\ (b) & & & & \\ w_1 & \leqslant & s_1 & \leqslant & s_1^* & = & \tau_1. \\ & & & & & (c) & \end{array}$$

Equality (a) has already been established (1.3, 1.10), equality (b) follows from (a) with the help of E.5.8, whereas equality (c) is proved by C. A. Akemann in [1].

The cases in which the equalities $wo = w$, $so = s$ hold, are discussed in Chapter 8.

For other results concerning topologies on von Neumann algebras, we refer to: J. F. Aarnes [2], C. A. Akemann [1], S. Sakai [6], [14], P. C. Shields [1].

C.5.2. One calls a *derivation* of an algebra $\mathscr{A}$ any linear mapping $\vartheta : \mathscr{A} \to \mathscr{A}$, such that

$$\vartheta(xy) = x\vartheta(y) + \vartheta(x)y, \quad x, y \in \mathscr{A}.$$

Any element $a \in \mathscr{A}$ determines an *inner derivation*

$$\vartheta_a : \mathscr{A} \ni x \mapsto ax - xa \in \mathscr{A}.$$

The study of the derivations of algebras of operators has been started by I. Kaplansky [18], [23], who proved that any derivation of a von Neumann algebra of type I is inner (aided by the fact that any derivation of a commutative C^*-algebra is identically zero, a result due to I. M. Singer) and made the conjecture that any derivation of a C^*-algebra is uniformly continuous. This conjecture has been positively solved by S. Sakai [9], and afterwards B. E. Johnson and A. M. Sinclair [1] showed that any derivation of a semi-simple Banach algebra is continuous. With the help of the results of I. Kaplansky and S. Sakai, already mentioned, R. V. Kadison [23] showed that any derivation of a C^*-algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ is *wo*-continuous and extends to an inner derivation of $\mathscr{B}(\mathscr{H})$; S. Sakai [17], with the help of Kadison's result, proved the following theorem:

Theorem 1. *Any derivation of a von Neumann algebra is inner.*

Other proofs of this theorem have been given by B. E. Johnson and J. R. Ringrose [1], and by W. B. Arveson [10]; D. Olesen [1] has shown that any derivation of an AW^*-algebra (see C.3.1) is inner, by extending the arguments of W. B. Arveson.

On the other hand, S. Sakai [22] has shown that any derivation of a simple C^*-algebra with the unit element is inner, whereas other results in this direction have been obtained by: S. Sakai [22], [31], [34], D. Olesen and G. K. Pedersen [1], G. A. Elliott [9], C. A. Akemann, G. A. Elliott, G. K. Pedersen and J. Tomiyama [1], and others.

As an extension of the study of the derivations, the theory of the cohomology of algebras of operators and of general Banach algebras was also developed: R. V. Kadison and J. R. Ringrose [4], B. E. Johnson, R. V. Kadison and J. R. Ringrose [1], B. E. Johnson [11], I. C. Craw [1], [2].

Along the study of the derivations, significant results were obtained in the theory of automorphisms of algebras of operators, for which we refer the reader to the works of R. V. Kadison and J. R. Ringrose [1], [3], [5], R. V. Kadison, E. C. Lance and J. R. Ringrose [1] and H. J. Borchers [4].

We recall, that any $*$-isomorphism between two von Neumann algebras is *w*-continuous, hence *s*-continuous (5.13). In connection with the continuity of the algebraic isomorphisms, we mention the following result of T. Okayasu [2], which we state in the form given by S. Sakai [32]:

Theorem 2. *Let $\Phi: \mathscr{A} \to \mathscr{B}$ be an algebraic isomorphism of C^*-algebras. Then there exists a derivation ϑ of $\mathscr{A}$ and a $*$-isomorphism $\Psi : \mathscr{A} \to \mathscr{B}$, such that*

$$\Phi = \Psi \exp (\vartheta).$$

In particular, one infers from this result that any pair of algebraically isomorphic C^*-algebras are $*$-isomorphic (L. T. Gardner [2]) and that any algebraic isomorphism between two C^*-algebras is uniformly (i.e., norm) continuous (C. E. Rickart [3]).

On the other hand, from Theorems 1 and 2 one obtains the following result

Corollary. *Let $\Phi : \mathscr{M} \to \mathscr{N}$ be an algebraic isomorphism between von Neumann algebras. Then there exists an invertible positive element $a \in \mathscr{M}$ and a $*$-isomorphism $\Psi : \mathscr{M} \to \mathscr{N}$, such that*

$$\Phi(x) = \Psi(axa^{-1}), \quad x \in \mathscr{M}.$$

Thus, any algebraic isomorphism between von Neumann algebras is continuous for the topologies w and s.

Let ϑ be a derivation of a von Neumann algebra $\mathcal{M}$. By virtue of Theorem 1, there exists an $a \in \mathcal{M}$, such that $\vartheta = \vartheta_a$. In connection with the selection of the element a, one knows that there exists a unique $a_\vartheta \in \mathcal{M}$, $\vartheta = \vartheta_{a_\vartheta}$, such that, for any central projection $p \in \mathcal{M}$ one has

$$\|p\vartheta\| = 2\,\|pa_\vartheta\|.$$

In particular, we have

$$\|\vartheta\| = 2 \inf \{\|a\|; \quad \vartheta = \vartheta_a\}.$$

This result has been obtained by J. G. Stampfli [1] for the case in which $\mathcal{M} = \mathcal{B}(\mathcal{H})$, by P. Gajendragadkar [1], for the case $\mathcal{M}$=von Neumann algebra with a separable predual, and by L. Zsidó [2], for the general case. For other information concerning the norm of the derivations, we refer the reader to C. Apostol and L. Zsidó [1].

The study of isometries between von Neumann algebras has been carried out by R. V. Kadison [2], [5], who obtained the following result:

Theorem 3. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a linear isomorphism between two C^*-algebras with the unit element, such that $\Phi(1) = 1$. The following assertions are equivalent*

(i) *Φ is an isometry;*

(ii) *for any $a \in \mathcal{A}^h$, one has $\Phi(a) \in \mathcal{B}^h$ and $\Phi(a^n) = \Phi(a)^n$, $n \in \mathbb{N}$;*

(iii) *for any $x \in \mathcal{M}$ one has: $x \geqslant 0 \Leftrightarrow \Phi(x) \geqslant 0$.*

If $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, then the preceding assertions are equivalent to the following one:

(iv) *There exist central projections $p \in \mathcal{A}$, $q \in \mathcal{B}$, such that Φ induces a $*$-isomorphism of $\mathcal{A}p$ onto $\mathcal{B}q$, and a $*$-antiisomorphism of $\mathcal{A}(1-p)$ onto $\mathcal{B}(1-q)$.*

Condition (ii) is equivalent to the fact that Φ is a *Jordan $*$-isomorphism*, i.e., Φ commutes with the $*$-operation and conserves the "Jordan product", $a \circ b = \frac{1}{2}(ab + ba)$. Condition (iii) is equivalent to the fact that Φ is a *bipositive* linear isomorphism, such that $\Phi(1) = 1$. Thus, the linear isometries which map 1 to 1, the bipositive linear isomorphisms which map 1 to 1 and the Jordan $*$-isomorphisms are equivalent notions, whereas in the case of von Neumann algebras, they are characterized by condition (iv) in terms of $*$-isomorphisms and $*$-antiisomorphisms. Consequently, Theorem 3 is a general, non-commutative, extension of the Banach-Stone theorem (see N. Dunford and J. Schwartz [1], Ch.V, 8.8).

R. V. Kadison's proof is based on the study of the extreme points of the closed unit ball and on some older results of N. Jacobson and C. E. Rickart [1]. Another proof of the equivalence (i) $\Leftrightarrow$ (ii), based on the notion of numerical range, was obtained by A. L. T. Paterson [1], whereas extensions of the theorem to C^*-algebras without the unit element, were given by L. A. Harris [1] and A. L. T. Paterson and A. M. Sinclair [1].

Other results concerning the Jordan structure of C^*-algebras are contained in some papers by E. Störmer, D. M. Topping et al.

With the help of the above theorem of Kadison and of the Tomita theory, A. Connes [7] obtained the characterization of von Neumann algebras as ordered linear spaces.

A linear mapping $\Phi : \mathscr{A} \to \mathscr{B}$, between the C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, is said to be *n-positive*, $n \in \mathbb{N}$, if the natural extension

$$\Phi_n : \mathrm{Mat}_n(\mathscr{A}) \to \mathrm{Mat}_n(\mathscr{B})$$

is a positive mapping. If Φ is n-positive, for any $n \in \mathbb{N}$, then Φ is called *completely positive.*

The fundamental result concerning the completely positive mappings is the following theorem of W. F. Stinespring [1]:

Theorem 4. *For any linear mapping $\Phi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ the following assertions are equivalent*

(i) *Φ is completely positive;*

(ii) *there exists a $*$-representation $\pi: \mathscr{A} \to \mathscr{B}(\mathscr{K})$ and a bounded operator $v: \mathscr{H} \to \mathscr{K}$, such that*

$$\Phi(a) = v^*\pi(a)v, \quad a \in \mathscr{A}.$$

Corollary. *If $\Phi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ is completely positive, then*

$$|\Phi(a)|^2 \leqslant \Phi(|a|^2), \quad a \in \mathscr{A}.$$

If either $\mathscr{A}$ or $\mathscr{B}$ is commutative, then any positive linear mapping $\Phi: \mathscr{A} \to \mathscr{B}$ is completely positive. Thus, from the preceding corollary trivially follows the "Schwarz type inequality" of R. V. Kadison [5].

For the study of positive and completely positive mappings we refer to W. B. Arveson [7], R. V. Kadison [1], [5], E. Störmer [1], [25].*) A detailed analysis of the n-positive mappings can be found in Man-Duen Choi [1]. Some applications of the completely positive mappings to the theory of operators and to the theory of algebras of operators can be found in W. B. Arveson [7], I. Suciu [1] and L. Zsidó [4].

Linear mappings between C^*-algebras, which map unitary elements to unitary elements, have been studied by B. Russo and H. A. Dye [1] and B. Russo [1].

C.5.3. We recall (C.2.1) that the C^*-algebras of operators possess an axiomatic description. The following theorem of S. Sakai [3] (see also [10], [32]) allows an axiomatic description of von Neumann algebras:

Theorem. *A C^*-algebra $\mathscr{M}$ is $*$-isomorphic to a von Neumann algebra iff it is the dual of a Banach space.*

A proof of this theorem has also been obtained by J. Tomiyama [1].

One calls a *W*-algebra* any C^*-algebra which is the dual of a Banach space. On account of the preceding theorem, von Neumann algebras are also called *concrete* W^*-algebras.

The proof of the preceding theorem required that some results, already known for von Neumann algebras, be obtained by non-spatial arguments. These essentially

*) See also the recent papers by D. E. Evans.

developed methods, used in the theory of abstract C^*-algebras, the compactness of the closed unit balls in W^*-algebras and the Krein-Šmulian theorem (C.1.1). Among the abstract C^*-algebra techniques, we mention the "Arens trick", presented in the proof of Lemma 2.5. A characteristic sample is the proof of Theorem 5.16. The advantage of this technique lies in the invariance with respect to $*$-isomorphisms of the results obtained with its help. Not incidentally, all the results, which are invariant with respect to $*$-isomorphisms, have proofs of this nature. The book of S. Sakai [32] is an excellent exposition of the theory based on these ideas.

If $\mathscr{A}$ is a C^*-algebra and $\varphi \in \mathscr{A}^*$, $\varphi \geqslant 0$, we have already defined the representation $\pi_\varphi : \mathscr{A} \to \mathscr{B}(\mathscr{H}_\varphi)$, of $\mathscr{A}$ into $\mathscr{B}(\mathscr{H}_\varphi)$ (5.18). If we denote by $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ the direct sum of all representations π_φ, one defines the *enveloping von Neumann algebra* of the C^*-algebras $\mathscr{A}$ as being the von Neumann algebra $\mathscr{R}(\pi(\mathscr{A})) \subset \mathscr{B}(\mathscr{H})$ (see J. Dixmier [42], § 12).

The bidual $\mathscr{A}^{**}$ of the C^*-algebra $\mathscr{A}$ can be organized, in a natural manner, as a W^*-algebra and it is $*$-isomorphic to the enveloping von Neumann algebra of $\mathscr{A}$ (see, for example, M. Tomita [9]). The multiplication that one introduces in $\mathscr{A}^{**}$ is a natural one (first defined by R. Arens [3]) and called the *Arens multiplication.*

Any continuous linear form on the C^*-algebra $\mathscr{A}$ can be extended in a unique manner, by continuity, as a w-continuous linear form on the bidual W^*-algebra $\mathscr{A}^{**}$; hence, $(\mathscr{A}^{**})_* = \mathscr{A}^*$. Any $*$-representation $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ can be extended in a unique manner to a normal $*$-representation $\tilde{\pi} : \mathscr{A}^{**} \to \mathscr{B}(\mathscr{H})$, and $\tilde{\pi}(\mathscr{A}^{**}) = \mathscr{R}(\pi(\mathscr{A}))$. These facts allow the extension of some results, known for w-continuous linear forms on von Neumann algebras, to the bounded linear forms on C^*-algebras.

In Ş. Strătilă, L. Zsidó [6] there is a unified exposition of the topics recorded in this section and in Section C.2.1.

C.5.4. The theory of operator algebras allowed the extension of the integration theory to a non-commutative framework. The first results in this direction were obtained by J. Dixmier [23] and I. E. Segal [11]. For further developments we recommend the works of S. K. Berberian, T. Ogasawara and K. Yoshinaga, H. Umegaki, L. Pukánszky, E. Nelson, A. R. Padmanabhan, K. Saitô, E. Christensen, F. J. Yeadon, E. C. Lance, U. Haagerup, A. Connes and I. Cuculescu. A presentation of the results and of the directions of research in the field of "abstract integration" can be found in the expository paper by I. E. Segal [26].

On the other hand, extensions of a different kind of the measure theory to the C^*-algebras have been obtained by G. K. Pedersen and F. Combes. An almost complete exposition of the results obtained in this direction can be found in Ş. Strătilă [1].

C.5.5. Bibliographical comments. Theorem 5.11 and Corollary 5.12 are due to M. Takesaki [2], but in our exposition we followed the proof of B. E. Johnson, R. V. Kadison and J. R. Ringrose [1]. Theorem 5.14 is due to C. A. Akemann [1]. Theorems 5.16 and 5.21 were obtained by S. Sakai [7], [15], whereas Theorem 5.17 by A. Grothendieck [2]. The construction from Section 5.18, with the help of which, to any positive form φ on a C^*-algebra $\mathscr{A}$, one associates a "cyclic representation" π_φ of $\mathscr{A}$, is sometimes called the *Gelfand-Naimark-Segal construction* or, briefly, the *GNS* construction. Theorem 5.23 is essentially due to F. J. Murray and J. von Neumann [1], whereas our proof follows that in the paper of J. Vowden [2] (see, also, the talk by Kadison [16]). Corollary 5.25 is due to F. J. Murray and J. von Neumann [2], and sometimes appears in the literature under the name *the spatial theorem of J. von Neumann.*

Crucial results in the theory of operator algebras are Radon-Nikodym type theorems. *Under the same hypotheses as those in Theorem* 5.21, S. Sakai ([10], Remark, p. 1.46; [32], Proposition 1.24.4) has also shown that *there exists an* $h \in \mathscr{M}$, $0 \leqslant h \leqslant 1$ *such that*

$$\varphi = \frac{1}{2}(L_h\psi + R_h\psi).$$

We shall come to the Radon-Nikodym type theorems again in C.6.1, C.6.2, and in Chapter 10.

In our exposition we also followed J. Dixmier [26], J. R. Ringrose [5] and S. Sakai [32].

6

Relationships between a von Neumann algebra and its commutant

In this chapter we shall show that the passage to the commutant of a von Neumann algebra is an operation which conserves the type and, afterwards, we shall give two applications of this result.

The following lemmas can be looked upon as being corollaries to Theorem 5.23. We shall use the notations already introduced in Sections 3.8 and 5.22.

6.1. Lemma. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\xi, \eta \in \mathcal{H}$. If $p_\xi \prec p_\eta$ (resp., $p_\xi \sim p_\eta$), in $\mathcal{M}$, then $p'_\xi \prec p'_\eta$ (resp., $p'_\xi \sim p'_\eta$) in $\mathcal{M}'$.*

Proof. By virtue of the Schröder-Bernstein type theorem (4.7), it is sufficient to prove the first assertion of the theorem.

Let $v \in \mathcal{M}$ be a partial isometry, such that:

$$v^*v = p_\xi, \qquad vv^* \leqslant p_\eta.$$

We denote $\eta_0 = v\xi$. Then $\xi = v^*\eta_0 \in \mathcal{M}\eta_0$. On the other hand, we have $\eta_0 \in [\mathcal{M}'\eta]$, hence $s(\omega_{\eta_0}) \leqslant p_\eta$. From Theorem 5.23, there exists a $\xi_0 \in [\mathcal{M}\eta]$, such that

$$\omega_{\eta_0} = \omega_{\xi_0}.$$

If we define

$$v'(x\xi_0) = x\eta_0, \qquad x \in \mathcal{M},$$

and if we observe that

$$\|x\xi_0\|^2 = \omega_{\xi_0}(x^*x) = \omega_{\eta_0}(x^*x) = \|x\eta_0\|^2,$$

it follows that we thus define a partial isometry $v' \in \mathcal{M}'$, such that

$$v'^*v' = p'_{\xi_0}, \qquad v'v'^* = p'_{\eta_0}.$$

Consequently, we have

$$p'_\xi \leqslant p'_{\eta_0} \sim p'_{\xi_0} \leqslant p'_\eta.$$

Q.E.D.

6.2. Lemma. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\xi \in \mathcal{H}$. Then p_ξ is an abelian projection in $\mathcal{M}$ iff p'_ξ is an abelian projection in $\mathcal{M}'$.*

Proof. Let us assume, for example, that p_ξ is an abelian projection in $\mathcal{M}$. We now consider the von Neumann algebra

$$\mathcal{N} = \mathcal{M}_{p'_\xi} \subset \mathcal{B}(p'_\xi\mathcal{H}),$$

whose commutant is

$$\mathcal{N}' = \mathcal{M}'_{p'_\xi} \subset \mathcal{B}(p'_\xi \mathcal{H}).$$

We must thus show that $\mathcal{N}'$ is abelian.

For any $\eta \in p'_\xi \mathcal{H}$ we denote

$$q_\eta = [\mathcal{N}'\eta] \in \mathcal{N}, \qquad q'_\eta = [\mathcal{N}\eta] \in \mathcal{N}'.$$

Since p_ξ is abelian in $\mathcal{M}$, the projection $q_\xi = (p_\xi)_{p'_\xi}$ is abelian in $\mathcal{N}$. Moreover, the vector ξ is cyclic for the abelian von Neumann algebra $\mathcal{N}_{q_\xi}$. With exercise E.3.10, we infer that

$$\mathcal{N}'_{q_\xi} = \mathcal{N}_{q_\xi},$$

hence $\mathcal{N}'_{q_\xi}$ is abelian.

Since $z(p_\xi) = z(p'_\xi)$, it is easy to see that, in the von Neumann algebra $\mathcal{N}$, the central support of the projection q_ξ is equal to the unit element. With Proposition 3.14, we infer that $\mathcal{N}'$ is $*$-isomorphic to $\mathcal{N}'_{q_\xi}$, hence $\mathcal{N}'$ is abelian.

Q.E.D.

6.3. Lemma. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\xi \in \mathcal{H}$. Then p_ξ is a finite projection in $\mathcal{M}$ iff p'_ξ is a finite projection in $\mathcal{M}'$.*

Proof. We shall get at a contradiction, if we suppose that p_ξ is finite, whereas p'_ξ is not. Without any loss of generality, we can assume that p'_ξ is properly infinite.

Let q be a central projection, such that qp_ξ be abelian. Then $p_{q\xi} = qp_\xi$ is abelian. With Lemma 6.2, we infer that $p'_{q\xi} = qp'_\xi$ is abelian, hence it is finite. Since p'_ξ is properly infinite, it follows that $qp'_\xi = 0$, and, since $z(p_\xi) = z(p'_\xi)$, we have $qp_\xi = 0$.

By taking into account exercise E.4.21, it follows that there exist $e, f \in \mathcal{P}_\mathcal{M}$, such that $p_\xi = e + f$, $ef = 0$, $z(e) = z(f) = z(p_\xi)$. If we denote $\alpha = e\xi$, $\beta = f\xi$ we have $e = p_\alpha$, $f = p_\beta$.

We now consider the projections $e' = p'_\alpha$, $f' = p'_\beta$. We shall show that e' and f' are finite projections in $\mathcal{M}'$. Indeed, if e' is not finite, then there exists a central projection $q \neq 0$, such that the projection qe' be properly infinite and $z(qe') = q \leq z(p'_\xi)$. Hence the projections qe' and qp'_ξ are properly infinite, of countable type (see exercise E.5.6) and their central supports are equal. From Proposition 4.13, it follows that

$$qe' \sim qp'_\xi,$$

hence, with Lemma 6.1, we have

$$qe \sim qp_\xi.$$

But $qp_\xi = qe + qf$ and $qf \neq 0$, because $0 \neq q \leq z(p'_\xi) = z(p_\xi) = z(f)$. Hence qp_ξ is not finite, but this contradicts the fact that p_ξ is finite. Analogously, one shows that f' is finite.

Since e' is finite, whereas p'_ξ is properly infinite, with the help of the comparison theorem (4.6), we find an $e'' \in \mathcal{P}_{\mathcal{M}'}$, such that $e'' \leq p'_\xi$, $e'' \sim e'$. By taking into account Proposition 4.15, it follows that $p'_\xi - e''$ is also a properly infinite projection, hence $f' \prec p'_\xi - e''$. Consequently, there exist $e'', f'' \in \mathscr{P}_{\mathscr{M}'}$ such that

$$e' \sim e'' \leqslant p'_\xi, \quad f' \sim f'' \leqslant p'_\xi, \quad e''f'' = 0.$$

We shall show that there exist $\gamma, \delta \in \mathscr{H}$, such that

$$e = p_\gamma, \quad e'' = p'_\gamma,$$
$$f = p_\delta, \quad f'' = p'_\delta.$$

Indeed, let $v' \in \mathscr{M}'$ be a partial isometry, such that

$$v'^* v' = e', \quad v'v'^* = e''.$$

Then we have

$$e = p_\alpha = [\mathscr{M}'\alpha] \supset [\mathscr{M}'v'\alpha] \supset [\mathscr{M}'v'^*v'\alpha] = [\mathscr{M}'\alpha] = e,$$

hence $e = p_{v'\alpha}$. On the other hand, we have

$$[\mathscr{M}v'\alpha] = v'[\mathscr{M}\alpha] = e'',$$

hence $e'' = p'_{v'\alpha}$. We have thus found γ and we can analogously find δ.

The following computations show that $p_\xi = p_{\gamma+\delta}$ and $e'' + f'' = p'_{\gamma+\delta}$:

$$p_\xi = e + f = p_\gamma + p_\delta = [\mathscr{M}'\gamma] + [\mathscr{M}'\delta] \supset [\mathscr{M}'(\gamma + \delta)]$$
$$\supset [\mathscr{M}'e''(\gamma + \delta)] + [\mathscr{M}'f''(\gamma + \delta)] = [\mathscr{M}'\gamma] + [\mathscr{M}'\delta] = p_\gamma + p_\delta,$$
$$e'' + f'' = p'_\gamma + p'_\delta = [\mathscr{M}\gamma] + [\mathscr{M}\delta] \supset [\mathscr{M}(\gamma + \delta)]$$
$$\supset [\mathscr{M}e(\gamma + \delta)] + [\mathscr{M}f(\gamma + \delta)] = [\mathscr{M}\gamma] + [\mathscr{M}\delta] = p'_\gamma + p'_\delta.$$

The contradiction we should arrive at is the following: since $p_\xi = p_{\gamma+\delta}$, from Lemma 6.1 we infer that $p'_\xi \sim p'_{\gamma+\delta}$, hence $p'_{\gamma+\delta}$ is properly infinite in $\mathscr{M}'$; on the other hand, since $p'_{\gamma+\delta} = e'' + f''$, from the above results and from Proposition 4.15, we infer that $p'_{\gamma+\delta}$ is finite in $\mathscr{M}'$.

Q.E.D.

6.4. Theorem. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. Then $\mathscr{M}$ is of type I (resp., of type II; resp., of type III) iff $\mathscr{M}'$ is of type I (resp., of type II; resp., of type III).*

Proof. It is sufficient to prove that if $\mathscr{M}'$ is discrete (resp., semifinite), then $\mathscr{M}$ is discrete (resp., semifinite) (see table 4.21). Let q be a non-zero central projection. Since $\mathscr{M}$ is discrete (resp., semifinite), there exists an abelian (resp., finite) non-zero projection $e' \in \mathscr{M}$, such that $e' \leqslant q$. Let $\xi \in e'(\mathscr{H})$. Then $p'_\xi \leqslant e'$ is an abelian (resp., finite) projection in $\mathscr{M}'$, such that $z(p'_\xi) \leqslant q$. From Lemma 6.2 (resp., 6.3), it follows that p_ξ is an abelian (resp., finite) projection in $\mathscr{M}$, such that $z(p_\xi) \leqslant q$. Thus, any non-zero central projection contains an abelian (resp., finite) non-zero projection in $\mathscr{M}$, hence $\mathscr{M}$ is, indeed, discrete (resp., semifinite).

Q.E.D.

6.5. Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. Then $\mathscr{M}$ is discrete (resp., semifinite) iff $\mathscr{M}$ is $*$-isomorphic to a von Neumann algebra whose commutant is abelian (resp., finite).*

Proof. If $\mathscr{M}$ is discrete (resp., semifinite), then, according to Theorem 6.4, $\mathscr{M}'$ is discrete (resp., semifinite). With Proposition 4.19, we infer that there exists an abelian (resp., finite) projection $e' \in \mathscr{M}'$, such that $z(e') = 1$. Then $(\mathscr{M}_{e'})' = (\mathscr{M}')_{e'}$ is an abelian (resp., finite) von Neumann algebra and $\mathscr{M}$ is $*$-isomorphic to $\mathscr{M}_{e'}$ (see Proposition 3.14 and Theorem 3.13).

The converse is an immediate consequence of Theorem 6.4 and of the evident fact that types of von Neumann algebras are conserved by $*$-isomorphic.

Q.E.D.

6.6. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra, $\mathscr{M}'$ its commutant and $\mathscr{Z} = \mathscr{M} \cap \mathscr{M}'$ its center. With the help of Corollary 3.3, it is easy to see that $\mathscr{Z}'$ is the smallest von Neumann algebra included in $\mathscr{B}(\mathscr{H})$, which contains $\mathscr{M}$ and $\mathscr{M}'$, i.e., it is the von Neumann algebra $\mathscr{R}(\mathscr{M}, \mathscr{M}')$, generated by $\mathscr{M}$ and $\mathscr{M}'$.

From Theorem 6.4 the following corollary obviously follows

Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. Then $\mathscr{R}(\mathscr{M}, \mathscr{M}')$ is a von Neumann algebra of type I.*

Exercises

!E.6.1. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra, $e' \in \mathscr{P}_{\mathscr{M}'}$ and $\xi \in \mathscr{H}$, such that $e' \sim p'_\xi$. Then there exists an $\eta \in \mathscr{H}$, such that

$$e' = p'_\eta, \qquad p_\xi = p_\eta.$$

E.6.2. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $e, f \in \mathscr{P}_{\mathscr{M}}$. If $\mathscr{M}_e \subset \mathscr{B}(e\mathscr{H})$ has a separating vector, $\mathscr{M}_f \subset \mathscr{B}(f\mathscr{H})$ has a cyclic vector and $z(e) \leqslant z(f)$, then $e \prec f$.

!E.6.3. If a von Neumann algebra has a cyclic vector and a separating vector, hen it has a vector which is both cyclic and separating (compare with C.3.5).

E.6.4. If $\mathscr{M}$ is a finite von Neumann algebra and if it has a finite totalizing family, then $\mathscr{M}'$ is also finite.

E.6.5. Let $\mathscr{H}$ be a separable Hilbert space and $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ a finite von Neumann algebra with a properly infinite commutant. Then there exists a sequence $\{\xi_n\} \subset \mathscr{H}$, such that the projections p'_{ξ_n} be mutually orthogonal, $\sum_{n=1}^{\infty} p'_{\xi_n} = 1$ and, for any n, $p_{\xi_n} = 1$ (Hint: see 7.18).

E.6.6. Let $\mathscr{H}$ be a separable Hilbert space and $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ a properly infinite von Neumann algebra with a properly infinite commutant. Then $\mathscr{M}$ has a separating cyclic vector.

E.6.7. Let $\mathscr{M}$ be a von Neumann algebra, with the center $\mathscr{Z}$. A projection $p \in \mathscr{Z}$ is the central support of a cyclic projection iff it is of countable type in $\mathscr{Z}$. Infer from this result the statement in E.3.8.

E.6.8. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. For any $x \in \mathscr{M}$ and any $\xi \in \mathscr{H}$, we have $p_{x\xi} \prec p_{\xi}$.

If, moreover, $\xi \in [x^*\mathscr{H}]$, then $p_{x\xi} \sim p_{\xi}$.

!E.6.9. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra, and $J : \mathscr{H} \to \mathscr{H}$ a conjugation, such that $x \mapsto Jx^*J$ is a $*$-antiisomorphism of $\mathscr{M}$ onto $\mathscr{M}'$. Then, for any $\xi \in \mathscr{H}$, we have

$$p'_{\xi} = Jp_{J\xi}J.$$

E.6.10. Let $\mathscr{M}$ be a von Neumann algebra of type I (resp., II; resp., III) and $e' \in \mathscr{P}_{\mathscr{M}'}$. Then $\mathscr{M}_{e'}$ is of type I (resp., II; resp., III).

E.6.11. Let $\mathscr{M}$ be a von Neumann algebra and $e \in \mathscr{M}$ a minimal projection in $\mathscr{M}$. Then $z(e)$ is a minimal projection in $\mathscr{Z}$, whereas $\mathscr{M}z(e)$ is a factor of type I. Infer from this result that the l.u.b. of the set of all minimal projections in $\mathscr{M}$ is a central projection.

Comments

C.6.1. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\xi \in \mathscr{H}$. By taking into account Theorem 5.23, which is the basis of all results in this chapter, it is only natural to inquire about the structure of the vectors in the space $[\mathscr{M}\xi]$. Since any such vector is the limit of a sequence of vectors of the form $x_n\xi$, where $x_n \in \mathscr{M}$, it is natural to search for the conditions which make true the following assertion

$$(T)\quad \left\{\begin{array}{l}\text{for any } \eta \in [\mathscr{M}\xi] \text{ there exists a closed*) operator } T \text{ in } \mathscr{H}, \text{ affiliated**)} \\ \text{to } \mathscr{M}, \text{ such that} \\ \qquad\qquad\qquad \eta = T\xi.\end{array}\right.$$

This deep problem was first considered by F. J. Murray and J. von Neumann [1] in connection with the results presented in this chapter. They gave a partially positive answer, by showing that the following statement is always true

$$(BT)\quad \left\{\begin{array}{l}\text{for any } \eta \in [\mathscr{M}\xi] \text{ there exists a closed operator } T \text{ in } \mathscr{H}, \text{ affiliated to } \mathscr{M}, \\ \text{and an operator } B \in \mathscr{M}, \text{ such that} \\ \qquad\qquad\qquad \eta = BT\xi.\end{array}\right.$$

The (BT)-theorem enabled F. J. Murray and J. von Neumann to obtain Lemma 6.1 from this chapter, for factors. For the proof of the (BT)-theorem we

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*) See Section 9.1.

**) See Section 9.7.

refer to S. Sakai [32], 2.7.14, or C.F. Skau [2], whereas for the proof of the results in this chapter, with the help of the (BT)-theorem, we refer to J. Dixmier [26], Ch. III, § 1.3, 1.4.

As far as statement (T) is concerned, H. A. Dye [1] has shown that the projection p_ζ, $\zeta \in \mathscr{H}$, is finite iff the following implication holds:

$$\xi \in [\mathscr{M}\zeta] \Rightarrow \text{for } \xi \text{ the statement } (T) \text{ is true.}$$

H. A. Dye called a von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ *essentially finite* if any cyclic projection $p_\xi \in \mathscr{M}$, $\xi \in \mathscr{H}$, is finite. Thus, the statement (T) is true for any vector in $\mathscr{H}$ iff $\mathscr{M}$ is essentially finite.

In particular, if $\mathscr{M}$ is finite, then statement (T) is true for any vector in $\mathscr{H}$, a result already known to F. J. Murray and J. von Neumann. The proof of this fact immediately follows from the (BT)-theorem, with the help of exercise E.9.26.

On the other hand, if $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra, $\xi \in \mathscr{H}$, and ψ is a normal form on $\mathscr{M}$, such that $\mathrm{s}(\psi) \leqslant \mathrm{s}(\omega_\xi)$, then Theorem 5.23 shows that there exists an $\eta \in [\mathscr{M}\xi]$, such that $\psi = \omega_\eta$. Having in mind the problem (T), there naturally arises the question whether the vector $\eta \in [\mathscr{M}\xi]$, such that $\psi = \omega_\eta$, can be chosen so that $\eta = T\xi$, where T is a closed operator in $\mathscr{H}$, affiliated to $\mathscr{M}$. Therefore, a new problem arises, namely to establish the conditions under which the following statement is true:

(RN) $\left\{\right.$ for any normal form ψ on $\mathscr{M}$, such that $\mathrm{s}(\psi) \leqslant \mathrm{s}(\omega_\xi)$, there exists a closed operator T in $\mathscr{H}$, affiliated to $\mathscr{M}$, such that

$$\psi = \omega_{T\xi}$$

Since the condition $\mathrm{s}(\psi) \leqslant \mathrm{s}(\omega_\xi)$ is a condition of "absolute continuity", and since the operator T plays the role of a "density", the statement (RT) is obviously analogous to the classical Radon-Nikodym theorem.

By taking into account Theorem 5.23, it is obvious that

$$(T) \Rightarrow (RN).$$

In particular, if $\mathscr{M}$ is essentially finite, then the statement (RN) is true for any vector in $\mathscr{H}$. In fact, the statement (RN) is true for any vector in $\mathscr{H}$, without any restriction on $\mathscr{M}$, as we shall see in Chapter 10, where we shall make more precise considerations concerning the density T.

C.6.2. Let us consider again a von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ and a vector $\xi \in \mathscr{H}$. Concerning the commutant $\mathscr{M}' \subset \mathscr{B}(\mathscr{H})$, the analogous statement to statement (T) is the following

(T') $\left\{\right.$ for any $\eta \in [\mathscr{M}'\xi]$ there exists a closed operator T' in $\mathscr{H}$, affiliated to $\mathscr{M}'$, such that

$$\eta = T'\xi.$$

If ψ is a normal form on $\mathscr{M}$, such that $\mathrm{s}(\psi) \leqslant \mathrm{s}(\omega_\xi)$, Theorem 5.23 shows that there exists an $\eta \in [\mathscr{M}'\xi]$, such that $\psi = \omega_\eta$. By taking into account statement (T'), there naturally arises the question whether the vector $\eta \in [\mathscr{M}'\xi]$, such that

$\psi = \omega_\eta$, can be chosen so that $\eta = T'\xi$, where T' is a closed operator in $\mathcal{H}$, affiliated to $\mathcal{M}'$. Consequently, a new problem arises, namely to establish the conditions under which the following statement is true:

$$(D)\quad \left\{\begin{array}{l}\text{for any normal form } \psi \text{ on } \mathcal{M}, \text{ such that } s(\psi) \leqslant s(\omega_\xi), \text{ there exists a} \\ \text{positive self-adjoint operator } A' \text{ in } \mathcal{H}, \text{ affiliated to } \mathcal{M}', \text{ such that} \\ \qquad\qquad \psi = \omega_{A'\xi}.\end{array}\right.$$

Together with this problem, one poses the analogous problem, concerning the statement:

$$(D')\quad \left\{\begin{array}{l}\text{for any normal form } \psi' \text{ on } \mathcal{M}', \text{ such that } s(\psi') \leqslant s(\omega'_\xi), \text{ there exists a} \\ \text{positive self-adjoint operator } A \text{ in } \mathcal{H}, \text{ affiliated to } \mathcal{M}, \text{ such that} \\ \qquad\qquad \psi' = \omega'_{A\xi}.\end{array}\right.$$

The solution to problems (D) and (D') was given by H. A. Dye [1], along with the solution to problem (T) (C.6.1). We remark that statements (D) and (D') are also of the Radon-Nikodym type, but the density now belongs to the commutant. A similar, but trivial, situation was considered in Lemma 5.19 (see, also, exercise E.9.33).

We now state the theorem of H. A. Dye and we also give a sketch of the proof, by using some of the results in Chapter 9.

Theorem. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\zeta \in \mathcal{H}$. The following statements are equivalent.*

(1) *p_ζ is a finite projection in $\mathcal{M}$;*

(2) *$\xi \in [\mathcal{M}\zeta] \Rightarrow$ for ξ the statement (T) is true;*

(3) *$\xi \in [\mathcal{M}'\zeta] \Rightarrow$ for ξ the statement (D) is true;*

(1') *p'_ζ is a finite projection in $\mathcal{M}'$;*

(2') *$\xi \in [\mathcal{M}'\zeta] \Rightarrow$ for ξ the statement (T') is true;*

(3') *$\xi \in [\mathcal{M}\zeta] \Rightarrow$ for ξ the statement (D') is true.*

The proof proceeds according to the following diagram

$$\begin{tikzcd}
(1) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (2) \arrow[dl, Rightarrow] \arrow[dr, Leftrightarrow] & (3) \\
(1') \arrow[r, Rightarrow] & (2') \arrow[ul, Rightarrow] \arrow[ur, Leftrightarrow] & (3')
\end{tikzcd}$$

The equivalence $(1) \Leftrightarrow (1')$ coincides with Lemma 6.3.

The implication $(1) \Rightarrow (2)$ immediately follows from the (BT)-theorem, with the help of exercise E.9.26.

The equivalence $(2') \Leftrightarrow (3)$ easily obtains from Theorem 5.23 and exercise E.9.32.

It is obvious that the implication $(1') \Rightarrow (2')$ and the equivalence $(2) \Leftrightarrow (3')$ obtain in a similar manner.

We have still to prove the implication $(2') \Rightarrow (1)$, since the implication $(2) \Rightarrow (1')$ is obviously similar to this one.

Let us assume that the projection p_ζ is not finite in $\mathcal{M}$. Then we can assume that p_ζ is properly infinite. It follows that there exist a projection $e \in \mathcal{M}$, $e \leqslant p_\zeta$, $e \neq p_\zeta$, and a partial isometry $v \in \mathcal{M}$, such that

$$v^*v = p_\zeta, \qquad vv^* = e, \qquad vp_\zeta = v = ev = p_\zeta v;$$

$$\mathbf{n}(1 - v) = \mathbf{n}(1 - v^*) = 0.$$

Therefore, the operator $1 - v$ is injective, and $(1 - v)\mathcal{H}$ is a dense subspace of $\mathcal{H}$. From exercise E.9.8 we infer that

$$A = \mathrm{i}(1 + v)(1 - v)^{-1}$$

is a closed (symmetric) linear operator in $\mathcal{H}$, such that $\mathcal{D}_A = (1 - v)\mathcal{H}$. Of course, A is affiliated to $\mathcal{M}$.

We have $\xi = (1 - v)\zeta \in \mathcal{D}_A$. It is easily verified that $\xi \in [\mathcal{M}'\zeta]$ and, since $\mathbf{n}(1 - v^*) = 0$, it follows that we even have the equality $p_\xi = p_\zeta$. Hence, if we again use the relations $\mathbf{n}(1 - v) = \mathbf{n}(1 - v^*) = 0$, we get

$$(*) \qquad \mathcal{D}_A \cap [\mathcal{M}'(p_\xi - e)\xi] = \{0\}.$$

Let us now assume that hypothesis (2′) holds.

We shall first show that for any projection $q \in \mathcal{M}$, $0 \neq q \leqslant p_\xi$, there exists a projection $r' \in \mathcal{M}'$, such that

$$r'\xi \neq 0 \quad \text{and} \quad r'\xi \in [\mathcal{M}'q\xi].$$

Indeed, we have $q\xi \in [\mathcal{M}'\xi]$ and, therefore, from hypothesis (2′), there exists a closed operator T' in $\mathcal{H}$, affiliated to $\mathcal{M}'$, such that

$$q\xi = T'\xi.$$

Then, for any $x' \in \mathcal{M}'$ we have

$$x'T'\xi \in [\mathcal{M}'q\xi].$$

By taking into account the polar decomposition Theorem (9.28) and the operational calculus with positive self-adjoint operators (9.11, 9.13), it is easily seen that there exist an $x' \in \mathcal{M}'$ and a projection $r' \in \mathcal{M}'$, such that

$$r'\xi = x'T'\xi \neq 0.$$

A familiar argument, based on the Zorn Lemma, shows that for any projection $q \in \mathcal{M}$, $0 \neq q \leqslant p_\xi$, there exists a projection $q' \in \mathcal{M}'$, such that

$$[\mathcal{M}'q\xi] = [\mathcal{M}'q'\xi].$$

In particular, let $q = p_\xi - e$. Since $\xi \in \mathcal{D}_A$ and since A is affiliated to $\mathcal{M}$, it follows that (see E.9.25):

$$\mathcal{M}'q'\xi \subset \mathcal{D}_A \cap [\mathcal{M}'q\xi];$$

hence

$$(**) \qquad \mathcal{D}_A \cap [\mathcal{M}'(p_\xi - e)\xi] \text{ is dense in } [\mathcal{M}'(p_\xi - e)\xi].$$

The contradiction between relations (*) and (**) proves the implication (2′) ⇒ (1) and, thus, the theorem is also proved.

Corollary. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. The following statements are equivalent:*

(1) *$\mathcal{M}$ is essentially finite;*

(2) *the statement (T) is true for any vector $\xi \in \mathcal{H}$;*

(3) *the statement (D) is true for any vector $\xi \in \mathcal{H}$.*

We stress the fact that, while the statement (*RN*) is always true, the statement (*D*), of Radon-Nikodym type, of H. A. Dye, depends on finiteness conditions.

C.6.3. From Lemma 6.3 it follows that the von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is essentially finite iff $\mathcal{M}' \subset \mathcal{B}(\mathcal{H})$ is essentially finite.

It is easily seen that any essentially finite von Neumann algebra is semifinite.

Conversely, according to Corollary 6.5, any semifinite von Neumann algebra is *-isomorphic to an essentially finite von Neumann algebra.

On the other hand, $\mathcal{B}(\mathcal{H})$ is essentially finite, but it is properly infinite if $\mathcal{H}$ is infinitely dimensional.

C.6.4. Bibliographical comments. Lemma 6.1 was proved, for the case of the factors, by F. J. Murray and J. von Neumann [1]. Lemmas 6.2, 6.3 and Theorem 6.4 are stated by I. Kaplansky [10], whereas proofs have been given by H. A. Dye [1], E. L. Griffin [2], J. Dixmier [24], R. Pallu de la Barrière [5] and R. V. Kadison [14].

In the proofs of Lemmas 6.2 and 6.3, we followed D. M. Topping [8] and, respectively, R. V. Kadison [14]. For another proof of Lemma 6.2, see L. Zsidó [3], I.7.7, whereas another proof of Lemma 6.3 is proposed in exercise E.7.20. A proof of Lemma 6.1 based directly on the polar decomposition theorem was given by R. Herman and M Takesaki [2].

7

Finite von Neumann algebras

In this chapter we study the lattice of the finite projections in a von Neumann algebra and we characterize the finite von Neumann algebras with the help of the traces.

7.1. Theorem. *Let $\mathscr{M}$ be a von Neumann algebra and $e, f, g \in \mathscr{P}_{\mathscr{M}}$. If $e \leqslant g$ and if the projection $(e \vee f) \wedge g$ is finite, then*

$$(e \vee f) \wedge g = e \vee (f \wedge g).$$

Proof. Let $h = (e \vee f) \wedge g$, $k = e \vee (f \wedge g)$. The relation $k \leqslant h$ is obvious in view of the hypothesis that $e \leqslant g$. On the other hand, we have

$$e \vee f = (e \vee (f \wedge g)) \vee f \leqslant ((e \vee f) \wedge g) \vee f \leqslant e \vee f,$$

hence

$$h \vee f = k \vee f = e \vee f,$$

and

$$g \wedge f \leqslant (e \vee (f \wedge g)) \wedge f \leqslant ((e \vee f) \wedge g) \wedge f \leqslant g \wedge f;$$

therefore, we have

$$h \wedge f = k \wedge f = g \wedge f.$$

By taking into account the parallelogram rule (4.4), we get

$$\begin{aligned} h - f \wedge g &= h - h \wedge f \sim h \vee f - f = e \vee f - f \\ &= k \vee f - f \sim k - k \wedge f = k - f \wedge g. \end{aligned}$$

It follows that $h \sim k$. But $k \leqslant h$ and, by hypothesis, h is finite. Consequently, we have $h = k$.

Q.E.D.

7.2. One says that a projection $e \in \mathscr{M}$ is *piecewise of countable type* if there exists a family $\{q_k\}_{k \in K}$ of mutually orthogonal central projections, such that $\sum_{k \in K} q_k = 1$, and eq_k is of countable type, for any $k \in K$.

Lemma. *Any finite projection e in a von Neumann algebra $\mathscr{M}$ is piecewise of countable type.*

Proof. Let φ be a normal form on $\mathscr{M}$, with $0 \neq e_0 = s(\varphi) \leqslant e$, and let $\mathscr{E}$ be a maximal family of mutually orthogonal subprojections of e, which are all equivalent to e_0. Since e is finite, this family is finite. Let $\mathscr{E} = \{e_1, \ldots, e_n\}$. By applying the comparison theorem (4.6) to the projections e_0, $e - \sum_{i=1}^{n} e_i$, and by taking into account the maximality of the family $\mathscr{E}$, it follows that there exists a central projection $q \neq 0$, such that

$$q\left(e - \sum_{i=1}^{n} e_i\right) \prec qe_0.$$

In accordance with exercise E.5.6, e_0 is of countable type. From the preceding results, it follows that qe is of countable type, $q \neq 0$.

Let now $\{q_k\}_{k \in K}$ be a maximal family of mutually orthogonal central non-zero projections, such that $q_k e$ is of countable type, for any $k \in K$. From the maximality of the family and from the first part of the proof it follows that $\sum_{k \in K} q_k = 1$.

Consequently, e is piecewise of countable type.

Q.E.D.

7.3. Lemma. *Let $\mathscr{M}$ be a von Neumann algebra, $f \in \mathscr{P}_{\mathscr{M}}$, and $\{e_n\}$ an increasing sequence of finite projections in $\mathscr{M}$. If $e_n \prec f$ for any n, then $\bigvee_n e_n \prec f$.*

Proof. We shall construct an increasing sequence $\{f_n\} \subset \mathscr{P}_{\mathscr{M}}$, such that

$$f_n \leqslant f,\ f_n \sim e_n; \quad n = 1, 2, \ldots$$

Then, by taking into account Proposition 4.2., and exercise E.4.9, we shall obtain:

$$\bigvee_n e_n \sim \bigvee_n f_n \leqslant f.$$

For the construction we shall proceed by induction. Let us assume that $f_1, \ldots, f_{n-1}$ have been already constructed. By hypothesis, there exists an equivalence between e_n and a subprojection of f. We deduce that there exists a $g \in \mathscr{P}_{\mathscr{M}}$, $g \leqslant f$, such that

$$e_{n-1} \sim g \text{ and } e_n - e_{n-1} \prec f - g.$$

But $e_{n-1} \sim f_{n-1}$, whence, in view of exercise E.4.9,

$$f - f_{n-1} \sim f - g.$$

Consequently, there exists an $h \in \mathscr{P}_{\mathscr{M}}$, such that

$$e_n - e_{n-1} \sim h \leqslant f - f_{n-1}.$$

We then define

$$f_n = f_{n-1} + h.$$

Q.E.D.

7.4. Theorem. *Let $\mathcal{M}$ be a von Neumann algebra, $f \in \mathcal{P}_{\mathcal{M}}$ and $\{e_i\}_{i\in I}$ an increasingly directed family of finite projections in $\mathcal{M}$. If $\bigvee_{i\in I} e_i$ is piecewise of countable type, and $e_i \prec f$, for any $i \in I$, then $\bigvee_{i\in I} e_i \prec f$.*

Proof. We shall first assume that $e = \bigvee_{i\in I} e_i$ is of countable type. In accordance with exercise E.5.6, there exists a normal form φ on $\mathcal{M}$, such that $e = \mathrm{s}(\varphi)$. It follows that

$$\varphi(e) = \sup_{i\in I} \varphi(e_i),$$

hence there exists a sequence $\{i_n\} \subset I$, such that

$$\varphi(e) = \sup_n \varphi(e_{i_n}).$$

We now define by induction the sequence $\{e_n\}$, in the following manner

$$e_1 = e_{i_1},$$

$$e_n = e_i, \text{ where } i \in I, e_i \geqslant e_{i_n} \text{ and } e_i \geqslant e_k \text{ for } k < n.$$

The definition is possible, because the family $\{e_i\}_{i\in I}$ is increasingly directed. Then $\{e_n\}$ is an increasing sequence of finite projections in $\mathcal{M}$, $e_n \prec f$, and $e = \bigvee_n e_n$, because $\varphi(e) = \sup_n \varphi(e_n)$ and $e = \mathrm{s}(\varphi)$.

With Lemma 7.3, we infer that $e \prec f$.

Let us now assume that e is piecewise of countable type and let $\{q_k\}_{k\in K}$ be a family of mutually orthogonal central projections, such that $\sum_{k\in K} q_k = 1$, such that eq_k is of countable type, for any $k \in K$. By virtue of the first part of the proof, it follows that $eq_k \prec fq_k$, for any $k \in K$. By taking into account Proposition 4.2, we infer that $e \prec f$.

Q.E.D.

7.5. Corollary. *Let $\mathcal{M}$ be a von Neumann algebra, $f \in \mathcal{P}_{\mathcal{M}}$ and $\{e_i\}_{i\in I}$ an increasingly directed family of projections in $\mathcal{M}$. If $\bigvee_{i\in I} e_i$ is finite and if $e_i \prec f$, for any $i \in I$, then $\bigvee_{i\in I} e_i \prec f$.*

7.6. Corollary. *Let $\mathcal{M}$ be a von Neumann algebra, $f \in \mathcal{P}_{\mathcal{M}}$ and $\{e_i\}_{i\in I}$ an increasingly directed family of projections in $\mathcal{M}$. If $\bigvee_{i\in I} e_i$ is finite, then*

$$\bigvee_{i\in I} (e_i \wedge f) = (\bigvee_{i\in I} e_i) \wedge f.$$

Proof. Let

$$e = \bigvee_{i\in I} e_i, \quad g = \bigvee_{i\in I} (e_i \wedge f).$$

We must show that $e \wedge f = g$. The relation $q \leqslant e \wedge f$ is obvious. We now consider the projection

$$h = e \wedge f - g.$$

The following computation, based on the parallelogram rule (4.4), shows that $h \prec e - e_\iota$:

$$h \leqslant e \wedge f - e_\iota \wedge f = e \wedge f - (e \wedge f) \wedge e_\iota \sim$$

$$\sim (e \wedge f) \vee e_\iota - e_\iota \leqslant e - e_\iota.$$

We now show that $e_\iota \prec e - h$, for any ι. If this is not true, by taking into account the comparison theorem (4.6), we would find a central projection $q \neq 0$ and a projection $g_\iota \leqslant qe_\iota$, $g_\iota \neq qe_\iota$, such that

$$q(e - h) \sim g_\iota.$$

On the other hand, from what we already proved, it follows that there exists a projection $h_\iota \leqslant q(e - e_\iota)$, such that

$$qh \sim h_\iota.$$

Consequently, we have

$$qe \sim g_\iota + h_\iota \leqslant qe, \quad g_\iota + h_\iota \neq qe,$$

and this result contradicts the finiteness of qe.

According to Corollary 7.5, it follows that

$$e = \bigvee_{\iota \in I} e_\iota \prec e - h.$$

Since e is finite, we infer that $h = 0$.

Q.E.D.

7.7. Let $\mathscr{L}$ be a lattice. One says that $\mathscr{L}$ is a *modular lattice* if

$$e, f, g \in \mathscr{L},\ e \leqslant g \Rightarrow (e \vee f) \wedge g = e \vee (f \wedge g).$$

One says that $\mathscr{L}$ is a *complemented lattice* if it has a smallest element 0, a greatest element 1 and if, for any $e \in \mathscr{L}$, there exists an $e' \in \mathscr{L}$, such that $e \wedge e' = 0$, $e \vee e' = 1$.

One says that $\mathscr{L}$ is *upper* (resp., *lower*) *continuous* if: $f \in \mathscr{L}$, $\{e_\iota\}_{\iota \in I} \subset \mathscr{L}$ increasingly (resp., decreasingly) directed and $\bigvee_{\iota \in I} e_\iota \in \mathscr{L}$ (resp., $\bigwedge_{\iota \in I} e_\iota \in \mathscr{L}$) $\Rightarrow$ $\bigvee_{\iota \in I} (e_\iota \wedge f) = (\bigvee_{\iota \in I} e_\iota) \wedge f$ (resp., $\bigwedge_{\iota \in I} (e_\iota \vee f) = (\bigwedge_{\iota \in I} e_\iota) \vee f$).

One says that $\mathscr{L}$ is a *continuous geometry* if $\mathscr{L}$ is an upper and lower continuous, complemented, modular lattice.

From Corollary 3.7, Theorem 7.1 and Corollary 7.6, the following theorem obtains

Theorem. *Let $\mathscr{M}$ be a finite von Neumann algebra. Then the lattice $\mathscr{P}_{\mathscr{M}}$ is a continuous geometry.*

7.8. We shall now construct the central trace on a finite von Neumann algebra.

Let $\mathscr{M}$ be a von Neumann algebra and φ a w-continuous form on $\mathscr{M}$. We shall consider the sets

$$\mathscr{Q}_\varphi = \{T_u\varphi;\ u \in \mathscr{M},\ \text{unitary}\} \subset \mathscr{M}_*,$$

$$\mathscr{K}_\varphi = \text{the norm closed convex hull of } \mathscr{Q}_\varphi \text{ in } \mathscr{M}_*.$$

Since $\mathscr{M}$ is the dual of the Banach space $\mathscr{M}_*$, from the Mackey theorem we now infer that the convex set $\mathscr{K}_\varphi$ is $\sigma(\mathscr{M}_*;\mathscr{M})$-closed.

Lemma. *Let $\mathscr{M}$ be a finite von Neumann algebra and φ a w-continuous form. Then the set $\mathscr{K}_\varphi$ is $\sigma(\mathscr{M}_*;\ \mathscr{M})$-compact.*

Proof. We must show that the set $\mathscr{K}_\varphi$ is $\sigma(\mathscr{M}_*;\ \mathscr{M})$-relatively compact. From the Akemann theorem (5.14), it is sufficient to show that, for any sequence $\{e_n\}$ of orthogonal projections in $\mathscr{M}$, we have

$$\lim_{n\to\infty} \psi(e_n) = 0, \quad \text{uniformly for } \psi \in \mathscr{K}_\varphi.$$

It is sufficient to prove the uniformity of this convergence with respect to $\psi \in \mathscr{Q}_\varphi$.

In order to prove this property, we shall assume that it is not true. Then there exists a $\delta > 0$, a subsequence $\{f_n\}$ of the sequence $\{e_n\}$ and a sequence $\{u_n\}$ of unitary operators in $\mathscr{M}$, such that, by denoting $\psi_n = T_{u_n}\varphi$, we should have

$$|\psi_n(f_n)| \geqslant \delta, \quad \text{for any } n = 1, 2, \dots$$

We denote $g_n = u_n^* f_n u_n \in \mathscr{P}_{\mathscr{M}}$. Then we have

(1) $$g_n \sim f_n \text{ and } |\varphi(g_n)| \geqslant \delta, \text{ for any } n = 1, 2, \dots$$

We shall define

$$h_{m,n} = \bigvee_{k=n}^{m} g_k; \quad 1 \leqslant n \leqslant m,$$

$$h_n = \bigvee_{k=n}^{\infty} g_k = \bigvee_{m=n}^{\infty} h_{m,n}; \quad n = 1, 2 \dots$$

$$h = \bigwedge_{n=1}^{\infty} h_n.$$

We shall show that

(2) $$h_{m,n} \prec \sum_{k=n}^{k=m} f_k, \text{ for any } m \geqslant n \geqslant 1.$$

We proceed by induction on $m \geqslant n$. For $m = n$, we have $h_{n,n} = g_n \sim f_n$. If relation (2) is true for $m = r$, then, since

$$g_{r+1} \vee h_{r,n} - h_{r,n} \sim g_{r+1} - g_{r+1} \wedge h_{r,n} \leqslant g_{r+1} \sim f_{r+1},$$

it follows that

$$h_{r+1,n} = g_{r+1} \vee h_{r,n} \prec \sum_{k=n}^{k=r} f_k + f_{r+1} = \sum_{k=n}^{k=r+1} f_k.$$

For any n, the sequence $\{h_{m,n}\}_m$ is increasing and, from relation (2), we infer that

$$h_{m,n} \prec \sum_{k=n}^{\infty} f_k.$$

According to Lemma 7.3, we now infer that

$$h_n \prec \sum_{k=n}^{\infty} f_k.$$

By taking into account exercise E.4.9, we get

$$1 - \sum_{k=n}^{\infty} f_k \prec 1 - h_n \leqslant 1 - h$$

and, by applying again Lemma 7.3, it follows that

$$1 = \bigvee_{n=1}^{\infty} \left(1 - \sum_{k=n}^{\infty} f_k\right) \prec 1 - h.$$

Consequently, $h = 0$, since $\mathscr{M}$ is finite.

Since the sequence $\{h_n\}$ is decreasing, $\bigwedge_{n=1}^{\infty} h_n = h = 0$ and $g_n \leqslant h_n$, it follows that the sequence $\{g_n\}$ is *wo*-convergent to 0. On the other hand, on the closed unit ball of $\mathscr{M}$ the *wo*-topology coincides with the *w*-topology, and, therefore, the sequence $\{g_n\}$ is *w*-convergent to 0.

Consequently, we have that $\lim_n \varphi(g_n) = 0$, which contradicts relation (1).

Q.E.D.

7.9. One calls a *central form* on an algebra $\mathscr{A}$ any form φ, such that $\varphi(xy) = \varphi(yx)$, for any $x, y \in \mathscr{A}$.

A form φ on a C^*-algebra $\mathscr{A} \ni 1$ is central iff it is *unitarily invariant*, i.e., $T_u\varphi = \varphi$, for any unitary $u \in \mathscr{A}$.

Lemma. *Let φ be a w-continuous central form on the von Neumann algebra $\mathscr{M}$, whose center is $\mathscr{Z}$. Then $\|\varphi\| = \|\varphi\,|\,\mathscr{Z}\|$. In particular, $\varphi \geqslant 0$ iff $\varphi\,|\,\mathscr{Z} \geqslant 0$.*

Proof. Let $\varphi = R_v|\varphi|$ be the polar decomposition (5.16) of φ. From the equality

$$\varphi = R_{uvu^*}T_u|\varphi|,$$

which is true for any unitary $u \in \mathscr{M}$, and from the uniqueness of the polar decomposition, it follows that v and $|\varphi|$ are unitarily invariant. It follows that $v \in \mathscr{Z}$ and $|\varphi|$ is central.

Consequently, we have

$$\|\varphi\| = \||\varphi|\| = |\varphi|(1) = \varphi(v^*) \leqslant \|\varphi|\mathscr{Z}\| \cdot \|v^*\| \leqslant \|\varphi\|,$$

hence $\|\varphi\| = \|\varphi|\mathscr{Z}\|$.

The second assertion follows from the first, by taking into account Proposition 5.4.

Q.E.D.

7.10. Lemma. *Let $\mathscr{M}$ be a finite von Neumann algebra, $\mathscr{Z}$ its center. Then any w-continuous form ω on $\mathscr{Z}$ uniquely extends to a bounded central form φ_ω on $\mathscr{M}$.*

Moreover, φ_ω is w-continuous, $\|\varphi_\omega\| = \|\omega\|$ and $\omega \geqslant 0$ implies $\varphi_\omega \geqslant 0$.

Proof. The uniqueness part of the lemma, as well as the two last assertions of the Lemma, follow from Lemma 7.9.

In order to prove the existence and the w-continuity of the form φ_ω, we first consider a w-continuous form φ on $\mathscr{M}$, such that $\varphi|\mathscr{Z} = \omega$ (see Theorem 1.10). We now apply the Ryll-Nardzewski fixed point theorem (see Theorem A.3 in the Appendix) for the following particular case

$\mathscr{X} = \mathscr{M}_*$, in the uniform (norm) topology is a separated locally convex vector space, whose dual is $\mathscr{M}$.

$\mathscr{K} = \mathscr{K}_\varphi$ is a weakly compact, convex, non-empty subset of $\mathscr{X}$ (in accordance with Lemma 7.8).

$\mathscr{S} = \{T_u|\mathscr{K};\ u \in \mathscr{M}, \text{unitary}\}$ is a non-contracting semi-group of weakly continuous affine mappings of $\mathscr{K}$ into $\mathscr{K}$, since any T_u is a linear isometry of $\mathscr{M}_*$ onto $\mathscr{M}_*$.

It follows that there exists a $\varphi_\omega \in \mathscr{K}_\varphi \subset \mathscr{M}_*$, such that $T_u\varphi_\omega = \varphi_\omega$, for any unitary $u \in \mathscr{M}$. Consequently, φ_ω is a w-continuous central form on $\mathscr{M}$.

On the other hand, since $\varphi|\mathscr{Z} = \omega$, it follows that $\psi|\mathscr{Z} = \omega$, for any $\psi \in \mathscr{K}_\varphi$; in particular, we have $\varphi_\omega|\mathscr{Z} = \omega$.

Q.E.D.

7.11. Theorem. *Let $\mathscr{M}$ be a von Neumann algebra, $\mathscr{Z}$ its center. Then $\mathscr{M}$ is finite iff there exists a mapping*

$$\natural : \mathscr{M} \ni x \mapsto x^\natural \in \mathscr{Z},$$

having the following properties:

(i) *$\natural$ is linear and bounded;*

(ii) *$(xy)^\natural = (yx)^\natural$, for any $x, y \in \mathscr{M}$;*

(iii) *$z^\natural = z$, for any $z \in \mathscr{Z}$;*

The mapping $\natural$ having properties (i)—(iii) *is unique.*

Moreover, the mapping $\natural$ also has the following properties:

(iv) *$\|\natural\| = 1$;*

(v) *$\natural$ is w-continuous;*

(vi) *$(zx)^\natural = zx^\natural$, for any $x \in \mathscr{M}$, $z \in \mathscr{Z}$;*

(vii) *$x \in \mathscr{M}$, $x \geqslant 0 \Rightarrow x^\natural \geqslant 0$;*

(viii) *$x \in \mathscr{M}$, $x \geqslant 0$, $x^\natural = 0 \Rightarrow x = 0$.*

(ix) *$x^\natural \in \overline{\mathrm{co}}^n\{uxu^*;\ u \in \mathscr{M}, \text{unitary}\}$ for every $x \in \mathscr{M}$.*

Proof. It is easy to see that the existence of the mapping $\natural$ implies the finiteness of the von Neumann algebra $\mathscr{M}$.

Let us now assume that the von Neumann algebra $\mathscr{M}$ is finite.

We shall first prove the uniqueness of the mapping $\natural$. In order to do this, it is sufficient to prove that for any w-continuous form ω on $\mathscr{Z}$, we have

$$\omega(x^\natural) = \varphi_\omega(x),$$

where φ_ω is the unique bounded central form on $\mathscr{M}$, such that $\varphi_\omega | \mathscr{Z} = \omega$ (7.10). But this fact is obvious, since from conditions (i), (ii), (iii) we infer that the mapping

$$x \mapsto \omega(x^\natural), \quad x \in \mathscr{M},$$

is a bounded central form on $\mathscr{M}$, which extends ω.

We now prove the existence of a mapping $\natural$, having properties (i) — (ix). In accordance with Lemma 7.10, the mapping $E: \mathscr{Z}_* \mapsto \mathscr{M}_*$, defined by

$$E\omega = \varphi_\omega, \quad \omega \in \mathscr{Z}_*,$$

is linear and isometric. By taking into account the canonical identifications $\mathscr{M} = (\mathscr{M}_*)^*$, $\mathscr{Z} = (\mathscr{Z}_*)^*$, we now define the mapping $\natural$ as being the transpose of the mapping E, $\natural = {}^tE : \mathscr{M} \to \mathscr{Z}$. In other words, the mapping $\natural$ is determined by the relations:

$$\omega(x^\natural) = \varphi_\omega(x), \quad \omega \in \mathscr{Z}_*, \quad x \in \mathscr{M}.$$

Properties (i), (ii), (iii), (iv), (vii) are easily verified. Since ${}^tE$ is weakly continuous, $\natural$ is w-continuous, hence property (v) is established.

It is now sufficient to prove property (vi) only for the unitary elements $z \in \mathscr{Z}$. Let $w \in \mathscr{Z}$ be unitary and let us define the mapping $W: \mathscr{M} \to \mathscr{Z}$ by $W(x) = w^*(wx)^\natural$, $x \in \mathscr{M}$. Then W satisfies conditions (i)—(iii), hence, in view of the uniqueness, we have $W = \natural$. Hence $w^*(wx)^\natural = x^\natural$, i.e., $(wx)^\natural = wx^\natural$, for any $x \in \mathscr{M}$.

Let us now prove property (viii). Let $x \in \mathscr{M}$, $x \geqslant 0$, $x \neq 0$. There then exists a non-zero positive normal form ω on $\mathscr{Z}$, such that $p = s(\omega) \leqslant z(x)$. Since φ_ω is a normal positive central form (7.10), it is easy to see that $s(\varphi_\omega)$ is unitarily invariant, hence it is a central projection, whence $s(\varphi_\omega) = p$. The relation $\varphi_\omega(x) = 0$ implies $xp = 0$, and this is not possible, because $0 \neq p \leqslant z(x)$. Consequently, we have $\varphi_\omega(x) \neq 0$, whence $x^\natural \neq 0$.

The last assertion (ix) follows using the fact that $\varphi(x) = \varphi(x^\natural)$ for every bounded central form φ on $\mathscr{M}$, the proof of Lemma 7.10 and the Hahn-Banach theorem.

Q.E.D.

7.12. If $\mathscr{M}$ is a finite von Neumann algebra, the mapping $\natural$, introduced by Theorem 7.11, is also called the *canonical central trace* on $\mathscr{M}$.

Let $e, f \in \mathscr{P}_\mathscr{M}$. Then $e \prec f$ (resp., $e \sim f$) iff $e^\natural \leqslant f^\natural$ (resp., $e^\natural = f^\natural$). Indeed, let us assume that $e^\natural \leqslant f^\natural$. From the comparison theorem (4.6), there exists a projection $p \in \mathscr{Z}$, such that $ep \prec fp$, $e(1-p) \succ f(1-p)$. By taking into account the properties of the mapping $\natural$ and, especially, property (viii), it follows that $e(1-p) \sim f(1-p)$, hence $e \prec f$.

In particular, if $\mathscr{M}$ is a finite factor, the mapping $\natural$ has scalar values. The restriction of the mapping $\natural$ to $\mathscr{P}_\mathscr{M}$ is also denoted by d and it is called the *norma-*

lized dimension function on $\mathscr{P}_{\mathscr{M}}$. *Two projections* $e, f \in \mathscr{M}$ *are equivalent iff they have the same dimension:* $d(e) = d(f)$!

7.13. Let $\mathscr{M}$ be a von Neumann algebra and $\mathscr{M}^+ = \{x \in \mathscr{M};\ x \geqslant 0\}$. One calls a *trace* on $\mathscr{M}^+$ any function $\mu : \mathscr{M}^+ \to [0, +\infty]$, having the properties

$$\mu(x + y) = \mu(x) + \mu(y), \quad x, y \in \mathscr{M}^+,$$

$$\mu(\lambda x) = \lambda\mu(x), \quad x \in \mathscr{M}^+,\ \lambda \geqslant 0,$$

$$\mu(x^*x) = \mu(xx^*), \quad x \in \mathscr{M}.$$

Then μ obviously is unitarily invariant.

One says that a trace μ on $\mathscr{M}^+$ is *faithful* if

$$x \in \mathscr{M}^+, \quad \mu(x) = 0 \Rightarrow x = 0.$$

One says that a trace μ on $\mathscr{M}^+$ is *normal* if for any family $\{x_\iota\}_{\iota \in I} \subset \mathscr{M}^+$, which is increasingly directed and bounded, one has that

$$\mu(\sup_{\iota \in I} x_\iota) = \sup_{\iota \in I} \mu(x_\iota).$$

One says that a trace μ on $\mathscr{M}^+$ is *finite* if $\mu(x) < +\infty$, for any $x \in \mathscr{M}$. The restriction to $\mathscr{M}^+$ of any positive central form on $\mathscr{M}$ is a finite trace on $\mathscr{M}^+$. Conversely, any finite trace on $\mathscr{M}^+$ uniquely extends to a positive central form on $\mathscr{M}$.

One says that a trace μ on $\mathscr{M}^+$ is *semifinite* if for any $0 \neq x \in \mathscr{M}^+$ there exists a $y \in \mathscr{M}^+$, $y \neq 0$, $y \leqslant x$, such that $\mu(y) < +\infty$. If μ is a normal semifinite trace on $\mathscr{M}^+$, then, for any $x \in \mathscr{M}^+$, we have

$$\mu(x) = \sup\{\mu(y);\ y \leqslant x,\ \mu(y) < +\infty\}.$$

Indeed, if $\mu(x) < +\infty$, the assertion if obvious. If $\mu(x) = +\infty$, one considers a maximal totally ordered family $\{y_\iota\}_{\iota \in I} \subset \mathscr{M}^+$, such that $0 \neq y_\iota \leqslant x$ and $\mu(y_\iota) < +\infty$, and then one easily proves that $\mu(\sup_{\iota \in I} y_\iota) = +\infty$.

One says that a family of traces $\{\mu_k\}_{k \in K}$ on $\mathscr{M}^+$ is *sufficient* if for any $x \in \mathscr{M}^+$, $x \neq 0$, there exists a $k \in K$, such that $\mu_k(x) \neq 0$.

One defines the *support* $s(\mu)$ of a normal trace μ on $\mathscr{M}^+$ as being the projection complementary to the greatest projection in $\mathscr{M}$, which is annihilated by μ. Since μ is unitarily invariant, $s(\mu)$ is a central projection. The normal trace μ is faithful iff $s(\mu) = 1$. A family $\{\mu_k\}_{k \in K}$ of normal traces is sufficient iff $\bigvee_{k \in K} s(\mu_k) = 1$.

If a von Neumann algebra possesses a sufficient family of semifinite normal traces, then it possesses a faithful semifinite normal trace.

7.14. Corollary. *A von Neumann algebra is finite iff it possesses a sufficient family of finite normal traces.*

7.15. Corollary. *A von Neumann algebra is semifinite iff it possesses a faithful semifinite normal trace.*

Proof. Let $\mathscr{M}$ be a von Neumann algebra and let μ be a faithful semifinite normal trace on $\mathscr{M}^+$. For any central projection $0 \neq p \in \mathscr{M}$ there exists an element $x \in \mathscr{M}^+$, $x \neq 0$, $x \leqslant p$, such that $\mu(x) < +\infty$. There then exists a projection $e \in \mathscr{M}$, $e \neq 0$, and an $\varepsilon > 0$, such that e commutes with x and $ex \geqslant \varepsilon e$ (see Corollary 2.22). Then $\mu(e) \leqslant \frac{1}{\varepsilon}\mu(xe) \leqslant \frac{1}{\varepsilon}\mu(x) < +\infty$. Thus $\mu(e) < +\infty$, and this implies that the projection e is finite. Hence, $\mathscr{M}$ is semifinite.

Conversely, let $\mathscr{M}$ be a semifinite von Neumann algebra. In order to show that $\mathscr{M}$ possesses a faithful semifinite normal trace it is sufficient to show that $\mathscr{M}$ possesses a sufficient family of semifinite normal traces. We can assume that $\mathscr{M}$ is a uniform von Neumann algebra, i.e., there exists a family $\{e_i\}_{i \in I}$ of equivalent finite mutually orthogonal projections, such that $\sum_{i \in I} e_i = 1$ (see exercise E.4.14). Let e_0 be one of these projections and, for any $i \in I$, let $v_i \in \mathscr{M}$ be a partial isometry such that

$$v_i^* v_i = e_0, \quad v_i v_i^* = e_i.$$

We now define, for any $x \in \mathscr{M}$,

$$x_{ik} = v_i^* x v_k \in e_0 \mathscr{M} e_0, \quad i, k \in I.$$

Then we have

$$x = \sum_{i,k \in I} e_i x e_k = \sum_{i,k \in I} v_i x_{ik} v_k^* = (x_{ik}),$$

where the last equality is a notation. One says that x_{ik} is the (i, k)-component in the matrix representation with respect to the basis $\{v_i v_k^*\}_{i,k \in I}$. It is easy to see that

$$(x^*)_{ik} = x_{ki}^* \text{ and } (xy)_{ik} = \sum_{l \in I} x_{il} y_{lk}.$$

The von Neumann algebra $e_0 \mathscr{M} e_0$ is finite. For any finite normal trace μ_0 on $(e_0 \mathscr{M} e_0)^+$ we define a function on $\mathscr{M}^+$ by

$$\mu(x) = \sum_{i \in I} \mu_0(x_{ii}), \quad x \in \mathscr{M}^+.$$

It is easily verified that μ is a semifinite normal trace on $\mathscr{M}^+$ and that the set of all semifinite normal traces on $\mathscr{M}^+$, obtained in this manner, is sufficient.

Q.E.D.

7.16. A von Neumann algebra $\mathscr{M}$ is said to be *homogeneous of type* I_n if there exists a family $\{e_i\}_{i \in I}$ of equivalent abelian mutually orthogonal projections, such that $\sum_{i \in I} e_i = 1$, and card $I = n$. In this case $\mathscr{M}$ is of type I and $\mathscr{M}$ is finite iff n is finite. Conversely, for any finite von Neumann algebra, of type I, there exists a family $\{p_n\}_{n=1,2,\ldots}$ of mutually orthogonal central projections, such that $\sum_{n=1}^{\infty} p_n = 1$, uniquely determined by the condition that Mp_n be of type I_n, $n = 1, 2, \ldots$ (see exercise E.4.14).

Proposition. *Let $\mathscr{M}$ be a von Neumann algebra of type I_n, n finite. Then $(\mathscr{P}_{\mathscr{M}})^\natural$ coincides with the set of all elements of the form*

$$\sum_{k=1}^{n} \frac{k}{n} q_k,$$

where $q_1, \ldots, q_n$ are mutually orthogonal central projections.

Proof. Let $e_0 \in \mathscr{M}$ be an abelian projection, such that $z(e_0) = 1$ (see Proposition 4.19). Since $\mathscr{M}$ is homogeneous, of type I_n, there exists a family of n abelian, mutually orthogonal projections, equivalent to e_0, whose sum equals 1. Consequently, we have

$$(e_0 q)^\natural = \frac{1}{n} q,$$

for any central projection q.

Therefore, it is obvious that any operator of the form given in the statement of the theorem belongs to $(\mathscr{P}_{\mathscr{M}})^\natural$.

Let now $e \in \mathscr{P}_{\mathscr{M}}$ and let $\{e_1, \ldots, e_k\}$ be a maximal family of subprojections of e, which are mutually orthogonal and equivalent to $e_0 z(e)$. There then exists a non-zero central projection $q \leqslant z(e)$, such that

$$qe = \sum_{i=1}^{k} qe_i$$

(see Proposition 4.10). It follows that

$$(qe)^\natural = \frac{k}{n} q$$

Consequently, there exists a family $\{q_i\}_{i\in I}$ of mutually orthogonal central projections, such that $\sum_{i\in I} q_i = z(e)$, and, for any $i \in I$, there exists a natural $1 \leqslant k_i \leqslant n$, such that:

$$(q_i e)^\natural = \frac{k_i}{n} q_i, \quad i \in I.$$

We define

$$q_k = \sum_{i\in I,\, k_i = k} q_i, \quad k = 1, 2, \ldots, n.$$

Then we have

$$e^\natural = \sum_{k=1}^{n} \frac{k}{n} q_k.$$

Q.E.D.

7.17. In the case of a von Neumann algebra of type II_1, we have the following "Darboux property" for the restriction of the mapping $\natural$ to $\mathcal{P}_{\mathcal{M}}$.

Proposition. *Let $\mathcal{M}$ be a von Neumann algebra of type II_1. For any $e, f \in \mathcal{P}_{\mathcal{M}}$, and any $z \in \mathcal{Z}$, such that $e^\natural \leqslant z \leqslant f^\natural$, there exists a $g \in \mathcal{P}_{\mathcal{M}}$, such that $e \leqslant g \leqslant f$ and $g^\natural = z$.*

Corollary. *Let $\mathcal{M}$ be a von Neumann algebra of type II_1. Then $(\mathcal{P}_{\mathcal{M}})^\natural$ coincides with the set of all elements:*

$$z \in \mathcal{Z}, \quad 0 \leqslant z \leqslant 1.$$

Proof. We shall first show that for any $e \in \mathcal{P}_{\mathcal{M}}$, $e \neq 0$, and any $\varepsilon > 0$, there exists an $e_\varepsilon \in \mathcal{P}_{\mathcal{M}}$, $0 \neq e_\varepsilon \leqslant e$, such that $e_\varepsilon^\natural \leqslant \varepsilon z(e_\varepsilon)$. Indeed, by taking into account Proposition 4.11, for any $n = 1,2,\ldots$ we can find a family $\{e_1^n, \ldots, e_{2^n}^n\}$ of equivalent, mutually orthogonal, non-zero subprojections of e, whose sum is e. Then $(e_1^n)^\natural = \frac{1}{2^n} e^\natural$ and so, it is sufficient to choose n, such that $1/2^n \leqslant \varepsilon$.

Let now $\mathcal{E}$ be a maximal totally ordered family of projections h in $\mathcal{M}$, such that

$$e \leqslant h \leqslant f, \quad h^\natural \leqslant z.$$

We denote by g the l.u.b of the family $\mathcal{E}$. It is obvious that $e \leqslant g \leqslant f$ and $g^\natural \leqslant z$.

If $z - g^\natural \neq 0$, then there exists an $\varepsilon > 0$ and a central projection $p \neq 0$, such that

$$(z - g^\natural)p \geqslant \varepsilon p.$$

It follows that $(f - g)p \neq 0$, since, if this is not true, then $g^\natural p = f^\natural p \geqslant zp$, and this relation contradicts the preceding one. In accordance with the first part of the proof, there exists an $e_\varepsilon \in \mathcal{P}_{\mathcal{M}}$, $0 \neq e_\varepsilon \leqslant (f-g)p$, such that $e_\varepsilon^\natural \leqslant \varepsilon p$. On the other hand, the existence of the element $g + e_\varepsilon$ contradicts the maximality of the family $\mathcal{E}$.

Consequently, we have $z = g^\natural$.

Q.E.D.

7.18. If $\mathcal{M}$ is a finite von Neumann algebra whose commutant $\mathcal{M}'$ is finite, then there exists a remarkable connection between the canonical central traces on $\mathcal{M}$ and $\mathcal{M}'$. In order to establish the connection, the following lemma is necessary.

Lemma. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra of countable type. Then there exist $p, q \in \mathcal{P}^{\mathcal{Z}}$, $pq = 0$, $p + q = 1$, and $\xi, \eta \in \mathcal{H}$, such that*

$$p = p_\xi, \; q = p'_\eta.$$

Proof. Let $\{\xi_n\}_{n \in I} \subset \mathcal{H}$, $\|\xi_n\| = 1$, be a maximal family such that the projections p_{ξ_n}, $n \in I$, are mutually orthogonal and the projections p'_{ξ_n}, $n \in I$, are also mutually orthogonal. Since $\mathcal{M}$ is of countable type, it follows that I is at most countable. We denote $e = \sum_n p_{\xi_n}$, $e' = \sum_n p'_{\xi_n}$. If we define $\xi_0 = \sum_n \frac{1}{2^n} \xi_n$, it follows that $e = p_{\xi_0}$ and $e' = p'_{\xi_0}$.

On the other hand, from the maximality of the family $\{\xi_n\}$, we infer that
$$(1-e)(1-e')=0.$$
By taking into account Corollary 3.9, we obtain
$$\mathbf{z}(1-e)\mathbf{z}(1-e')=0.$$
Let us denote $p=1-\mathbf{z}(1-e)$, $q=\mathbf{z}(1-e)$. Then, from the relations
$$p=1-\mathbf{z}(1-e)\leqslant e$$
$$q=\mathbf{z}(1-e)\leqslant 1-\mathbf{z}(1-e')\leqslant e'$$
it follows that there exist $\xi,\eta\in\mathscr{H}$, such that
$$p=p_\xi,\quad q=p'_\eta.$$
Q.E.D.

7.19. Theorem *(of coupling). Let $\mathscr{M}\subset\mathscr{B}(\mathscr{H})$ be a finite von Neumann algebra, whose commutant $\mathscr{M}'\subset\mathscr{B}(\mathscr{H})$ is also finite. Let $\natural$, $\natural'$ be the canonical central traces on $\mathscr{M}$, $\mathscr{M}'$, respectively. Then, for any $\xi,\eta\in\mathscr{H}$, we have*
$$(p_\xi)^\natural(p'_\eta)^{\natural'}=(p_\eta)^\natural(p'_\xi)^{\natural'}.$$

Proof. If $\{q_\iota\}_{\iota\in I}$ is a family of mutually orthogonal central projections, such that $\sum_{\iota\in I}q_\iota=1$, then it is sufficient to prove the result in the statement of the theorem for each of the von Neumann algebras $\mathscr{M}q_\iota\subset\mathscr{B}(q_\iota\mathscr{H})$.

In accordance with Theorem 4.17, and with exercise E.4.14, we can assume that $\mathscr{M}$ and $\mathscr{M}'$ are either of type II_1, or of homogeneous type $\mathrm{I}_{\mathrm{fin}}$.

By Lemma 7.2. we may assume that $\mathscr{M}$ is of countable type.

Since $\mathscr{M}$ is of countable type, Lemma 7.18 can be applied. Consequently, without any loss of generality, we can assume that there exists a vector $\xi_0\in\mathscr{H}$, such that
$$p_{\xi_0}=1.$$

In this case, we shall show that, for any $\xi\in\mathscr{H}$, we have
$$(p'_\xi)^{\natural'}=(p'_{\xi_0})^{\natural'}(p_\xi)^\natural$$
and the theorem will be proved.

Since $\mathbf{s}((p'_{\xi_0})^{\natural'})=\mathbf{z}(p'_{\xi_0})=\mathbf{z}(p_{\xi_0})=1$, from Corollary 2.22 we infer that there exists a sequence $\{q_n\}$ of central projections, $q_n\uparrow 1$, such that $(p'_{\xi_0})^{\natural'}q_n\geqslant\dfrac{1}{n}q_n$, for any n. Without any loss of generality, we can assume that there exists an $\varepsilon>0$, such that
$$(p'_{\xi_0})^{\natural'}\geqslant\varepsilon.$$

We denote by $\mathscr{C}$ (resp. by $\mathscr{C}'$) the set of all elements $z\in\mathscr{Z}$, such that there exists a $\xi\in\mathscr{H}$, for which $z=(p_\xi)^\natural$(resp., $z=(p'_\xi)^{\natural'}$). Since any projection which is dominated by a cyclic projection is cyclic and since any cyclic projection in $\mathscr{M}'$

is dominated by p'_{ξ_0}(since $p_{\xi_0} = 1$ and as a result of Lemma 6.1), from 7.12 we infer that

$$\mathscr{C} = (\mathscr{P}_{\mathscr{M}})^{\natural} \text{ and } \mathscr{C}' = \{z \in (\mathscr{P}_{\mathscr{M}'})^{\natural'};\ z \leqslant (p'_{\xi_0})^{\natural'}\}.$$

By taking into account Propositions 7.16 and 7.17, the structure of the sets $\mathscr{C}$ and $\mathscr{C}'$ becomes evident in the case of the type II_1, as well as in the case of the homogeneous type $\mathrm{I}_{\mathrm{fin}}$.

We now define a mapping Ψ_0: $\mathscr{C} \to \mathscr{C}'$ by the relations

$$\Psi_0((p_\xi)^{\natural}) = (p'_\xi)^{\natural'}, \quad \xi \in \mathscr{H}.$$

By virtue of Lemma 6.1 and of Section 7.12, this mapping is correctly defined. injective, surjective and both it, as well as its inverse, preserve the order relation, Since $z(p'_\xi) = z(p_\xi)$, it follows that $s(\Psi_0(z)) = s(z)$, $z \in \mathscr{C}$.

In what follows we shall need to note: if $e \in \mathscr{P}_{\mathscr{M}}$, $e' \in \mathscr{P}_{\mathscr{M}'}$, and $\Psi_0(e^{\natural}) = (e')^{\natural'}$, then there exists a $\xi \in \mathscr{H}$, such that $e = p_\xi$, $e' = p'_\xi$. Indeed, there exists an $\eta \in \mathscr{H}$, such that $e = p_\eta$; hence, $(p'_\eta)^{\natural'} = \Psi_0((p_\eta)^{\natural}) = \Psi_0(e^{\natural}) = (e')^{\natural'}$. According to 7.12, we infer that $e' \sim p'_\eta$, and now the desired result can be obtained by applying exercise E.6.1.

If z_1, z_2, $z_1 + z_2 \in \mathscr{C}$, and $\Psi_0(z_1) + \Psi_0(z_2) \in \mathscr{C}'$, then

$$\Psi_0(z_1 + z_2) = \Psi_0(z_1) + \Psi_0(z_2).$$

Indeed, there exist $e_1, e_2 \in \mathscr{P}_{\mathscr{M}}$, $e'_1, e'_2 \in \mathscr{P}_{\mathscr{M}'}$, such that

$$\begin{aligned} e_1 e_2 &= 0, \quad e'_1 e'_2 = 0, \\ z_1 &= (e_1)^{\natural}, \quad \Psi_0(z_1) = (e'_1)^{\natural'}, \\ z_2 &= (e_2)^{\natural}, \quad \Psi_0(z_2) = (e'_2)^{\natural'}. \end{aligned}$$

In view of the preceding remark, there exist ξ_1, $\xi_2 \in \mathscr{H}$, such that

$$\begin{aligned} e_1 &= p_{\xi_1}, \quad e'_1 = p'_{\xi_1}, \\ e_2 &= p_{\xi_2}, \quad e'_2 = p'_{\xi_2}. \end{aligned}$$

It is now easy to see that, if we denote $\xi = \xi_1 + \xi_2$, we have

$$e_1 + e_2 = p_\xi, \quad e'_1 + e'_2 = p'_\xi.$$

Consequently, we have

$$\begin{aligned} \Psi_0(z_1 + z_2) &= \Psi_0((e_1)^{\natural} + (e_2)^{\natural}) = \Psi_0((e_1 + e_2)^{\natural}) \\ &= \Psi_0((p_\xi)^{\natural}) = (p'_\xi)^{\natural'} = (e'_1 + e'_2)^{\natural'} \\ &= (e'_1)^{\natural'} + (e'_2)^{\natural'} = \Psi_0(z_1) + \Psi_0(z_2). \end{aligned}$$

If $\lambda \geqslant 0$, $z \in \mathscr{C}$, $\lambda z \in \mathscr{C}$ and $\lambda \Psi_0(z) \in \mathscr{C}'$, then

$$\Psi_0(\lambda z) = \lambda \Psi_0(z).$$

Indeed, for rational λ, the result can be obtained from the additivity property we have just proved. Then, for an arbitrary $\lambda > 0$, the desired result can be obtained by taking into account the monotony property of the mapping Ψ_0.

Let $\mathscr{L}$ (resp., $\mathscr{L}'$) be the vector space generated in $\mathscr{Z}$ by $\mathscr{C}$ (resp. by $\mathscr{C}'$). In view of the additivity and homogeneity properties of the mapping Ψ_0, it follows that there exists a unique linear mapping

$$\Psi: \mathscr{L} \to \mathscr{L}'$$

which extends Ψ_0. We note that $s(\Psi(z)) \leqslant s(z)$, $z \in \mathscr{L}$.

Since $\Psi(1) = (p'_{\xi_0})^{\natural'} \geqslant \varepsilon$, it follows that $\Psi(1)$ is invertible. We now define the mapping

$$\Phi: \mathscr{L} \to \mathscr{Z}$$

by the relations

$$\Phi(z) = \frac{1}{\Psi(1)} \Psi(z), \quad z \in \mathscr{L}.$$

In the case of the homogeneous type I_{fin}, the set $\mathscr{L}$ consists of all linear combinations with rational coefficients, which have the same denominator, of central projections, and Φ is a linear mapping such that $\Phi(1) = 1$. In the case of the type II_1, we have $\mathscr{L}=\mathscr{Z}$, and Φ is positive linear mapping of $\mathscr{Z}$ into $\mathscr{Z}$, such that $\Phi(1)= 1$; it is easily verified that Φ is bounded ($0 \leqslant z \leqslant 1 \Rightarrow 0 \leqslant \Phi(z) \leqslant 1$, whence $\|\Phi\| \leqslant 4$). In each case, we have $s(\Phi(z)) \leqslant s(z)$.

Let q be a central projection. Since $1 = q + (1 - q)$, it follows that

$$1 = \Phi(q) + \Phi(1 - q).$$

If we multiply this relation by q we get

$$\Phi(q) = q.$$

Consequently, Φ is the identity mapping. Then, for any vector $\xi \in \mathscr{H}$, we have

$$(p'_\xi)^{\natural'} = \Psi((p_\xi)^\natural) = \Psi(1)\Phi((p_\xi)^\natural) = (p'_{\xi_0})^{\natural'}(p_\xi)^\natural.$$

Q.E.D.

7.20. Let $\mathfrak{S}$ be the set of all families $\{(z_\iota, q_\iota)\}_{\iota \in I}$, where q_ι are mutually orthogonal central projections, such that $\sum_{\iota \in I} q_\iota = 1$, $z_\iota \in \mathscr{Z}$, and $s(|z_\iota|) \leqslant q_\iota$, $\iota \in I$. We shall say that the elements $\{(z_\iota, q_\iota)\}_{\iota \in I}$ and $\{(z'_k, q'_k)\}_{k \in K}$ are equivalent if for any $\iota \in I$ and $k \in K$ we have $z_\iota q'_k = z'_k q_\iota$ (they "coincide on intersections"). We shall denote by $\tilde{\mathscr{Z}}$ the quotient set of $\mathfrak{S}$ by the preceding equivalence relation, and by $\{(z_\iota, q_\iota)\}\tilde{}_{\iota \in I}$ the equivalence class of $\{(z_\iota, q_\iota)\}_{\iota \in I}$.

We shall define the following operations on $\tilde{\mathscr{Z}}$:

$$\{(z_\iota, q_\iota)\}\tilde{}_{\iota \in I} + \{(z'_k, q'_k)\}\tilde{}_{k \in K} = \{(z_\iota q'_k + z'_k q_\iota, q_\iota q'_k)\}\tilde{}_{(\iota, k) \in I \times K}$$

$$\lambda\{(z_\iota, q_\iota)\}\tilde{}_{\iota \in I} = \{(\lambda z_\iota, q_\iota)\}\tilde{}_{\iota \in I}.$$

$$\{(z_\iota, q_\iota)\}\tilde{}_{\iota \in I} \{(z'_k, q'_k)\}\tilde{}_{k \in K} = \{(z_\iota z'_k, q_\iota q'_k)\}\tilde{}_{(\iota, k) \in I \times K}$$

$$(\{(z_\iota, q_\iota)\}\tilde{}_{\iota \in I})^* = \{(z_\iota^*, q_\iota)\}\tilde{}_{\iota \in I}.$$

It is easily verified that $\widetilde{\mathscr{Z}}$, endowed with the operations already defined, is an involutive algebra with the unit element. Thus, the notions of a positive element and of an invertible element make sense.

The mapping $z \mapsto \{(z, 1)\}^{\sim}$ is an injective $*$-homomorphism of $\mathscr{Z}$ into $\widetilde{\mathscr{Z}}$, hence $\widetilde{\mathscr{Z}}$ is an extension of $\mathscr{Z}$.

From the coupling theorem (7.19), the following corollary easily follows.

Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a finite von Neumann algebra, whose commutant $\mathscr{M}' \subset \mathscr{B}(\mathscr{H})$ is also finite. Then there exists an element $c_{\mathscr{M},\mathscr{M}'} \in \widetilde{\mathscr{Z}}$, which is positive and invertible, such that for any $\xi \in \mathscr{H}$ we have*

$$(p'_\xi)^{\natural'} = c_{\mathscr{M},\mathscr{M}'}(p_\xi)^{\natural}.$$

The element $c_{\mathscr{M},\mathscr{M}'}$ is uniquely determined by this condition.

If $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a finite factor, whose commutant $\mathscr{M}' \subset \mathscr{B}(\mathscr{H})$ is also finite, then $\mathscr{Z}$ and $\widetilde{\mathscr{Z}}$ coincide with the field $\mathbb{C}$ of the scalars, hence $c_{\mathscr{M},\mathscr{M}'}$ is a non-zero positive number.

The element $c_{\mathscr{M},\mathscr{M}'}$ is called the *coupling element* or the *coupling function;* in the case of factors, it is called the *coupling constant.*

Obviously, we have

$$c_{\mathscr{M}',\mathscr{M}} = (c_{\mathscr{M},\mathscr{M}'})^{-1}.$$

If p is a central projection, then the center of $\mathscr{M}p$ obviously identifies with $\mathscr{Z}p$. If we assume that this identification is already performed, we have

$$c_{\mathscr{M}p,\mathscr{M}'p} = (c_{\mathscr{M},\mathscr{M}'})\,p.$$

7.21. Let $\mathscr{M}$ be a finite von Neumann algebra, whose commutant $\mathscr{M}'$ is also finite, and let $\mathscr{Z}$ be their center. Let $e' \in \mathscr{M}'$ be a projection whose central support $z(e') = 1$.

In accordance with Proposition 3.14, the canonical induction $\mathscr{M} \to \mathscr{M}_{e'}$, is a $*$-isomorphism, and this fact allows for the canonical identification of the common center of the algebras $\mathscr{M}_{e'}$, and $\mathscr{M}'_{e'}$, with $\mathscr{Z}$. This identification induces a canonical identification of the extension of the common center of the algebras $\mathscr{M}_{e}$, $\mathscr{M}'_{e'}$, with $\widetilde{\mathscr{Z}}$ (see 7.20). We shall assume that these identifications have been performed.

Let now $\natural$, $\natural'$, $\natural_{e'}$, $\natural'_{e'}$ be the canonical central traces on $\mathscr{M}$, $\mathscr{M}'$, $\mathscr{M}_{e'}$, $\mathscr{M}'_{e'}$. Then for any $x \in \mathscr{M}$, we have

$$(x_{e'})^{\natural_{e'}} = x^{\natural}$$

and, for any $x' \in \mathscr{M}'$,

$$(x'_{e'})^{\natural'_{e'}} = ((e')^{\natural'})^{-1}(e'x'e')^{\natural'}.$$

The first relation is obvious. For the second, we first remark that, since $z(e') = 1$, $(e')^{\natural'}$ is an invertible element in $\widetilde{\mathscr{Z}}$ (see 2.22). Then, one sees that, for any $x' \in \mathscr{M}'$, we have

$$((e')^{\natural'})^{-1}(e'x'e')^{\natural'} \in \mathscr{Z}.$$

Finally, the mapping

$$\mathcal{M}'_{e'} \ni x'_{e'} \mapsto ((e')^{\natural'})^{-1}(e'x'e')^{\natural'} \in \mathcal{Z}$$

satisfies the conditions (i), (ii), (iii) from Theorem 7.11, and therefore, it coincides with the canonical central trace on $\mathcal{M}'_{e'}$.

With these preparations, we can now state the following

Proposition. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra, whose commutant $\mathcal{M}' \subset \mathcal{B}(\mathcal{H})$ is finite. If $e' \in \mathcal{M}'$ is a projection whose central support $z(e') = 1$, then*

$$c_{\mathcal{M}_{e'}, \mathcal{M}'_{e'}} = ((e')^{\natural'})^{-1} c_{\mathcal{M}, \mathcal{M}'}.$$

Proof. Let $\xi \in e'\mathcal{H} \subset \mathcal{H}$ and let us consider the projections

$$p_\xi = [\mathcal{M}'\xi] \in \mathcal{M} \quad , \; p'_\xi = [\mathcal{M}\xi] \in \mathcal{M}',$$

$$q_\xi = [M'_{e'}\xi] \in \mathcal{M}_{e'}, \; q'_\xi = [\mathcal{M}_{e'}\xi] \in \mathcal{M}'_{e'}.$$

Then

$$q_\xi = (p_\xi)_{e'} \text{ and } q'_\xi = (p'_\xi)_{e'}.$$

Consequently, we have

$$(q'_\xi)^{\natural'_{e'}} = ((p'_\xi)_{e'})^{\natural'_{e'}} = ((e')^{\natural'})^{-1}(p'_\xi)^{\natural'} = ((e')^{\natural'})^{-1} c_{\mathcal{M}, \mathcal{M}'}(p_\xi)^{\natural}$$

$$= ((e')^{\natural'})^{-1} c_{\mathcal{M}, \mathcal{M}'}((p_\xi)_{e'})^{\natural_{e'}} = ((e')^{\natural'})^{-1} c_{\mathcal{M}, \mathcal{M}'}(q_\xi)^{\natural_{e'}}.$$

The uniqueness of the coupling element now implies the formula in the statement of the proposition.

Q.E.D.

7.22. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra whose commutant $\mathcal{M}' \subset \mathcal{B}(\mathcal{H})$ is also finite, and let n be a natural number. Let $\mathcal{H}_n$ be a Hilbert space of dimension n. We now consider the von Neumann algebras

$$\widetilde{\mathcal{M}}_n = \mathcal{M} \overline{\otimes} \mathcal{C}(\mathcal{H}_n) \subset \mathcal{B}(\mathcal{H} \overline{\otimes} \mathcal{H}_n), \quad \widetilde{\mathcal{M}}'_n = \mathcal{M}' \overline{\otimes} \mathcal{B}(\mathcal{H}_n) \subset \mathcal{B}(\mathcal{H} \overline{\otimes} \mathcal{H}_n).$$

Then $\widetilde{\mathcal{M}}_n$ and $\widetilde{\mathcal{M}}'_n$ are finite and there exists a projection $e'_n \in \widetilde{\mathcal{M}}'_n$, $z(e'_n) = 1$, such that the canonical induction $\widetilde{\mathcal{M}}_n \to (\widetilde{\mathcal{M}}_n)_{e'_n}$ be the inverse of the canonical amplification $\mathcal{M} \to \widetilde{\mathcal{M}}_n$ (see 3.18 and E.4.20).

From Proposition 7.12 we deduce the following

Corollary. $c_{\widetilde{\mathcal{M}}_n, \widetilde{\mathcal{M}}'_n} = n c_{\mathcal{M}, \mathcal{M}'}$.

7.23. Finally we present a topological criterion of finiteness:

Proposition. *A von Neumann algebra $\mathcal{M}$ is finite iff the $*$-operation is s-continuous on the closed unit ball of $\mathcal{M}$.*

Proof. Let us first assume that $\mathcal{M}$ is finite. We shall show that the set

$$\{L_a\varphi;\ \varphi \text{ a finite normal trace on } \mathcal{M},\ a \in \mathcal{M}\}$$

is total in $\mathcal{M}_*$. Indeed, if $x \in \mathcal{M}$, and $\varphi(ax) = 0$ for any finite normal trace φ on $\mathcal{M}$, and any $a \in \mathcal{M}$, then $\varphi(x^*x) = 0$, for any finite normal trace φ on $\mathcal{M}$; hence, in view of Corollary 7.14, we have $x = 0$. Since $(\mathcal{M}_*)^* = \mathcal{M}$, the assertion is now obvious.

Let $\{x_i\}$ be a net in the closed unit ball of $\mathcal{M}$, which is s-convergent to 0. Then, for any normal trace φ on $\mathcal{M}$, and any $a \in \mathcal{M}$, we have

$$|\varphi(ax_ix_i^*)| = |\varphi(x_i^*ax_i)| \leqslant \varphi(x_i^*aa^*x_i)^{1/2}\varphi(x_i^*x_i)^{1/2} \leqslant \|a\|\ \varphi(x_i^*x_i) \to 0.$$

By taking into account the first part of the proof, it follows that the net $\{x_i^*\}$ is s-convergent to 0.

Consequently, the $*$-operation is s-continuous on the closed unit ball of $\mathcal{M}$.

Conversely, let us assume that $\mathcal{M}$ is not finite. According to Proposition 4.12, there exists a sequence $\{e_n\}$ of equivalent mutually orthogonal, non-zero projections in $\mathcal{M}$. Let $\{v_n\}$ be a sequence of partial isometries in $\mathcal{M}$, such that

$$v_n^*v_n = e_n,\ v_nv_n^* = e_1, \quad n = 1, 2, \ldots$$

Then it is easy to see that

$$v_n \xrightarrow{s} 0, \text{ but } v_n^* \overset{s}{\not\to} 0$$

and this shows that the $*$-operation is not s-continuous on the closed unit ball of $\mathcal{M}$.

Q.E.D.

Exercises

E.7.1. A von Neumann algebra is finite iff

$$x, y \in \mathcal{M}, \quad xy = 1 \Rightarrow yx = 1.$$

E.7.2. Let $\mathcal{M}$ be a von Neumann algebra and $x, y \in \mathcal{M}$. If $\mathbf{n}(xy)$ is a finite projection, then

$$\mathbf{n}(xy) - \mathbf{n}(y) \sim \mathbf{n}(x) \wedge \mathbf{l}(y).$$

E.7.3. Let $\mathcal{M}$ be a von Neumann algebra, $\mathcal{Z}$ its center, and let $d: \mathcal{P}_{\mathcal{M}} \to \mathcal{Z}^+$ be a mapping having the properties

(i) $d(e+f) = d(e) + d(f)$, for any $e, f \in \mathcal{P}_{\mathcal{M}}$, $ef = 0$;
(ii) $d(e) = d(f)$, for any $e, f \in \mathcal{P}_{\mathcal{M}}$, $e \sim f$;
(iii) $d(1) = 1$;
(iv) $d(pe) = p\, d(e)$, for any $e \in \mathcal{P}_{\mathcal{M}}$, $p \in \mathcal{P}_{\mathcal{Z}}$;
(v) $d(e) = 0 \Rightarrow e = 0$, for any $e \in \mathcal{P}_{\mathcal{M}}$.

Show that $\mathscr{M}$ is finite and, for any $e \in \mathscr{P}_{\mathscr{M}}$ we have

$$d(e) = e^{\natural}.$$

E.7.4. Let $\mathscr{M}$ be a finite von Neumann algebra and $\mathscr{Z}$ its center. The following assertions are equivalent

(i) $\mathscr{M}$ is of countable type,

(ii) $\mathscr{Z}$ is of countable type,

(iii) there exists a faithful finite normal trace on $\mathscr{M}^+$.

E.7.5. Let $\{\xi_\iota\}_{\iota \in I}$ be an orthonormal basis in $\mathscr{H}$. For any $x \in \mathscr{B}(\mathscr{H})$, $x \geqslant 0$, the number

$$tr(x) = \sum_{\iota \in I} (x\xi_\iota | \xi_\iota) = \sum_{\iota \in I} \|x^{1/2}\xi_\iota\|^2$$

does not depend on the chosen basis in $\mathscr{H}$, and the mapping $x \mapsto tr(x)$ is a faithful semifinite normal trace on $\mathscr{B}(\mathscr{H})^+$; it is called the *canonical trace* on $\mathscr{B}(\mathscr{H})^+$.

E.7.6. Let us denote

$$\mathscr{T}r(\mathscr{H}) = \{a \in \mathscr{B}(\mathscr{H});\ \ tr(|a|) < +\infty\},$$

$$\|a\|_{tr} = tr(|a|), \quad a \in \mathscr{T}r(\mathscr{H}).$$

Then $\mathscr{T}r(\mathscr{H})$ is a Banach space for the norm $\|\cdot\|_{tr}$, and tr extends by linearity to a bounded linear form on the Banach space $\mathscr{T}r(\mathscr{H})$, which is also denoted by tr.

***E.7.7.** The set $\mathscr{T}r(\mathscr{H})$ is a two-sided ideal in $\mathscr{B}(\mathscr{H})$, contained in the closed two-sided ideal $\mathscr{K}(\mathscr{H})$ of all the compact operators on $\mathscr{H}$. Thus, any element $a \in \mathscr{T}r(\mathscr{H})$ determines a form φ_a on $\mathscr{B}(\mathscr{H})$, given by the formula

$$\varphi_a(x) = tr(ax) = tr(xa), \ x \in \mathscr{B}(\mathscr{H}).$$

Show that the mapping

$$a \mapsto \varphi_a|_{\mathscr{K}(\mathscr{H})}$$

is an isometric isomorphism of the Banach space $\mathscr{T}r(\mathscr{H})$ onto the Banach space $\mathscr{K}(\mathscr{H})^*$, whereas the mapping

$$a \mapsto \varphi_a$$

is an isometric isomorphism of the Banach space $\mathscr{T}r(\mathscr{H})$ onto the Banach space $\mathscr{B}(\mathscr{H})_*$.

In particular, $\mathscr{B}(\mathscr{H})$ identifies canonically with $\mathscr{K}(\mathscr{H})^{**}$.

E.7.8. With the help of E.7.7, show that for any normal form φ on $\mathscr{B}(\mathscr{H})$, there exists an orthogonal sequence $\{\xi_n\} \subset \mathscr{H}$, $\sum_{n=1}^{\infty} \|\xi_n\|^2 < +\infty$, such that

$$\varphi(x) = \sum_{n=1}^{\infty} (x\xi_n | \xi_n), \quad x \in \mathscr{B}(\mathscr{H}).$$

With the help of a polar decomposition theorem, infer then that, for any w-continuous form φ on $\mathscr{B}(\mathscr{H})$, there exist orthogonal sequences $\{\xi_n\}$, $\{\eta_n\} \subset \mathscr{H}$, $\sum_{n=1}^{\infty} \|\xi_n\|^2 < +\infty$, $\sum_{n=1}^{\infty} \|\eta_n\|^2 < +\infty$, such that

$$\varphi(x) = \sum_{n=1}^{\infty} (x\xi_n | \eta_n), \quad x \in \mathscr{B}(\mathscr{H}),$$

$$\|\varphi\| = \sum_{n=1}^{\infty} \|\xi_n\| \, \|\eta_n\|.$$

E.7.9. Let $\mathscr{M}$ be a semifinite von Neumann algebra. Then $\mathscr{M}$ is continuous iff there exists a decreasing sequence $\{e_n\}$ of finite projections in $\mathscr{M}$, such that, for any n,

$$\mathrm{z}(e_n) = 1,$$

$$e_n - e_{n+1} \sim e_{n+1}.$$

E.7.10. Show that a von Neumann algebra $\mathscr{M}$ is properly infinite (resp., of type III) iff any finite (resp., semifinite) normal trace on $\mathscr{M}^+$ is identically zero.

Infer from this result that if $\mathscr{M}$ is a von Neumann algebra of type III, and if μ is a non-zero normal trace on $\mathscr{M}^+$, then $\mu(x) = +\infty$, for any $x \in \mathscr{M}^+$, $x \neq 0$.

E.7.11. Let $\mathscr{M}$ be a von Neumann algebra and μ a normal trace on $\mathscr{M}^+$. For any family $\{e_\iota\}_{\iota \in I} \subset \mathscr{M}$ of mutually orthogonal projections, such that $\sum_{\iota \in I} e_\iota = 1$, we have

$$\mu(x) = \sum_{\iota \in I} \mu(e_\iota x e_\iota), \quad x \in \mathscr{M}^+.$$

Infer from this result that for any semifinite normal trace μ on $\mathscr{M}^+$, there exists a family $\{\varphi_\iota\}_{\iota \in I}$ of normal forms on $\mathscr{M}$, whose supports $\mathrm{s}(\varphi_\iota)$ are mutually orthogonal, and such that

$$\mu(x) = \sum_{\iota \in I} \varphi_\iota(x), \quad x \in \mathscr{M}^+.$$

E.7.12. Let $\mathscr{M}$ be a finite von Neumann algebra with center $\mathscr{Z}$ and μ a normal semifinite trace on $\mathscr{M}^+$. Then

$$\mu(x^\natural) \leqslant \mu(x), \quad x \in \mathscr{M}^+.$$

Infer that $\mu | \mathscr{Z}^+$ is a normal semifinite trace on $\mathscr{Z}^+$, there exists a family $\{\mu_\iota\}_{\iota \in I}$ of finite normal traces on $\mathscr{M}$ with mutually orthogonal supports, such that

$$\mu(x) = \sum_{\iota \in I} \mu_\iota(x), \quad x \in \mathscr{M}^+,$$

and hence

$$\mu(x^\natural) = \mu(x), \quad x \in \mathscr{M}^+.$$

E.7.13. Let $\mathscr{M}$ be a semifinite von Neumann algebra of a countable type with center $\mathscr{Z}$, μ a normal semifinite faithful trace on $\mathscr{M}^+$ and $e, f \in \mathscr{P}_{\mathscr{M}}$. Show that:

(1) $e \prec f \Leftrightarrow \mu(ep) \leqslant \mu(fp)$ for all $p \in \mathscr{P}_{\mathscr{Z}}$;

(2) e is finite $\Leftrightarrow$ for every $0 \neq p \in \mathscr{P}_{\mathscr{Z}}$ there exists $0 \neq q \in \mathscr{P}_{\mathscr{Z}}$, $q \leqslant p$, such that $\mu(qe) < +\infty \Leftrightarrow$ there exists a sufficient family $\{\mu_\iota\}_{\iota \in I}$ of normal semifinite traces on $\mathscr{M}^+$ such that $\mu_\iota(e) < +\infty$, $\iota \in I$.

E.7.14. Show that any two normal traces μ, ν on a factor $\mathscr{M}$ are proportional. Extend this result for two normal semifinite faithful traces on a von Neumann algebra. (Hint: if $\mathscr{M}$ is a finite factor, then $\mu(x) = \alpha x^\natural$, $x \in \mathscr{M}$, for some $\alpha \in [0, +\infty]$. The required statement for the general case is given as a consequence of Corollary 1 in C.10.4; for the proof here use 5.21 and 7.11.)

!E.7.15. One says that a von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is *standard* if there exists a conjugation $J: \mathscr{H} \to \mathscr{H}$, such that the mapping $x \mapsto Jx^*J$ be a $*$-anti-isomorphism of $\mathscr{M}$ onto $\mathscr{M}'$, which acts identically on the center.

Show that any standard von Neumann algebra of countable type has a separating cyclic vector.

!E.7.16. Any maximal abelian von Neumann algebra is standard. Consequently, any abelian von Neumann algebra is $*$-isomorphic to a standard von Neumann algebra.

!E.7.17. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a finite von Neumann algebra of countable type. The following assertions are equivalent:

(i) $\mathscr{M}$ is standard,

(ii) $\mathscr{M}$ has a separating cyclic trace vector,

(iii) $\mathscr{M}'$ is finite and $c_{\mathscr{M},\mathscr{M}'} = 1$.

!E.7.18. Any finite von Neumann algebra is $*$-isomorphic to a standard von Neumann algebra. (Hint: If $\mathscr{M}$ is of countable type, apply the construction given in Section 5.18, to a faithful w-continuous central positive form).

E.7.19. Any semifinite von Neumann algebra is $*$-isomorphic to a standard von Neumann algebra.

E.7.20. With the help of E.7.17, give a simpler proof to Lemma 6.3.

E.7.21. Let $\mathscr{M}$ be a homogeneous von Neumann algebra of type I_m, whose commutant is homogeneous, and of type I_n, where m and n are natural numbers. Show that

$$c_{\mathscr{M},\mathscr{M}'} = m/n.$$

E.7.22. Let $\mathcal{M}$ be a von Neumann algebra and $e \in \mathcal{P}_{\mathcal{M}}$. The following assertions are equivalent:

(i) e is the l.u.b. of a finite family of minimal projections in $\mathcal{M}$,

(ii) the mapping

$$\mathcal{M} \ni x \mapsto exe \in e\mathcal{M}e$$

is continuous for the w-topology in $\mathcal{M}$ and the uniform (norm) topology in $e\mathcal{M}e$.

***E.7.23.** Let $\mathcal{M}$ be a finite von Neumann algebra and $\mathcal{G} \subset \mathcal{M}$ a multiplicative group of invertible elements such that $\sup\{\|g\|;\ g \in \mathcal{G}\} < +\infty$. With the help of the Ryll-Nardzewski fixed point theorem, show that there exists an invertible element $a \in \mathcal{M}^+$, such that for any $g \in \mathcal{G}$ the element aga^{-1} be unitary.

Comments

C.7.1. Let $\mathcal{M}$ be a finite von Neumann algebra and $x \in \mathcal{M}$. With the notations from C.4.4, it is easy to see that

$$z \in \mathcal{C}(x) \Rightarrow z = x^{\natural}.$$

According to Theorem 1 from C.4.4 (or by Theorem 7.11. (ix)), it follows that

$$(*) \qquad \mathcal{K}(x) = \mathcal{C}(x) = \{x^{\natural}\}.$$

The first proof of Theorem 7.11 was given by J. Dixmier [12] who extended the arguments of F. J. Murray and J. von Neumann, used by them for the case of factors. The culminating point of J. Dixmier's proof consists in showing that the set $\mathcal{K}(x)$ reduces to a single element. Although this proof is much longer than that given above, we consider it to be very illuminating (see J. Dixmier [26], Ch. III, § 8).

The shortening of the classical proof of Theorem 7.11 has been an open problem for a long time (see, for example, R. V. Kadison [7], [20]). The proof given above has recently been obtained by F. J. Yeadon [1].

A description of the sets $\mathcal{K}(x)$ and $\mathcal{C}(x)$, analogous to that given by the relation $(*)$, for properly infinite von Neumann algebras, has been obtained by H. Halpern [13] and Ş. Strătilă and L. Zsidó (see also Ş. Strătilă [2]).

C.7.2. Some phenomena which take place in the lattice of all the projections in a von Neumann algebra also appear in the abstract frame of lattice theory (see, for example, J. von Neumann [8], S. Maeda [3], F. Maeda and S. Maeda [1], L. H. Loomis [2], L. A. Skorniakov [1]). We also mention the remarkable result contained in the title of I. Kaplansky's paper [21].

C.7.3. Let $\mathcal{M}$ be a von Neumann algebra and μ a trace on $\mathcal{M}^+$. Then the set

$$\{x \in \mathcal{M}^+;\ \mu(x) < +\infty\}$$

is the positive part of a two-sided ideal $\mathfrak{M}_\mu$ of $\mathcal{M}$ and there exists a unique linear form on $\mathfrak{M}_\mu$, which coincides with μ on $\mathfrak{M}_\mu^+$. This linear form will be denoted also by μ.

One can show that for any $a \in \mathfrak{M}_\mu$, one has

$$\mu(ax) = \mu(xa), \quad x \in \mathcal{M},$$

and, if μ is normal, then the linear form

$$\mathcal{M} \ni x \mapsto \mu(ax)$$

is w-continuous.

Obviously, μ is finite iff $\mathfrak{M}_\mu = \mathcal{M}$. If μ is normal, then μ is semifinite iff $\mathfrak{M}_\mu$ is w-dense in $\mathcal{M}$.

The proofs of these results can easily be obtained by using the results in Section 3.21 (see also 10.14 and J. Dixmier [26], Ch. I, § 6).

C.7.4. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. According to the types of the algebras $\mathcal{M}$ and $\mathcal{N}$, the type of the tensor product $\mathcal{M} \overline{\otimes} \mathcal{N}$ is completely described by the following table

$\mathcal{M}$	$\mathcal{N}$	$\mathcal{M} \overline{\otimes} \mathcal{N}$
type I_m	type I_n	type I_{mn}
type I	type I	type I
finite	finite	finite
semifinite	semifinite	semifinite
continuous	arbitrary	continuous
properly infinite	arbitrary	properly infinite
type III	arbitrary	type III

For the proofs we refer the reader to J. Dixmier [26], Ch. III, § 8.7, and S. Sakai [32], 2.6. We mention the fact that only the last implication in this table offers some difficulties. This implication has been proved by S. Sakai [6], who, to this end, also obtained the topological criterion of finiteness (7.23).

C.7.5. Bibliographical comments. The results in this section are essentially due to F. J. Murray and J. von Neumann, who proved them in the case of factors. The globalization of these results to arbitrary von Neumann algebras was begun by J. Dixmier [12] and I. Kaplansky [10], and continued by H. A. Dye, E. L. Griffin, R. V. Kadison, R. Pallu de la Barrière, and others. We mention the fact that Lemma 7.18 is due to E. L. Griffin [2].

In writing this chapter we referred to I. Kaplansky [22], J. R. Ringrose [5], R. V. Kadison [14] and J. Dixmier [26].

8

Spatial isomorphisms and relations between topologies

In this chapter we consider the cases in which a $*$-isomorphism between von Neumann algebras is implemented by a unitary operator, as well as the cases in which the w-topology coincides with the wo-topology.

8.1. Let $\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1)$, $\mathcal{M}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be von Neumann algebras. One says that a $*$-isomorphism $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is a *spatial* isomorphism (or that it is *unitarily implemented*) if there exists a unitary operator $u : \mathcal{H}_1 \to \mathcal{H}_2$, such that

$$\pi(x_1) = u \circ x_1 \circ u^*, \ x_1 \in \mathcal{M}_1.$$

While any $*$-isomorphism is w-continuous (5.13), but it is not necessarily wo-continuous, it is obvious that any spatial isomorphism is wo-continuous. This is one of the main properties which distinguish the two kinds of isomorphisms.

The problem which we consider in this chapter is to find some sufficiently simple and general conditions under which a given $*$-isomorphism is spatial.

For example, according to Corollary 5.25, *any $*$-isomorphism between two von Neumann algebras with vectors which are both cyclic and separating, is spatial.*

8.2. In what follows we shall essentially use the following simple proposition, which establishes a "canonical form" for the $*$-homomorphisms of a von Neumann algebra onto another one, and makes more precise the problem of the unitary implementation of a $*$-isomorphism.

Proposition. *Let $\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1)$, $\mathcal{M}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be von Neumann algebras and $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ a w-continuous $*$-homomorphism, such that $\pi(\mathcal{M}_1) = \mathcal{M}_2$. Then there exist*

a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$
two projections $e'_1, e'_2 \in \mathcal{M}' \subset \mathcal{B}(\mathcal{H})$ and
two spatial isomorphisms

$$\pi_1 : \mathcal{M}_1 \to \mathcal{M}_{e'_1} \subset \mathcal{B}(e'_1\mathcal{H})$$

$$\pi_2 : \mathcal{M}_2 \to \mathcal{M}_{e'_2} \subset \mathcal{B}(e'_2\mathcal{H})$$

such that

$$z(e'_1) = 1 \ \textit{and}$$

$$(\pi_2 \circ \pi \circ \pi_1^{-1})(x_{e'_1}) = x_{e'_2}, \ x \in \mathcal{M}.$$

Moreover

$$\pi \textit{ is a } *\textit{-isomorphism} \Leftrightarrow z(e_2') = 1,$$
$$\pi \textit{ is a spatial isomorphism} \Leftrightarrow e_1' \sim e_2' \textit{ in } \mathcal{M}'.$$

Proof. Let $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. The mapping

$$\hat{\pi}_1 : \mathcal{M}_1 \ni x_1 \to x_1 \oplus \pi(x_1) \in \mathscr{B}(\mathscr{H})$$

is an injective w-continuous $*$-homomorphism; hence, in accordance with Corollary 3.12, we have

$$\mathcal{M} = \hat{\pi}_1(\mathcal{M}_1) = \{x_1 \oplus \pi(x_1);\ x_1 \in \mathcal{M}_1\} \subset \mathscr{B}(\mathscr{H})$$

is a von Neumann algebra, whereas $\hat{\pi}_1 : \mathcal{M}_1 \to \mathcal{M}$ is a $*$-isomorphism.

We now consider the canonical isometries

$$u_1 : \mathscr{H}_1 \to \mathscr{H},\ u_1(\xi_1) = \xi_1 \oplus 0,\ \xi_1 \in \mathscr{H}_1,$$
$$u_2 : \mathscr{H}_2 \to \mathscr{H},\ u_2(\xi_2) = 0 \oplus \xi_2,\ \xi_2 \in \mathscr{H}_2,$$

the projections

$$e_1' = u_1 \circ u_1^*,\ e_2' = u_2 \circ u_2^* \in \mathcal{M}' \subset \mathscr{B}(\mathscr{H})$$

and the spatial isomorphisms

$$\pi_1 : \mathcal{M}_1 \ni x_1 \mapsto u_1 \circ x_1 \circ u_1^* \in \mathcal{M}_{e_1'} \subset \mathscr{B}(e_1'\mathscr{H}),$$
$$\pi_2 : \mathcal{M}_2 \ni x_2 \mapsto u_2 \circ x_2 \circ u_2^* \in \mathcal{M}_{e_2'} \subset \mathscr{B}(e_2'\mathscr{H}).$$

Since the canonical induction $\mathcal{M} \to \mathcal{M}_{e_1'}$ coincides with the $*$-isomorphism $\pi_1 \circ \hat{\pi}_1^{-1}$, from Proposition 3.14 we infer that $z(e_1') = 1$. Consequently, the mapping

$$\hat{\pi} : \mathcal{M}_{e_1'} \ni x_{e_1'} \mapsto x_{e_2'} \in \mathcal{M}_{e_2'}$$

is correctly defined. It is immediately verified that $\hat{\pi} = \pi_2 \circ \pi \circ \pi_1^{-1}$.

The other assertions in the statement of the proposition easily follow.

The proof can be sketched by the following commutative diagram

$$\begin{tikzcd}
 & & \mathcal{M} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{M}_{e_1'} \arrow[rr, "\hat{\pi}"] & & \mathcal{M}_{e_2'} & \\
\mathcal{M}_1 \arrow[uurr, "\hat{\pi}_1"] \arrow[ur, "\pi_1"'] \arrow[rrrr, "\pi"] & & & & \mathcal{M}_2 \arrow[ul, "\pi_2"']
\end{tikzcd}$$

Q.E.D.

8.3. *Let now* $\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1)$, $\mathcal{M}_2 \subset \mathcal{B}(\mathcal{H}_2)$ *be finite von Neumann algebras, whose commutants* $\mathcal{M}_1' \subset \mathcal{B}(\mathcal{H}_1)$, $\mathcal{M}_2' \subset \mathcal{B}(\mathcal{H}_2)$ *are also finite.*

According to Section 7.20, let $\tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_2$ be the extensions of the centers $\mathcal{Z}_1, \mathcal{Z}_2$ of the algebras $\mathcal{M}_1, \mathcal{M}_2$, and let $c_{\mathcal{M}_1, \mathcal{M}_1'} \in \tilde{\mathcal{Z}}_1$, $c_{\mathcal{M}_2, \mathcal{M}_2'} \in \tilde{\mathcal{Z}}_2$ be the corresponding coupling elements.

Any $*$-isomorphism $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ induces a $*$-isomorphism $\tilde{\pi} : \tilde{\mathcal{Z}}_1 \to \tilde{\mathcal{Z}}_2$, uniquely determined by the condition that the restriction of $\tilde{\pi}$ to $\mathcal{Z}_1$ coincide with the restriction of π to $\mathcal{Z}_1$.

Theorem. *Let* $\mathcal{M}_1, \mathcal{M}_2$ *be finite von Neumann algebras, whose commutants are also finite. A* $*$*-isomorphism* $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ *is spatial iff* $\tilde{\pi}(c_{\mathcal{M}_1, \mathcal{M}_1'}) = c_{\mathcal{M}_2, \mathcal{M}_2'}$.

Proof. If π is spatial, then obviously, $\tilde{\pi}(c_{\mathcal{M}_1, \mathcal{M}_1'}) = c_{\mathcal{M}_2, \mathcal{M}_2'}$. Conversely, let us assume that this condition is satisfied.

According to Proposition 8.2, we can assume that there exists a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, and two projections $e_1', e_2' \in \mathcal{M}'$, such that:

$$z(e_1') = z(e_2') = 1,$$

$$\pi : \mathcal{M}_1 = \mathcal{M}_{e_1'} \ni x_{e_1'} \mapsto x_{e_2'} \in \mathcal{M}_{e_2'} = \mathcal{M}_2.$$

On one hand, $\mathcal{M}$ is finite since it is $*$-isomorphic to $\mathcal{M}_1$ and to $\mathcal{M}_2$. On the other hand, since $\mathcal{M}'_{e_1'} = \mathcal{M}_1'$, $\mathcal{M}'_{e_2'} = \mathcal{M}_2'$ are finite, it follows that $e_1', e_2' \in \mathcal{M}'$ are finite projections. In accordance with Proposition 4.15, $e_1' \vee e_2' \in \mathcal{M}'$ is a finite projection. Therefore, we can assume that $\mathcal{M}'$ is finite.

The centers of the algebras $\mathcal{M}_{e_1'}$, $\mathcal{M}_{e_2'}$ canonically identify with the center of the algebra $\mathcal{M}$, and these identifications induce canonical identifications of the corresponding extensions. Assuming that these identifications have been performed, the condition $\tilde{\pi}(c_{\mathcal{M}_1, \mathcal{M}_1'}) = c_{\mathcal{M}_2, \mathcal{M}_2'}$ becomes

$$c_{\mathcal{M}_1, \mathcal{M}_1'} = c_{\mathcal{M}_2, \mathcal{M}_2'}.$$

Let now $\natural'$ be the canonical central trace on $\mathcal{M}'$. According to Proposition 7.21, we have

$$c_{\mathcal{M}, \mathcal{M}'} = (e_1')^{\natural'} c_{\mathcal{M}_1, \mathcal{M}_1'},$$

$$c_{\mathcal{M}, \mathcal{M}'} = (e_2')^{\natural'} c_{\mathcal{M}_2, \mathcal{M}_2'}.$$

It follows that

$$(e_1')^{\natural'} = (e_2')^{\natural'};$$

hence, according to 7.12,

$$e_1' \sim e_2',$$

and this implies that the $*$-isomorphism π is spatial.

Q.E.D.

8.4. In order to define the spatial invariants which will be used in the other theorems of unitary implementation, we need the following

Lemma. *Let $\mathscr{M}$ be a properly infinite von Neumann algebra, and let, for any $j = 1, 2$, $\{e_{j,i_j}\}_{i_j \in I_j}$ be a family of equivalent, mutually orthogonal projections in $\mathscr{M}$, which are piecewise of countable type, such that $\sum_{i_j \in I_j} e_{j,i_j} = 1$. Then* card $I_1 =$ card I_2.

Proof. Without any loss of generality, we can assume that e_{j,i_j} are of countable type, $i_j \in I_j$, $j = 1, 2$.

For each $i_1 \in I_1$ we denote

$$I_{2,i_1} = \{i_2 \in I_2;\ e_{1,i_1} e_{2,i_2} e_{1,i_1} \neq 0\}.$$

Obviously, we have

$$(*) \qquad I_2 = \bigcup_{i_1 \in I_1} I_{2,i_1}.$$

Since e_{1,i_1} is of countable type, according to exercise E.5.6, we infer that there exists a normal form φ_{i_1} on $\mathscr{M}$, such that $s(\varphi_{i_1}) = e_{1,i_1}$. Then, we have

$$\begin{aligned} +\infty > \varphi_1(e_{1,i_1}) &= \varphi_{i_1}(e_{1,i_1} (\sum_{i_2 \in I_2} e_{2,i_2}) e_{1,i_1}) \\ &= \sum_{i_2 \in I_2} \varphi_{i_1}(e_{1,i_1} e_{2,i_2} e_{1,i_1}) \\ &= \sum_{i_2 \in I_{2,i_1}} \varphi_{i_1}(e_{1,i_1} e_{2,i_2} e_{1,i_1}), \end{aligned}$$

hence I_{2,i_1} is at most countable. Since I_2 is an infinite set, from relation $(*)$ it follows that

$$\text{card } I_2 \leqslant \text{card } I_1.$$

The reversed inequality can be obtained analogously.

Q.E.D.

8.5. Let $\mathscr{M}$ be a properly infinite von Neumann algebra and let γ be an infinite cardinal. One says that $\mathscr{M}$ is *uniform of type* γ if there exists a family $\{e_i\}_{i \in I}$ of equivalent, mutually orthogonal projections in $\mathscr{M}$, piecewise of countable type, such that $\sum_{i \in I} e_i = 1$ and card $I = \gamma$.

According to Lemma 7.2, any finite projection is piecewise of countable type. Consequently, a semifinite properly infinite von Neumann algebra, which is *uniform of type* S_γ (see exercise E.4.14) is also uniform of type γ.

By taking into account exercise E.4.14, Lemma 8.4 shows that for any semifinite von Neumann algebra $\mathscr{M}$ there exist a family Γ of distinct infinite cardinals, and a family $\{p_1, p_\gamma\}_{\gamma \in \Gamma}$ of mutually orthogonal central projections, uniquely determined by the conditions

$$p_1 + \sum_{\gamma \in \Gamma} p_\gamma = 1,$$

$\mathscr{M} p_1$ is finite,

$\mathscr{M} p_\gamma$ is uniform, of type S_γ.

On the other hand, since any abelian projection is finite (4.8), any properly infinite von Neumann algebra, which is *homogeneous of type* I_γ (see exercise E.4.14) is also uniform of type γ.

By taking into account exercise E.4.14, Lemma 8.4 and exercise E.4.15 show that for any discrete von Neumann algebra $\mathcal{M}$ there exists a family Γ of distinct cardinals, and a family $\{p_\gamma\}_{\gamma\in\Gamma}$ of mutually orthogonal, central, non-zero projections, uniquely determined by the conditions

$$\sum_{\gamma\in\Gamma} p_\gamma = 1,$$

$$\mathcal{M}p_\gamma \text{ is homogeneous, of type } I_\gamma.$$

In the general case of a properly infinite von Neumann algebra we have the following result:

Proposition. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra. Then there exist a family Γ of distinct cardinals and a family $\{p_\gamma\}_{\gamma\in\Gamma}$ of non-zero, mutually orthogonal, central projections, uniquely determined by the conditions*

$$\sum_{\gamma\in\Gamma} p_\gamma = 1,$$

$$\mathcal{M}p_\gamma \textit{ is uniform, of type } \gamma.$$

Proof. The existence part of the proposition easily follows with the help of a usual argument based on the Zorn lemma and on the comparison theorem (4.6), whereas the uniqueness part of the proposition follows from Lemma 8.4.

Q. E. D.

8.6. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra *whose commutant $\mathcal{M}' \subset \mathcal{B}(\mathcal{H})$ is properly infinite.* Let Γ' and $\{p'_{\gamma'}\}_{\gamma'\in\Gamma'} \subset \mathcal{M}' \cap \mathcal{M}$ be the families canonically associated to $\mathcal{M}'$ in accordance with Proposition 8.5. One can then define the symbol

$$\mathfrak{u}_{\mathcal{M}'} = (\Gamma', \{p'_{\gamma'}\}_{\gamma'\in\Gamma'})$$

which can be called the *uniformity* of $\mathcal{M}$.

If $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is a $*$-isomorphism between two von Neumann algebras, whose commutants are properly infinite, and if

$$\mathfrak{u}_{\mathcal{M}'_j} = (\Gamma'_j, \{p'_{j,\gamma'}\}_{\gamma'\in\Gamma'_j}), \quad j = 1, 2,$$

then we shall write that

$$\tilde{\pi}(\mathfrak{u}_{\mathcal{M}'_1}) = \mathfrak{u}_{\mathcal{M}'_2},$$

if $\Gamma'_1 = \Gamma'_2$ and $\pi(p'_{1,\gamma'}) = p'_{2,\gamma'}$, for any $\gamma' \in \Gamma'_1 = \Gamma'_2$.

The following theorem of spatial isomorphism contains the case of the *semi-finite algebras whose commutants are properly infinite,* as well as the case of the *algebras of type III* (see Theorem 6.4).

Theorem. *Let $\mathcal{M}_1, \mathcal{M}_2$ be von Neumann algebras whose commutants are properly infinite. A $*$-isomorphism $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is spatial iff $\tilde{\pi}(u_{\mathcal{M}_1'}) = u_{\mathcal{M}_2'}$.*

Proof. If π is spatial, then, obviously $\tilde{\pi}(u_{\mathcal{M}_1'}) = u_{\mathcal{M}_2'}$. Conversely, let us assume that this condition is satisfied.

Then we can assume that $\mathcal{M}_1'$, as well as $\mathcal{M}_2'$, are uniform, of the same type γ.

According to Proposition 8.2, we can assume that there exist a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and two projections $e_1', e_2' \in \mathcal{M}'$, such that

$$z(e_1') = z(e_2') = 1,$$

$$\pi : \mathcal{M}_1 = \mathcal{M}_{e_1'} \ni x_{e_1'} \mapsto x_{e_2'} \in \mathcal{M}_{e_2'} = \mathcal{M}_2.$$

Since, by assumption, $\mathcal{M}_1'$ and $\mathcal{M}_2'$ are uniform, of the same type, it follows that, for each $j = 1, 2$, there exists an infinite family

$$\{e_{j,i}'\}_{i \in I} \subset \mathcal{M}_j' = \mathcal{M}_{e_j'}',$$

of equivalent, mutually orthogonal projections, piecewise of countable type, and such that

$$\sum_{i \in I} e_{j,i}' = e_j', \quad j = 1, 2$$

(i.e., the unit element of the corresponding algebra).

By performing a partition by countable subsets of the set I, and by considering the corresponding sums of those projections $e_{j,i}'$ whose indices i belong to the same subset of the partition, it is easy to see that we can assume that the projections $e_{j,i}'$ are properly infinite.

For any $i \in I$, $e_{1,i}'$ and $e_{2,i}'$ are then properly infinite projections, piecewise of countable type, in $\mathcal{M}'$, and $z(e_{1,i}') = z(e_{2,i}') = 1$. According to Proposition 4.13, it follows that

$$e_{1,i}' \sim e_{2,i}', \quad i \in I.$$

Therefore, we have

$$e_1' \sim e_2',$$

and, hence, π is a spatial isomorphism.

Q.E.D.

8.7. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a *discrete* von Neumann algebra. Its commutant $\mathcal{M}' \subset \mathcal{B}(\mathcal{H})$ is again a discrete von Neumann algebra (according to Theorem 6.4). By taking into account Section 8.5, we obtain a family Γ' of distinct cardinals, and a family $\{p_{\gamma'}'\}_{\gamma' \in \Gamma'}$ of mutually orthogonal central projections, uniquely determined by the conditions

$$\sum_{\gamma' \in \Gamma'} p_{\gamma'}' = 1,$$

$$\mathcal{M}' p_{\gamma'}' \text{ is homogeneous, of type } I_{\gamma'}.$$

We can then define the symbol

$$\mathfrak{o}_{\mathcal{M}'} = (\Gamma', \{p'_{\gamma'}\}_{\gamma' \in \Gamma'}),$$

which can be called the *homogeneity* of $\mathcal{M}'$.

If $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is a $*$-isomorphism between two discrete von Neumann algebras and if

$$\mathfrak{o}_{\mathcal{M}'_j} = (\Gamma'_j, \{p'_{j,\gamma'}\}_{\gamma' \in \Gamma'_j}), \quad j = 1, 2,$$

then we shall write that

$$\tilde{\pi}(\mathfrak{o}_{\mathcal{M}'_1}) = \mathfrak{o}_{\mathcal{M}'_2}$$

if $\Gamma'_1 = \Gamma'_2$ and $\pi(p'_{1,\gamma'}) = p'_{2,\gamma'}$, for any $\gamma' \in \Gamma'_1 = \Gamma'_2$.

Theorem. *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be discrete von Neumann algebras. Then a $*$-isomorphism $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is spatial iff $\tilde{\pi}(\mathfrak{o}_{\mathcal{M}'_1}) = \mathfrak{o}_{\mathcal{M}'_2}$.*

Proof. The proof is similar to that of Theorem 8.6. Instead of Proposition 4.13, one uses Proposition 4.10.

Q.E.D.

8.8. Let $\pi : \mathcal{M} \to \mathcal{M}$ be a $*$-automorphism of the von Neumann algebra $\mathcal{M}$, whose center is $\mathcal{Z}$. Obviously, we have $\pi(\mathcal{Z}) = \mathcal{Z}$. We shall say that π *acts identically on the center* if $\pi(z) = z$, for any $z \in \mathcal{Z}$. It is obvious that *if $\mathcal{M}$ is a factor, then any $*$-automorphism of $\mathcal{M}$ acts identically on the center.*

If π acts identically on the center, then π conserves the invariants $\mathfrak{c}_{\mathcal{M},\mathcal{M}'}$, $\mathfrak{u}_{\mathcal{M}'}$, $\mathfrak{o}_{\mathcal{M}'}$, already introduced, and according to the case.

In the following sections we state some obvious consequences of Theorems 8.3, 8.6, 8.7.

8.9. Corollary. *Any $*$-automorphism of a von Neumann algebra, whose commutant is properly infinite, which acts identically on the center, is spatial.*

8.10. Corollary. *Any $*$-automorphism of a finite von Neumann algebra, which acts identically on the center, is spatial.*

8.11. Corollary. *Any $*$-automorphism of a discrete von Neumann algebra, which acts identically on the center, is inner.*

Proof. It follows from 8.11 and 6.5.

8.12. Corollary. *Any $*$-isomorphism between von Neuman algebras, whose commutants are properly infinite and of countable type, is spatial.*

8.13. Corollary. *Any $*$-isomorphism between von Neumann algebras of type III, which operate in separable Hilbert spaces, is spatial.*

8.14. In what follows we shall study the relations existing between the various topologies already defined in a von Neumann algebra.

We recall that, besides the norm (uniform) topology and the topologies *wo*, *so* and *w*, which were defined in Section 1.3, we have also considered the topology *s*, which has been defined in exercise E.5.5. Between these topologies we have the following relations of strength

$$\begin{array}{ccccc} wo & \leqslant & so & & \\ \wedge\!\!| & & \wedge\!\!| & & \\ w & \leqslant & s & \leqslant & n, \end{array}$$

where by n we have denoted the norm (uniform) topology.

We are now concerned with the precise relations existing between the topologies w and wo, and between the topologies s and so. On the closed unit ball of $\mathscr{M}$ the restrictions of the topologies w and wo (resp., s and so) coincide (see 1.2, 1.10 and E.5.8).

The s-continuous (resp., the so-continuous) linear forms coincide with the w-continuous (resp., the wo-continuous) linear forms (see 1.4 and E.5.8). It follows that the w-topology (resp., the wo-topology) is the weakened topology associated to the s-topology (resp., the so-topology). On the other hand, a net $\{x_i\}$ in $\mathscr{M}$ is s-convergent to 0 (resp., so-convergent to 0) iff the net $\{x_i^*x_i\}$ is w-convergent to 0 (resp., wo-convergent to 0) (see E.5.8).

Consequently, *the s-topology coincides with the so-topology iff the w-topology coincides with the wo-topology.* Obviously, these conditions are equivalent to the condition that any w-continuous linear form be wo-continuous; hence (see 5.16), to the condition that *any normal form be wo-continuous.*

The fundamental result of the problem we are concerned with is contained in Corollary 5.24: *if the von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ has a separating vector, then any normal form on $\mathscr{M}$ is an ω_ξ, $\xi \in \mathscr{H}$*; in particular, in this case, the w-topology coincides with the wo-topology.

It is therefore only natural to begin our investigations with the study of the conditions under which a von Neumann algebra has separating vectors.

8.15. Lemma. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra whose commutant $\mathscr{M}' \subset \mathscr{B}(\mathscr{H})$ is properly infinite. Then $\mathscr{M}$ has a separating vector iff $\mathscr{M}$ is of countable type.*

Proof. If $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ has a separating vector $\xi \in \mathscr{H}$, then $s(\omega_\xi) = p_\xi = 1$ and, according to exercise E.5.6, $\mathscr{M}$ is of countable type. Conversely, let $\mathscr{M}$ be of countable type and $\mathscr{M}'$ properly infinite. In order to show that $\mathscr{M}$ has a separating vector, we can assume that, in accordance with Lemma 7.18, there exists an $\eta \in \mathscr{H}$, such that $p'_\eta = 1$. Then the projections p_η and 1 in $\mathscr{M}$ are of countable type, properly infinite (see Lemma 6.3) and they have the same central support. Proposition 4.13 now implies that $p_\eta \sim 1$. Let $v \in \mathscr{M}$, such that $v^*v = p_\eta$, $vv^* = 1$ and $\xi = v\eta$. Then we get $p_\xi = 1$; hence ξ is a separating vector for $\mathscr{M}$.

Q.E.D.

8.16. Theorem. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra whose commutant $\mathscr{M}' \subset \mathscr{B}(\mathscr{H})$ is properly infinite. For any w-continuous linear form φ on $\mathscr{M}$ there exist $\xi, \eta \in \mathscr{H}$, such that*

$$\varphi = \omega_{\xi,\eta}.$$

In particular, the w-topology coincides with the wo-topology, and the s-topology coincides with the so-topology.

Proof. Let first ψ be a normal form on $\mathscr{M}$ and $e = s(\psi)$. Then $\mathscr{M}_e$ is a von Neumann algebra of countable type, whose commutant is properly infinite. According to Lemma 8.15, there exists a $\zeta \in \mathscr{H}$, such that $e = p_\zeta$. Since $s(\psi) = p_\zeta$, from Theorem 5.23 we infer that there exists an $\eta \in \mathscr{H}$, such that $\psi = \omega_\eta$.

Let now φ be a w-continuous linear form on $\mathscr{M}$, and let $\varphi = R_v\psi$ be its polar decomposition (5.16). Since ψ is a normal form, there exists an $\eta \in \mathscr{H}$, such that $\psi = \omega_\eta$. Then $\varphi = \omega_{v\eta, \eta}$.

Q.E.D.

8.17. Corollary. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be an arbitrary von Neumann algebra. For any w-continuous linear form φ on $\mathscr{M}$ there exist two sequences $\{\xi_n\}, \{\eta_n\} \subset \mathscr{H}_\infty$, such that*

$$\varphi = \sum_{n=1}^{\infty} \omega_{\xi_n, \eta_n}, \quad \sum_{n=1}^{\infty} \|\xi_n\|^2 < +\infty, \quad \sum_{n=1}^{\infty} \|\eta_n\|^2 < +\infty.$$

Proof. Let us consider the separable Hilbert space l^2 and the von Neumann algebra $\tilde{\mathscr{M}} = \mathscr{M} \,\overline{\otimes}\, \mathscr{C}(l^2) \subset \mathscr{B}(\mathscr{H} \,\overline{\otimes}\, l^2)$. Then $\mathscr{H} \,\overline{\otimes}\, l^2$ identifies with the Hilbert space

$$\tilde{\mathscr{H}} = \left\{\{\xi_n\} \subset \mathscr{H};\ \sum_{n=1}^{\infty} \|\xi_n\|^2 < +\infty\right\}$$

and, for any $\tilde{x} = x \,\overline{\otimes}\, 1 \in \tilde{\mathscr{M}}$ and any $\tilde{\xi} = \{\xi_n\} \in \tilde{\mathscr{H}}$, we have

$$\tilde{x}\tilde{\xi} = \{x\xi_n\}.$$

According to Section 3.18, the amplification

$$\mathscr{M} \ni x \mapsto \tilde{x} \in \tilde{\mathscr{M}}$$

is a $*$-isomorphism, whereas, according to Proposition 3.17. and to exercise E.4.20, $(\tilde{\mathscr{M}})' = \mathscr{M}' \,\overline{\otimes}\, \mathscr{B}(l^2) \subset \mathscr{B}(\mathscr{H} \,\overline{\otimes}\, l^2)$ is a properly infinite von Neumann algebra.

Let φ be a w-continuous linear form on $\mathscr{M}$ and let us define the w-continuous linear form $\tilde{\varphi}$ on $\tilde{\mathscr{M}}$ by the relation

$$\tilde{\varphi}(\tilde{x}) = \varphi(x), \quad \tilde{x} = x \,\overline{\otimes}\, 1 \in \tilde{\mathscr{M}}.$$

According to Theorem 8.16, there exist $\tilde{\xi} = \{\xi_n\}$, $\tilde{\eta} = \{\eta_n\} \in \tilde{\mathscr{H}}$, such that

$$\tilde{\varphi} = \omega_{\tilde{\xi}, \tilde{\eta}}.$$

It follows that

$$\varphi = \sum_{n=1}^{\infty} \omega_{\xi_n, \eta_n}.$$

Q.E.D.

8.18. Lemma. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a finite von Neumann algebra of countable type, whose commutant $\mathscr{M}' \subset \mathscr{B}(\mathscr{H})$ is finite. Then $\mathscr{M}$ has a separating vector iff $c_{\mathscr{M}, \mathscr{M}'} \leqslant 1$.*

Proof. Let $\xi \in \mathscr{H}$ be a separating vector for $\mathscr{M}$. Then we have $p_\xi = 1$, $p'_\xi \leqslant 1$, hence

$$1 \geqslant (p'_\xi)^{\natural'} = c_{\mathscr{M},\mathscr{M}'} (p_\xi)^{\natural} = c_{\mathscr{M},\mathscr{M}'} .$$

Conversely, let us assume that $c_{\mathscr{M},\mathscr{M}'} \leqslant 1$ and then we shall prove that $\mathscr{M}$ has a separating vector. According to Lemma 7.18, we can assume that there exists a $\xi \in \mathscr{H}$, such that $p'_\xi = 1$. Then

$$1 = (p'_\xi)^{\natural'} = c_{\mathscr{M},\mathscr{M}'} (p_\xi)^{\natural} \leqslant (p_\xi)^{\natural} \leqslant 1,$$

hence $p_\xi = 1$. Thus, ξ is a separating vector for $\mathscr{M}$.

Q.E.D.

8.19. Theorem. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a finite von Neumann algebra, whose commutant $\mathscr{M}' \subset \mathscr{B}(\mathscr{H})$ is finite. Then the w-topology (resp., the s-topology) coincides with the wo-topology (resp., the so-topology) iff $c_{\mathscr{M},\mathscr{M}'} \in \mathscr{Z}$.*

Moreover, if n is a natural number, then the following conditions are equivalent

(i) *for any normal form ψ on $\mathscr{M}$ there exist n vectors $\eta_1, \ldots, \eta_n$, such that*

$$\psi = \sum_{k=1}^{n} \omega_{\eta_k}$$

(ii) *for any w-continuous linear form φ on $\mathscr{M}$ there exist n pairs of vectors $(\xi_1, \eta_1), \ldots, (\xi_n, \eta_n)$, such that*

$$\varphi = \sum_{k=1}^{n} \omega_{\xi_k,\eta_k}$$

(iii) $c_{\mathscr{M},\mathscr{M}'} \leqslant n$.

Proof. It is easy to see that assertion (i) (resp., (ii); resp., (iii)) for the algebra $\mathscr{M}$ and $n = n_0$ is equivalent to assertion (i) (resp., (ii); resp., (iii)) for the algebra $\tilde{\mathscr{M}}_{n_0} = \mathscr{M} \overline{\otimes} \mathscr{C}(\mathscr{H}_{n_0})$ and $n = 1$, where $\mathscr{H}_{n_0}$ is a Hilbert space of dimension n_0 (see Corollary 7.22 and the proof of Corollary 8.17).

Consequently, in order to prove the equivalence of the assertions (i), (ii), (iii), we can assume that $n = 1$. Then (i) $\Rightarrow$ (ii), according to the polar decomposition theorem (5.16); (ii) $\Rightarrow$ (i), according to exercise E.5.2, whereas the equivalence (i) $\Leftrightarrow$ (iii) follows by taking into account Lemma 8.18 and Proposition 7.21. We remark that the equivalence (i) $\Leftrightarrow$ (ii) holds in any von Neumann algebra.

If $c_{\mathscr{M},\mathscr{M}'} \in \mathscr{Z}$, then there exists a natural number n such that $c_{\mathscr{M},\mathscr{M}'} \leqslant n$; from what we have already proved it clearly follows that the w-topology coincides with the wo-topology.

If $\tilde{z} \in \tilde{\mathscr{Z}}$, $\tilde{z} \geqslant 0$ (see 7.20) and if there exists a $z \in \mathscr{Z}$, such that $\tilde{z} \leqslant z$, then $\tilde{z} \in \mathscr{Z}$.

Consequently, if $c_{\mathscr{M},\mathscr{M}'} \notin \mathscr{Z}$, then there exists a sequence $\{q_n\}$ of non-zero central projections of countable type, which are mutually orthogonal, and such that, for any n, we have $c_{\mathscr{M},\mathscr{M}'} q_n \geqslant n q_n$. Then $q = \sum_{n=1}^{\infty} q_n$ is of countable type and $c_{\mathscr{M},\mathscr{M}'} q \notin \mathscr{Z} q$.

Thus, we can assume that $\mathcal{M}$ is of countable type. Let φ be a faithful normal form on $\mathcal{M}$. If the w-topology coincides with the wo-topology, then there exist a natural number n and vectors $\xi_1, \ldots, \xi_n \in \mathcal{H}$, such that

$$\varphi = \sum_{k=1}^{n} \omega_{\xi_k}.$$

Then $\tilde{\xi} = \{\xi_1, \ldots, \xi_n\} \in \tilde{\mathcal{H}}_n$ is a separating vector for the von Neumann algebra $\tilde{\mathcal{M}}_n$. According to Corollary 5.24, any normal form on $\tilde{\mathcal{M}}_n$ is an $\omega_{\tilde{\zeta}}$, $\tilde{\zeta} \in \tilde{\mathcal{M}}_n$. In particular (see the proof of Corollary 8.17), any normal form on $\mathcal{M}$ is equal to a sum $\sum_{k=1}^{n} \omega_{\zeta_k}$. From the first part of the proof we infer that $c_{\mathcal{M},\mathcal{M}'} \leqslant n$, hence $c_{\mathcal{M},\mathcal{M}'} \in \mathcal{Z}$.

Q.E.D.

8.20. Corollary. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a finite factor. Then the w-topology (resp., the s-topology) coincides with the wo-topology (resp.,the so-topology).*

8.21. Our study of the relationship existing between the topologies w and wo (resp., s and so) is completed by the following:

Theorem. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a properly infinite von Neumann algebra, whose commutant $\mathcal{M}' \subset \mathcal{B}(\mathcal{H})$ is finite. Then, for any wo-continuous positive form φ on $\mathcal{M}$, its support* $s(\varphi)$ *is a finite projection in $\mathcal{M}$.*

In particular, the w-topology (resp., the s-topology) is strictly stronger than the wo-topology (resp., the so-topology).

Proof. Since $\mathcal{M}'$ is finite, for any $\xi \in \mathcal{H}$, p'_ξ is a finite projection in $\mathcal{M}'$, hence (see 6.3), p_ξ is a finite projection in $\mathcal{M}$.

If φ is a wo-continuous positive form on $\mathcal{M}$, then there exist $\xi_1, \ldots, \xi_n \in \mathcal{H}$, such that

$$\varphi = \sum_{k=1}^{n} \omega_{\xi_k},$$

whence

$$s(\varphi) = \bigvee_{k=1}^{n} p_{\xi_k}.$$

By taking into account Proposition 4.15, it follows that $s(\varphi)$ is finite.

The final part of the theorem follows from the obvious remark that, on a properly infinite von Neumann algebra, there exist normal forms whose supports are infinite.

Q.E.D.

8.22. Corollary. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be an infinite factor. Then the w-topology (resp., the s-topology) coincides with the wo-topology (resp., the so-topology) iff the commutant $\mathcal{M}' \subset \mathcal{B}(\mathcal{H})$ is infinite.*

Exercises

*E.8.1. Let $\mathcal{M}$ be a von Neumann algebra, whose commutant is $\mathcal{M}'$ and whose center is $\mathcal{Z}$. One says that $e \in \mathcal{P}_{\mathcal{M}}$ is a *maximally cyclic projection* in $\mathcal{M}$ if it is cyclic and if

$$f \in \mathcal{P}_{\mathcal{M}} \text{ cyclic, } e \prec f \Rightarrow e \sim f.$$

If $e \in \mathcal{P}_{\mathcal{M}}$ is a maximally cyclic projection in $\mathcal{M}$ and if $p \in \mathcal{P}_{\mathcal{Z}}$, then $ep \in \mathcal{P}_{\mathcal{M}_p}$ is maximally cyclic in $\mathcal{M}p$. With the help of the comparison theorem, one infers that the maximally cyclic projections in $\mathcal{M}$ are mutually equivalent.

Show that if $\mathcal{Z}$ is of countable type, then any cyclic projection in $\mathcal{M}$ is contained in a maximally cyclic projection.

If, moreover, $\mathcal{M}$ and $\mathcal{M}'$ are properly infinite, then the set of all maximally cyclic projections in $\mathcal{M}$ coincides with the set of all properly infinite projections of countable type, whose central support is equal to 1.

E.8.2. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with the cyclic vector $\xi \in \mathcal{H}$ and $e \in \mathcal{P}_{\mathcal{M}}$. The following assertions are equivalent

(i) e is maximally cyclic,

(ii) $e \sim p_\xi$,

(iii) $e \sim p_\eta \Rightarrow p'_\eta = 1$, $\eta \in \mathcal{H}$.

E.8.3. Let $\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1)$, $\mathcal{M}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be von Neumann algebras with the cyclic vectors $\xi_1 \in \mathcal{H}_1$, $\xi_2 \in \mathcal{H}_2$, respectively. A $*$-isomorphism $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is spatial iff *it conserves the maximal cyclicity of the projections* (i.e., $e_1 \in \mathcal{P}_{\mathcal{M}_1}$ maximally cyclic $\Rightarrow \pi(e_1) \in \mathcal{P}_{\mathcal{M}_2}$ maximally cyclic).

E.8.4. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be properly infinite von Neumann algebras, whose commutants are finite and whose centers are of countable type. A $*$-isomorphism $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is spatial iff it conserves the maximal cyclicity of the projections.

!E.8.5. Any $*$-isomorphism between two standard von Neumann algebras is spatial.

E.8.6. Prove the assertion from E.7.18 with the help of Theorem 8.16.

E.8.7. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then the w-topology on $\mathcal{M}$ is determined by the family of seminorms:

$$x \mapsto \left| \sum_{n=1}^{\infty} (x\xi_n | \eta_n) \right|,$$

where $\{\xi_n\}, \{\eta_n\} \subset \mathcal{H}$, $\sum\limits_{n=1}^{\infty} (\|\xi_n\|^2 + \|\eta_n\|^2) < +\infty$, whereas the s-topology on $\mathcal{M}$ is determined by the family of seminorms

$$x \mapsto \left(\sum_{n=1}^{\infty} \| x\xi_n \|^2 \right)^{1/2},$$

where $\{\xi_n\} \subset \mathcal{H}$, $\sum\limits_{n=1}^{\infty} \|\xi_n\|^2 < +\infty$.

In particular, the s-topology coincides with the ultrastrong topology.

!E.8.8. Let $\mathscr{M}_1$, $\mathscr{M}_2$ be von Neumann algebras and $\pi : \mathscr{M}_1 \to \mathscr{M}_2$ a w-continuous $*$-homomorphism, such that $\pi(\mathscr{M}_1) = \mathscr{M}_2$. Then there exist

an amplification	$\pi_1 : \mathscr{M}_1 \to \tilde{\mathscr{M}}_1$,
an induction	$\pi_2 : \tilde{\mathscr{M}}_1 \to (\tilde{\mathscr{M}}_1)_{e'}$, $e' \in (\tilde{\mathscr{M}}_1)'$
a spatial isomorphism	$\pi_3 : (\tilde{\mathscr{M}}_1)_{e'} \to \mathscr{M}_2$,

such that

$$\pi = \pi_3 \circ \pi_2 \circ \pi_1.$$

Hint: if $\mathscr{M}_2$ has a cyclic vector ξ_2, then, with a suitable amplification, we have $(\omega_{\xi_2} \circ \pi)^{\sim} = \omega_{\tilde{\xi}_1}$ and one defines $e' = [\tilde{\mathscr{M}}_1 \tilde{\xi}_1]$).

E.8.9. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a finite factor and μ a finite normal trace on $\mathscr{M}^+$. From Corollary 8.20, one obviously infers that there exists a finite family $\{\xi_1, \ldots, \xi_n\} \subset \mathscr{H}$, such that

$$\mu = \left(\sum_{k=1}^{n} \omega_{\xi_k}\right) | \mathscr{M}^+.$$

With the help of E.5.9 show that the preceding representation can be chosen in such a manner, that the projections $p_{\xi_1}, \ldots, p_{\xi_n}$ be mutually orthogonal.

E.8.10. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and let μ be a normal trace on $\mathscr{M}^+$. Then there exists a family $\{\xi_i\}_{i \in I} \subset \mathscr{H}$, such that

$$\mu = (\sum_{i \in I} \omega_{\xi_i}) | \mathscr{M}^+,$$

whereas if μ is, moreover, semifinite, then this representation can be obtained in such a manner, that the projections $p_{\xi_i}, i \in I$, be mutually orthogonal.

In particular, if $\mathscr{M}$ is a von Neumann algebra of countable type, then for any semifinite normal trace μ on $\mathscr{M}^+$, there exists a sequence $\{\xi_k\} \subset \mathscr{H}$, such that $\{p_{\xi_k}\}$ be mutually orthogonal and such that

$$\mu = (\sum_{k=1}^{\infty} \omega_{\xi_k}) | \mathscr{M}^+.$$

E.8.11. Show that, on a von Neumann algebra $\mathscr{M}$, the s-topology coincides with the so-topology iff any projection, which is the l.u.b. of a countable family of cyclic projections in $\mathscr{M}$ is equal to the l.u.b. of a finite family of cyclic projections in $\mathscr{M}$.

E.8.12. Let $\mathscr{M}$ be a von Neumann algebra. Show that if any w-continuous form on $\mathscr{M}$ is a finite sum of forms $\omega_{\xi,\eta}$, $\xi, \eta \in \mathscr{H}$, then there exists a natural number n, such that any w-continuous form on $\mathscr{M}$ is the sum of n forms $\omega_{\xi,\eta}$.

Infer from this result that if any projection of countable type in $\mathcal{M}$ is the l.u.b. of a finite family of cyclic projections in $\mathcal{M}$, then there exists a natural number n such that any projection of countable type in $\mathcal{M}$ is the l.u.b. of n cyclic projections in $\mathcal{M}$.

E.8.13. Prove the assertion from E.3.8, with the help of Lemma 7.18.

E.8.14. Let π_1 (resp., π_2) be a w-continuous $*$-homomorphism of the von Neumann algebra $\mathcal{M}_1$ (resp., $\mathcal{M}_2$) onto the von Neumann algebra $\mathcal{N}_1$ (resp., $\mathcal{N}_2$). Show that there exists a unique w-continuous $*$-homomorphism π of the von Neumann algebra $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ onto the von Neumann algebra $\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2$, such that

$$\pi(x_1 \overline{\otimes} x_2) = \pi_1(x_1) \overline{\otimes} \pi_2(x_2), \; x_1 \in \mathcal{M}_1, \; x_2 \in \mathcal{M}_2.$$

If π_1 and π_2 are $*$-isomorphisms, then π is a $*$-isomorphism (Hint: use exercise E.8.8).

E.8.15. Let φ_1 (resp., φ_2) be a w-continuous linear form on the von Neumann algebra $\mathcal{M}_1$ (resp., $\mathcal{M}_2$). Show that there exists a unique w-continuous linear form φ on the von Neumann algebra $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$, such that

$$\varphi(x_1 \overline{\otimes} x_2) = \varphi_1(x_1)\varphi_2(x_2), \; x_1 \in \mathcal{M}_1, \; x_2 \in \mathcal{M}_2.$$

if φ_1 and φ_2 are positive, then φ is positive. If φ_1 and φ_2 are faithful, then φ is faithful (Hint: use Corollary 8.17).

Comments

C.8.1. We have not yet discussed the unitary implementation of the $*$-isomorphisms between von Neumann algebras of type II_∞, whose commutants are of type II_1 (see the table 4.21).

In the cases dealt with by Theorems 8.3, 8.6, 8.7, the invariants $c_{\mathcal{M},\mathcal{M}'}$, $u_{\mathcal{M}'}$, $o_{\mathcal{M}'}$, which decide on the unitary implementability of the $*$-isomorphisms, are expressed in terms of cardinal numbers and central elements. In particular, in these cases, any $*$-automorphism of a factor is spatial.

In contrast to these cases, R. V. Kadison [10] showed that *there exist factors of type II_∞, whose commutants are of type II_1, which possess $*$-automorphisms, which are not spatial.* It follows that the conceivable invariants which would decide on the unitary implementability of the $*$-isomorphisms between von Neumann algebras of type II_∞, whose commutants are of type II_1, are not of the same type as those for the cases already studied. R. V. Kadison [14] indicated such a system of invariants.

Independently of the type of the von Neumann algebras, we have the fundamental result given by Corollary 5.25, and recalled in Section 8.1. Extensions of this result are contained in exercises E.8.3. (R. V. Kadison) and E.8.5 (J. Dixmier and I. E. Segal) (see also E.8.4).

C.8.2. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, G a locally compact group and $g \mapsto \pi_g$ a *wo*-measurable representation (with respect to the Haar measure) of G, by $*$-automorphisms of $\mathcal{M}$.

The problem arises whether there exists a *so*-continuous unitary representation $g \mapsto u_g$ of G, in $\mathcal{H}$ such that we have

$$\pi_g(x) = u_g x u_g^*, \quad x \in \mathcal{M}, g \in G.$$

In general, the answer to this problem is negative, even if each $*$-automorphism π_g is spatial. The following theorem gives a positive result in this direction.

Theorem 1. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, G a locally compact group, and $g \mapsto \pi_g$ a wo-measurable representation of G by $*$-automorphisms of $\mathcal{M}$. If*

(i) *$\mathcal{H}$ is separable and G is separable,*

(ii) *the commutant $\mathcal{M}' \subset \mathcal{B}(\mathcal{H})$ is properly infinite,*

then there exists a so-continuous unitary representation $g \mapsto u_g$ of G in $\mathcal{H}$, such that

$$\pi_g(x) = u_g x u_g^*, \ x \in \mathcal{M}, \ g \in G.$$

R. R. Kallman [13] proved this theorem by additionally assuming that $\mathcal{M}$ is semifinite and that the representation $g \to \pi_g$ is *wo*-continuous. Under the above stated, more general, conditions the theorem has been formulated and proved by M. Henle [1]. The simple and elegant proof of M. Henle reduces the problem to the result contained in Corollary 8.12.

A $*$-automorphism of $\mathcal{M}$ is said to be *inner* if it is implemented by a unitary operator in $\mathcal{M}$. Another positive result in connection with the above stated problem has been obtained by R. R. Kallman [14] and C. C. Moore [4, III, IV]:

Theorem 2. *Let $\{\pi_t\}_{t\in\mathbb{R}}$ be a wo-continuous one-parameter group of inner $*$-automorphisms of the von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$. If $\mathcal{H}$ is separable, then there exists a so-continuous one-parameter group $\{u_t\}_{t\in\mathbb{R}}$ of unitary operators in $\mathcal{M}$, such that*

$$\pi_t(x) = u_t x u_t^*, \quad x \in \mathcal{M}, \quad t \in \mathbb{R}.$$

A particular case of this theorem was previously proved by R. V. Kadison [22]. A simple proof, in the case of factors, has recently been given by F. Hansen [1].

An assertion, equivalent to the fact that any derivation of a von Neumann algebra is inner, is that for any one-parameter group $\{\pi_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms of the von Neumann algebra $\mathcal{M}$, which, moreover, is *norm*-continuous, there exists an invertible operator $a \in \mathcal{M}$, $0 \leqslant a \leqslant 1$ such that

$$\pi_t(x) = a^{it} x a^{-it}, \quad x \in \mathcal{M}, \quad t \in \mathbb{R}.$$

H. J. Borchers [4] gave a condition for the inner implementability of *wo*-continuous groups, with several real parameters, of spatial automorphisms. We state his result only in the case of a single parameter.

Theorem 3. *Let $\{\pi_t\}_{t\in\mathbb{R}}$ be a wo-continuous one-parameter group of $*$-automorphisms of the von Neumann algebra $\mathscr{M}$. Then the following conditions are equivalent:*

(i) *there exists a $b \in \mathscr{B}(\mathscr{H})$, $b \geqslant 0$, such that*

$$\pi_t(x) = b^{it}xb^{-it}, \quad x \in \mathscr{M}, \quad t \in \mathbb{R}\,;$$

(ii) *there exists an $a \in \mathscr{M}$, $a \geqslant 0$, $a \leqslant 1$, such that*

$$\pi_t(x) = a^{it}xa^{-it}, \quad x \in \mathscr{M}, \quad t \in \mathbb{R}.$$

Variants of the proof of H. J. Borchers appear in G. Dell'Antonio [1], R. V. Kadison [27], and S. Sakai [32]. Another proof has been given by W. B. Arveson [10] (see also L. Zsidó [8]).

In connection with the conditions under which a $*$-automorphism of a von Neumann algebra is inner, we mention the following remarkable result due to R. V. Kadison and J. R. Ringrose [3]:

Theorem 4. *Any $*$-automorphism π, of a von Neumann algebra, such that $\|\pi - 1\| < 2$, is inner.*

A simple proof of this theorem, in which the bound 2 is replaced by $\sqrt{3}$, can be found in J. Dixmier's book [26]. For other criteria we refer the reader to S. Sakai ([32], p. 167—168).

C.8.3. Bibliographical comments. Theorem 8.3 is due to F. J. Murray and J. von Neumann [2], [3], for the case of factors, and to H. A. Dye [2], E. L. Griffin [2], and R. Pallu de la Barrière [4], [5], for the general case.

Theorem 8.6 is due to E. L. Griffin [2], whereas Theorem 8.7 to I. Kaplansky [17].

The results on the comparison of the topologies *wo* and *w* are due to J. Dixmier [23], J. A. Dye [1], I. Kaplansky [10], R. Pallu de la Barrière [5], and others.

The decomposition given in exercise E.8.8 is due to J. Dixmier [24]. The result given in exercise E.8.14 is due to F. J. Murray and J. von Neumann [3], Y. Misonou [4] and T. Turumaru [2].

In our exposition of these results we used: J. Dixmier [26], R. V. Kadison [14], J. R. Ringrose [3] and S. Sakai [32].

9

Unbounded linear operators in Hilbert spaces

This chapter contains the fundamental results from the theory of (unbounded) linear operators in Hilbert spaces, results which will be used in the next chapter.

9.1. Let $\mathscr{H}$, $\mathscr{K}$ be Hilbert spaces. One says that T is a *linear operator* from $\mathscr{H}$ into $\mathscr{K}$ if T is a linear mapping of a vector subspace $\mathscr{D}_T$ of $\mathscr{H}$ into the vector space $\mathscr{K}$; in this case, $\mathscr{D}_T$ is called the *domain of definition* of T. If $\mathscr{H} = \mathscr{K}$, one also says that T is a linear operator in $\mathscr{H}$. When no danger of confusion could arise, we shall not indicate the spaces between which the linear operators act.

Let S, T be linear operators. One says that they are *equal* and one denotes by $T = S$ this relation, if $\mathscr{D}_T = \mathscr{D}_S$ and $T\xi = S\xi$, for any $\xi \in \mathscr{D}_T = \mathscr{D}_S$. One says that T is an *extension* of S (or, that S is a *restriction* of T), and one denotes by $T \supset S$ (or $S \subset T$) this relation, if $\mathscr{D}_T \supset \mathscr{D}_S$ and $T\xi = S\xi$, for any $\xi \in \mathscr{D}_S$.

For the linear operators T, S one defines

the multiplication by scalars $\lambda \in \mathbb{C}$, λT:

$$\mathscr{D}_{\lambda T} = \mathscr{D}_T,$$

$$(\lambda T)\xi = \lambda(T\xi), \qquad \xi \in \mathscr{D}_{\lambda T};$$

the addition $T + S$

$$\mathscr{D}_{T+S} = \mathscr{D}_T \cap \mathscr{D}_S,$$

$$(T + S)\,\xi = T\xi + S\xi, \qquad \xi \in \mathscr{D}_{T+S};$$

the composition $S \circ T = ST$:

$$\mathscr{D}_{ST} = \{\xi \in \mathscr{D}_T;\ T\xi \in \mathscr{D}_S\},$$

$$(ST)\xi = S(T\xi), \qquad \xi \in \mathscr{D}_{ST};$$

the inverse T^{-1} (if the mapping $T\colon \mathscr{D}_T \to \mathscr{K}$ is injective):

$$\mathscr{D}_{T^{-1}} = T\mathscr{D}_T,$$

$$T^{-1}\eta = \xi \Leftrightarrow T\xi = \eta, \qquad \eta \in \mathscr{D}_{T^{-1}}.$$

It is easy to verify the associativity of the addition and of the composition, as well as the following distributivity relations

$$(S_1 + S_2)T = S_1T + S_2T,$$
$$S(T_1 + T_2) \supset ST_1 + ST_2!$$

We recall that by the Hilbert sum of the Hilbert spaces $\mathscr{H}$, $\mathscr{K}$ the following vector space is meant

$$\mathscr{H} \oplus \mathscr{K} = \{(\xi, \eta);\ \xi \in \mathscr{H},\ \eta \in \mathscr{K}\},$$

in which the scalar product is given by

$$((\xi_1, \eta_1)(\xi_2, \eta_2)) = (\xi_1 \mid \xi_2) + (\eta_1 \mid \eta_2), \quad \xi_1, \xi_2 \in \mathscr{H},\ \eta_1, \eta_2 \in \mathscr{K}.$$

Let T be a linear operator from $\mathscr{H}$ into $\mathscr{K}$. The set

$$\mathscr{G}_T = \{(\xi, T\xi);\ \xi \in \mathscr{D}_T\} \subset \mathscr{H} \oplus \mathscr{K}$$

is a vector subspace, called the *graph* of T. A vector subspace $\mathscr{G}$ of $\mathscr{H} \oplus \mathscr{K}$ is the graph of a linear operator from $\mathscr{H}$ into $\mathscr{K}$ iff

$$(0, \eta) \in \mathscr{G} \Rightarrow \eta = 0.$$

Obviously, we have $T = S$ (resp., $T \supset S$) iff $\mathscr{G}_T = \mathscr{G}_S$ (resp., $\mathscr{G}_T \supset \mathscr{G}_S$).

One says that T is densely defined if $\mathscr{D}_T$ is dense in $\mathscr{H}$.

One says that T is *preclosed* if it is densely defined and if the closure of $\mathscr{G}_T$ in $\mathscr{H} \oplus \mathscr{K}$ is the graph of a linear operator, denoted by $\overline{T}$ and called the *closure* of T. Thus, T is preclosed iff it is densely defined and

$$\{\xi_n\} \subset \mathscr{D}_T,\ \xi_n \to 0,\ \{T\xi_n\} \text{ converges } \Rightarrow T\xi_n \to 0.$$

One says that T is *closed* if it is preclosed and $T = \overline{T}$, i.e., if T is densely defined and $\mathscr{G}_T$ is closed in $\mathscr{H} \oplus \mathscr{K}$. Thus, T is closed iff it is densely defined and

$$\{\xi_n\} \subset \mathscr{D}_T,\ \xi_n \to \xi_0,\ T\xi_n \to \eta_0 \Rightarrow \xi_0 \in \mathscr{D}_T,\ T\xi_0 = \eta_0.$$

One says that T is *bounded* if

$$\sup\{\|T\xi\|;\ \xi \in \mathscr{D}_T,\ \|\xi\| \leqslant 1\} < +\infty.$$

If this condition is not satisfied, one says that T is *unbounded* and then T is continuous at no point of its domain of definition.

If T is densely defined and bounded, then T is preclosed and $\overline{T}$ is *everywhere defined* ($\mathscr{D}_{\overline{T}} = \mathscr{H}$). Conversely, if T is closed and everywhere defined, then T is bounded, in accordance with the closed graph theorem.

If T is closed and T^{-1} exists, then T^{-1} is closed.

For any closed operator T, from $\mathscr{H}$ into $\mathscr{K}$, the kernel of T is a closed vector subspace of $\mathscr{H}$. We denote the projection of $\mathscr{H}$ onto this subspace by $\mathfrak{n}(T)$ and

$\mathbf{r}(T) = 1 - \mathbf{n}(T)$. We shall also denote by $\mathbf{l}(T)$ the projection of $\mathscr{K}$ onto the closure of the vector subspace $T\mathscr{D}_T$. One says that $\mathbf{r}(T)$ (resp., $\mathbf{l}(T)$) is the *right* (resp., *left*) *support* of T. If T is bounded, these notations are in accordance with those already introduced in Section 2.13.

9.2. Let T be a densely defined linear operator from $\mathscr{H}$ into $\mathscr{K}$. The set

$$\mathscr{D} = \{\eta \in \mathscr{K};\ \text{the form } \mathscr{D}_T \ni \xi \mapsto (T\xi \mid \eta) \text{ is bounded}\}$$

is a vector subspace of $\mathscr{K}$. Since $\mathscr{D}_T$ is dense in $\mathscr{H}$, from the Riesz theorem one infers that, for any $\eta \in \mathscr{D}$ there exists a unique element $\eta^* \in \mathscr{H}$, such that

$$(T\xi \mid \eta) = (\xi \mid \eta^*), \qquad \xi \in \mathscr{D}_T.$$

We now define a linear operator T^* from $\mathscr{K}$ into $\mathscr{H}$, called the *adjoint* of T, by the relations

$$\mathscr{D}_{T^*} = \mathscr{D},$$

$$T^*\eta = \eta^*, \qquad \eta \in \mathscr{D}_{T^*}.$$

Thus, T^* is determined by the relations

$$(T\xi \mid \eta) = (\xi \mid T^*\eta), \quad \xi \in \mathscr{D}_T, \quad \eta \in \mathscr{D}_{T^*}.$$

It is easily verified that if the operators $T, S, T + S, ST$ are densely defined and $\lambda \in \mathbf{C}$, then

$$(\lambda T)^* = \bar{\lambda} T^*,$$

$$T \supset S \Rightarrow T^* \subset S^*,$$

$$(T + S)^* \supset T^* + S^*,$$

$$(ST)^* \supset T^*S^*,$$

and, if T^{-1} exists and is densely defined, then

$$(T^{-1})^* = (T^*)^{-1}.$$

Proposition. *If T is a densely defined linear operator and if x is a bounded, everywhere defined linear operator, then*

$$(T + x)^* = T^* + x^*,$$

$$(xT)^* = T^*x^*.$$

Proof. Since x is everywhere defined, $\mathscr{D}_{T+x} = \mathscr{D}_T$. Hence, from the fact that x is bounded and from the relation

$$((T + x)\xi \mid \eta) = (T\xi \mid \eta) + (x\xi \mid \eta)$$

it follows that $\mathscr{D}_{T^*} = \mathscr{D}_{(T+x)^*}$. Thus, the first relation from the statement of the proposition follows by observing that, for any $\xi \in \mathscr{D}_T = \mathscr{D}_{T+x}$, $\eta \in \mathscr{D}_{T^*} = \mathscr{D}_{(T+x)^*}$ we have

$$(\xi \mid (T+x)^*\eta) = ((T+x)\xi \mid \eta) = (T\xi \mid \eta) + (x\xi \mid \eta)$$
$$= (\xi \mid T^*\eta) + (\xi \mid x^*\eta) = (\xi \mid (T^* + x^*)\eta).$$

Since x is everywhere defined, we have $\mathscr{D}_{xT} = \mathscr{D}_T$, and as a result of a remark made just above the proposition, we have $(xT)^* \supset T^*x^*$. Let $\eta \in \mathscr{D}_{(xT)^*}$ and $\xi \in \mathscr{D}_T$. From the relation

$$(\xi \mid (xT)^*\eta) = (xT\xi \mid \eta) = (T\xi \mid x^*\eta)$$

it follows that $x^*\eta \in \mathscr{D}_{T^*}$, hence $\eta \in \mathscr{D}_{T^*x^*}$, and

$$T^*x^*\eta = (xT)^*\eta.$$

Thus, we have $(xT)^* \subset T^*x^*$. Consequently, $(xT)^* = T^*x^*$.

Q.E.D.

9.3. In order to study more thoroughly the adjoint operator, we consider the unitary operator

$$V_{\mathscr{K}\mathscr{H}} : \mathscr{H} \oplus \mathscr{K} \ni (\xi, \eta) \mapsto (\eta, -\xi) \in \mathscr{K} \oplus \mathscr{H}.$$

Obviously, we have

$$V_{\mathscr{K}\mathscr{H}}^{-1} = - V_{\mathscr{H}\mathscr{K}}.$$

If $\mathscr{H} = \mathscr{K}$, we shall denote $V_{\mathscr{H}} = V_{\mathscr{H}\mathscr{H}}$.

It is easily verified that for any densely defined linear operator T, from $\mathscr{H}$ into $\mathscr{K}$, we have

$$\mathscr{G}_{T^*} = (V_{\mathscr{K}\mathscr{H}}\mathscr{G}_T)^\perp.$$

In particular, G_{T^*} is closed.

Proposition. *If T is a preclosed linear operator from $\mathscr{H}$ into $\mathscr{K}$, then T^* is closed and $T^{**} = \bar{T}$.*

Proof. Let $\eta \in \mathscr{K}$, $\eta \perp \mathscr{D}_{T^*}$. Then

$$(0, \eta) \in (\mathscr{G}_{T^*})^\perp = ((V_{\mathscr{K}\mathscr{H}}\mathscr{G}_T)^\perp)^\perp = V_{\mathscr{K}\mathscr{H}}((\mathscr{G}_T)^{\perp\perp}) = V_{\mathscr{K}\mathscr{H}}\mathscr{G}_{\bar{T}},$$

hence $(0, \eta) \in \mathscr{G}_{\bar{T}}$, whence $\eta = 0$. Thus, $\mathscr{D}_{T^*}$ is dense in $\mathscr{K}$. Since $\mathscr{G}_{T^*}$ is closed, T^* is closed.

Finally, we have

$$\mathscr{G}_{T^{**}} = (V_{\mathscr{H}\mathscr{K}}\mathscr{G}_{T^*})^\perp = (V_{\mathscr{H}\mathscr{K}}(V_{\mathscr{K}\mathscr{H}}\mathscr{G}_T)^\perp)^\perp = \mathscr{G}_{\bar{T}};$$

hence, $T^{**} = \bar{T}$.

Q.E.D.

It is easy to verify that

$$\mathfrak{r}(T)^* = \mathfrak{l}(T).$$

9.4. A linear operator T in $\mathscr{H}$ is said to be *symmetric* if it is densely defined and $T \subset T^*$. In other words, T is symmetric iff it is densely defined and

$$(T\xi \mid \eta) = (\xi \mid T\eta), \qquad \xi, \eta \in \mathscr{D}_T.$$

Obviously, any symmetric operator is preclosed. If T is symmetric, then, for any $\xi \in \mathscr{D}_T$, the number $(T\xi \mid \xi)$ is real.

A symmetric operator T is said to be *lower* (resp., *upper*) *semibounded* if it is densely defined and there exists a real number α, such that

$$(T\xi \mid \xi) \geqslant \alpha(\xi \mid \xi), \qquad \xi \in \mathscr{D}_T,$$

$$(\text{resp., } (T\xi \mid \xi) \leqslant \alpha(\xi \mid \xi), \qquad \xi \in \mathscr{D}_T).$$

In this case, the greatest (resp., the smallest) $\alpha \in \mathbb{R}$ with this property is called the g.l.b. (resp., l.u.b.) of T.

A linear operator T in $\mathscr{H}$ is said to be *positive* if it is densely defined and

$$(T\xi \mid \xi) \geqslant 0, \qquad \xi \in \mathscr{D}_T.$$

It is easy to see that any positive operator is symmetric and lower semibounded, with g.l.b. $\geqslant 0$.

A linear operator T in $\mathscr{H}$ is said to be *self-adjoint* if it is densely defined and $T = T^*$. Obviously, any self-adjoint operator is closed and symmetric. It is easily verified that any symmetric operator everywhere defined is self-adjoint.

If T is a self-adjoint operator in $\mathscr{H}$, then one denotes

$$\mathrm{s}(T) = \mathrm{r}(T) = \mathrm{l}(T),$$

and the projection $\mathrm{s}(T)$ is called the *support* of T.

9.5. Let A be a positive linear operator in $\mathscr{H}$. Then, for any $\xi \in \mathscr{D}_A$, we have

$$\|(1 + A)\xi\|^2 = \|\xi\|^2 + 2(A\xi \mid \xi) + \|A\xi\|^2 \geqslant \|\xi\|^2;$$

hence the mapping $1 + A$ is injective. Therefore, $(1 + A)^{-1}$ is a linear operator in $\mathscr{H}$, whose domain of definition is $(1 + A)\mathscr{D}_A$. Moreover, $(1 + A)^{-1}$ is bounded, of norm $\leqslant 1$, and, obviously, positive.

Lemma. *Let A be a positive linear operator in $\mathscr{H}$. Then A is self-adjoint iff $(1 + A)\mathscr{D}_A = \mathscr{H}$.*

Proof. Let us first assume that A is self-adjoint. Then A is closed. If $\{\xi_n\} \subset \mathscr{D}_A$ and $(1 + A)\xi_n \to \eta_0$, then, from the inequality

$$\|\xi_n - \xi_m\| \leqslant \|(1 + A)(\xi_n - \xi_m)\| = \|(1 + A)\xi_n - (1 + A)\xi_m\|,$$

we infer that the sequence $\{\xi_n\}$ converges to a vector $\xi_0 \in \mathscr{H}$. Since $\xi_n \to \xi_0$, $A\xi_n \to \eta_0 - \xi_0$ and A is closed, we get that $\xi_0 \in \mathscr{D}_A$ and $\eta_0 - \xi_0 = A\xi_0$, i.e., $\eta_0 = (1 + A)\xi_0$. Consequently, $(1 + A)\mathscr{D}_A$ is a closed vector subspace of $\mathscr{H}$.

Let now $\eta \in \mathscr{H}$, $\eta \perp (1 + A)\mathscr{D}_A$. Then

$$(0, \eta) \in (\mathscr{G}_{1+A})^{\perp} = (\mathscr{G}_{(1+A)^*})^{\perp} = V_{\mathscr{H}}\mathscr{G}_{1+A},$$

hence $(\eta, 0) \in \mathscr{G}_{1+A}$, i.e., $\eta \in \mathscr{D}_A$ and $(1 + A)\eta = 0$. But $1 + A$ is an injective mapping, hence $\eta = 0$. Thus, $(1 + A)\mathscr{D}_A$ is dense in $\mathscr{H}$.

Consequently, we have $(1 + A)\mathscr{D}_A = \mathscr{H}$.

Conversely, let us assume that $(1 + A)\mathscr{D}_A = \mathscr{H}$ and let us consider an element, $(\eta_0, \xi_0) \in \mathscr{G}_{A^*}$. For any $\xi \in \mathscr{D}_A$ we have

$$(A\xi \mid \eta_0) = (\xi \mid \xi_0),$$

$$((1 + A)\xi \mid \eta_0) = (\xi \mid \xi_0 + \eta_0);$$

hence, for any $\eta \in \mathscr{H}$, we have

$$(\eta \mid \eta_0) = ((1 + A)^{-1}\eta \mid \xi_0 + \eta_0).$$

Since $(1 + A)^{-1}$ is everywhere defined and symmetric, it is self-adjoint. Thus for any $\eta \in \mathscr{H}$, we have

$$(\eta \mid \eta_0) = (\eta \mid (1 + A)^{-1}(\xi_0 + \eta_0)),$$

whence we infer that

$$\eta_0 = (1 + A)^{-1}(\xi_0 + \eta_0).$$

Consequently, $\eta_0 \in \mathscr{D}_A$ and $(1 + A)\eta_0 = \xi_0 + \eta_0$, $A\eta_0 = \xi_0$, i.e.,

$$(\eta_0, \xi_0) \in \mathscr{G}_A.$$

Therefore, A is self-adjoint.

Q.E.D.

9.6. If a linear operator A in $\mathscr{H}$ has a symmetric extension, then A is symmetric and any symmetric extension of A is a restriction of A^*. These assertions immediately follow, if we take into account the implication

$$T \supset S \Rightarrow T^* \subset S^*.$$

An interesting problem in the theory of symmetric operators is that of the existence and the classification of the self-adjoint extensions. The semibounded symmetric operators have a canonical self-adjoint extension, with the conservation of the bound (lower, or upper, as the case may be). In the following theorem we describe the corresponding construction for positive operators.

Theorem. *Let B be a positive operator in $\mathscr{H}$. We define a linear operator A in $\mathscr{H}$ by the relations*

$$\mathscr{D}_A = \left\{ \begin{array}{c} \xi \in \mathscr{D}_{B^*};\ \text{there exists a } \{\xi_n\} \subset \mathscr{D}_B \text{ such that:} \\ \xi_n \to \xi \text{ and } (B(\xi_n - \xi_m) \mid \xi_n - \xi_m) \to 0, \end{array} \right\}$$

$$A\xi = B^*\xi, \quad \xi \in \mathscr{D}_A.$$

Then A is a positive self-adjoint extension of B.

Proof. We consider, on $\mathscr{D}_B$, the scalar product

$$(\xi \mid \eta)_B = ((1 + B)\xi \mid \eta),$$

and we define

$$\mathscr{D} = \left\{ \begin{array}{l} \xi \in \mathscr{H}; \text{ there exists a } \{\xi_n\} \subset \mathscr{D}_B, \text{ such that:} \\ \quad \xi_n \to \xi \text{ and } (\xi_n - \xi_m \mid \xi_n - \xi_m)_B \to 0. \end{array} \right\}$$

Then $\mathscr{D}$ is a vector subspace of $\mathscr{H}$, $\mathscr{D} \supset \mathscr{D}_B$ and $\mathscr{D}_A = \mathscr{D} \cap \mathscr{D}_{B^*}$.

If $\xi \in \mathscr{D}$ and $\{\xi_n\} \subset \mathscr{D}_B$, $\xi_n \to \xi$, $(\xi_n - \xi_m \mid \xi_n - \xi_m)_B \to 0$, then the sequence $\{(\xi_n \mid \xi_n)_B\}$ converges and its limit does not depend on the choice of the sequence $\{\xi_n\}$. We denote this limit by

$$(\xi \mid \xi)_B,$$

and for any $\xi, \eta \in \mathscr{D}$ we define

$$(\xi \mid \eta)_B = \frac{1}{4}(\xi + \eta \mid \xi + \eta)_B - \frac{1}{4}(\xi - \eta \mid \xi - \eta)_B + \frac{i}{4}(\xi + i\eta \mid \xi + i\eta)_B - \frac{i}{4}(\xi - i\eta \mid \xi - i\eta)_B.$$

We easily verify that $(\xi \mid \eta)_B$ is a scalar product on $\mathscr{D}$ and that $\mathscr{D}$, endowed with this scalar product, is a Hilbert space. By definition, $\mathscr{D}_B$ is dense in the Hilbert space $\mathscr{D}$. We can now easily prove the following relations

$$(\xi \mid \xi)_B \geqslant \|\xi\|^2, \qquad \xi \in \mathscr{D},$$

$$(\xi \mid \eta)_B = ((1 + B)\xi \mid \eta), \quad \xi \in \mathscr{D}_B, \quad \eta \in \mathscr{D}.$$

Let $\xi \in \mathscr{D}_A = \mathscr{D} \cap \mathscr{D}_{B^*}$ and let $\{\xi_n\} \subset \mathscr{D}_B$ be a sequence, such that

$$(\xi - \xi_n \mid \xi - \xi_n)_B \to 0.$$

Then we have

$$\begin{aligned}(A\xi \mid \xi) &= (B^*\xi \mid \xi) = \lim_{n\to\infty} (B^*\xi \mid \xi_n) = \lim_{n\to\infty} (\xi \mid B\xi_n) \\ &= \lim_{n\to\infty} ((\xi \mid (1 + B)\xi_n) - (\xi \mid \xi_n)) = \lim_{n\to\infty} ((\xi \mid \xi_n)_B - (\xi \mid \xi_n)) \\ &= (\xi \mid \xi)_B - \|\xi\|^2 \geqslant 0.\end{aligned}$$

Consequently, A is positive.

Let $\eta_0 \in \mathscr{H}$ be arbitrary. The mapping $\xi \mapsto (\xi \mid \eta_0)$ is a bounded form on the Hilbert space $\mathscr{D}$; hence there exists a $\zeta_0 \in \mathscr{D}$, such that

$$(\xi \mid \eta_0) = (\xi \mid \zeta_0)_B, \qquad \xi \in \mathscr{D}.$$

In particular, for any $\xi \in \mathscr{D}_B$, we have

$$(\xi \mid \eta_0) = (\xi \mid \zeta_0)_B = ((1 + B)\xi \mid \zeta_0),$$

$$(\xi \mid \eta_0 - \zeta_0) = (B\xi \mid \zeta_0);$$

hence

$$\xi_0 \in \mathscr{D}_{B^*}, \qquad \eta_0 - \xi_0 = B^*\xi_0,$$
$$\xi_0 \in \mathscr{D}_A, \qquad \eta_0 = (1 + A)\,\xi_0.$$

Consequently, we have $(1 + A)\,\mathscr{D}_A = \mathscr{H}$ and, according to lemma 9.5, A is self-adjoint.

Obviously, A is an extension of B.

Q.E.D.

The positive self-adjoint extension of the positive operators, which has just been constructed, is called the *Friedrichs extension.*

9.7. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and T a linear operator in $\mathscr{H}$. One says that T is *affiliated* to $\mathscr{M}$ if, for any unitary operator $u' \in \mathscr{M}'$, one has

$$u'^*Tu' = T.$$

From Corollary 3.4 we infer that if T is everywhere defined and bounded, then T is affiliated to $\mathscr{M}$ iff $T \in \mathscr{M}$.

If T is densely defined and affiliated to $\mathscr{M}$, then T^* is affiliated to $\mathscr{M}$.

If T is preclosed and affiliated to $\mathscr{M}$, then $\overline{T}$ is affiliated to $\mathscr{M}$.

In what follows we shall use the notations $\mathrm{Mat}_2(\mathscr{M}) \subset \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ and $(\tilde{\mathscr{M}}')_2 \subset \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, already introduced in Section 2.32. In accordance with Lemma 3.16, the commutant of $\mathrm{Mat}_2(\mathscr{M})$ is $(\tilde{\mathscr{M}}')_2$.

Let T be a closed linear operator in $\mathscr{H}$. Then the graph $\mathscr{G}_T$ is a closed linear subspace of $\mathscr{H} \oplus \mathscr{H}$ and, in order not to complicate the notations, the orthogonal projection on $\mathscr{G}_T$ will be denoted again by $\mathscr{G}_T$. Thus, we have $\mathscr{G}_T \subset \mathscr{H} \oplus \mathscr{H}$ and, at the same time, $\mathscr{G}_T \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$.

Lemma. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and T a closed linear operator in $\mathscr{H}$. Then T is affiliated to $\mathscr{M}$ iff $\mathscr{G}_T \in \mathrm{Mat}_2(\mathscr{M})$.*

Proof. It is easy to verify the following equivalences: T is affiliated to $\mathscr{M} \Leftrightarrow$ for any $x' \in \mathscr{M}'$ and $(\xi, \eta) \in \mathscr{G}_T$ we have $(x'\xi, x'\eta) \in \mathscr{G}_T \Leftrightarrow$ for any $\tilde{x}' \in (\tilde{\mathscr{M}}')_2$ we have $\tilde{x}'(\mathscr{G}_T) \subset \mathscr{G}_T \Leftrightarrow \mathscr{G}_T \in ((\tilde{\mathscr{M}}')_2)' = \mathrm{Mat}_2(\mathscr{M})$.

Q.E.D.

9.8. We recall (E.4.20) that if $\mathscr{M}$ is finite, then $\mathrm{Mat}_2(\mathscr{M})$ is also finite. In this case we shall denote by $\natural$ the canonical central trace (7.12) on $\mathrm{Mat}_2(\mathscr{M})$.

Lemma. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a finite von Neumann algebra and T a closed linear operator in $\mathscr{H}$, affiliated to $\mathscr{M}$. Then*

$$(\mathscr{G}_T)^\natural = 1/2.$$

Proof. We consider the projections

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{Mat}_2(\mathscr{M})$. It is easy to see that P_1, P_2 are equivalent orthogonal projections, whose sum is 1. Consequently,

$$P_1^{\natural} = P_2^{\natural} = 1/2.$$

On the other hand, it is easy to verify that

$$\mathfrak{n}(P_1\mathscr{G}_T) = (\mathscr{G}_T)^{\perp}, \qquad \mathfrak{n}(\mathscr{G}_T P_1) = P_2,$$

whence

$$\mathfrak{r}(P_1\mathscr{G}_T) = \mathscr{G}_T, \qquad \mathfrak{l}(P_1\mathscr{G}_T) = P_1.$$

Consequently, in accordance with Theorem 4.3, we have

$$\mathscr{G}_T \sim P_1.$$

As a conclusion, we have

$$(\mathscr{G}_T)^{\natural} = P_1^{\natural} = 1/2.$$

Q.E.D.

Theorem. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a finite von Neumann algebra. If T and S are closed linear operators, affiliated to $\mathscr{M}$, and if*

$$T \subset S,$$

then

$$T = S.$$

Proof. The theorem follows from the preceding lemma and from the properties of the mapping $\natural$ (7.11).

Q.E.D.

For a symmetric operator, affiliated to a finite von Neumann algebra, the above theorem completely solves the problem of the existence and the classification of self-adjoint extensions:

Corollary. *Let T be a symmetric operator in $\mathscr{H}$, affiliated to a finite von Neumann algebra. Then $\bar{T}$ is self-adjoint, and it is the unique self-adjoint extension of T.*

Proof. Since the operator T is symmetric, we have $\bar{T} \subset T^*$. The theorem implies that $\bar{T} = T^*$, hence $\bar{T}$ is self-adjoint.

On the other hand, if S is a self-adjoint extension of T, then

$$\bar{T} \subset S \subset T^* = \bar{T},$$

whence

$$S = \bar{T}.$$

Q.E.D.

9.9. In this section we describe the *operational calculus* for positive self-adjoint operators, with the help of Lemma 9.5, of the operational calculus for bounded self-adjoint operators (2.20) and of a natural passage to the limit process.

Let A be a positive self-adjoint linear operator in the Hilbert space $\mathscr{H}$. From Lemma 9.5 we infer that

$$a = (1 + A)^{-1} \in \mathscr{B}(\mathscr{H}),$$
$$0 \leqslant a \leqslant 1, \qquad s(a) = 1.$$

For any natural number n, let χ_n be the characteristic function of the set $((n + 1)^{-1}, +\infty)$. Let us define

$$e_n = \chi_n(a) \in \mathscr{R}(\{a\}).$$

There exists a unique $a_n \in \mathscr{R}(\{a_n\})$, such that

$$e_n \leqslant a_n \leqslant (n + 1)\, e_n$$
$$e_n = aa_n.$$

Since

$$a\mathscr{H} = (1 + A)^{-1}\mathscr{H} = \mathscr{D}_A,$$

it follows that, for any n, we have

$$e_n\mathscr{H} = aa_n\mathscr{H} \subset \mathscr{D}_A.$$

Hence, the operator Ae_n is everywhere defined. Moreover, we have

$$Ae_n = A(1 + A)^{-1} a_n = (1 - (1 + A)^{-1})\, a_n = (1 - a)\, a_n = a_n - e_n.$$

In particular,

$$Ae_n \in \mathscr{R}(\{a\}) \subset \mathscr{B}(\mathscr{H}),$$
$$0 \leqslant Ae_n \leqslant ne_n.$$

It is easy to verify that

$$e_nA \subset Ae_n.$$

We shall denote by $\mathscr{B}([0, +\infty))$ the $*$-algebra of all Borel measurable complex functions, which are defined on $[0, +\infty)$ and bounded on compact sets. For any $f \in \mathscr{B}([0, +\infty))$ we have

$$f \mid \sigma(Ae_n) \in \mathscr{B}(\sigma(Ae_n)),$$

hence we can consider the operator

$$f(Ae_n) \in \mathscr{B}(\mathscr{H}).$$

For any $f \in \mathscr{B}([0, +\infty))$, we define a linear operator $f(A)$ in $\mathscr{H}$, by the relations

$$\mathscr{D}_{f(A)} = \{\xi \in \mathscr{H};\ \text{the sequence } \{f(Ae_n)\xi\}_n \text{ converges}\}$$
$$f(A)\xi = \lim_{n\to\infty} f(Ae_n)\xi, \qquad \xi \in \mathscr{D}_{f(A)}.$$

It is easy to see that

$$f(0)\,(1 - e_n) + f(A)\, e_n = f(Ae_n).$$

On the other hand, if $f \in \mathscr{B}([0, +\infty))$, $f(0) = 0$, then for any $\xi \in \mathscr{D}_{f(A)}$ and any natural number n,

$$e_n f(A)\xi = \lim_{k\to\infty} e_n f(Ae_k)\xi = f(Ae_n)\xi,$$

hence

$$e_n f(A) \subset f(Ae_n).$$

We now introduce the notation

$$\mathscr{S}_A = \bigcup_{n=1}^{\infty} e_n \mathscr{H}.$$

Then it is easy to verify that, for any $f \in \mathscr{B}([0, +\infty))$, we have

$$\mathscr{S}_A \subset \mathscr{D}_{f(A)},$$
$$f(A)\mathscr{S}_A \subset \mathscr{S}_A.$$

If A is an everywhere defined, positive and bounded linear operator, then $e_n = 1$, for n sufficiently great. Hence, for any $f \in \mathscr{B}([0, +\infty))$, the operator $f(A)$, we have just defined, coincides with that introduced by Theorem 2.20.

If A is a positive self-adjoint linear operator in $\mathscr{H}$, and if $f \in \mathscr{B}([0, +\infty))$, then the linear operator $f(A)$, we have just defined, is affiliated to the von Neumann algebra $\mathscr{R}(\{a\})$ generated by $a = (1 + A)^{-1}$. Thus, if A is affiliated to a von Neumann algebra $\mathscr{M}$, then, for any $f \in \mathscr{B}([0, +\infty))$, $f(A)$ is affiliated to $\mathscr{M}$.

9.10. Let A, a and e_n be as in the preceding section. For any bounded $f \in \mathscr{B}([0, +\infty))$ we consider the function $F_f \in \mathscr{B}([0, 1])$, defined by

$$F_f(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0, \\ f((1-\lambda)/\lambda) & \text{if } \lambda \in (0, 1]. \end{cases}$$

Let n be a fixed natural number. We recall that $Ae_n \in \mathscr{B}(\mathscr{H})$. It is easy to verify that the mapping

$$\mathscr{B}(\sigma(Ae_n)) \ni f \mapsto F_f(ae_n) \in \mathscr{B}(\mathscr{H})$$

has the properties (i) and (ii) from Theorem 2.20, relatively to $x = Ae_n$. From Theorem 2.20 it then follows that for any bounded $f \in \mathscr{B}([0, +\infty))$ we have

$$f(Ae_n) = F_f(ae_n) = F_f(a)\, e_n.$$

By using the definition of $f(A)$ and the fact that $e_n \uparrow s(a) = 1$, we infer that for any bounded $f \in \mathscr{B}([0, +\infty))$ we have the relation

$$f(A) = F_f(a).$$

In particular, we found that if f is bounded, then $f(A) \in \mathscr{B}(\mathscr{H})$, and we have

$$\|f(A)\| = \|F_f(a)\| \leqslant \sup\{|F_f(\lambda)|;\ \lambda \in (0, 1]\}$$
$$= \sup\{|f(\lambda)|;\ \lambda \in [0, +\infty)\}.$$

9.11. The following theorem states the main rules of the operational calculus for positive self-adjoint operators.

Theorem. *Let A be a positive self-adjoint linear operator in the Hilbert space $\mathscr{H}$. Then*

(i) *for $f_0(\lambda) = c \in \mathbf{C}$, $\lambda \in [0, +\infty)$, we have*

$$f_0(A) = c;$$

for $f_1(\lambda) = \lambda$, $\lambda \in [0, +\infty)$, we have

$$f_1(A) = A;$$

(ii) *for any $f \in \mathscr{B}([0, +\infty))$, we have*

$$D_{f(A)} = \{\xi \in \mathscr{H};\ \sup_n \|f(Ae_n)\xi\| < +\infty\},$$

the linear operator $f(A)$ is closed and

$$\overline{f(A)\,|\,\mathscr{S}_A} = f(A);$$

(iii) *for any $f \in \mathscr{B}([0, +\infty))$, we have*

$$f(A)^* = \bar{f}(A);$$

(iv) *for any $f, g \in \mathscr{B}([0, +\infty))$, we have*

the linear operator $f(A) + g(A)$ is preclosed and

$$\overline{f(A) + g(A)} = (f + g)(A);$$

(v) *for any $f, g \in \mathscr{B}([0, +\infty))$ we have*

the linear operator $f(A)\,g(A)$ is preclosed,

$$\mathscr{D}_{f(A)\,g(A)} = \mathscr{D}_{(fg)(A)} \cap \mathscr{D}_{g(A)} \quad \textit{and}$$

$$\overline{f(A)\,g(A)} = (fg)(A);$$

(vi) *for any sequence $\{f_k\}_k \subset \mathscr{B}([0, +\infty))$, which is uniformly bounded on compact sets and pointwise convergent to $f_0 \in \mathscr{B}([0, +\infty))$ we have*

$$f_0(A)\,\xi = \lim_{k\to\infty} f_k(A)\,\xi, \qquad \xi \in \mathscr{S}_A.$$

Proof. (i) For any n, we have $f_0(Ae_n) = c$; hence, indeed, $f_0(A) = c$.

For any n, we have $f_1(Ae_n) = Ae_n$. If $\xi \in \mathscr{D}_A$, then

$$f_1(Ae_n)\,\xi = e_n A\xi \to A\xi;$$

hence $\xi \in \mathscr{D}_{f_1(A)}$ and $f_1(A)\,\xi = A\xi$. If $\xi \in \mathscr{D}_{f_1(A)}$, then

$$e_n\xi \to \xi,$$

$$Ae_n\xi = f_1(Ae_n)\,\xi \to f_1(A)\,\xi,$$

hence, since A is closed, $\xi \in \mathscr{D}_A$ and $A\xi = f_1(A)\,\xi$. Thus, we have $f_1(A) = A$.

If $f \in \mathscr{B}([0, +\infty))$ and $f_0(\lambda) = f(0)$, $\lambda \in [0, +\infty)$, it is easy to verify that

$$f(A) = (f - f_0)(A) + f(0).$$

Since $(f - f_0)(0) = 0$, this remark will enable us to assume, without any essential loss of generality, that the considered functions vanish at 0.

(ii) We can assume that $f(0) = 0$. Let

$$\alpha = \sup_n \|f(Ae_n)\xi\| < +\infty.$$

Since $f(0) = 0$, for any n we have

$$\|f(Ae_n)\xi\| = \|e_n f(Ae_{n+1})\xi\| \leqslant \|f(Ae_{n+1})\xi\|.$$

Consequently, we have

$$\|f(Ae_n)\xi\|^2 \uparrow \alpha^2.$$

On the other hand, for any n, k, we have

$$\|f(Ae_{n+k})\xi - f(Ae_n)\xi\|^2 = \|(e_{n+k} - e_n) f(Ae_{n+k})\xi\|^2$$
$$= \|e_{n+k} f(Ae_{n+k})\xi\|^2 - \|e_n f(Ae_{n+k})\xi\|^2 = \|f(Ae_{n+k})\xi\|^2 - \|f(Ae_n)\xi\|^2.$$

Consequently, the sequence $\{f(Ae_n)\xi\}$ is fundamental, hence convergent.

Therefore we have

$$\mathscr{D}_{f(A)} = \{\xi \in \mathscr{H};\ \sup_n \|f(Ae_n)\xi\| < +\infty\}.$$

Since $e_n \uparrow s(a) = 1$, $\mathscr{S}_A \subset \mathscr{D}_{f(A)}$ is a dense subset of $\mathscr{H}$; hence $f(A)$ is densely defined. If $\{\xi_k\} \subset \mathscr{D}_{f(A)}$, $\xi_k \to \xi_0$ and $f(A)\xi_k \to \eta_0$, then, for any n we have

$$f(Ae_n)\xi_0 = \lim_{k\to\infty} f(Ae_n)\xi_k = \lim_{k\to\infty} e_n f(A)\xi_k = e_n\eta_0.$$

It follows that $f(Ae_n)\xi_0 \to \eta_0$; hence $\xi_0 \in \mathscr{D}_{f(A)}$ and $f(A)\xi_0 = \eta_0$.

Consequently, $f(A)$ is closed.

If $(\xi, f(A)\xi) \in \mathscr{G}_{f(A)}$ and is orthogonal to the graph of the operator $f(A) | \mathscr{S}_A$, then, for any n, we have

$$(\xi \mid e_n\xi) = (f(A)\xi \mid f(Ae_n)\xi) = 0.$$

By tending to the limit for $n \to +\infty$, we find that

$$\|\xi\|^2 + \|f(A)\xi\|^2 = 0;$$

hence $\xi = f(A)\xi = 0$.

Therefore, we have $\overline{f(A) | \mathscr{S}_A} = f(A)$.

(iii) By taking into account the remark we made in (i) and Proposition 9.2, we can assume that $f(0) = 0$. It is then easy to verify that

$$\bar{f}(A) | \mathscr{S}_A \subset f(A)^*,$$

whence, in accordance with (ii),

$$\bar{f}(A) \subset f(A)^*.$$

Let now $\eta \in \mathscr{D}_{f(A)^*}$. For any $\xi \in \mathscr{H}$ and any n, we have

$$(\xi \mid \bar{f}(Ae_n)\eta) = (f(Ae_n)\xi \mid \eta) = (f(A)e_n\xi \mid \eta) = (\xi \mid e_n f(A)^*\eta);$$

hence

$$\bar{f}(Ae_n)\eta = e_n f(A)^*\eta.$$

It follows that $\bar{f}(Ae_n)\eta \to f(A)^*\eta$, i.e., $\eta \in \mathscr{D}_{\bar{f}(A)}$ and $\bar{f}(A)\eta = f(A)^*\eta$.

Consequently, we have $f(A)^* = \bar{f}(A)$.

(iv) It is easy to verify that

$$f(A) + g(A) \subset (f+g)(A),$$

hence $f(A) + g(A)$ is preclosed. Since on $\mathscr{S}_A$ the operators $f(A) + g(A)$ and $(f+g)(A)$ coincide, from (ii) we infer that

$$\overline{f(A) + g(A)} = (f+g)(A)$$

(v) We can assume that $f(0) = 0$. It is easy to verify that

$$f(A)g(A) \subset (fg)(A);$$

hence $f(A)g(A)$ is preclosed. Since on $\mathscr{S}_A$ the operators $f(A)g(A)$ and $(fg)(A)$ coincide, from (ii) we infer that

$$\overline{f(A)g(A)} = (fg)(A).$$

In accordance with the preceding results, we have

$$\mathscr{D}_{f(A)g(A)} \subset \mathscr{D}_{(fg)(A)} \cap \mathscr{D}_{g(A)}.$$

In order to prove the reversed inclusion, we must prove the implication

$$\xi \in \mathscr{D}_{(fg)(A)} \cap \mathscr{D}_{g(A)} \Rightarrow g(A)\xi \in \mathscr{D}_{f(A)}.$$

Indeed, since $f(0) = 0$, for any n we have

$$f(Ae_n)g(A)\xi = f(Ae_n)g(Ae_n)\xi = (fg)(Ae_n)\xi;$$

hence the sequence $\{f(Ae_n)g(A)\xi\}$ is convergent.

(vi) We can assume that $f_k(0) = 0$, for any $k \geqslant 0$. Let $\xi \in \mathscr{S}_A$; then there exists an n, such that $\xi \in e_n\mathscr{H}$. Then, for any $k \geqslant 0$, we have

$$f_k(A)\xi = f_k(A)e_n\xi = f_k(Ae_n)\xi.$$

From Theorem 2.20 we infer that

$$f_k(Ae_n)\xi \to f_0(Ae_n)\xi,$$

i.e.,

$$f_k(A)\xi \to f_0(A)\xi.$$

Q.E.D.

9.12. Corollary. *Let A be a positive self-adjoint linear operator in $\mathscr{H}$. If $f, g \in \mathscr{B}([0, +\infty))$, and $|f| \leqslant |g|$, then*

$$\mathscr{D}_{g(A)} \subset \mathscr{D}_{f(A)},$$
$$\|f(A)\xi\| \leqslant \|g(A)\xi\|, \qquad \xi \in \mathscr{D}_{g(A)}.$$

In particular, if f is bounded, then

$$f(A) \in \mathscr{B}(\mathscr{H})$$
$$\|f(A)\| \leqslant \sup\{|f(\lambda)|;\ \lambda \in [0, +\infty)\}.$$

Proof. If $|f| \leqslant |g|$, then, for any $\xi \in \mathscr{D}_{g(A)}$ and any natural number n, we have

$$\|f(Ae_n)\xi\|^2 = (|f|^2(Ae_n)\xi \mid \xi) \leqslant (|g|^2(Ae_n)\xi \mid \xi) = \|g(Ae_n)\xi\|^2;$$

hence

$$\sup_n \|f(Ae_n)\xi\| \leqslant \sup_n \|g(Ae_n)\xi\| = \|g(A)\xi\| < +\infty.$$

In accordance with Theorem 9.11 (ii), we infer that $\xi \in \mathscr{D}_{f(A)}$; obviously, we have $\|f(A)\xi\| \leqslant \|g(A)\xi\|$, for any $\xi \in \mathscr{D}_{g(A)}$.

If f is bounded, we have

$$|f| \leqslant c = \sup\{|f(\lambda)|;\ \lambda \in [0, +\infty)\};$$

then $\mathscr{D}_{f(A)} \supset \mathscr{D}_c = \mathscr{H}$ and

$$\|f(A)\| = \sup\{\|f(A)\xi\|;\ \|\xi\| \leqslant 1\} \leqslant \sup\{\|c\xi\|;\ \|\xi\| \leqslant 1\} = c.$$

Q.E.D.

If $f, g \in \mathscr{B}([0, +\infty))$ and at least one of the functions f, g is bounded, then it is easy to show that we have

$$(f+g)(A) = f(A) + g(A),$$
$$(fg)(A) = f(A)g(A).$$

If the sequence $\{f_k\} \subset \mathscr{B}([0, +\infty))$ converges uniformly to $f_0 \in \mathscr{B}([0, +\infty))$, then, for a sufficiently great k, we have $\mathscr{D}_{f_k(A)} = \mathscr{D}_{f_0(A)}$, $f_0(A) - f_k(A) \subset (f_0 - f_k)(A)$ is bounded and

$$\|f_0(A) - f_k(A)\| \to 0.$$

9.13. Corollary. *Let A be a positive self-adjoint linear operator in $\mathscr{H}$. Then*

(i) *for any real $f \in \mathscr{B}([0, +\infty))$, $f(A)$ is self-adjoint;*

(ii) *for any positive $f \in \mathscr{B}([0, +\infty))$, $f(A)$ is self-adjoint and positive;*

(iii) *for any characteristic function $f \in \mathscr{B}([0, +\infty))$, $f(A) \in \mathscr{B}(\mathscr{H})$ is a projection; moreover, $s(A) = \chi_{(0, +\infty)}(A)$;*

(iv) *for any $f \in \mathscr{B}([0, +\infty))$, such that $|f| = 1$, $f(A) \in \mathscr{B}(\mathscr{H})$ is unitary.*

A linear operator T in $\mathscr{H}$ is said to be *normal* if T is closed and $TT^* = T^*T$ It is easy to see that T is normal iff $D_T = D_{T^*}$ and $\|T\xi\| = \|T^*\xi\|$, $\xi \in \mathscr{D}_T$.

For any positive self-adjoint linear operator A in $\mathscr{H}$ and any $f \in \mathscr{B}([0, +\infty))$, the linear operator $f(A)$ is normal.

9.14. Corollary. *For any positive self-adjoint linear operator A in $\mathscr{H}$, there exists a unique positive self-adjoint linear operator $\mathscr{B}$ in $\mathscr{H}$, such that $B^2 = A$.*

Proof. We consider the continuous functions f, g, defined on $[0, +\infty)$ by the formulas

$$f(\lambda) = \lambda^{1/2},$$

$$g(\lambda) = 1 + \lambda.$$

According to Corollary 9.13, $f(A)$ is self-adjoint and positive. Since $0 \leqslant f \leqslant g$, from Corollary 9.12 we infer that

$$\mathscr{D}_A = \mathscr{D}_{g(A)} \subset \mathscr{D}_{f(A)}.$$

By taking into account Theorem 9.11 (v), we get

$$\mathscr{D}_{f(A)^2} = \mathscr{D}_A \cap \mathscr{D}_{f(A)} = \mathscr{D}_A$$

and

$$f(A)^2 = A.$$

Let now B be a positive self-adjoint operator in $\mathscr{H}$, such that $B^2 = A$. We denote

$$b = (1 + B)^{-1},$$

$$f_n = \chi_{((1+n)^{-1}, +\infty)}(b).$$

We consider the continuous functions h, k, defined on $[0, 1]$ by the formulas

$$h(\lambda) = \lambda^2/(\lambda^2 + (1 - \lambda)^2),$$

$$k(\lambda) = \lambda^{1/2}/(\lambda^{1/2} + (1 - \lambda)^{1/2}).$$

It is easy to verify that, for any n, we have

$$h(b)f_n = h(bf_n) = (1 + B^2)^{-1}f_n = (1 + A)^{-1}f_n,$$

whence

$$h(b) = (1 + A)^{-1}.$$

Since $(k \circ h)(\lambda) = \lambda$, $\lambda \in [0, 1]$, by taking into account Corollary 2.7, we get

$$k((1 + A)^{-1}) = k(h(b)) = (k \circ h)(b) = b = (1 + B)^{-1}.$$

From this equality we infer that B is determined by A in a unique manner.

Q.E.D.

This unique positive self-adjoint linear operator B in $\mathscr{H}$, such that $B^2 = A$, will be denoted

$$B = A^{1/2}.$$

9.15. Let $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha \geqslant 0$. We now consider the mapping

$$f_\alpha : [0, +\infty) \ni \lambda \mapsto \lambda^\alpha \in \mathbb{C}$$

as in Section 2.30. Then $f_\alpha \in \mathscr{B}([0, +\infty))$. For any positive self-adjoint linear operator A in $\mathscr{H}$ we define the operator

$$A^\alpha = f_\alpha(A).$$

Corollary. *Let A be a positive self-adjoint linear operator in $\mathscr{H}$, $\xi \in \mathscr{H}$ and $\varepsilon \geqslant 0$. The following assertions are equivalent:*

(i) $\xi \in \mathscr{D}_{A^\varepsilon}$;

(ii) $\xi \in \mathscr{D}_{A^\alpha}$ *for any* $\alpha \in \mathbb{C}$, $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon$, *and the mapping*

$$\alpha \mapsto A^\alpha \xi$$

is continuous on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ *and analytic in* $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$;

(iii) *the mapping*

$$it \mapsto A^{it}\xi,$$

defined on the imaginary axis, has a continuous extension to the set $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$, *which is analytic in* $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$.

Proof. (i) $\Rightarrow$ (ii). We define the continuous functions f_α and g on $[0, +\infty)$ by the relations

$$f_\alpha(\lambda) = \lambda^\alpha,$$

$$g(\lambda) = 1 + \lambda^\varepsilon.$$

Then, for any $\alpha \in \mathbb{C}$, such that $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon$, we have $|f_\alpha| \leqslant g$. In accordance with Corollary 9.12, we have

$$\mathscr{D}_{A^\varepsilon} = \mathscr{D}_{g(A)} \subset \mathscr{D}_{f_\alpha(A)} = \mathscr{D}_{A^\alpha}.$$

For any natural number n, we consider the projection e_n defined in Section 9.9. If $\xi \in e_n\mathscr{H}$, then, in accordance with Corollary 2.30, the mapping

$$\alpha \mapsto A^\alpha \xi = (Ae_n)^\alpha \xi$$

is continuous on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ and analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$.

Let $\xi \in \mathscr{D}_{A^\varepsilon}$ be arbitrarily chosen. We denote $\xi_n = e_n\xi$. By taking into account Corollary 9.12, for any $\alpha \in \mathbb{C}$, $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon$, and any natural number n, we get

$$\|A^\alpha\xi - A^\alpha\xi_n\| = \|A^\alpha(\xi - \xi_n)\| \leqslant \|(1 + A^\varepsilon)(\xi - \xi_n)\| \leqslant \|\xi - \xi_n\| + \|A^\varepsilon\xi - A^\varepsilon\xi_n\|.$$

Thus, the mappings

$$\alpha \mapsto A^\alpha \xi_n$$

are uniformly convergent on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ to the mapping

$$\alpha \mapsto A^{\alpha}\xi.$$

It follows that the mapping $\alpha \mapsto A^{\alpha}\xi$ is continuous on $\{\alpha \in \mathbb{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ and analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). We shall denote by F a continuous extension on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$, which is analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$, of the mapping

$$i\mathbb{R} \ni it \mapsto A^{it}\xi.$$

In accordance with the implication (i) $\Rightarrow$ (ii), for any $\xi, \eta \in \mathscr{S}_A$, the mapping

$$\alpha \mapsto (A^{\alpha}\xi \mid \eta) = (\xi \mid A^{\bar{\alpha}}\eta)$$

is continuous on $\{\alpha \in \mathbb{C};\ \operatorname{Re}\alpha \geqslant 0\}$, and analytic in $\{\alpha \in \mathbb{C};\ \operatorname{Re}\alpha > 0\}$.

Let $\{\xi_k\} \subset \mathscr{S}_A$, $\xi_k \to \xi$. For any $\alpha \in \mathbb{C}$, $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon$, in accordance with Corollary 9.12, we have

$$|(\xi_k \mid A^{\bar{\alpha}}\eta) - (\xi \mid A^{\bar{\alpha}}\eta)|^2 \leqslant \|\xi_k - \xi\|^2\, \|A^{\operatorname{Re}\alpha}\eta\|^2 \leqslant \|\xi_k - \xi\|^2\, \|(1 + A^{\varepsilon})\eta\|^2.$$

It follows that the mappings

$$\alpha \mapsto (\xi_k \mid A^{\bar{\alpha}}\eta)$$

converge uniformly on $\{\alpha \in \mathbb{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ to the mapping

$$\alpha \mapsto (\xi \mid A^{\bar{\alpha}}\eta).$$

Thus, the mapping $\alpha \mapsto (\xi \mid A^{\bar{\alpha}}\eta)$ is continuous on $\{\alpha \in \mathbb{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ and analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$.

For any $\eta \in \mathscr{S}_A$, the mappings

$$\alpha \mapsto (F(\alpha) \mid \eta),$$
$$\alpha \mapsto (\xi \mid A^{\bar{\alpha}}\eta),$$

which are continuous on $\{\alpha \in \mathbb{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ and analytic in $\{\alpha \in \mathbb{C}; 0 < \operatorname{Re}\alpha < \varepsilon\}$, coincide on the imaginary axis $i\mathbb{R}$. From the symmetry principle we infer that they coincide on $\{\alpha \in \mathbb{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$. In particular,

$$(F(\varepsilon) \mid \eta) = (\xi \mid A^{\varepsilon}\eta), \quad \eta \in \mathscr{S}_A,$$

whence, in accordance with Theorem 9.11 (ii), we get

$$(F(\varepsilon) \mid \eta) = (\xi \mid A^{\varepsilon}\eta), \quad \eta \in \mathscr{D}_{A^{\varepsilon}}.$$

Consequently, we have

$$\xi \in \mathscr{D}_{(A^{\varepsilon})^*} = \mathscr{D}_{A^{\varepsilon}}.$$

Q.E.D.

9.16. One calls a *(one-parameter) group of unitary operators* in a Hilbert space $\mathscr{H}$ a family $\{u_t; t \in \mathbb{R}\} \subset \mathscr{B}(\mathscr{H})$ of unitary operators, such that

$$u_0 = 1,$$

$$u_{t+s} = u_t u_s, \quad t, s \in \mathbb{R}.$$

One says that the group $\{u_t\}$ is *so-continuous* (resp. *wo-continuous*) if the mapping

$$\mathbb{R} \ni t \mapsto u_t \in \mathscr{B}(\mathscr{H})$$

is *so*-continuous (*wo*-continuous).

Let A be a positive self-adjoint linear operator in $\mathscr{H}$, such that $s(A) = 1$. Then, by taking into account Theorem 9.11 and Corollary 9.13, it follows that

$$(A^{it})^* A^{it} = A^{it}(A^{it})^* = \chi_{(0, +\infty)}(A) = s(A) = 1,$$

hence the operators A^{it}, $t \in \mathbb{R}$, are unitary. By taking into account Theorem 9.11 and Corollary 9.15, it follows that $\{A^{it}\}$ is a *so*-continuous group of unitary operators.

Conversely, we shall prove in what follows that any *wo*-continuous group of unitary operators is of the preceding form (the representation theorem of M. H. Stone). In particular, we shall infer that any *wo*-continuous group of unitary operators is *so*-continuous.

9.17. A mapping into $\mathscr{H}$ is said to be *weakly continuous* (resp., *weakly analytic*; resp., *weakly entire*) if it is continuous (resp., analytic; resp., entire) for the weak topology in $\mathscr{H}$.

Lemma. *Let $\{u_t\}$ be a wo-continuous group of unitary operators in $\mathscr{H}$. Then the set*

$$\{\xi \in \mathscr{H};\ \text{the mapping } it \mapsto u_t\xi \text{ has a weakly entire extension}\}$$

is a dense vector subspace of $\mathscr{H}$.

Proof. Let $\xi \in \mathscr{H}$ and let n be a natural number. For any $\alpha \in \mathbb{C}$, the mapping

$$\mathscr{H} \ni \eta \mapsto (n/\pi)^{1/2} \int_{-\infty}^{+\infty} \exp(-n(s + i\alpha)^2)(u_s\xi \mid \eta)\, ds$$

is a bounded antilinear form on $\mathscr{H}$, hence, there exists an $F_{\xi,n}(\alpha) \in \mathscr{H}$, such that

$$(F_{\xi,n}(\alpha) \mid \eta) = (n/\pi)^{1/2} \int_{-\infty}^{+\infty} \exp(-n(s + i\alpha)^2)(u_s\xi \mid \eta)\, ds.$$

It is obvious that the mapping $\alpha \mapsto F_{\xi,n}(\alpha)$ is weakly entire.

Let $\xi_n = F_{\xi,n}(0)$. For any $\eta \in \mathscr{H}$ and any $t \in \mathbb{R}$ we have

$$(F_{\xi,n}(\mathrm{i}t) \mid \eta) = (n/\pi)^{1/2}\int_{-\infty}^{+\infty} \exp(-n(s-t)^2)\,(u_s\xi \mid \eta)\,\mathrm{d}s$$

$$= (n/\pi)^{1/2}\int_{-\infty}^{+\infty} \exp(-ns^2)\,(u_{t+s}\xi \mid \eta)\,\mathrm{d}s$$

$$= (n/\pi)^{1/2}\int_{-\infty}^{+\infty} \exp(-ns^2)\,(u_s\xi \mid u_t^*\eta)\,\mathrm{d}s = (\xi_n \mid u_t^*\eta) = (u_t\xi_n \mid \eta),$$

hence

$$F_{\xi,n}(\mathrm{i}t) = u_t\xi_n.$$

Consequently, the mapping $\mathrm{i}t \mapsto u_t\xi_n$ has a weakly entire extension.

It is easy to verify that $\xi_n \to \xi$ weakly.

Q.E.D.

9.18. Let $\{u_t\}$ be a *wo*-continuous group of unitary operators in $\mathscr{H}$. For any $\varepsilon \geqslant 0$ we denote

$$\mathscr{D}_\varepsilon = \left\{\begin{array}{l} \xi \in \mathscr{H};\ \text{the mapping } \mathrm{i}t \mapsto u_t\xi \text{ has a weakly continuous extension to} \\ \{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}, \text{ which is weakly analytic in } \{\alpha \in \mathbb{C}; \\ 0 < \operatorname{Re}\alpha < \varepsilon\}. \end{array}\right\}$$

For any $\xi \in \mathscr{D}_\varepsilon$, the weakly continuous extension to the set $\{\alpha \in \mathbb{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$, which is weakly analytic in $\{\alpha \in \mathbb{C}; 0 < \operatorname{Re}\alpha < \varepsilon\}$, of the mapping $\mathrm{i}t \mapsto u_t\xi$, is determined in a unique manner and will be denoted by F_ξ.

It is easy to verify that, for any $\xi_1, \xi_2 \in \mathscr{D}_\varepsilon$ we have

$$F_{\xi_1+\xi_2} = F_{\xi_1} + F_{\xi_2}$$

and that, for any $\xi \in \mathscr{D}_\varepsilon$ and any $\lambda \in \mathbb{C}$ we have

$$F_{\lambda\xi} = \lambda F_\xi.$$

Lemma. *Let $\{u_t\}$ be a wo-continuous group of unitary operators in $\mathscr{H}$, and $\varepsilon \geqslant 0$. Then*

(i) *for any $\xi \in \mathscr{D}_\varepsilon$ and any $\beta \in \mathbb{C}$, $0 \leqslant \operatorname{Re}\beta \leqslant \varepsilon$, we have*

$$F_\xi(\beta) \in \mathscr{D}_{\varepsilon-\operatorname{Re}\beta},$$

$$F_{F_\xi(\beta)}(\alpha) = F_\xi(\alpha+\beta), \quad \alpha \in \mathbb{C},\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon - \operatorname{Re}\beta;$$

(ii) *for any $\xi, \eta \in \mathscr{D}_\varepsilon$ and any $\alpha \in \mathbb{C}$, $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon$, we have*

$$(F_\xi(\alpha) \mid \eta) = (\xi \mid F_\eta(\bar{\alpha})).$$

Proof. (i) Let $t \in \mathbb{R}$. The mappings

$$\gamma \mapsto u_t F_\xi(\gamma),$$
$$\gamma \mapsto F_\xi(it + \gamma),$$

are both weakly continuous extensions, to the set $\{\gamma \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\gamma \leqslant \varepsilon\}$, of the mapping

$$is \mapsto u_{t+s}\xi,$$

and they are weakly analytic on $\{\gamma \in \mathbb{C};\ 0 < \operatorname{Re}\gamma < \varepsilon\}$; hence, they coincide.

Thus, for any $t \in \mathbb{R}$, we have

$$u_t F_\xi(\beta) = F_\xi(it + \beta).$$

It follows that the mapping

$$\alpha \mapsto F_\xi(\alpha + \beta)$$

is a weakly continuous extension to $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon - \operatorname{Re}\beta\}$ of the mapping

$$it \mapsto u_t F_\xi(\beta),$$

and it is weakly analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon - \operatorname{Re}\beta\}$. Consequently, $F_\xi(\beta) \in \mathscr{D}_{\varepsilon - \operatorname{Re}\beta}$ and, for $\alpha \in \mathbb{C}$, $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon - \operatorname{Re}\beta$, we have

$$F_{F_\xi(\beta)}(\alpha) = F_\xi(\alpha + \beta).$$

(ii) The mappings

$$\alpha \mapsto (F_\xi(\alpha) \mid \eta),$$
$$\alpha \mapsto (\xi \mid F_\eta(\alpha)) = \overline{(F_\eta(\bar{\alpha}) \mid \xi)},$$

are continuous extensions to $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ of the mapping

$$it \mapsto (u_t\xi \mid \eta)$$

and they are analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$; hence they coincide.

Q.E.D.

9.19. Let $\{u_t\}$ be a *wo*-continuous group of unitary operators in $\mathscr{H}$. For any $\varepsilon \geqslant 0$ we define a linear operator A_ε in $\mathscr{H}$ by the relations

$$\mathscr{D}_{A_\varepsilon} = \mathscr{D}_\varepsilon,$$
$$A_\varepsilon\xi = F_\xi(\varepsilon), \quad \xi \in \mathscr{D}_{A_\varepsilon}.$$

Lemma. *For any $\varepsilon \geqslant 0$, the operator A_ε is self-adjoint and positive. For any $\varepsilon_1, \varepsilon_2 \geqslant 0$, the relation*

$$A_{\varepsilon_1 + \varepsilon_2} = A_{\varepsilon_1} + A_{\varepsilon_2}$$

holds.

Proof. Let $\varepsilon > 0$. In accordance with Lemma 9.17, the operator A_ε is densely defined. By taking into account Lemma 9.18 (ii) for any ξ, $\eta \in \mathscr{D}_{A_\varepsilon}$, we get

$$(A_\varepsilon\xi \mid \eta) = (F_\xi(\varepsilon) \mid \eta) = (\xi \mid F_\eta(\varepsilon)) = (\xi \mid A_\varepsilon\eta),$$

hence the linear operator A_ε is symmetric.

Let $\eta \in \mathscr{D}_{(A_\varepsilon)^*}$ and $\xi \in \mathscr{D}_{A_\varepsilon}$. The mapping

$$\alpha \mapsto (F_\xi(\alpha) \mid \eta)$$

is continuous on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ and analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$. With the help of Lemma 9.18 (i), it is easy to see that F_ξ is bounded on all lines parallel to the imaginary axis; hence it is bounded. For any $t \in \mathbb{R}$ we have

$$|(F_\xi(it) \mid \eta)| = |(u_t\xi \mid \eta)| \leqslant \|\xi\| \cdot \|\eta\|$$

and, by applying Lemma 9.18 (i), we get

$$\begin{aligned} |(F_\xi(\varepsilon + it) \mid \eta)| &= |(F_{F_\xi(it)}(\varepsilon) \mid \eta)| = |(A_\varepsilon F_\xi(it) \mid \eta)| \\ &= |(u_t\xi \mid (A_\varepsilon)^*\eta)| \leqslant \|\xi\| \cdot \|(A_\varepsilon)^*\eta\|. \end{aligned}$$

In accordance with Phragmen-Lindelöf principle (see N. Dunford and J. Schwartz [1], III.14), for any $\alpha \in \mathbb{C}$, $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon$, we have

$$|(F_\xi(\alpha) \mid \eta)| \leqslant \|\xi\| \max\{\|\eta\|, \|(A_\varepsilon)^*\eta\|\}.$$

From the preceding results we infer that, for any $\alpha \in \mathbb{C}$, $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon$, the mapping

$$\mathscr{D}_{A_\varepsilon} \ni \xi \mapsto (F_\xi(\alpha) \mid \eta)$$

is a bounded form on $\mathscr{H}$, whose norm is $\leqslant \max\{\|\eta\|, \|(A_\varepsilon)^*\eta\|\}$. Thus, there exists a $G_\eta(\bar{\alpha}) \in \mathscr{H}$, such that

$$\|G_\eta(\bar{\alpha})\| \leqslant \max\{\|\eta\|, \|(A_\varepsilon)^*\eta\|\},$$

$$(\xi \mid G_\eta(\bar{\alpha})) = (F_\xi(\alpha) \mid \eta), \quad \xi \in \mathscr{D}_{A_\varepsilon}.$$

It is easy to verify that the mapping

$$\alpha \mapsto G_\eta(\alpha)$$

is a weakly continuous extension to $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ of the mapping

$$it \mapsto u_t\eta,$$

and it is weakly analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$; hence $\eta \in \mathscr{D}_{A_\varepsilon}$.

Consequently, A_ε is self-adjoint.

Let now $\varepsilon_1, \varepsilon_2 \geqslant 0$. By taking into account Lemma 9.18 (i), it follows that if $\xi \in D_{A_{\varepsilon_1+\varepsilon_2}}$, then

$$A_{\varepsilon_2}\xi = F_\xi(\varepsilon_2) \in \mathscr{D}_{(\varepsilon_1+\varepsilon_2)-\varepsilon_2} = \mathscr{D}_{\varepsilon_1},$$

$$A_{\varepsilon_1}A_{\varepsilon_2}(\xi) = F_{F_\xi(\varepsilon_2)}(\varepsilon_1) = F_\xi(\varepsilon_1 + \varepsilon_2) = A_{\varepsilon_1+\varepsilon_2}(\xi).$$

Hence

$$A_{\varepsilon_1+\varepsilon_2} \subset A_{\varepsilon_1}A_{\varepsilon_2}.$$

Let now $\xi \in \mathscr{D}_{A_{\varepsilon_1}A_{\varepsilon_2}}$. For any $\eta \in \mathscr{D}_{A_{\varepsilon_1+\varepsilon_2}}$ we have

$$(A_{\varepsilon_1}A_{\varepsilon_2}\xi \mid \eta) = (\xi \mid A_{\varepsilon_2}A_{\varepsilon_1}\eta) = (\xi \mid A_{\varepsilon_1+\varepsilon_2}\eta),$$

hence

$$\xi \in \mathscr{D}_{(A_{\varepsilon_1+\varepsilon_2})^*} = D_{A_{\varepsilon_1+\varepsilon_2}},$$

$$A_{\varepsilon_1+\varepsilon_2}\xi = (A_{\varepsilon_1+\varepsilon_2})^*\xi = A_{\varepsilon_1}A_{\varepsilon_2}\xi.$$

Thus, we have

$$A_{\varepsilon_1+\varepsilon_2} = A_{\varepsilon_1}A_{\varepsilon_2}.$$

Finally, for any $\varepsilon \geqslant 0$ and any $\xi \in \mathscr{D}_{A_\varepsilon}$ we have

$$(A_\varepsilon\xi \mid \xi) = (A_{\varepsilon/2}A_{\varepsilon/2}\xi \mid \xi) = \|A_{\varepsilon/2}\xi\|^2 \geqslant 0,$$

hence A_ε is positive.

Q.E.D.

9.20. We now prove the representation theorem of M. H. Stone:
Theorem. *Let $\{u_t;\ t \in \mathbb{R}\} \subset \mathscr{B}(\mathscr{H})$. The following assertions are equivalent*

(i) *$\{u_t\}$ is a wo-continuous group of unitary operators;*

(ii) *$\{u_t\}$ is a so-continuous group of unitary operators;*

(iii) *there exists a positive self-adjoint linear operator A in $\mathscr{H}$, such that* $\mathbf{s}(A) = I$ *and*

$$u_t = A^{it}, \quad t \in \mathbb{R};$$

A is given by the equivalence

$$(\xi, \eta) \in \mathscr{G}_A \Leftrightarrow \left[\begin{array}{l}\text{the mapping } it \mapsto u_t\xi \text{ has a weakly continuous extension to} \\ \{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}, \text{ which is weakly analytic in } \{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < 1\}, \\ \text{and has the value } \eta \text{ at } 1.\end{array}\right]$$

Moreover, the relation

$$u_t = A^{it}, \quad t \in \mathbb{R}$$

establishes a one-to-one correspondence between the so-continuous groups $\{u_t\}$ of unitary operators on $\mathscr{H}$ and the positive self-adjoint linear operators in $\mathscr{H}$, such that $\mathbf{s}(A) = 1$.

Proof. The implication (iii) $\Rightarrow$ (ii) follows from Corollary 9.15, whereas the implication (ii) $\Rightarrow$ (i) is obvious.

Let us now assume that $\{u_t\}$ is a wo-continuous group of unitary operators on $\mathscr{H}$.

By using the notations from Section 9.19, we define

$$A = A_1.$$

In accordance with Lemma 9.19, A is a positive self-adjoint linear operator. If $\xi \in \mathscr{D}_A = \mathscr{D}_1$ and $F_\xi(1) = A\xi = 0$, then, by taking into account Lemma 9.18 (i) we get

$$F_\xi(1 + it) = F_{F_\xi(1)}(it) = u_t F_\xi(1) = 0, \quad t \in \mathbb{R}.$$

Consequently, we have $F_\xi = 0$, whence

$$\xi = F_\xi(0) = 0.$$

Therefore, we have $\mathfrak{n}(A) = 0$, i.e., $\mathrm{s}(A) = 1$.

In accordance with Lemma 9.19, and from the uniqueness part of Corollary 9.14, we obtain successively

$$A_{1/2} = A^{1/2},$$
$$A_{1/4} = A^{1/4},$$
$$A_{3/4} = A_{1/4}A_{1/2} = A^{1/4}A^{1/2} = A^{3/4}, \text{ etc.}$$

Thus for any natural number n and any integer k, $0 \leqslant k \leqslant 2^n$, we have

$$A_{k/2^n} = A^{k/2^n},$$

hence

$$F_\xi(k/2^n) = A_{k/2^n}\xi = A^{k/2^n}\xi, \quad \xi \in \mathscr{D}_A.$$

By taking into account Corollary 9.15, we infer that for any $\xi \in \mathscr{D}_A$, the mappings

$$\alpha \mapsto F_\xi(\alpha),$$
$$\alpha \mapsto A^\alpha \xi,$$

coincide on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$. In particular, we have

$$u_t\xi = F_\xi(it) = A^{it}\xi, \quad t \in \mathbb{R}.$$

The implication (i) $\Rightarrow$ (iii) is thus established.

The second part of the theorem immediately follows from the first part and from Corollary 9.15.

Q.E.D.

9.21. Let A be a positive self-adjoint operator in $\mathscr{H}$, such that $\mathrm{s}(A) = 1$. As we have seen in the preceding section, A^{it} is a unitary operator, and

$$(A^{it})^{-1} = A^{-it}.$$

On the other hand, A^{-1} is also a positive self-adjoint operator, such that $\mathrm{s}(A) = 1$. Stone's theorem leads to the following

Proposition. *For any positive self-adjoint operator A, such that* $\mathrm{s}(A) = 1$, *we have*

$$(A^{-1})^{it} = A^{-it}, \quad t \in \mathbb{R}.$$

Proof. Since $\{A^{it}\}$ is a *wo*-continuous group of unitary operators, from Theorem 9.20 we infer that there exists a positive self-adjoint operator B, such that $\mathrm{s}(B) = 1$ and

$$B^{it} = A^{-it}, \quad t \in \mathbb{R}.$$

In order to prove the proposition, it is sufficient to show that

$$B = A^{-1}.$$

Let $\xi \in \mathscr{D}_B$ and $\eta = B\xi$. In accordance with Corollary 9.15, the mapping

$$F: \{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant 1\} \ni \alpha \mapsto B^{1-\alpha}\xi \in \mathscr{H}$$

is continuous on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$ and analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < 1\}$. Since

$$B^{it}F(it) = B^{it}B^{1-it}\xi = B\xi = \eta, \quad t \in \mathbb{R},$$

it follows that

$$F(it) = B^{-it}\eta = A^{it}\eta, \quad t \in \mathbb{R}.$$

Obviously, we have

$$F(1) = \xi.$$

If we use again Corollary 9.15, we infer that

$$\eta \in \mathscr{D}_A \text{ and } A\eta = \xi,$$

i. e.,

$$\xi \in \mathscr{D}_{(A^{-1})} \text{ and } A^{-1}\xi = \eta.$$

Consequently, we have

$$B \subset A^{-1}.$$

Conversely, let $\xi \in \mathscr{D}_{(A^{-1})}$ and $\eta = A^{-1}\xi$. Then $\eta \in \mathscr{D}_A$ and $\xi = A\eta$. In accordance with Corollary 9.15, the mapping

$$G: \{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant 1\} \ni \alpha \mapsto A^{1-\alpha}\eta \in \mathscr{H}$$

is continuous on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$ and analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < 1\}$. Since we have

$$A^{it}G(it) = A^{it}A^{1-it}\eta = A\eta = \xi, \quad t \in \mathbb{R},$$

it follows that

$$G(it) = A^{-it}\xi = B^{it}\xi, \quad t \in \mathbb{R}.$$

Since

$$G(1) = \eta,$$

by using again Corollary 9.15, we infer that

$$\xi \in \mathscr{D}_B \text{ and } B\xi = \eta.$$

Consequently, we have

$$A^{-1} \subset B$$

and we infer that

$$B = A^{-1}.$$

Q.E.D.

Thus far we have defined (9.15) the operator A^α for any positive self-adjoint A and $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha \geqslant 0$. If $s(A)=1$, then it is natural to define A^α, for any $\alpha \in \mathbb{C}$. The preceding proposition allows the formulation of the following definition

$$A^\alpha = \begin{cases} A^\alpha & \text{if } \operatorname{Re}\alpha \geqslant 0, \\ (A^{-1})^{-\alpha} & \text{if } \operatorname{Re}\alpha \leqslant 0. \end{cases}$$

Indeed, if $\operatorname{Re}\alpha = 0$, then from this proposition we infer that

$$A^\alpha = (A^{-1})^{-\alpha}.$$

It is clear that this relation holds for any $\alpha \in \mathbb{C}$.

With this definition, Corollary 9.15 can be extended in the following manner

Corollary. *Let A be a positive self-adjoint linear operator in $\mathscr{H}$, such that $s(A) = 1$, $\xi \in \mathscr{H}$ and $\varepsilon_1 \leqslant 0$, $\varepsilon_2 \geqslant 0$. The following assertions are equivalent*

(i) $\xi \in \mathscr{D}_{(A^{\varepsilon_1})} \cap \mathscr{D}_{(A^{\varepsilon_2})}$;

(ii) $\xi \in \mathscr{D}_{A^\alpha}$ *for any* $\alpha \in \mathbb{C}$, $\varepsilon_1 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon_2$, *and the mapping*

$$\alpha \mapsto A^\alpha\xi$$

is continuous on $\{\alpha \in \mathbb{C};\ \varepsilon_1 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon_2\}$ *and analytic in* $\{\alpha \in \mathbb{C};\ \varepsilon_1 < \operatorname{Re}\alpha < \varepsilon_2\}$;

(iii) *the mapping*

$$it \mapsto A^{it}\xi,$$

defined on the imaginary axis, has a continuous extension to the set $\{\alpha \in \mathbb{C};\ \varepsilon_1 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon_2\}$, *analytic in* $\{\alpha \in \mathbb{C};\ \varepsilon_1 < \operatorname{Re}\alpha < \varepsilon_2\}$.

Proof. The implication (ii) $\Rightarrow$ (iii) is trivial, whereas the implication (iii) $\Rightarrow$ (i) directly follows from Corollary 9.15.

Let us now assume that $\xi \in \mathscr{D}_{(A^{\varepsilon_1})} \cap \mathscr{D}_{(A^{\varepsilon_2})}$. From Corollary 9.15, we infer that $\xi \in \mathscr{D}_{A^\alpha}$, for any $\alpha \in \mathbb{C}$, such that $\varepsilon_1 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon_2$, whereas the mapping

$$\alpha \mapsto A^\alpha\xi$$

is continuous on $\{\alpha\in\mathbb{C}; \varepsilon_1\leqslant \operatorname{Re}\alpha\leqslant\varepsilon_2\}$ and analytic in $\{\alpha\in\mathbb{C}; \varepsilon_1<\operatorname{Re}\alpha<\varepsilon_2; \operatorname{Re}\alpha\neq 0\}$. With the help of a classical argument, based on the Cauchy integral, we can infer that the preceding mapping is analytic in the set $\{\alpha\in\mathbb{C}; \varepsilon_1 < \operatorname{Re}\alpha<\varepsilon_2\}$.

Q.E.D.

9.22. Proposition 9.21 indicated a connection between the operational calculus for A and that for A^{-1}, a connection which we shall now explain:

Proposition. *Let A be a positive self-adjoint operator, such that* $\mathrm{s}(A) = 1$, *and let* $f, g\in \mathscr{B}([0, +\infty))$ *be bounded functions. If*

$$f(\lambda) = g(\lambda^{-1}),\quad \lambda\in(0, +\infty),$$

then

$$f(A) = g(A^{-1}).$$

Proof. We denote

$$a = (1 + A)^{-1},\quad b = (1 + A^{-1})^{-1}.$$

It is easy to see that

$$a + b = 1.$$

Since $\mathrm{s}(a) = 1 = \mathrm{s}(b)$, we infer that

$$\chi_{(0,1)}(a) = 1 = \chi_{(0,1)}(b).$$

With the notations from Section 9.10, we denote

$$F = F_f,\qquad G = F_g.$$

Then, for any $\lambda\in(0, 1)$ we have

$$F(\lambda) = f((1-\lambda)/\lambda) = g(\lambda/(1-\lambda)) = G(1-\lambda).$$

By taking into account Section 9.10, we infer that

$$\begin{aligned} f(A) &= F(a) = (F\chi_{(0,1)})(a) = (G\chi_{(0,1)})(1-a)\\ &= (G\chi_{(0,1)})(b) = G(b) = g(A^{-1}). \end{aligned}$$

Q.E.D.

In accordance with the preceding proposition, if A is a positive self-adjoint operator, such that $\mathrm{s}(A) = 1$ and if $0\leqslant\lambda_1<\lambda_2\leqslant+\infty$, then

$$\chi_{(\lambda_1,\lambda_2)}(A) = \chi_{(\lambda_2^{-1},\lambda_1^{-1})}(A^{-1}).$$

In particular, for any λ, $1<\lambda\leqslant+\infty$, we have

$$\chi_{(\lambda^{-1},\lambda)}(A) = \chi_{(\lambda^{-1},\lambda)}(A^{-1}).$$

9.23. In this section we present an integral formula which will be one of the main instruments for the development of Tomita's theory, in Chapter 10.

If $F\colon \mathbb{R} \to \mathscr{B}(\mathscr{H})$ is a *wo*-continuous mapping, and if the function $t \mapsto \|F(t)\|$ is dominated by a Lebesgue integrable function, then

$$(\xi, \eta) \mapsto \int_{-\infty}^{+\infty} (F(t)\xi \mid \eta)\, \mathrm{d}t$$

is a bounded sesquilinear form on $\mathscr{H} \times \mathscr{H}$. With the Riesz theorem we infer that there exists a (unique) operator $x_F \in \mathscr{B}(\mathscr{H})$, such that

$$(x_F\xi \mid \eta) = \int_{-\infty}^{+\infty} (F(t)\,\xi \mid \eta)\, \mathrm{d}t, \quad \xi, \eta \in \mathscr{H}.$$

In what follows we shall denote

$$x_F = \int_{-\infty}^{+\infty} (F(t)\, \mathrm{d}t.$$

Let now A and B be positive self-adjoint operators in $\mathscr{H}$, such that $\mathrm{s}(A) = \mathrm{s}(B) = 1$, and $\lambda > 0$. For any $x \in \mathscr{B}(\mathscr{H})$ we shall denote

$$\Phi_\lambda(x) = \int_{-\infty}^{+\infty} \frac{\lambda^{it-\frac{1}{2}}}{e^{\pi t} + e^{-\pi t}} A^{it} x B^{-it}\, \mathrm{d}t.$$

Proposition. *Let A and B be positive self-adjoint operators in $\mathscr{H}$, such that $\mathrm{s}(A) = \mathrm{s}(B) = 1$ and $\lambda > 0$. For any $x \in \mathscr{B}(\mathscr{H})$ there exists a unique $y \in \mathscr{B}(\mathscr{H})$ such that*

$$(x\eta \mid \xi) = \lambda(yB^{-1/2}\eta \mid A^{1/2}\xi) + (yB^{1/2}\eta \mid A^{-1/2}\xi),$$

$$\xi \in \mathscr{D}_{(A^{1/2})} \cap \mathscr{D}_{(A^{-1/2})}, \quad \eta \in \mathscr{D}_{(B^{1/2})} \cap \mathscr{D}_{(B^{-1/2})},$$

and it is given by

$$y = \Phi_\lambda(x).$$

Proof. Let us define, for any natural number n,

$$e_n = \chi_{(1/n,\, n)}(A) = \chi_{(1/n,\, n)}(A^{-1}),$$

$$f_n = \chi_{(1/n,\, n)}(B) = \chi_{(1/n,\, n)}(B^{-1}),$$

where by $\chi_{(1/n,\, n)}$ we denoted the characteristic function of the interval $(1/n, n)$.

We now consider the mapping

$$\alpha \mapsto F_{n,m}(\alpha) = \frac{-i\lambda^{i\alpha}}{e^{\pi\alpha} - e^{-\pi\alpha}} A^{i\alpha} e_n x B^{-i\alpha} f_m,$$

which is analytic in the set $\{\alpha \in \mathbb{C};\ \alpha \neq ik,\ k \in \mathbb{Z}\}$ for the norm topology. At the point $\alpha = 0$, the function $F_{n,m}$ has a first order pole, with the residue

$$\lim_{\alpha \to 0} \alpha F_{n,m}(\alpha) = -\frac{i}{2\pi} e_n x f_m.$$

From the Cauchy residue theorem we infer that

$$\int_{-\infty}^{+\infty} F_{n,m}\left(t-\frac{\mathrm{i}}{2}\right)\mathrm{d}t-\int_{-\infty}^{+\infty} F_{n,m}\left(t+\frac{\mathrm{i}}{2}\right)\mathrm{d}t = e_n x f_m.$$

Thus

$$\int_{-\infty}^{+\infty}\frac{\lambda^{\mathrm{i}t+\frac{1}{2}}}{\mathrm{e}^{\pi t}+\mathrm{e}^{-\pi t}}A^{1/2}e_n(A^{\mathrm{i}t}xB^{-\mathrm{i}t})B^{-1/2}f_m\mathrm{d}t$$

$$+\int_{-\infty}^{+\infty}\frac{\lambda^{\mathrm{i}t-\frac{1}{2}}}{\mathrm{e}^{\pi t}+\mathrm{e}^{-\pi t}}A^{-1/2}e_n(A^{\mathrm{i}t}xB^{-\mathrm{i}t})B^{1/2}f_m\mathrm{d}t = e_n x f_m,$$

whence

$$\lambda A^{1/2}e_n\Phi_\lambda(x)B^{-1/2}f_m + A^{-1/2}e_n\Phi_\lambda(x)B^{1/2}f_m = e_n x f_m.$$

Let $\xi\in\mathscr{D}_{(A^{1/2})}\cap\mathscr{D}_{(A^{-1/2})}$ and $\eta\in\mathscr{D}_{(B^{1/2})}\cap\mathscr{D}_{(B^{-1/2})}$. From the above equality we infer that, for any n and m,

$$\lambda(\Phi_\lambda(x)B^{-1/2}f_m\eta\,|\,A^{1/2}e_n\xi)+(\Phi_\lambda(x)B^{1/2}f_m\eta\,|\,A^{-1/2}e_n\xi) = (xf_m\eta\,|\,e_n\xi).$$

Tending to the limit for n and $m\to\infty$, we get

$$\lambda(\Phi_\lambda(x)B^{-1/2}\eta\,|\,A^{1/2}\xi)+(\Phi_\lambda(x)B^{1/2}\eta\,|\,A^{-1/2}\xi) = (x\eta\,|\,\xi).$$

Let now $y\in\mathscr{B}(\mathscr{H})$ be an arbitrary operator which satisfies the relations in the statement of the proposition. By denoting

$$z = \Phi_\lambda(x)-y,$$

for any n and m we have

$$\lambda A^{1/2}e_n zB^{-1/2}f_m + A^{-1/2}e_n zB^{1/2}f_m = 0.$$

If we multiply this equality by $f_m z^* A^{1/2}e_n$ to the left and by $B^{1/2}f_m$ to the right, we get

$$\lambda f_m z^* A e_n z f_m + f_m z^* e_n z B f_m = 0,$$

i.e.,

$$(f_m z^* e_n z f_m)(Bf_m) = -\lambda f_m z^* A e_n z f_m \leqslant 0.$$

Thus, the operators $f_m z^* e_n z f_m \geqslant 0$ and $Bf_m \geqslant 0$ commute, hence

$$(f_m z^* e_n z f_m)(Bf_m)\geqslant 0.$$

Consequently,

$$f_m z^* e_n z B f_m = 0,$$

whence we successively infer that

$$f_m z^* e_n z f_m = (f_m z^* e_n z B f_m)(B^{-1} f_m) = 0,$$

$$e_n z f_m = 0.$$

By tending to the limit for n and $m \to \infty$, we infer that $z = 0$, i.e.,

$$y = \Phi_\lambda(x).$$

Q.E.D.

With the help of Lemma 9.5, it is easy to see that if A is a positive self-adjoint operator and $\lambda > 0$, then $(\lambda + A)^{-1} \in \mathscr{B}(\mathscr{H})$.

Corollary. *Let A be a positive self-adjoint operator in $\mathscr{H}$, such that* $\mathrm{s}(A) = 1$ *and* $\lambda > 0$. *Then*

$$A^{-1/2}(\lambda + A^{-1})^{-1} = \int_{-\infty}^{+\infty} \frac{\lambda^{it - \frac{1}{2}}}{\mathrm{e}^{\pi t} + \mathrm{e}^{-\pi t}} A^{it} \mathrm{d}t.$$

Proof. Let

$$y = A^{-1/2}(\lambda + A^{-1})^{-1} \in \mathscr{B}(\mathscr{H}).$$

It is easy to verify that y satisfies the relation from the statement of the preceding proposition, with $B = 1$ and $x = 1$, i.e.,

$$(\eta \mid \xi) = \lambda(y\eta \mid A^{1/2}\xi) + (y\eta \mid A^{-1/2}\xi),$$

$$\xi \in \mathscr{D}_{(A^{1/2})} \cap \mathscr{D}_{(A^{-1/2})}, \quad \eta \in \mathscr{H}.$$

Thus the assertion in the corollary obviously follows from the preceding proposition.

Q.E.D.

9.24. In this section we shall prove an analogue of Corollary 9.21, which we shall use in Chapter 10. We shall first state the following

Lemma. *Let $\Omega \subset \mathbb{C}$ an open set and $F: \Omega \to \mathscr{B}(\mathscr{H})$. Then the following assertions are equivalent:*

(i) *F is analytic for the norm topology;*

(ii) *for any $\xi, \eta \in \mathscr{H}$, the function $\Omega \ni \alpha \mapsto (F(\alpha)\xi \mid \eta)$ is analytic.*

Proof. Obviously, (i) $\Rightarrow$ (ii).

Let us now assume that assertion (ii) is true. Let $\alpha \in \Omega$ and $V \subset \overline{V} \subset \Omega$ be a relatively compact neighbourhood of α. For any $\beta, \gamma \in V$, $\beta \neq \alpha$, $\gamma \neq \alpha$, $\beta \neq \gamma$, we define

$$G(\alpha; \beta, \gamma) = \frac{1}{\beta - \gamma}\left[\frac{1}{\beta - \alpha}(F(\beta) - F(\alpha)) - \frac{1}{\gamma - \alpha}(F(\gamma) - F(\alpha))\right].$$

According to the hypothesis, for any $\xi, \eta \in \mathscr{H}$ we have

$$\sup\{|(G(\alpha; \beta, \gamma)\xi \mid \eta)|;\ \beta, \gamma \in V, \beta \neq \alpha,\ \gamma \neq \alpha,\ \beta \neq \gamma\} < +\infty.$$

From the Banach-Steinhauss theorem we infer that

$$c = \sup\{\|G(\alpha; \beta, \gamma)\|;\ \beta, \gamma \in V,\ \beta \neq \alpha,\ \gamma \neq \alpha, \beta \neq \gamma\} < +\infty.$$

Consequently, for any $\beta, \gamma \in V$, $\beta \neq \alpha$, $\gamma \neq \alpha$, $\beta \neq \gamma$, we have

$$\left\| \frac{1}{\beta - \alpha}(F(\beta) - F(\alpha)) - \frac{1}{\gamma - \alpha}(F(\gamma) - F(\alpha)) \right\| \leqslant c\,|\beta - \gamma|.$$

With the help of the Cauchy criterion, it follows that F is differentiable with respect to the norm topology.

Consequently, assertion (i) follows from assertion (ii).

Q.E.D.

A mapping F, which satisfies the equivalent assertions of this lemma will be called, briefly, an analytic mapping.

Proposition. *Let A, B be positive self-adjoint operators in $\mathscr{H}$, such that $\mathrm{s}(A) = \mathrm{s}(B) = 1$, $x \in \mathscr{B}(\mathscr{H})$ and $\varepsilon_1 \leqslant 0 \leqslant \varepsilon_2$. Then the following assertions are equivalent*

(i) *there exist vector subspaces $\mathscr{D}_1 \subset \mathscr{D}_{(A^{\varepsilon_1}xB^{-\varepsilon_1})}$, $\mathscr{D}_2 \subset \mathscr{D}_{(A^{\varepsilon_2}xB^{-\varepsilon_2})}$, such that*

$$\overline{B^{-\varepsilon_1}|\mathscr{D}_1} = B^{-\varepsilon_1}, \quad \overline{B^{-\varepsilon_2}|\mathscr{D}_2} = B^{-\varepsilon_2}$$

and the operators

$$A^{\varepsilon_1}xB^{-\varepsilon_1}|\mathscr{D}_1, \quad A^{\varepsilon_2}xB^{-\varepsilon_2}|\mathscr{D}_2$$

are bounded;

(ii) *for any $\alpha \in \mathbf{C}$, $\varepsilon_1 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon_2$, we have $\mathscr{D}_{(A^{\alpha}xB^{-\alpha})} = \mathscr{D}_{B^{-\alpha}}$ the operator $A^{\alpha}xB^{-\alpha}$ is bounded and the mapping*

$$\alpha \mapsto A^{\alpha}xB^{-\alpha}$$

is so-continuous on $\{\alpha \in \mathbf{C};\ \varepsilon_1 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon_2\}$ and analytic in $\{\alpha \in \mathbf{C};\ \varepsilon_1 < \operatorname{Re}\alpha < \varepsilon_2\}$;

(iii) *the mapping*

$$it \mapsto A^{it}xB^{-it}, \qquad t \in \mathbf{R},$$

has a wo-continuous extension to the set $\{\alpha \in \mathbf{C};\ \varepsilon_1 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon_2\}$, which is analytic in $\{\alpha \in \mathbf{C};\ \varepsilon_1 < \operatorname{Re}\alpha < \varepsilon_2\}$.

Proof. It is obvious that we can consider only the case in which $\varepsilon_1 = 0$, $\varepsilon_2 = \varepsilon$ (see the proof of Corollary 9.21).

Let us assume that assertion (i) is true. We denote $\mathscr{D} = \mathscr{D}_2$ and $c = \|A^{\varepsilon}xB^{-\varepsilon}|\mathscr{D}\|$. For any $\xi \in \mathscr{D}$ and any $\eta \in \mathscr{D}_{A^{\varepsilon}}$ we have

$$|(xB^{-\varepsilon}\xi\,|\,A^{\varepsilon}\eta)| = |(A^{\varepsilon}xB^{-\varepsilon}\xi\,|\,\eta)| \leqslant c\|\xi\|\,\|\eta\|.$$

Since $\overline{B^{-\varepsilon}|\mathscr{D}} = B^{-\varepsilon}$, we infer that, for any $\xi \in \mathscr{D}_{B^{-\varepsilon}}$, $\eta \in \mathscr{D}_{A^{\varepsilon}}$, we have

$$|(xB^{-\varepsilon}\xi\,|\,A^{\varepsilon}\eta)| \leqslant c\|\xi\|\,\|\eta\|.$$

If we replace ξ by $B^{-it}\xi$ and η by $A^{-it}\eta$, we obtain that for any $\xi \in \mathscr{D}_{B^{-\varepsilon}}$ and any $\eta \in \mathscr{D}_{A^{\varepsilon}}$ we have

$$|(xB^{-\varepsilon-it}\xi \mid A^{\varepsilon-it}\eta)| \leqslant c\|\xi\| \, \|\eta\|, \quad t \in \mathbb{R}.$$

On the other hand, for any ξ and η we trivially have

$$|(xB^{-it}\xi \mid A^{-it}\eta)| \leqslant \|x\| \, \|\xi\| \, \|\eta\|, \quad t \in \mathbb{R}.$$

Let $\xi \in \mathscr{D}_{B^{-\varepsilon}}$ and $\eta \in \mathscr{D}_{A^{\varepsilon}}$. Then

$$\alpha \mapsto (xB^{-\alpha}\xi \mid A^{\bar{\alpha}}\eta)$$

is a bounded continuous function on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$, which is analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$. By taking into account the results we have already obtained, with the help of the Phragmen-Lindelöf principle (see N. Dunford and J. Schwartz [1], Ch. III, 14), we get that, for any $\alpha \in \mathbb{C}$, such that $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon$, we have

$$|(xB^{-\alpha}\xi \mid A^{\bar{\alpha}}\eta)| \leqslant \max\{c, \|x\|\} \, \|\xi\| \, \|\eta\|.$$

Obviously, this inequality extends for any $\xi \in \mathscr{D}_{B^{-\alpha}}$ and any $\eta \in \mathscr{D}_{A^{\bar{\alpha}}}$.

Let now $\alpha \in \mathbb{C}$, $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon$. From the preceding relation it follows that for any $\xi \in \mathscr{D}_{B^{-\alpha}}$ we have

$$xB^{-\alpha}\xi \in \mathscr{D}_{(A^{\bar{\alpha}})^*} = \mathscr{D}_{A^{\alpha}}$$

and

$$|(A^{\alpha}xB^{-\alpha}\xi \mid \eta)| \leqslant \max\{c, \|x\|\}, \ \|\xi\| \, \|\eta\|, \quad \eta \in \mathscr{D}_{A^{\bar{\alpha}}}.$$

In other words, we have

$$\mathscr{D}_{(A^{\alpha}xB^{-\alpha})} = \mathscr{D}_{B^{-\alpha}},$$

the operator $A^{\alpha}xB^{-\alpha}$ is bounded, and its norm is uniformly bounded with respect to α:

$$\|\overline{A^{\alpha}xB^{-\alpha}}\| \leqslant \max\{c, \|x\|\}.$$

With the help of the equality

$$((\overline{A^{\alpha}xB^{-\alpha}})\,\xi \mid \eta) = (xB^{-\alpha}\xi \mid A^{\bar{\alpha}}\eta), \quad 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon, \quad \xi \in \mathscr{D}_{B^{-\varepsilon}}, \quad \eta \in \mathscr{D}_{A^{\varepsilon}},$$

it is easy to infer that the mapping

$$\alpha \mapsto \overline{A^{\alpha}xB^{-\alpha}}$$

is *wo*-continuous on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ and analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$.

It is easy to verify that for any natural number n, any vector $\xi \in \chi_{(1/n, n)}(B)\mathscr{H}$ and any α, such that $0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon$, we have

$$xB^{-\alpha}\xi \in \mathscr{D}_{A^{\varepsilon}},$$

whence, for any β, such that $0 \leqslant \operatorname{Re}\beta \leqslant \varepsilon$, we have

$$\begin{aligned}\|\overline{(A^\beta x B^{-\beta})}\,\xi - \overline{(A^\alpha x B^{-\alpha})}\,\xi\| &\leqslant \|A^\beta x B^{-\beta}\,\xi - A^\beta x B^{-\alpha}\xi\| + \|A^\beta x B^{-\alpha}\xi - A^\alpha x B^{-\alpha}\xi\| \\ &= \|A^\beta x B^{-\beta}(\xi - B^{\beta-\alpha}\xi)\| + \|A^\beta x B^{-\alpha}\xi - A^\alpha x B^{-\alpha}\xi\| \\ &\leqslant \max\{c, \|x\|\}\|\xi - B^{\beta-\alpha}\xi\| + \|A^\beta(xB^{-\alpha}\xi) - A^\alpha(xB^{-\alpha}\xi)\|.\end{aligned}$$

With the help of Corollary 9.21, we infer from here that

$$\lim_{\beta\to\alpha} \|\overline{(A^\beta x B^{-\beta})}\,\xi - \overline{(A^\alpha x B^{-\alpha})}\,\xi\| = 0.$$

Since $\bigcup_n \chi_{(1/n,n)}(B)\mathscr{H}$ is dense in $\mathscr{H}$ and since

$$\sup\{\|\overline{A^\gamma x B^{-\gamma}}\|;\quad 0 \leqslant \operatorname{Re}\gamma \leqslant \varepsilon\} < +\infty,$$

the preceding equality extends for any $\xi \in \mathscr{H}$. Consequently, the mapping

$$\alpha \mapsto \overline{A^\alpha x B^{-\alpha}}$$

is *so*-continuous on $\{\alpha \in \mathbf{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$.

We have thus proved that (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) is trivial.

Finally, let us assume that assertion (iii) is true. We denote by F the *wo*-continuous extension to $\{\alpha \in \mathbf{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ of the mapping

$$it \mapsto A^{it} x B^{-it}, \qquad t \in \mathbb{R},$$

which is analytic in $\{\alpha \in \mathbf{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$.

For any $\xi \in \mathscr{D}_{B^{-\varepsilon}}$ and $\eta \in \mathscr{D}_{A^\varepsilon}$ the functions

$$\begin{aligned}&\alpha \mapsto (F(\alpha)\xi \mid \eta),\\ &\alpha \mapsto (xB^{-\alpha}\,\xi \mid A^{\bar\alpha}\eta),\end{aligned}$$

are continuous on $\{\alpha \in \mathbf{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$ and analytic in $\{\alpha \in \mathbf{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$; moreover, they coincide on the imaginary axis. Hence they coincide on $\{\alpha \in \mathbf{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$. In particular, we have

$$(xB^{-\varepsilon}\xi \mid A^\varepsilon\eta) = (F(\varepsilon)\xi \mid \eta), \quad \xi \in \mathscr{D}_{B^{-\varepsilon}}, \quad \eta \in \mathscr{D}_{A^\varepsilon},$$

whence, for any $\xi \in \mathscr{D}_{B^{-\varepsilon}}$ we have

$$xB^{-\varepsilon}\xi \in \mathscr{D}_{(A^\varepsilon)^*} = \mathscr{D}_{A^\varepsilon} \text{ and } A^\varepsilon x B^{-\varepsilon}\xi = F(\varepsilon)\xi.$$

Hence

$$\mathscr{D}_{(A^\varepsilon x B^{-\varepsilon})} = \mathscr{D}_{B^{-\varepsilon}}$$

and the operator $A^{\varepsilon}xB^{-\varepsilon}$ is bounded.

Consequently, (iii) $\Rightarrow$ (i).

Q.E.D.

9.25. The following proposition shows that the operational calculus is invariant with respect to $*$-isomorphisms and provides a natural method of *transfer by $*$-isomorphism* of the positive self-adjoint operators, which are affiliated to a von Neumann algebra.

Proposition. *Let π be a $*$-isomorphism of the von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ onto the von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{K})$. Obviously, π establishes a one-to-one correspondence*

$$\mathcal{M}^+ \ni a \mapsto b = \pi(a) \in \mathcal{N}^+.$$

This correspondence extends to a one-to-one correspondence

$$A \mapsto B = \pi(A)$$

between the positive self-adjoint operators A in $\mathcal{H}$, affiliated to $\mathcal{M}$, and the positive self-adjoint operators B in $\mathcal{K}$, affiliated to $\mathcal{N}$, which is unique, subject to the condition

$$\pi((1 + A)^{-1}) = (1 + B)^{-1}.$$

Moreover, for any positive $f \in \mathcal{B}([0, +\infty))$, we have

$$\pi(f(A)) = f(\pi(A)).$$

Proof. Let $a \in \mathcal{M}$, $a \geqslant 0$. By taking into account Corollary 5.13 it is easy to verify that the mapping

$$\mathcal{B}(\sigma(a)) \ni f \mapsto \pi(f(a))$$

satisfies conditions (i) and (ii) from Theorem 2.20, for $x = \pi(a)$. According to Theorem 2.20, it follows that for any $f \in \mathcal{B}(\sigma(a))$ we have

$$\pi(f(a)) = f(\pi(a)).$$

Let A be a positive self-adjoint operator in $\mathcal{H}$, which is affiliated to $\mathcal{M}$. We write $a = (1 + A)^{-1}$. Then

$$a \in \mathcal{M},\ 0 \leqslant a \leqslant 1,\ \mathrm{s}(a) = 1.$$

Consequently, if we define $b = \pi(a)$, we have

$$b \in \mathcal{N},\ 0 \leqslant b \leqslant 1,\ \mathrm{s}(b) = 1.$$

The operator $B = b^{-1} - 1$ is a positive self-adjoint operator in $\mathcal{K}$, affiliated to $\mathcal{N}$, and $b = (1 + B)^{-1}$.

From the preceding results, it follows that the correspondence

$$A \mapsto B = \pi(A),$$

we have thus defined, is a one-to-one correspondence between the positive self-adjoint operators A in $\mathscr{H}$, which are affiliated to $\mathscr{M}$, and the positive self-adjoint operators B in $\mathscr{K}$, which are affiliated to $\mathscr{N}$, and, moreover,

$$\pi((1 + A)^{-1}) = (1 + B)^{-1}.$$

Let now $f \in \mathscr{B}([0, +\infty))$ be positive and $g = (1 + f)^{-1} \in \mathscr{B}([0, +\infty))$, which obviously is positive and bounded. By taking into account Section 9.10 and by using the notations we have introduced there, we have

$$\pi((1 + f(A))^{-1}) = \pi(g(A)) = \pi(F_g(a)) = F_g(\pi(a)) = F_g(b)$$
$$= g(B) = (1 + f(B))^{-1} = (1 + f(\pi(A)))^{-1},$$

hence

$$\pi(f(A)) = f(\pi(A)).$$

Q.E.D.

With the preceding notations, if we also write

$$e_n = \chi_{[0,n)}(A), \quad f_n = \chi_{[0,n)}(B),$$

from the preceding proposition we also obtain, in particular, that

$$\pi(Ae_n) = Bf_n.$$

9.26. Let T be a linear operator in $\mathscr{H}$. Its *resolvent set* is defined by

$$\rho(T) = \{\lambda \in \mathbb{C};\, (\lambda - T)^{-1} \in \mathscr{B}(\mathscr{H}) \text{ exists}\}$$

and its *spectrum*, by

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

As in the case in which $T \in \mathscr{B}(\mathscr{H})$, it is easy to prove that $\rho(T)$ is an open set, whereas the function

$$\rho(T) \ni \lambda \mapsto (\lambda - T)^{-1} \in \mathscr{B}(\mathscr{H})$$

is analytic for the norm topology in $\mathscr{B}(\mathscr{H})$. In contrast to the case in which $T \in \mathscr{B}(\mathscr{H})$, in general $\sigma(T)$ can be either the empty set, or it can coincide with $\mathbb{C}$.

Proposition. *Let A be a self-adjoint linear operator in $\mathscr{H}$. Then $\sigma(A) \subset \mathbb{R}$ and, for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we have*

$$\|(\lambda - A)^{-1}\| \leqslant 1/|\operatorname{Im} \lambda|.$$

Proof. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. For any $\xi \in \mathscr{D}_A$ we have

$$\|(\lambda - A)\xi\|^2 = ((\lambda - A)\xi \mid (\lambda - A)\xi)$$
$$= |\operatorname{Im} \lambda|^2 \|\xi\|^2 + ((\operatorname{Re} \lambda - A)\xi \mid (\operatorname{Re} \lambda - A)\xi)$$
$$\geqslant |\operatorname{Im} \lambda|^2 \|\xi\|^2,$$

hence $\lambda - A$ is injective, its range is closed and $(\lambda - A)^{-1}$ is bounded, of norm $\leqslant 1/|\operatorname{Im} \lambda|$.

Let $\eta \in \mathscr{H}$ be orthogonal to $(\lambda - A)\mathscr{D}_A$. Then

$$(0, \eta) \in (\mathscr{G}_{(\lambda - A)})^{\perp} = (\mathscr{G}_{(\bar{\lambda} - A)^*})^{\perp} = V_{\mathscr{H}}\mathscr{G}_{(\bar{\lambda} - A)},$$

hence

$$(\eta, 0) \in \mathscr{G}_{(\bar{\lambda} - A)},$$

i.e.,

$$\eta \in \mathscr{D}_A, \quad (\bar{\lambda} - A)\eta = 0.$$

Since $\operatorname{Im} \bar{\lambda} = - \operatorname{Im} \lambda \neq 0$, from the first part of the proof we infer that the operator $\bar{\lambda} - A$ is injective, hence

$$\eta = 0.$$

Consequently, $(\lambda - A)^{-1}$ is everywhere defined and bounded, i.e., $\lambda \in \rho(A)$ and

$$\|(\lambda - A)^{-1}\| \leqslant 1/|\operatorname{Im} \lambda|.$$

Q.E.D.

For any $\lambda \in \mathbb{C}$, we define

$$d(\lambda, \mathbb{R}^+) = \inf \{|\lambda - \mu|;\ \mu \in \mathbb{R}^+\}.$$

Corollary. *Let A be a positive self-adjoint linear operator in $\mathscr{H}$. Then $\sigma(A) \subset \mathbb{R}^+$, and, for any $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$, we have*

$$\|(\lambda - A)^{-1}\| \leqslant 1/d(\lambda, \mathbb{R}^+).$$

Proof. From the preceding proposition and from Lemma 9.5 we infer that $\sigma(A) \subset \mathbb{R}^+$.

Let $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$. For any $\xi \in \mathscr{D}_A$ we have

$$\begin{aligned}
\|(\lambda - A)\xi\|^2 &= ((\lambda - A)\xi \mid (\lambda - A)\xi) \\
&= |\operatorname{Im} \lambda|^2 \|\xi\|^2 + ((\operatorname{Re} \lambda - A)\xi \mid (\operatorname{Re} \lambda - A)\xi) \\
&\geqslant \begin{cases} |\operatorname{Im} \lambda|^2 \|\xi\|^2, & \text{if } \operatorname{Re} \lambda > 0 \\ |\operatorname{Im} \lambda|^2 \|\xi\|^2 + |\operatorname{Re} \lambda|^2 \|\xi\|^2, & \text{if } \operatorname{Re} \lambda \leqslant 0 \end{cases} \\
&= d(\lambda, \mathbb{R}^+)^2 \|\xi\|^2;
\end{aligned}$$

hence

$$\|(\lambda - A)^{-1}\| \leqslant 1/d(\lambda, \mathbb{R}^+).$$

Q.E.D.

9.27. Proposition. *Let A be a positive self-adjoint linear operator in $\mathscr{H}$, f a bounded analytic function, defined on an open convex neighbourhood of the interval $[0, +\infty)$ and $\Gamma: \mathbb{R} \to \mathbb{C}$ a locally rectifiable Jordan curve, contained in the domain of definition of f, which contains in its "interior" the interval $[0, +\infty)$ with respect to which it is positively oriented. We assume that*

$$\int_{\Gamma} |f(\lambda)|\, d(\lambda, \mathbb{R}^+)^{-1}\, \mathrm{d}\lambda < +\infty.$$

Then, for any $\xi \in \mathscr{H}$, we have

$$f(A)\xi = (2\pi i)^{-1} \int_\Gamma f(\lambda)(\lambda - A)^{-1}\xi \, d\lambda.^{*)}$$

Proof. Let $e_n = \chi_{[0,n)}(A)$ (see 9.9). It is easy to verify that for any $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$ and any n we have

$$(\lambda - A)^{-1} e_n = (\lambda - Ae_n)^{-1} e_n.$$

Since Γ contains the interval $[0, +\infty)$ in its "interior" with respect to which it is positively oriented, there exist real numbers

$$-\infty \leftarrow \ldots < t_{-n} < \ldots < t_{-1} < t_1 < \ldots < t_n < \ldots \rightarrow +\infty$$

such that the curve Γ_n, obtained by composing the restriction of Γ to $[t_{-n}, t_n]$ with the segment $[\Gamma(t_n), \Gamma(t_{-n})] \subset \mathbb{C}$, should "contain" the interval $[0, n]$ in its interior.

By taking into account Theorem 2.29 and Cauchy's integral theorem, we infer that, for any $\xi \in e_n\mathscr{H}$, we have

$$\begin{aligned} f(A)\xi = f(Ae_n)\xi &= (2\pi i)^{-1} \int_{\Gamma_n} f(\lambda)(\lambda - Ae_n)^{-1}\xi \, d\lambda \\ &= (2\pi i)^{-1} \int_\Gamma f(\lambda)(\lambda - Ae_n)^{-1}\xi \, d\lambda^{*)} \\ &= (2\pi i)^{-1} \int_\Gamma f(\lambda)(\lambda - A)^{-1}\xi \, d\lambda. \end{aligned}$$

Let now $\xi \in \mathscr{H}$ be arbitrary. We denote $\xi_n = e_n\xi$. Then

$$\|\xi - \xi_n\| \to 0;$$

hence

$$\|f(A)\xi - f(A)\xi_n\| \to 0,$$
$$\|(\lambda - A)^{-1}\xi - (\lambda - A)^{-1}\xi_n\| \to 0.$$

Since, for any n, we have

$$f(A)\xi_n = (2\pi i)^{-1} \int_\Gamma f(\lambda)(\lambda - A)^{-1}\xi_n \, d\lambda,$$

with the help of the Lebesgue dominated convergence theorem we obtain

$$f(A)\xi = (2\pi i)^{-1} \int_\Gamma f(\lambda)(\lambda - A)^{-1}\xi \, d\lambda.$$

Q.E.D.

9.28. Proposition. *If T is a closed linear operator from $\mathscr{H}$ into $\mathscr{K}$, then T^*T is self-adjoint and positive. Moreover,*

$$\overline{T|\mathscr{D}_{T^*T}} = T.$$

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*) Some extra-conditions are necessary in order to insure that the integral over the segment $[\Gamma(t_n), \Gamma(t_{-n})]$ converges to zero for $n \to \infty$. This is automatically satisfied for the specific aplication of Proposition 9.27 in the proof of Lemma 1 from Section 10.19 [Translator's Note].

Proof. Let $\zeta \in \mathscr{H}$. Since $\mathscr{G}_T$ is closed in $\mathscr{H} \oplus \mathscr{K}$, we can write

$$(\zeta, 0) = (\xi, \eta) + (\xi_0, \eta_0), \quad (\xi, \eta) \in \mathscr{G}_T, \quad (\xi_0, \eta_0) \in (\mathscr{G}_T)^{\perp}.$$

Then

$$\xi \in \mathscr{D}_T, \ T\xi = \eta \text{ and } \eta_0 \in \mathscr{D}_{T^*}, \ T^*\eta_0 = -\xi_0,$$

hence

$$\zeta = \xi - T^*\eta_0, \quad 0 = T\xi + \eta_0,$$

whence

$$T\xi = -\eta_0 \in \mathscr{D}_{T^*}, \quad \zeta = \xi + T^*T\xi = (1 + T^*T)\xi.$$

Thus

$$(1 + T^*T)\mathscr{D}_{T^*T} = \mathscr{H}.$$

Let now $\zeta \in \mathscr{H}$, $\zeta \perp \mathscr{D}_{T^*T}$. In accordance with what we have already proved, there exists a $\xi \in \mathscr{D}_{T^*T}$, such that $\zeta = (1 + T^*T)\xi$.
Then

$$0 = (\zeta \mid \xi) = (\xi + T^*T\xi \mid \xi) = \|\xi\|^2 + \|T\xi\|^2,$$

whence $\xi = 0$ and $\zeta = 0$. It follows that $\mathscr{D}_{T^*T}$ is dense in $\mathscr{H}$. For any $\xi \in \mathscr{D}_{T^*T}$ we have

$$(T^*T\xi \mid \xi) = \|T\xi\|^2 \geqslant 0.$$

Thus, T^*T is positive.

Since $(1 + T^*T)\mathscr{D}_{T^*T} = \mathscr{H}$, from Lemma 9.5 we infer that T^*T is self-adjoint.

Finally, let $(\zeta, T\zeta) \in \mathscr{G}_T$ be orthogonal to the graph of the operator $T \mid \mathscr{D}_{T^*T}$. Then, for any $\xi \in \mathscr{D}_{T^*T}$,

$$(\zeta \mid (1 + T^*T)\xi) = (\zeta \mid \xi) + (T\zeta \mid T\xi) = 0$$

and, since $(1 + T^*T)\mathscr{D}_{T^*T} = \mathscr{H}$, it follows that $\zeta = 0$.

Consequently, we have $\overline{T \mid \mathscr{D}_{T^*T}} = T$.

Q.E.D.

For any closed linear operator T its *absolute value* (or *modulus*) $|T|$ is defined by

$$|T| = (T^*T)^{1/2}.$$

9.29. The notion of *partial isometry* extends to the case of two different Hilbert spaces: it is a bounded linear operator

$$v : \mathscr{H} \to \mathscr{K}$$

such that

$$\|v\xi\| = \|\xi\|, \quad \xi \in \mathbf{r}(v)\mathscr{H}.$$

If v is a partial isometry, then v^* is also a partial isometry, $v^*v = \mathrm{r}(v)$, $vv^* = \mathrm{l}(v)$. A bounded linear operator $v : \mathscr{H} \to \mathscr{K}$ is a partial isometry iff v^*v is a projection.

The following theorem extends Theorem 2.14.

Theorem *(of polar decomposition). Let T be a closed linear operator from $\mathscr{H}$ into $\mathscr{K}$. Then there exists a positive self-adjoint linear operator A in $\mathscr{H}$, and a partial isometry $v : \mathscr{H} \to \mathscr{K}$, such that*

$$T = vA,$$
$$v^*v = \mathrm{s}(A).$$

These conditions determine in a unique manner the operators A and v.

Moreover,

$$A = |T|,$$
$$v^*v = \mathrm{r}(T), \quad vv^* = \mathrm{l}(T).$$

Proof. In accordance with Corollary 9.14, we have $|T|^2 = T^*T$. Hence, for any $\xi \in \mathscr{D}_{T^*T}$,

$$\||T|\xi\|^2 = (|T|^2\xi \mid \xi) = (T^*T\xi \mid \xi) = \|T\xi\|^2.$$

Consequently, the relations

$$v(|T|\xi) = T\xi, \quad \xi \in \mathscr{D}_{T^*T},$$
$$v(\eta) = 0, \quad \eta \in (|T|\mathscr{D}_{T^*T})^{\perp},$$

determine a partial isometry $v : \mathscr{H} \to \mathscr{K}$.

Let $\xi \in \mathscr{D}_{|T|}$. By taking into account Proposition 9.28, we infer that there exists a sequence $\{\xi_n\} \subset \mathscr{D}_{T^*T}$, such that

$$\xi_n \to \xi, \quad |T|\xi_n \to |T|\xi.$$

Since T is closed and

$$T\xi_n = v(|T|\xi_n) \to v(|T|\xi),$$

it follows that

$$\xi \in \mathscr{D}_T, \quad T\xi = v(|T|\xi).$$

Moreover, since $\|v(|T|\xi)\| = \||T|\xi\|$, it follows that

$$|T|\xi \in v^*v\mathscr{H}.$$

Thus,

$$v^*v\mathscr{H} = \overline{|T|\mathscr{D}_{T^*T}} \subset \overline{|T|\mathscr{D}_{|T|}} \subset v^*v\mathscr{H},$$
$$v^*v\mathscr{H} = \overline{|T|\mathscr{D}_{|T|}},$$

i.e.,

$$v^*v = \mathrm{s}(|T|).$$

Let now $\xi \in \mathscr{D}_T$. In accordance with Proposition 9.28, there exists a sequence $\{\xi_n\} \subset \mathscr{D}_{T^*T}$, such that

$$\xi_n \to \xi, \quad T\xi_n \to T\xi.$$

Since

$$\||T|\xi_n - |T|\xi_m\| = \|T\xi_n - T\xi_m\|,$$

the sequence $\{|T|\xi_n\}$ is convergent. Since $|T|$ is closed, $\xi \in \mathscr{D}_{|T|}$.

We have thus already proved that

$$\mathscr{D}_{|T|} = \mathscr{D}_T,$$
$$T\xi = v(|T|\xi), \quad \xi \in \mathscr{D}_T.$$

Consequently, we have

$$T = v|T|.$$

Let A' be a positive self-adjoint operator in $\mathscr{H}$ and $v' : \mathscr{H} \to \mathscr{K}$ a partial isometry, such that

$$T = v'A',$$
$$v'^*v' = \mathrm{s}(A').$$

Then $v'^*v'A' = A'$, and therefore,

$$|T|^2 = T^*T = A'v'^*v'A' = (A')^2.$$

From Corollary 9.14 we infer that

$$A' = |T|.$$

On the other hand, the relations

$$v'(|T|\xi) = v'A'\xi = T\xi, \quad \xi \in \mathscr{D}_T,$$
$$v'(\eta) = 0, \quad \eta \in (|T|\mathscr{D}_{|T|})^\perp = (A'\mathscr{D}_{A'})^\perp,$$

show that v' is determined by T in a unique manner.

Q.E.D.

Let T be a closed linear operator in $\mathscr{H}$ and let

$$T = v|T|, \quad v^*v = \mathrm{s}(|T|),$$

be its polar decomposition. If T is affiliated to a von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$, then $v \in \mathscr{M}$ and $|T|$ is affiliated to $\mathscr{M}$. In particular, $\mathrm{r}(T), \mathrm{l}(T) \in \mathscr{M}$.

9.30. Let T be a closed linear operator from $\mathscr{H}$ into $\mathscr{K}$ and let

$$T = v|T|, \quad v^*v = \mathrm{s}(|T|)$$

be its polar decomposition. Since the operator $v|T|v^*$ is self-adjoint and positive, from

$$T^* = v^*(v|T|v^*),$$
$$vv^* = s(v|T|v^*),$$

we infer that these two relations yield the polar decomposition of T^*.

In particular,

$$|T^*| = v|T|v^*,$$

and

$$T = |T^*|v, \quad vv^* = s(|T^*|).$$

9.31. Corollary. *Let A be a self-adjoint linear operator in $\mathscr{H}$. Then there exist two positive, self-adjoint linear operators A^+, A^- in $\mathscr{H}$, such that*

$$A = A^+ - A^-,$$
$$s(A^+)\,s(A^-) = 0.$$

Moreover, these conditions determine the operators A^+, A^- in a unique manner.

Proof. Let $A = v|A|$ the polar decomposition of A. In accordance with Section 9.30, the polar decomposition of A^* is $A^* = v^*(v|A|v^*)$. Since $A = A^*$, from the uniqueness of the polar decomposition we infer that

$$v = v^*,$$
$$|A| = v|A|v^*.$$

Consequently, $v = e - f$, where $e, f \in \mathscr{B}(\mathscr{H})$ are projections, $ef = 0$ and

$$A = (e-f)|A| = |A|(e-f).$$

Since $e + f = v^*v = s(A)$, it follows that

$$|A| = (e+f)|A| = |A|(e+f).$$

Consequently,

$$\frac{1}{2}(|A| + A) = e|A| = |A|e|\mathscr{D}_A,$$

$$\frac{1}{2}(|A| - A) = f|A| = |A|f|\mathscr{D}_A.$$

In particular

$$e\mathscr{D}_A \subset \mathscr{D}_A, \quad f\mathscr{D}_A \subset \mathscr{D}_A.$$

We define

$$A^+ = |A|e, \quad A^- = |A|f.$$

It is easy to see that A^+, A^- are positive operators.

For any $\eta \in \mathscr{H}$, there exists a $\xi \in \mathscr{D}_A$, such that

$$\eta = \xi + |A|\xi.$$

Then $\zeta = e\xi + (1-e)\eta \in \mathscr{D}_{A^+}$ and

$$\zeta + A^+\zeta = e\xi + (1-e)\eta + |A|\,e\xi = e(\xi + |A|\,\xi) + (1-e)\eta = \eta.$$

Then, in accordance with Lemma 9.5, A^+ is self-adjoint. In an analogous manner one shows that A^- is self-adjoint.

Obviously,

$$A \subset A^+ - A^-.$$

On the other hand,

$$\begin{aligned}\mathscr{D}_{(A^+-A^-)} &= \{\xi \in \mathscr{H};\ e\xi \in \mathscr{D}_A,\ f\xi \in \mathscr{D}_A\}\\ &\subset \{\xi \in \mathscr{H};\ (e-f)\,\xi \in \mathscr{D}_A\}\\ &= \mathscr{D}_{(|A|(e-f))} = \mathscr{D}_A.\end{aligned}$$

Thus

$$A = A^+ - A^-.$$

Finally, since $s(A^+) \leqslant e$ and $s(A^-) \leqslant f$, we have

$$s(A^+)\,s(A^-) = 0.$$

The uniqueness part of the corollary can be easily obtained from the uniqueness of the polar decomposition of A.

Q.E.D.

If A is a self-adjoint linear operator in $\mathscr{H}$, which is affiliated to a von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$, then A^+ and A^- are affiliated to $\mathscr{M}$.

9.32. In this section we indicate a method for extending the operational calculus, defined for positive self-adjoint operators (9.9), to arbitrary self-adjoint operators.

We shall denote by $\mathscr{B}((-\infty, +\infty))$ the $*$-algebra of all Borel measurable complex functions, defined on $(-\infty, +\infty)$, which are bounded on compact subsets. For any $f \in \mathscr{B}((-\infty, +\infty))$, we define the function $\hat{f} \in \mathscr{B}((-\infty, +\infty))$ by the relation

$$\hat{f}(\lambda) = f(-\lambda), \quad \lambda \in \mathbb{R}.$$

Let A be a self-adjoint linear operator in $\mathscr{H}$. For any $f \in \mathscr{B}((-\infty, +\infty))$ we define

$$f(A) = (f\chi_{(0,+\infty)})(A^+) + (\hat{f}\chi_{(0,+\infty)})(A^-) + f(0)(1 - s(A)).$$

For the operational calculus with self-adjoint operators, defined in this manner, it is easy to verify that properties, analogous to those already established for the case of the positive self-adjoint operators, hold, too (see 9.11—9.13).

9.33. In the final sections of this chapter we shall study the tensor product of linear operators.

Let T_j be a linear operator from $\mathscr{H}_j$ into $\mathscr{K}_j$, $j = 1, 2$. We define a linear operator $T_1 \otimes T_2$ from $\mathscr{H}_1 \overline{\otimes} \mathscr{H}_2$ into $\mathscr{K}_1 \overline{\otimes} \mathscr{K}_2$ by the following relations

$\mathscr{D}_{T_1 \otimes T_2}$ = the vector subspace of $\mathscr{H}_1 \overline{\otimes} \mathscr{H}_2$ generated by

$$\{\xi_1 \otimes \xi_2;\ \xi_1 \in \mathscr{D}_{T_1},\ \xi_2 \in \mathscr{D}_{T_2}\},$$

$$(T_1 \otimes T_2)(\xi_1 \otimes \xi_2) = (T_1\xi_1) \otimes (T_2\xi_2), \quad \xi_1 \in \mathscr{D}_{T_1}, \quad \xi_2 \in \mathscr{D}_{T_2}.$$

If the operator $T_1 \otimes T_2$ is preclosed, then one denotes

$$T_1 \overline{\otimes} T_2 = \overline{T_1 \otimes T_2},$$

and the operator $T_1 \overline{\otimes} T_2$ is called the *tensor product* of the operators T_1 and T_2.

For example, let us assume that the operators T_1 and T_2 are *closed*. Then the operators T_1, T_2, T_1^*, T_2^* are densely defined and, for any $\xi_1 \in \mathscr{D}_{T_1}$, $\xi_2 \in \mathscr{D}_{T_2}$, $\eta_1 \in \mathscr{D}_{T_1^*}$, $\eta_2 \in \mathscr{D}_{T_2^*}$ we have

$$((T_1 \otimes T_2)(\xi_1 \otimes \xi_2) \mid \eta_1 \otimes \eta_2) = (\xi_1 \otimes \xi_2 \mid (T_1^* \otimes T_2^*)(\eta_1 \otimes \eta_2)),$$

whence we immediately infer that the operator $T_1 \otimes T_2$ is preclosed and, therefore, it makes sense to consider the *closed* operator $T_1 \overline{\otimes} T_2$.

Obviously, if T_1 and T_2 are bounded operators, then $T_1 \overline{\otimes} T_2$ is bounded, whereas if $T_1 \in \mathscr{B}(\mathscr{H}_1)$ and $T_2 \in \mathscr{B}(\mathscr{H}_2)$, then $T_1 \overline{\otimes} T_2$ coincides with the tensor product already defined in Section 2.33.

Proposition. *Let T_j be a closed linear operator from $\mathscr{H}_j$ into $\mathscr{K}_j$, $j = 1, 2$. Then*

$$(T_1 \overline{\otimes} T_2)^* = T_1^* \overline{\otimes} T_2^*.$$

Proof. From the preceding argument, which allowed the definition of $T_1 \overline{\otimes} T_2$, we also infer the relation

$$T_1^* \overline{\otimes} T_2^* \subset (T_1 \overline{\otimes} T_2)^*.$$

Let

$$(\sigma, \tau) \in \mathscr{G}_{(T_1 \overline{\otimes} T_2)^*}, \quad (\sigma, \tau) \perp \mathscr{G}_{(T_1^* \overline{\otimes} T_2^*)}$$

Then, for any $\xi_1 \in \mathscr{D}_{T_1}$, $\xi_2 \in \mathscr{D}_{T_2}$, we have

$$(T_1\xi_1 \otimes T_2\xi_2 \mid \sigma) = (\xi_1 \otimes \xi_2 \mid \tau),$$

and, for any $\eta_1 \in \mathscr{D}_{T_1^*}$, $\eta_2 \in \mathscr{D}_{T_2^*}$,

$$(\eta_1 \otimes \eta_2 \mid \sigma) + (T_1^*\eta_1 \otimes T_2^*\eta_2 \mid \tau) = 0.$$

Consequently, for any $\xi_1 \in \mathscr{D}_{T_1^*T_1}$, $\xi_2 \in \mathscr{D}_{T_2^*T_2}$, we have

$$\begin{aligned}
&((1 + (T_1^*T_1 \overline{\otimes} T_2^*T_2))(\xi_1 \otimes \xi_2) \mid \tau) \\
&= (\xi_1 \otimes \xi_2 \mid \tau) + (T_1^*T_1\xi_1 \otimes T_2^*T_2\xi_2 \mid \tau) \\
&= (T_1\xi_1 \otimes T_2\xi_2 \mid \sigma) + (T_1^*(T_1\xi_1) \otimes T_2^*(T_2\xi_2) \mid \tau) = 0.
\end{aligned}$$

Consequently, we have

$$\tau \perp (1 + (T_1^*T_1 \,\overline{\otimes}\, T_2^*T_2))\mathscr{D}_{T_1^*T_1 \overline{\otimes} T_2^*T_2}.$$

In accordance with Proposition 9.28, the operators $T_1^*T_1$ and $T_2^*T_2$ are positive and self-adjoint. If we write, for any natural number n,

$$e_n^1 = \chi_{(1/(1+n),\, +\infty)}((1 + T_1^*T_1)^{-1}),$$
$$e_n^2 = \chi_{(1/(1+n),\, +\infty)}((1 + T_2^*T_2)^{-1}),$$

then the operators

$$(T_1^*T_1 \,\overline{\otimes}\, T_2^*T_2)(e_n^1 \otimes e_n^2) = (e_n^1 \,\overline{\otimes}\, e_n^2)(T_1^*T_1 \,\overline{\otimes}\, T_2^*T_2)(e_n^1 \,\overline{\otimes}\, e_n^2)$$
$$= (T_1^*T_1e_n^1) \,\overline{\otimes}\, (T_2^*T_2e_n^2)$$

are defined everywhere, bounded and positive. Hence

$$\bigcup_{n=1}^{\infty} (e_n^1 \,\overline{\otimes}\, e_n^2)(\mathscr{H}_1 \,\overline{\otimes}\, \mathscr{H}_2)$$

$$= \bigcup_{n=1}^{\infty} (e_n^1 \,\overline{\otimes}\, e_n^2)(1 + (T_1^*T_1 \,\overline{\otimes}\, T_2^*T_2)(e_n^1 \,\overline{\otimes}\, e_n^2))(\mathscr{H}_1 \,\overline{\otimes}\, \mathscr{H}_2)$$

$$= \bigcup_{n=1}^{\infty} (1 + (T_1^*T_1 \,\overline{\otimes}\, T_2^*T_2))(e_n^1 \,\overline{\otimes}\, e_n^2)(\mathscr{H}_1 \,\overline{\otimes}\, \mathscr{H}_2)$$

$$\subset (1 + (T_1^*T_1 \,\overline{\otimes}\, T_2^*T_2))\,\mathscr{D}_{T_1^*T_1 \overline{\otimes} T_2^*T_2}.$$

Consequently,

$$(1 + (T_1^*T_1 \,\overline{\otimes}\, T_2^*T_2))\,\mathscr{D}_{T_1^*T_1 \overline{\otimes} T_2^*T_2}$$

is a dense vector subspace of $\mathscr{H}$.

From the preceding argument we infer that $\tau = 0$.

Since, for any $\eta_1 \in \mathscr{D}_{T_1^*}$, $\eta_2 \in \mathscr{D}_{T_2^*}$,

$$(\eta_1 \,\overline{\otimes}\, \eta_2 \mid \sigma) = -(T_1^*\eta_1 \,\overline{\otimes}\, T_2^*\eta_2 \mid \tau) = 0,$$

it follows that $\sigma = 0$.

Consequently, we have $\mathscr{G}_{(T_1 \overline{\otimes} T_2)^*} = \mathscr{G}_{(T_1^* \overline{\otimes} T_2^*)}$, hence

$$(T_1 \,\overline{\otimes}\, T_2)^* = T_1^* \,\overline{\otimes}\, T_2^*.$$

Q.E.D.

9.34. By taking into account Proposition 9.33 and Corollary 9.14, one obtains the following

Corollary. *If T_1, T_2 are self-adjoint (resp. positive self-adjoint) linear operators, then $T_1 \overline{\otimes} T_2$ is a self-adjoint (resp., positive self-adjoint) linear operator.*

9.35. Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. One says that S is an *antilinear* operator from $\mathscr{H}$ into $\mathscr{K}$ if S is a mapping from a vector subspace $\mathscr{D}_S \subset \mathscr{H}$ into $\mathscr{K}$, such that

$$S(\xi + \eta) = S\xi + S\eta, \quad \xi, \eta \in \mathscr{D}_S,$$

$$S(\lambda\xi) = \bar{\lambda} S\xi, \quad \lambda \in \mathbf{C}, \quad \xi \in \mathscr{D}_S.$$

The notions and the results from Sections 9.1, 9.2, 9.3, 9.28, 9.30, and 9.33 obviously extend to the antilinear operators.

The antilinear operators will be used often in Chapter 10. For this reason we formulate below some statements about antilinear operators, which we have proved for linear operators.

The product of two antilinear operators is linear. The product of an antilinear operator by a linear operator is antilinear.

The adjoint S^* of a densely defined antilinear operator S is an antilinear operator and it is defined by the relations

$$(S\xi \mid \eta) = (S^*\eta \mid \xi), \quad \xi \in \mathscr{D}_S, \quad \eta \in \mathscr{D}_{S^*}.$$

If S is preclosed, then S^* is closed and $S^{**} = \bar{S}$. If S is closed, then S^*S is a positive self-adjoint linear operator.

If S is a closed antilinear operator from $\mathscr{H}$ into $\mathscr{K}$, then there exist a positive self-adjoint linear operator A, in $\mathscr{H}$, and an antilinear partial isometry $v : \mathscr{H} \to \mathscr{K}$, such that

$$S = vA,$$

$$v^*v = \mathrm{s}(A).$$

These conditions determine in a unique manner the operators A and v, and the above relations give the polar decomposition of S.

An antilinear operator $J : \mathscr{H} \to \mathscr{K}$ is called a *conjugation* if $J = J^* = J^{-1}$. Conjugations are the antilinear analogues of the self-adjoint unitary operators.

Exercises

In the exercises in which the symbols $\mathscr{H}$, $\mathscr{K}$ are not explained they will denote Hilbert spaces.

E.9.1. Let T be a closed linear operator in $\mathscr{H}$, and $\mathscr{D}$ a vector subspace of $\mathscr{D}_T$. The following assertions are equivalent

(i) $\overline{T \mid \mathscr{D}} = T$;

(ii) $(1 + |T|)\mathscr{D}$ is dense in $\mathscr{H}$.

E.9.2. Let T_1, T_2 be linear operators from $\mathscr{H}$ into $\mathscr{K}_1$, $\mathscr{K}_2$, respectively. Show that if T_1 is closed, T_2 is preclosed and $\mathscr{D}_{T_1} \subset \mathscr{D}_{T_2}$, then there exists a constant $c > 0$, such that

$$\|T_2\xi\|^2 \leqslant c(\|T_1\xi\|^2 + \|\xi\|^2), \quad \xi \in \mathscr{D}_{T_1}.$$

E.9.3. Let T be a closed linear operator from $\mathscr{H}$ into $\mathscr{K}$. With the help of the Hahn-Banach and Banach-Steinhauss theorems, show that the following assertions are equivalent

(i) $T(\mathscr{D}_T) = \mathscr{K}$;

(ii) there exists a constant $c > 0$, such that

$$\|\eta\| \leqslant c\|T^*\eta\|, \quad \eta \in \mathscr{D}_{T^*}.$$

By assuming that condition (ii) is satisfied, show that for any $\xi \in \mathscr{H}$, $\xi \perp \mathfrak{n}(T)$, here exists an $\eta \in \mathscr{D}_{T^*}$, such that

$$T^*\eta = \xi \quad \text{and} \quad \|\eta\| \leqslant c\|\xi\|.$$

E.9.4. Let T be a linear operator in $\mathscr{H}$. Show that if $\mathscr{D}_T = \mathscr{H}$, and $\mathscr{D}_{T^*} = \mathscr{H}$, then $T \in \mathscr{B}(\mathscr{H})$.

E.9.5. Let T, S be normal linear operators in $\mathscr{H}$. Show that if $T \subset S$, then $T = S$.

Infer from this result that any normal symmetric operator is self-adjoint.

E.9.6. Let A be a symmetric linear operator in $\mathscr{H}$. Then A is self-adjoint iff $(\mathrm{i} + A)\mathscr{D}_A = \mathscr{H}$.

E.9.7. Let A be a linear operator in $\mathscr{H}$. Show that the following assertions are equivalent

(1) A is self-adjoint;

(2) for any non-zero $t \in \mathbb{R}$, we have $it \in \rho(A)$ and

$$\|(it + A)^{-1}\| \leqslant \frac{1}{|t|}.$$

(Hint: by assuming that (2) is satisfied, show that

$$\|A\xi\|^2 \geqslant 2t\,\mathrm{Im}\,(A\xi \mid \xi), \quad t \in \mathbb{R},$$

whence $(A\xi \mid \xi) \in \mathbb{R}$; if $\zeta \in \mathscr{D}_A$ and $\xi = (\mathrm{i} + A)^{-1}\zeta$, then $\|\zeta\|^2 = \|A\xi\|^2 + \mathrm{i}(A\xi \mid \xi)$, whence $\zeta = 0$).

E.9.8. Let A be a symmetric linear operator in $\mathscr{H}$. Then the linear operator $(A + \mathrm{i})$ is injective, hence one can define a linear operator $V(A)$ in $\mathscr{H}$ by

$$V(A)\xi = (A - \mathrm{i})(A + \mathrm{i})^{-1}\xi, \quad \xi \in \mathscr{D}_{V(A)} = (A + \mathrm{i})\mathscr{D}_A.$$

Show that $V(A)$ is an isometric linear operator, i.e.,

$$\|V(A)\xi\| = \|\xi\|, \quad \xi \in \mathscr{D}_{V(A)},$$

and that $(1 - V(A))\mathscr{D}_{V(A)}$ is a dense vector subspace of $\mathscr{H}$.

Let V be an isometric linear operator in $\mathscr{H}$, such that the vector subspace $(1 - V)\mathscr{D}_V$ is dense in $\mathscr{H}$. Then the linear operator $(1 - V)$ is injective, hence one can define a linear operator $A(V)$ in $\mathscr{H}$ by

$$A(V)\,\xi = \mathrm{i}(1 + V)(1 - V)^{-1}\xi, \quad \xi \in \mathscr{D}_{A(V)} = (1 - V)\,\mathscr{D}_V.$$

Show that $A(V)$ is a symmetric linear operator.

With the foregoing notations, we have

(1) $A(V(A)) = A$,
$V(A(V)) = V$,

(2) A is closed iff $V(A)$ is closed,
V is closed iff $A(V)$ is closed,

(3) A is self-adjoint iff $V(A)$ is unitary,
V is unitary iff $A(V)$ is self-adjoint,

(4) $A_1 \subset A_2$ iff $V(A_1) \subset V(A_2)$,
$V_1 \subset V_2$ iff $A(V_1) \subset A(V_2)$.

The operator $V(A)$ is called the *Cayley transform* of the symmetric linear operator A.

E.9.9. Let S be a closed symmetric operator in $\mathscr{H}$.

Show that there exist an isometry v of $\mathscr{H}$ into a Hilbert space $\mathscr{K}$, and a self-adjoint operator A in $\mathscr{K}$, such that

$$\mathscr{D}_S = \{\xi \in \mathscr{H};\ v\xi \in \mathscr{D}_A\},$$
$$S = v^* A v\xi, \quad \xi \in \mathscr{D}_S.$$

E.9.10. Let A be a positive self-adjoint operator in $\mathscr{H}$.

For any $\lambda \in (0, +\infty)$ we define the *spectral projection*

$$e_\lambda = \chi_{[0,\lambda)}(A).$$

Show that

(i) $e_\lambda \in \mathscr{R}(\{(1 + A)^{-1}\})$;

(ii) $\lambda_1 \leqslant \lambda_2 \Rightarrow e_{\lambda_1} \leqslant e_{\lambda_2}$;

(iii) $\lambda_n \uparrow \lambda \Rightarrow e_{\lambda_n} \uparrow e_\lambda$;

(iv) $e_\lambda \underset{\lambda\to 0}{\downarrow} 1 - \mathrm{s}(A)$, $\quad e_\lambda \underset{\lambda\to\infty}{\uparrow} 1$;

(v) $Ae_\lambda \leqslant \lambda e_\lambda$, $\quad A(1 - e_\lambda) \geqslant \lambda(1 - e_\lambda)$;

(vi) $\mathscr{D}_A = \left\{\xi \in \mathscr{H};\ \int_0^\infty \lambda^2\,\mathrm{d}(e_\lambda\xi \mid \xi) < +\infty\right\}$, and, for any $\xi \in \mathscr{D}_A$, we have

$$A\xi = \int_0^\infty \lambda\,\mathrm{d}e_\lambda\xi,$$

with norm convergent vector Stieltjes integral.

These assertions make up the contents of the so-called *spectral theorem* for the positive self-adjoint operator A, and the family of projections $\{e_\lambda\}_{\lambda\in[0,+\infty)}$ is called the *spectral scale* of A.

E.9.11. Let A be a positive self-adjoint operator in $\mathscr{H}$, and $f\in\mathscr{B}([0,+\infty))$. Show that

$$\mathscr{D}_{f(A)}=\left\{\xi\in\mathscr{H};\int_0^\infty|f(\lambda)|^2\,\mathrm{d}(e_\lambda\xi\mid\xi)<+\infty\right\}$$

and that, for any $\xi\in\mathscr{D}_{f(A)}$,

$$f(A)\xi=\int_0^\infty f(\lambda)\,\mathrm{d}e_\lambda\xi,$$

with a norm convergent vector Stieltjes integral.

E.9.12. Let A be a self-adjoint operator in $\mathscr{H}$, and $A=A^+-A^-$ the decomposition given by Corollary 9.31. We define the spectral scale of A by

$$e_\lambda=\begin{cases}\chi_{[0,\lambda)}(A^+), & \text{if } \lambda>0,\\ \mathrm{s}(A^-), & \text{if } \lambda=0,\\ \chi_{(-\lambda,+\infty)}(A^-), & \text{if } \lambda<0.\end{cases}$$

Extend to this case the assertions from exercises E.9.10, E.9.11.

In this manner one obtains the spectral theorem and the integral formula of the operational calculus for self-adjoint operators.

E.9.13. Let A be a positive self-adjoint operator in $\mathscr{H}$. For any Borel measurable subset D of the spectrum of A we define the *spectral projection* of A, which corresponds to D, by the formula

$$e(D)=\chi_D(A).$$

Then $e(\sigma(A))=1$. Show that for any $f\in\mathscr{B}([0,+\infty))$ we have

$$\|f(A)\|=\inf_{\substack{D\subset\sigma(A)\\ e(D)=1}}\ \sup_{\lambda\in D}|f(\lambda)|$$

and

$$\sigma(f(A))=\bigcap_{\substack{D\subset\sigma(A)\\ e(D)=1}}\overline{\{f(\lambda);\ \lambda\in D\}}\subset\overline{\{f(\lambda);\ \lambda\in\sigma(A)\}}.$$

E.9.14. Let A be a positive self-adjoint operator in $\mathscr{H}$ and $f\in\mathscr{B}([0,+\infty))$, real and positive. Then, for any $g\in\mathscr{B}([0,+\infty])$, we have

$$g(f(A))=(g\circ f)(A).$$

Infer from this result a new proof for the uniqueness of the positive square root of A (cf. Corollary 9.14).

E.9.15. Let A be a positive self-adjoint operator in $\mathscr{H}$ and $\mathscr{H}_\lambda$ the range of the projection $\chi_{[0,\lambda]}(A)$, $\lambda \in [0, +\infty)$. Show that, for any $\lambda \in [0, +\infty)$,

$$\mathscr{H}_\lambda = \left\{\xi \in \bigcap_{n=1}^{\infty} \mathscr{D}_{A^n};\ \overline{\lim_{n\to\infty}}\ \|A^n\xi\|^{1/n} \leqslant \lambda\right\}.$$

Infer from this result a new proof for the uniqueness of the positive square root of A.

E.9.16. Let A be a positive self-adjoint operator in $\mathscr{H}$, such that $s(A) = 1$, and let $0 \neq \xi \in \bigcap_{n=-\infty}^{\infty} \mathscr{D}_{A^n}$. Then

$$\overline{\lim_{n\to\infty}}\ \|A^n\xi\|^{1/n}\ \overline{\lim_{m\to\infty}}\ \|A^{-m}\xi\|^{1/m} \geqslant 1,$$

and the equality holds iff

$$A\xi = (\overline{\lim_{n\to\infty}}\ \|A^n\xi\|^{1/n})\xi.$$

(Hint: one can use Corollary 9.15 and the "theorem of the three lines", from N. Dunford and J. Schwartz, [1], VI, 10.3.)

E.9.17. For any function $f \in \mathscr{L}^1(\mathbb{R})$ we denote by $\hat{f}$ the "inverse Fourier transform" of f:

$$\hat{f}(s) = \int_{-\infty}^{+\infty} f(t)\, e^{ist}\, dt, \quad s \in \mathbb{R}.$$

With the same hypotheses and notations as in E.9.15, show that, for any $\lambda \in [0, +\infty)$,

$$\mathscr{H}_\lambda = \left\{\xi \in \mathscr{H};\ \text{for } f \in \mathscr{L}^1(\mathbb{R}), \text{ such that support } \hat{f} \subset (\ln \lambda, +\infty),\right.$$

$$\left.\text{we have } \int_{-\infty}^{+\infty} f(t) A^{it}\, dt = 0\right\}.$$

E.9.18. For any linear operator T in $\mathscr{H}$ and any $\xi \in \mathscr{D}_T$ we denote

$$E_\xi(T) = (T\xi \mid \xi), \quad \sigma_\xi(T) = \|(T - E_\xi(T))\xi\|.$$

Let A and B be self-adjoint operators in $\mathscr{H}$, such that the intersection

$$\mathscr{D} = \mathscr{D}_{AB} \cap \mathscr{D}_{BA}$$

be dense in $\mathscr{H}$. Show that

$$\sigma_\xi(A)\, \sigma_\xi(B) \geqslant \frac{1}{2} |E_\xi(AB - BA)|, \quad \xi \in \mathscr{D}.$$

This inequality is a variant of Heisenberg's "uncertainty principle".

E.9.19. Let A be a self-adjoint operator in $\mathscr{H}$. Show that the following assertions are equivalent

(i) A is positive;

(ii) $\sigma(A) \subset [0, +\infty)$.

E.9.20. One says that a linear operator T in $\mathscr{H}$ *commutes* with an operator $x \in \mathscr{B}(\mathscr{H})$ if

$$xT \subset Tx.$$

Show that if T is closed, then the set $\{T\}'$ of all operators $x \in \mathscr{B}(\mathscr{H})$ which commute with T is a *wo*-closed subalgebra of $\mathscr{B}(\mathscr{H})$ and

$$(\{T\}')^* = \{T^*\}'.$$

In particular, if T is self-adjoint, then $\{T\}'$ is a von Neumann algebra (we mention the fact that $\{T\}'$ is a von Neumann algebra even if T is normal, this being an extension of Fuglede's theorem 2.31).

E.9.21. Let T be a closed linear operator in $\mathscr{H}$ and $x \in \mathscr{B}(\mathscr{H})$ an operator which commutes with T. Show that

$$\overline{xT} = Tx.$$

E.9.22. Let T be a closed linear operator and $T = v|T|$ its polar decomposition. Show that if T is normal, then $\mathbf{l}(T) = \mathbf{r}(T)$ and

$$v|T| = |T|v.$$

Conversely, if $\mathbf{l}(T) = \mathbf{r}(T)$ and if $|T|$ commutes with v, then T is normal.

E.9.23. Let A be a positive self-adjoint operator in $\mathscr{H}$ and $x \in \mathscr{B}(\mathscr{H})$. Then the following assertions are equivalent

(i) A commutes with x;

(ii) $(1 + A)^{-1}$ commutes with x;

(iii) e_λ commutes with x, $\lambda \in (0, +\infty)$;

(iv) A^{it} commutes with x, $t \in \mathbb{R}$.

Show that if A commutes with x, then, for any $f \in \mathscr{B}([0, +\infty))$, the operator $xf(A)$ is preclosed and

$$\overline{xf(A)} = f(A)x.$$

Show that if e is a projection which commutes with A and if $f \in \mathscr{B}([0, +\infty))$, $f(0) = 0$, then

$$f(Ae) = f(A)e.$$

E.9.24. Let A and B be positive self-adjoint operators in $\mathscr{H}$, and $\{e_\lambda\}$ (resp., $\{f_\mu\}$) the spectral scale of A (resp., B). Show that the following assertions are equivalent

(i) $(1 + A)^{-1}$ commutes with $(1 + B)^{-1}$;

(ii) e_λ commutes with f_μ, $\lambda, \mu \in (0, +\infty)$;

(iii) A^{it} commutes with B^{is}, $t, s \in \mathbb{R}$.

In this case one says that A and B *commute*.

!E.9.25. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and A a positive self-adjoint operator in $\mathscr{H}$. Show that the following assertions are equivalent

(i) A is affiliated to $\mathscr{M}$;

(ii) $(1 + A)^{-1} \in \mathscr{M}$;

(iii) $e_\lambda \in \mathscr{M}$, $\lambda \in (0, +\infty)$;

(iv) $A^{it} \in \mathscr{M}$, $t \in \mathbb{R}$;

(v) A commutes with any $x' \in \mathscr{M}'$;

(vi) for any $x' \in \mathscr{M}'$ we have $x'(\mathscr{D}_A) \subset \mathscr{D}_A$ and

$$(Ax'\xi \mid A\eta) = (A\eta \mid Ax'^*\eta), \quad \xi, \eta \in \mathscr{D}_A.$$

E.9.26. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a finite von Neumann algebra and $\mathscr{C}(\mathscr{M})$ the set of all closed linear operators in $\mathscr{H}$, which are affiliated to $\mathscr{M}$.

Let $T, S \in \mathscr{C}(\mathscr{M})$. With the help of the polar decomposition (9.28, 9.29) and of Corollary 7.6, show that $T + S$ and TS are densely defined. Then, with the help of the inclusions

$$T + S \subset (T^* + S^*)^*, \quad TS \subset (S^*T^*)^*,$$

show that $T + S$ and TS are preclosed.

Infer from these results and from Theorem 9.8, that $\mathscr{C}(\mathscr{M})$ is a $*$-algebra for the operations

$$(T, S) \mapsto \overline{(T + S)}, \quad (\lambda, T) \mapsto \lambda T,$$

$$(T, S) \mapsto \overline{TS},$$

$$T \mapsto T^*.$$

!E.9.27. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and B a positive operator in $\mathscr{H}$, which is affiliated to $\mathscr{M}$. Show that the Friedrichs extension of B is affiliated to $\mathscr{M}$.

E.9.28. Let A be a positive operator in $\mathscr{H}$. Then the following assertions are equivalent

(i) A is self-adjoint;

(ii) B is a positive linear operator in $\mathscr{H}$, $B \supset A \Rightarrow B = A$.

E.9.29. Let T be a closed linear operator in $\mathscr{H}$. Then, with the scalar product

$$(\xi \mid \eta)_T = (\xi \mid \eta) + (T\xi \mid T\eta),$$

$\mathscr{D}_T$ becomes a Hilbert space. Show that $\mathscr{D}_T = \mathscr{D}_{(1+|T|^2)^{1/2}}$ and that

$$(\xi \mid \eta)_T = ((1 + |T|^2)^{1/2}\xi \mid (1 + |T|^2)^{1/2}\eta), \quad \xi, \eta \in \mathscr{D}_T.$$

E.9.30. Let B be a positive operator in $\mathscr{H}$, A its Friedrichs extension and

$$\mathscr{D} = \left\{ \begin{matrix} \xi \in \mathscr{H}; \text{ there exists } \{\xi_n\} \subset \mathscr{D}_B \text{ such that} \\ \xi_n \to \xi \text{ and } (B(\xi_n - \xi_m) \mid \xi_n - \xi_m) \to 0 \end{matrix} \right\}.$$

Show that

$$\mathscr{D} = \mathscr{D}_{(A^{1/2})}.$$

(Hint: $\mathscr{D}_{(A^{1/2})} = \mathscr{D}_{((1+A)^{1/2})}$ is a Hilbert space for the scalar product $(\xi, \eta) \mapsto ((1 + A)^{1/2}\xi \mid (1 + A)^{1/2}\eta)$, and $\mathscr{D}$ is a closed vector subspace of the latter).

E.9.31. Let B be a positive operator in $\mathscr{H}$ and A its Friedrichs extension. Show that any self-adjoint extension of B, whose domain of definition is included in $\mathscr{D}_{(A^{1/2})}$, coincides with A.

E.9.32. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\xi, \eta \in \mathscr{H}$. The following assertions are equivalent

(i) there exists a closed linear operator T' in $\mathscr{H}$, which is affiliated to $\mathscr{M}'$, such that

$$\eta = T'\xi.$$

(ii) there exists a positive self-adjoint operator A' in $\mathscr{H}$, which is affiliated to $\mathscr{M}'$, such that

$$\omega_\eta = \omega_{A'\xi}.$$

Moreover, the operator A' from condition (ii) can be chosen in such a manner that

$$s(A') \leqslant p'_\xi.$$

E.9.33. Let $\mathscr{M}$ be a von Neumann algebra and φ, ψ normal forms on $\mathscr{M}$. We denote by $\pi_\varphi : \mathscr{M} \mapsto \mathscr{B}(\mathscr{H}_\varphi)$ (resp., $\pi_\psi : \mathscr{M} \to \mathscr{B}(\mathscr{H}_\psi)$) the $*$-representation associated to the form φ (resp., to the form ψ), whose corresponding cyclic vector is 1_φ (resp., 1_ψ) (see 5.18).

(1) The correspondence

$$\mathscr{H}_\psi \supset \pi_\psi(\mathscr{M})1_\psi \ni \pi_\psi(x)1_\psi \mapsto \pi_\varphi(x)1_\varphi \in \mathscr{H}_\varphi$$

is a correctly defined linear operator iff

$$s(\varphi) \leqslant s(\psi).$$

(2) The correspondence

$$\mathscr{H}_\psi \supset \pi_\psi(\mathscr{M})1_\psi \ni \pi_\psi(x)1_\psi \mapsto \pi_\varphi(x)1_\varphi \in \mathscr{H}_\varphi$$

is a preclosed linear operator iff the implication

$$x_n \in \mathscr{M}, \quad \psi(|x_n|^2) \to 0, \quad \varphi(|x_n - x_m|^2) \to 0 \Rightarrow \varphi(|x_n|^2) \to 0$$

holds. In this case one says that φ is *almost dominated* by ψ,

(3) The correspondence

$$\mathscr{H}_\psi \supset \pi_\psi(\mathscr{M})1_\psi \ni \pi_\psi(x)1_\psi \mapsto \pi_\varphi(x)1_\varphi \in \mathscr{H}_\varphi$$

is a bounded linear operator iff there exists a $\lambda > 0$, such that

$$\varphi \leqslant \lambda\psi.$$

In this case one says that φ is *dominated* by ψ.

E.9.34. Let $\{\varphi_k\}_k$ be a sequence of normal forms on the von Neumann algebra $\mathscr{M}$, which are almost dominated by the normal form ψ on $\mathscr{M}$, and such that $\sum_{k=1}^{\infty} \varphi_k(1) < +\infty$. Show that $\varphi = \sum_{k=1}^{\infty} \varphi_k$ is a normal form on $\mathscr{M}$, which is almost dominated by ψ.

E.9.35. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra, φ a normal form on $\mathscr{M}$, and $\xi \in \mathscr{H}$. Then the following assertions are equivalent:

(i) there exists a positive self-adjoint operator A' in $\mathscr{H}$, which is affiliated to $\mathscr{M}'$, such that

$$\varphi = \omega_{A'\xi};$$

(ii) φ is almost dominated by ω_ξ.

(Hint for the implication (ii) $\Rightarrow$ (i): one can assume ξ to be cyclic and separating and, in this case, $\mathscr{H}_{\omega_\xi}$ identifies with $[\mathscr{M}\xi] = \mathscr{H}$; one denotes by T' the closure of the operator which one obtains from E.9.33 (2) for $\psi = \omega_\xi$; if $T' = v'A'$ is the polar decomposition of T', show that A' is affiliated to $\mathscr{M}'$, with the help of exercise E.9.25 (vi)).

E.9.36. Let A, B be positive self-adjoint operators in $\mathscr{H}$, such that $s(A) = s(B) = 1$.

For any $\varepsilon > 0$ we consider the set $\mathscr{D}_\varepsilon$ of all operators $x \in \mathscr{B}(\mathscr{H})$, such that the mapping $it \mapsto A^{it}xB^{-it}$ has a w-continuous extension to $\{\alpha \in \mathbf{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant \varepsilon\}$, which is analytic in $\{\alpha \in \mathbf{C};\ 0 < \operatorname{Re}\alpha < \varepsilon\}$. If $x \in \mathscr{D}_\varepsilon$, the preceding extension is unique, and we denote it by F_x.

We define the operator T in $\mathscr{B}(\mathscr{H})$ by

$$T(x) = F_x(1), \quad x \in \mathscr{D}_T = \mathscr{D}_1.$$

Show that, for any $\lambda > 0$, the operator $\lambda + T$ is injective. Show that, for any $x \in \mathscr{D}_T$ and any $\lambda > 0$, $Tx \in \mathscr{D}_{(\lambda+T)^{-1}}$ and

$$(\lambda + T)^{-1}Tx = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} \frac{\lambda^{-\alpha}}{\sin \pi\alpha} F_x(\alpha)\, d\alpha, \quad 0 < c < 1.$$

By making in this formula $c = 1/2$, infer Proposition 9.23.

Prove that, for $x \in \mathscr{D}_T$ and $\alpha \in \mathbb{C}$, $0 < \operatorname{Re}\alpha < 1$, the following inversion formula holds

$$F_x(\alpha) = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \lambda^{\alpha-1}(\lambda + T)^{-1}Tx \, d\lambda.$$

E.9.37. Let A be a positive self-adjoint operator in $\mathscr{H}$, such that $s(A) = 1$. Show that for any $\xi \in \mathscr{D}_A$ and $\alpha \in \mathbb{C}$, $0 < \operatorname{Re}\alpha < 1$, we have

$$A^\alpha \xi = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \lambda^{\alpha-1}(\lambda + A)^{-1}A\xi \, d\lambda.$$

Comments

C.9.1. In our presentation of the theory we have developed the operational calculus only for positive self-adjoint operators, since this is the essential tool in Chapter 10, and in most of operator algebras theory.

In the case of arbitrary self-adjoint operators, one can make an analogous construction, by replacing the transform $a = (1 + A)^{-1}$ (9.9) by the Cayley transform (E.9.8) and by using the operational calculus for unitary operators. Another method is that which was indicated in Section 9.32 (see, also, exercise E.9.12).

As in the case of bounded operators, one can develop an operational calculus for normal operators, for which we refer the reader to: C. Ionescu Tulcea [2], F. Riesz and B. Sz.-Nagy [1], N. Dunford and J. Schwartz [1], R. P. Halmos [1], B. Sz.-Nagy [1], M. A. Naimark [6].

An analytic operational calculus for closed operators in Banach spaces was also developed (see N. Dunford and J. Schwartz [1], Ch. VII, § 9).

From the theory of self-adjoint extensions of symmetric operators we presented only the theorem of Friedrichs. This theory has important applications in the theory of differential operators, and its basic results are exposed in F. Riesz and B. Sz.-Nagy [1], N. Dunford and J. Schwartz [1], M. G. Krein [2], as well as in M. A. Naimark's book on linear differential operators.

C.9.2. The aim of the theory of generators of one-parameter groups of operators is to characterize these groups with the help of a single mathematical object, which is usually an unbounded operator. The theory of generators uses differential and integral calculus techniques, Fourier analysis and complex analysis and is usually developed for operators in Banach spaces.

Let $\{u_t\}_{t\in\mathbb{R}}$ be a strongly continuous one-parameter group of operators in a Banach space $\mathscr{X}$. The classical *infinitesimal generator* G of $\{u_t\}$ is defined in the following manner

$$\mathscr{D}_G = \left\{\xi \in \mathscr{X};\ \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(u_\varepsilon\xi - \xi) \text{ exists}\right\},$$

$$G\xi = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(u_\varepsilon\xi - \xi), \quad \xi \in \mathscr{D}_G.$$

One proves that G is a closed linear operator and, in a certain sense, u_t is the exponential of tG, $t \in \mathbb{R}$.

If $\mathscr{H}$ is a Hilbert space and if u_t are unitary operators, then $B = -\mathrm{i}G$ is a self-adjoint operator and

$$u_t = \exp(\mathrm{i}tB), \quad t \in \mathbb{R},$$

where the exponential is meant in the sense of the operational calculus. This is the form in which Stone's representation theorem is usually stated.

For details regarding the theory of the infinitesimal generator, which also applies to the one-parameter semigroups, see E. Hille and R. Phillips [1] and N. Dunford and J. Schwartz [1].

Another type of generator, which is more suitable for the applications we have in mind in Chapter 10, can be defined with the help of the analytic continuation. More precisely, we define the *analytic generator* A of u_t by

$$\mathscr{D}_A = \left\{\xi \in \mathscr{H} \;\middle|\; \begin{array}{l}\text{the mapping } \mathrm{i}t \mapsto u_t\xi \text{ has an extension } F_\xi, \text{ which is continuous}\\ \text{on } \{\alpha \in \mathbb{C}; 0 \leqslant \mathrm{Re}\,\alpha \leqslant 1\} \text{ and analytic in } \{\alpha \in \mathbb{C}; 0 < \mathrm{Re}\,\alpha < 1\}.\end{array}\right\}$$

$$A\xi = F_\xi(1), \quad \xi \in \mathscr{D}_A.$$

One shows that A is a closed linear operator and that u_t is the $(\mathrm{i}t)$th power of A, in the sense of V. Balakrishnan's "fractional powers".

If $\mathscr{H}$ is a Hilbert space and if u_t are unitary operators, then Theorem 9.20 shows that A is self-adjoint and positive and

$$u_t = A^{\mathrm{i}t}, \quad t \in \mathbb{R}.$$

For details regarding the theory of the analytic generator see I. Ciorănescu and L. Zsidó [1], and L. Zsidó [8].

We note that Propositions 9.23, 9.24 and exercises E.9.36, E.9.37 are particular cases of some results of the theory of the analytic generator.

C.9.3. Bibliographical comments. Corollary 9.15 was first mentioned by G. K. Pedersen and M. Takesaki ([2], Lemma 3.2), and it lies at the basis of the methods of analytic continuation which we use in Chapters 9 and 10. Proposition 9.23 is due to A. van Daele [4] and it is the principal argument in the proof given by him to the fundamental theorem of M. Tomita (10.12). The other material, concerning the theory of operators, is classic.

Theorem 9.8 and exercise E.9.26 are due to F. J. Murray and J. von Neumann [1].

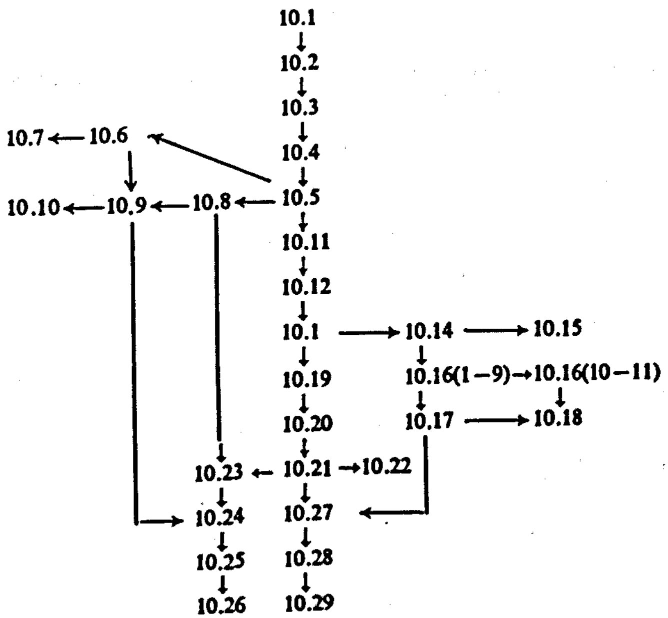
10

The theory of standard von Neuman algebras

In the preceding chapters we have presented that part of the theory of operator algebras which is based on the original ideas of F. J. Murray and J. von Neumann. A turning point in the development of the theory of von Neumann algebras was produced in 1967 by M. Tomita.

In this chapter we present Tomita's theory, which enables us to obtain canonical forms for the von Neumann algebras, forms which are called standard von Neumann algebras.

The logical dependence of the sections of Chapter 10



10.1. Let $\mathfrak{A}$ be a complex algebra with involution, which is also endowed with a scalar product $(\cdot|\cdot)$. We denote by $\xi \mapsto \xi^{\#}$ the involution in $\mathfrak{A}$ and by $\mathscr{H}$ the Hilbert space obtained by the completion of $\mathfrak{A}$. We denote by $\mathfrak{A}^2$ the vector space generated by the elements of the form $\xi\eta$, $\xi, \eta \in \mathfrak{A}$. One says that $\mathfrak{A}$ is a *left Hilbert algebra* if

(i) $\mathfrak{A} \ni \eta \mapsto \xi\eta \in \mathfrak{A}$ is continuous, for any $\xi \in \mathfrak{A}$;

(ii) $(\xi\eta_1|\eta_2) = (\eta_1|\xi^{\#}\eta_2)$, for any $\xi, \eta_1, \eta_2 \in \mathfrak{A}$;

(iii) $\mathfrak{A}^2$ is dense in $\mathfrak{A}$;

(iv) $\mathscr{H} \supset \mathfrak{A} \ni \xi \mapsto \xi^{\#} \in \mathscr{H}$ is a preclosed antilinear operator.

In accordance with (i), for any $\xi \in \mathfrak{A}$ one defines a $L_\xi \in \mathscr{B}(\mathscr{H})$ by the formula

$$L_\xi(\eta) = \xi\eta, \qquad \eta \in \mathfrak{A}.$$

With the help of (ii) it is easy to see that the mapping

$$L\colon \mathfrak{A} \ni \xi \mapsto L_\xi \in \mathscr{B}(\mathscr{H})$$

is a $*$-representation of A, i.e.,

$$L_{\xi_1\xi_2} = L_{\xi_1}L_{\xi_2}, \quad \xi_1, \xi_2 \in \mathfrak{A},$$

$$(L_\xi)^* = L_{\xi^{\#}}, \quad \xi \in \mathfrak{A}.$$

We define

$$\mathfrak{L}(\mathfrak{A}) = \mathscr{R}(\{L_\xi;\ \xi \in \mathfrak{A}\}) = \{L_\xi;\ \xi \in \mathfrak{A}\}''.$$

With the help of (iii) it is easy to see that

$$\mathfrak{L}(\mathfrak{A}) = \overline{\{L_\xi;\ \xi \in \mathfrak{A}\}}^{so}.$$

Conditions (iii) and (iv) allow the definition of the closed antilinear operator S, as being the closure of the operator

$$\mathscr{H} \supset \mathfrak{A}^2 \ni \sum_i \xi_i\eta_i \mapsto \Big(\sum_i \xi_i\eta_i\Big)^{\#} \in \mathscr{H}.$$

We recall that the adjoint antilinear operator S^* is defined by

$$(S^*\eta|\xi) = (S\xi|\eta), \quad \xi \in \mathscr{D}_S, \quad \eta \in \mathscr{D}_{S^*}.$$

Since $\#$ is an involution, we have

$$\xi \in \mathscr{D}_S \Rightarrow S\xi \in \mathscr{D}_S, \qquad SS\xi = \xi, \qquad \text{hence } S^2 \subset 1,$$

$$\eta \in \mathscr{D}_{S^*} \Rightarrow S^*\eta \in \mathscr{D}_{S^*}, \quad S^*S^*\eta = \eta, \quad \text{hence } (S^*)^2 \subset 1.$$

In other words,

$$S = S^{-1}, \quad S^* = (S^*)^{-1}.$$

We consider the positive self-adjoint linear operator

$$\Delta = S^*S,$$

which is called the *modular operator* associated to the left Hilbert algebra $\mathfrak{A}$. Then $s(\Delta) = 1$ and

$$\Delta^{-1} = SS^*.$$

We also consider the polar decomposition of S

$$S = J\Delta^{1/2}.$$

Since $S = S^{-1}$, we get successively

$$J\Delta^{1/2} = \Delta^{-1/2}J^{-1},$$

$$\Delta^{-1/2} = J\Delta^{1/2}J = J^2(J^*\Delta^{1/2}J),$$

whence, with the help of the polar decomposition, we obtain

$$J^2 = 1, \quad J = J^* = J^{-1}.$$

Consequently, J is a conjugation in $\mathscr{H}$, which is called the *canonical conjugation* associated to the left Hilbert algebra $\mathfrak{A}$.

We have thus obtained the following relations

$$S = J\Delta^{1/2} = \Delta^{-1/2}J, \quad \mathscr{D}_S = \mathscr{D}_{(\Delta^{1/2})},$$

$$S^* = J\Delta^{-1/2} = \Delta^{1/2}J, \quad \mathscr{D}_{S^*} = \mathscr{D}_{(\Delta^{-1/2})}.$$

The data concerning J are synthetized in the following relations

$$J^2 = 1,$$

$$(J\eta\,|\,\xi) = (J\xi\,|\,\eta), \quad \xi, \eta \in \mathscr{H}.$$

Finally, we mention the fact that for any $f \in \mathscr{B}([0, +\infty))$ we have

$$Jf(\Delta)J = \bar{f}(\Delta^{-1}),$$

this equality being a consequence of the relations

$$J\Delta J = \Delta^{-1},$$

$$J(\lambda\Delta)J = \bar{\lambda}J\Delta J = \bar{\lambda}\Delta^{-1}.$$

We also mention the following equivalent forms of the preceding formula

$$f(\Delta)J = J\bar{f}(\Delta^{-1}),$$

$$Jf(\Delta) = \bar{f}(\Delta^{-1})J.$$

In particular,

$$J\Delta^{it} = \Delta^{it}J, \quad t \in \mathbb{R}.$$

The *so*-continuous group $\{\Delta^{it}\}$ of unitary operators is called the *modular group* associated to the left Hilbert algebra $\mathfrak{A}$.

10.2. In this section we consider some operators which are naturally associated to a left Hilbert algebra $\mathfrak{A} \subset \mathscr{H}$.

Let $\eta \in \mathscr{H}$. We define the linear operator R_η^0 by the relations

$$R_\eta^0(\xi) = L_\xi(\eta), \quad \xi \in \mathscr{D}_{(R_\eta^0)} = \mathfrak{A}.$$

If R_η^0 is preclosed, we denote its closure by R_η.

Lemma 1. *Let $\eta \in \mathscr{H}$. If R_η^0 is preclosed, then R_η is affiliated to the von Neumann algebra $\mathfrak{L}(\mathfrak{A})'$.*

Proof. For any $\xi \in \mathfrak{A}$ we have

$$L_\xi R_\eta^0 \subset R_\eta^0 L_\xi,$$
$$L_\xi R_\eta \subset R_\eta L_\xi,$$

whence, for any $x \in \mathfrak{L}(\mathfrak{A})$,

$$x R_\eta \subset R_\eta x.$$

Q.E.D.

Lemma 2. *If $\eta \in \mathscr{D}_{S^*}$, then R_η^0 is preclosed and*

$$\mathfrak{A} \subset \mathscr{D}_{(R_\eta)^*},$$
$$(R_\eta)\xi^* = L_\xi S^*\eta, \quad \xi \in \mathfrak{A}.$$

Proof. Since

$$R_\eta^0 \subset (R_{S^*\eta}^0)^*,$$

the operator R_η^0 is preclosed. A trivial argument now completes the proof.

Q.E.D.

10.3. In this section we consider the "multiplications to the right", which are associated to a left Hilbert algebra $\mathfrak{A} \subset \mathscr{H}$.

We define

$$\mathfrak{A}' = \{\eta \in \mathscr{D}_{S^*};\ R_\eta \text{ is bounded}\}.$$

Lemma 1 from 10.2 now implies that

$$\eta \in \mathfrak{A}' \Rightarrow R_\eta \in \mathfrak{L}(\mathfrak{A})'.$$

Proposition. *Let $\eta \in \mathscr{H}$, $\zeta \in \mathscr{H}$ and $x' \in \mathscr{B}(\mathscr{H})$. Then the following assertions are equivalent*

(i) $\eta \in \mathfrak{A}'$ *and* $S^*(\eta) = \zeta$, $R_\eta = x'$;

(ii) *for any* $\xi \in \mathfrak{A}$ *we have* $L_\xi(\eta) = x'\xi$, $L_\xi(\zeta) = x'^*\xi$.

Proof. Assuming (i) to be true, for any $\xi \in \mathfrak{A}$ we have

$$L_\xi(\eta) = R_\eta(\xi) = x'\xi$$

and, from Lemma 2, Section 10.2, we get

$$L_\xi(\zeta) = L_\xi S^*\eta = (R_\eta)^*\xi = x'^*\xi.$$

Let us now assume that (ii) is true. For any $\xi_1, \xi_2 \in \mathfrak{A}$ we have

$$(\eta| S(\xi_1\xi_2)) = (\eta|\xi_2^{\#}\xi_1^{\#}) = (\eta| L_{\xi_2^{\#}}(\xi_1^{\#})) = (\eta|(L_{\xi_2})^*\xi_1^{\#})$$
$$= (L_{\xi_2}(\eta)|\xi_1^{\#}) = (x'\xi_2|\xi_1^{\#}) = (\xi_2|x'^*\xi_1^{\#})$$
$$= (\xi_2| L_{\xi_1^{\#}}(\zeta)) = (L_{\xi_1}(\xi_2)|\zeta) = (\xi_1\xi_2|\zeta),$$

whence $\eta \in \mathscr{D}_{S^*}$ and $S^*(\eta) = \zeta$. Obviously, $R_\eta = x'$ is bounded; hence $\eta \in \mathfrak{A}'$.
Q.E.D.

Let $\eta \in \mathscr{D}_{S^*}$ and let

$$R_\eta = u_\eta A_\eta = B_\eta u_\eta$$

be the polar decompositions of R_η. For any $f \in \mathscr{B}([0, +\infty))$ we have the relation

$$u_\eta f(A_\eta) = f(B_\eta)u_\eta.$$

We observe that

$$\eta \in \overline{R_\eta \mathscr{D}_{R_\eta}} = s(B_\eta)\mathscr{H},$$
$$S_\eta^* \in \overline{(R_\eta)^* \mathscr{D}_{(R_\eta)^*}} = s(A_\eta)\mathscr{H}.$$

Indeed, in accordance with Section 10.1, there exists a net $\{\xi_i\} \subset \mathfrak{A}$, such that $L_{\xi_i} \xrightarrow{so} 1$, hence, from Lemma 2, Section 10.2, we get

$$R_\eta \mathscr{D}_{R_\eta} \ni R_\eta(\xi_i) = L_{\xi_i}(\eta) \to \eta,$$
$$(R_\eta)^* \mathscr{D}_{(R_\eta)^*} \ni (R_\eta)^*(\xi_i) = L_{\xi_i}(S^*\eta) \to S^*\eta.$$

With the help of the preceding proposition it is easy to prove the following assertions:

Corollary 1. *If $\eta \in \mathfrak{A}'$, then*

$$S^*\eta \in \mathfrak{A}', \textit{ and}$$
$$S^*(S^*\eta) = \eta, \quad R_{S^*\eta} = (R_\eta)^*.$$

Corollary 2. *If $\eta_1, \eta_2 \in \mathfrak{A}'$, then*

$$R_{\eta_1}(\eta_2) \in \mathfrak{A}' \textit{ and}$$
$$S^*R_{\eta_1}(\eta_2) = R_{S^*\eta_2}(S^*\eta_1), \quad R_{R_{\eta_1}(\eta_2)} = R_{\eta_1}R_{\eta_2}.$$

More generally, we have the

Corollary 3. *If $\eta_1, \eta_2 \in \mathfrak{A}'$ and $x' \in \mathfrak{L}(\mathfrak{A})'$, then*

$$R_{\eta_1}x'(\eta_2) \in \mathfrak{A}' \textit{ and}$$
$$S^*R_{\eta_1}x'(\eta_2) = R_{S^*\eta_2}(x')^*S^*\eta_1, \quad R_{R_{\eta_1}x'(\eta_2)} = R_{\eta_1}x'R_{\eta_2}.$$

The following two corollaries enable us to obtain elements in $\mathfrak{A}'$.

Corollary 4. *If* $\eta \in \mathscr{D}_{S^*}$ *and* $f \in \mathscr{B}([0, +\infty))$, $\sup \{\lambda^2 |f(\lambda)|;\ \lambda \in [0, +\infty)\} < +\infty$, *then*

$$R_\eta f(A_\eta) S^* \eta \in \mathfrak{A}' \text{ and}$$

$$S^*(R_\eta f(A_\eta) S^* \eta) = R_\eta \bar{f}(A_\eta) S^* \eta, \quad R_{R_\eta f(A_\eta) S^* \eta} = B_\eta^2 f(B_\eta).$$

Corollary 5. *If* $\eta \in \mathscr{D}_{S^*}$ *and* $f \in \mathscr{B}([0, +\infty))$, $\sup \{\lambda |f(\lambda)|;\ \lambda \in [0, +\infty)\} < +\infty$, *then*

$$f(B_\eta)\eta \in \mathfrak{A}' \text{ and}$$

$$S^*(f(B_\eta)\eta) = \bar{f}(A_\eta) S^* \eta, \quad R_{f(B_\eta)\eta} = B_\eta f(B_\eta) u_\eta.$$

Finally, the last corollary in this series shows that $\mathfrak{A}'$ contains enough elements.

Corollary 6. *If* $\eta \in \mathscr{D}_{S^*}$, *then there exists sequences* $\{\eta_n\}, \{\zeta_n\} \subset \mathfrak{A}'$, *such that*

$$R_{\eta_n}(\zeta_n) \to \eta,$$

$$S^* R_{\eta_n}(\zeta_n) \to S^* \eta.$$

Proof. Let us consider the functions $f_n \in \mathscr{B}([0, +\infty))$, $n \in \mathbb{N}$, defined by

$$f_n(\lambda) = \lambda^{-1} \chi_{(n^{-1}, n)}(\lambda).$$

We define

$$\eta_n = R_\eta f_n(A_\eta) S^* \eta, \quad \zeta_n = f_n(B_\eta)\eta.$$

With the help of Corollaries 2, 4 and 5 it is easy to see that, for any natural number n, we have

$$\eta_n, \zeta_n, R_{\eta_n}(\zeta_n) \in \mathfrak{A}' \text{ and}$$

$$R_{\eta_n}(\zeta_n) = B_\eta^2 f_n^2(B_\eta)\eta, \quad S^* R_{\eta_n}(\zeta_n) = A_\eta^2 f_n^2(A_\eta) S^* \eta.$$

Since $\eta \in s(B_\eta)\mathscr{H}$ and $S^*\eta \in s(A_\eta)\mathscr{H}$, by taking into account Theorem 9.11 (vi) and Corollary 9.13 (iii), it follows that

$$R_{\eta_n}(\zeta_n) \to \eta,$$

$$S^* R_{\eta_n}(\zeta_n) \to S^* \eta.$$

Q.E.D.

10.4. Let $\mathfrak{B}$ be a complex algebra with involution, endowed also with a scalar product $(\cdot | \cdot)$. We denote by $\eta \mapsto \eta^b$ the involution in $\mathfrak{B}$ and by $\mathscr{H}$ the Hilbert space obtained by the completion of $\mathfrak{B}$. One says that $\mathfrak{B}$ is a *right Hilbert algebra* if

(j) $\mathfrak{B} \ni \xi \mapsto \xi\eta \in \mathfrak{B}$ is continuous for any $\eta \in \mathfrak{B}$;

(jj) $(\xi_1 \eta | \xi_2) = (\xi_1 | \xi_2 \eta^b)$, for any $\eta, \xi_1, \xi_2 \in \mathfrak{B}$;

(jjj) $\mathfrak{B}^2$ is dense in $\mathfrak{B}$;

(jw) $\mathscr{H} \supset \mathfrak{B} \ni \eta \mapsto \eta^b \in \mathscr{H}$ is a preclosed antilinear operator.

Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra. Then $\mathfrak{A}'$, endowed with the operations

$$\eta_1\eta_2 = R_{\eta_2}(\eta_1), \quad \eta_1, \eta_2 \in \mathfrak{A}',$$
$$\eta^\flat = S^*\eta, \quad \eta \in \mathfrak{A}',$$

and with the scalar product of $\mathscr{H}$, is a right Hilbert algebra (see Corollaries 1, 2, 6 from Section 10.3).

If $\mathfrak{B} \subset \mathscr{H}$ is a right Hilbert algebra, then the closed antilinear operator F is defined to be the closure of the operator

$$\mathscr{H} \supset \mathfrak{B}^2 \ni \sum_i \eta_i\xi_i \mapsto (\sum_i \eta_i\xi_i)^\flat \in \mathscr{H}.$$

Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra and $\mathfrak{B} = \mathfrak{A}'$. Then, in accordance with Corollary 6 from Section 10.3, we have

(1) $$S^* = \overline{S^*|_{(\mathfrak{A}')^2}},$$

i.e.,

$$F = S^*.$$

For a right Hilbert algebra $\mathfrak{B} \subset \mathscr{H}$ the operators $R_\eta \in \mathscr{B}(\mathscr{H})$, $\eta \in \mathfrak{B}$, are defined by the formula

$$R_\eta(\xi) = \xi\eta, \quad \xi \in \mathfrak{B}.$$

If $\mathfrak{B} = \mathfrak{A}'$, then, for any $\eta \in \mathfrak{B}$, R_η is just the operator defined in Sections 10.2 and 10.3.

If $\mathfrak{B} \subset \mathscr{H}$ is a right Hilbert algebra, then the mapping

$$R: \mathfrak{B} \ni \eta \mapsto R_\eta \in \mathscr{B}(\mathscr{H})$$

is a $*$-antirepresentation of $\mathfrak{B}$, i.e.,

$$R_{\eta_1\eta_2} = R_{\eta_2}R_{\eta_1}, \quad \eta_1, \eta_2 \in \mathfrak{B},$$
$$(R_\eta)^* = R_{\eta^\flat}, \quad \eta \in \mathfrak{B}.$$

One defines

$$\mathfrak{R}(\mathfrak{B}) = \mathscr{R}(\{R_\eta;\ \eta \in \mathfrak{B}\}) = \{R_\eta;\ \eta \in \mathfrak{B}\}'' = \overline{\{R_\eta;\ \eta \in \mathfrak{B}\}}^{so}.$$

If $\mathfrak{B} = \mathfrak{A}'$, then

(2) $$\mathfrak{R}(\mathfrak{A}') = \mathfrak{L}(\mathfrak{A})'.$$

Indeed, the inclusion $\mathfrak{R}(\mathfrak{A}') \subset \mathfrak{L}(\mathfrak{A})'$ is obvious. Conversely, let $x' \in \mathfrak{L}(\mathfrak{A})'$ and $\{\eta_i\} \subset \mathfrak{A}'$ be a net, such that $\|R_{\eta_i}\| \leqslant 1$ and $R_{\eta_i} \xrightarrow{so} 1$. Then, in accordance with Corollary 3 from Section 10.3, we have

$$R_{\eta_i}x'(\eta_i) \in \mathfrak{A}' \quad \text{and} \quad R_{R_{\eta_i}x'(\eta_i)} = R_{\eta_i}x'R_{\eta_i}.$$

Thus, we have

$$\mathfrak{R}(\mathfrak{A}') \ni R_{\eta_i}x'R_{\eta_i} \xrightarrow{so} x',$$

and relation (2) is proved.

Let $\mathfrak{B} \subset \mathscr{H}$ be a right Hilbert algebra. For any $\xi \in \mathscr{D}_{F^*}$ one defines the closed linear operator L_ξ in $\mathscr{H}$ by analogy with Section 10.2, as being the closure of the operator

$$\mathscr{H} \supset \mathfrak{B} \ni \eta \mapsto R_\eta(\xi) \in \mathscr{H}.$$

Then L_ξ is affiliated to $\mathscr{R}(\mathfrak{B})'$, $\mathfrak{B} \subset \mathscr{D}_{(L_\xi)^*}$ and

$$(L_\xi)^*\eta = R_\eta F^*\xi, \quad \eta \in \mathfrak{B}.$$

One also defines

$$\mathfrak{B}' = \{\xi \in \mathscr{D}_{F^*};\ L_\xi \text{ is bounded}\}$$

and a proposition analogous to Proposition 10.3 holds. By a dualization of the preceding discussion $\mathfrak{B}'$ canonically becomes a left Hilbert algebra and the following relation holds

$$\mathfrak{L}(\mathfrak{B}') = \mathscr{R}(\mathfrak{B})'.$$

If $\mathfrak{B} = \mathfrak{A}'$, we have $F^* = S$. Thus, for any $\xi \in \mathscr{D}_S$, we defined a closed linear operator L_ξ, which is affiliated to $\mathscr{R}(\mathfrak{A}')' = \mathfrak{L}(\mathfrak{A})$, as being the closure of the operator

$$\mathscr{H} \supset \mathfrak{A}' \ni \eta \mapsto R_\eta(\xi) \in \mathscr{H}.$$

We mention that $\mathfrak{A}' \subset \mathscr{D}_{(L_\xi)^*}$ and we have the relation

(3) $$(L_\xi)^*(\eta) = R_\eta S\xi.$$

We denote

$$\mathfrak{A}'' = \mathfrak{B}' = \{\xi \in \mathscr{D}_S;\ L_\xi \text{ is bounded}\}.$$

Then $\mathfrak{A}''$ becomes a left Hilbert algebra, with the operations

$$\xi_1\xi_2 = L_{\xi_1}(\xi_2), \qquad \xi_1, \xi_2 \in \mathfrak{A}'',$$

$$\xi^{\#} = S\xi, \quad \xi \in \mathfrak{A}''.$$

(We shall see later that these notations are compatible with those already introduced for the operations in $\mathfrak{A}$).

The right Hilbert algebras were only introduced in order to systematize the discussions about $\mathfrak{A}, \mathfrak{A}', \mathfrak{A}''$. Thus, if $\mathfrak{A} \subset \mathscr{H}$ is a left Hilbert algebra, then the proposition analogous to Proposition 10.3 is the following:

Proposition. *Let $\xi \in \mathscr{H}$, $\zeta \in \mathscr{H}$ and $x \in \mathscr{B}(\mathscr{H})$. Then the following assertions are equivalent*

(i) $\xi \in \mathfrak{A}''$ *and* $S(\xi) = \zeta$, $L_\xi = x$;

(ii) *for any* $\eta \in \mathfrak{A}'$ *we have* $R_\eta(\xi) = x\eta$, $R_\eta(\zeta) = x^*\eta$.

With the help of the preceding proposition it is easy to obtain the following

Corollary 1. *If $\xi \in \mathfrak{A}$, then*

$$\xi \in \mathfrak{A}'' \text{ and}$$

$$S\xi = \xi^{\#}, \ L_{\xi} = \text{the operator defined in Section 10.1.}$$

In particular, $\mathfrak{A} \subset \mathscr{D}_S$ and $S\xi = \xi^{\#}$, for any $\xi \in \mathfrak{A}$. Thus, S is the closure of the operator

$$\mathscr{H} \supset \mathfrak{A} \ni \xi \mapsto \xi^{\#} \in \mathscr{H}.$$

Let $\mathfrak{A}_1 \subset \mathscr{H}$ be a left Hilbert algebra. One calls a *left Hilbert subalgebra* of $\mathfrak{A}_1$ any involutive subalgebra $\mathfrak{A}_2$ of $\mathfrak{A}_1$, which, endowed with the scalar product of $\mathscr{H}$, becomes a left Hilbert algebra, which is dense in $\mathscr{H}$.

It is easy to verify the following

Corollary 2. *$\mathfrak{A}$ is a left Hilbert subalgebra of $\mathfrak{A}''$ and the following relations hold*

$$(\mathfrak{A}'')' = \mathfrak{A}',$$

$$(\mathfrak{A}'')'' = \mathfrak{A}'',$$

$$\mathfrak{L}(\mathfrak{A}'') = \mathfrak{L}(\mathfrak{A}).$$

We conclude this section with a "dominated convergence" result:

Corollary 3. *Let $x \in \mathfrak{L}(\mathfrak{A})$ and $\{\xi_\iota\} \subset \mathfrak{A}''$ be a net having the following properties*

$$L_{\xi_\iota} \xrightarrow{wo} x, \sup_\iota \|\xi_\iota\| < +\infty, \ \sup_\iota \|S\xi_\iota\| < +\infty.$$

Then there exists a $\xi \in \mathfrak{A}''$, such that

$$x = L_\xi, \ \xi_\iota \to \xi \text{ weakly}, \quad S\xi_\iota \to S\xi \text{ weakly}.$$

Proof. We consider the linear form φ defined on $(\mathfrak{A}')^2$ by the formula

$$\varphi(R_{S^*\eta}(\zeta)) = (\zeta \mid x\eta) = \lim_\iota (\zeta \mid L_{\xi_\iota}(\eta)) = \lim_\iota (R_{S^*\eta}(\zeta) \mid \xi_\iota), \ \eta, \zeta \in \mathfrak{A}'.$$

Since $\sup_\iota \|\xi_\iota\| < +\infty$, φ is bounded; hence, there exists a $\xi \in \mathscr{H}$, such that

$$(\zeta \mid x\eta) = (R_{S^*\eta}(\zeta) \mid \xi) = (\zeta \mid R_\eta(\xi)), \quad \eta, \zeta \in \mathfrak{A}',$$

whence

$$R_\eta(\xi) = x(\eta), \quad \eta \in \mathfrak{A}'.$$

Since

$$(R_{S^*\eta}(\zeta) \mid \xi_\iota) \to (R_{S^*\eta}(\zeta) \mid \xi), \quad \eta, \zeta \in \mathfrak{A}',$$

from the facts that $(\mathfrak{A}')^2$ is dense in $\mathscr{H}$ and that $\sup_\iota \|\xi_\iota\| < +\infty$, it follows that

$$\xi_\iota \to \xi, \text{ weakly.}$$

By using the fact that $L_{S\xi_i}=(L_{\xi_i})^* \xrightarrow{wo} x^*$, one can analogously show that there exists a $\zeta \in \mathscr{H}$, such that

$$R_\eta(\zeta) = x^*(\eta), \quad \eta \in \mathfrak{A}',$$

$$S\xi_i \to \zeta, \text{ weakly.}$$

By applying the preceding proposition, it follows that

$$\xi \in \mathfrak{A}'', \quad S\xi = \zeta, \qquad L_\xi = x.$$

Q.E.D.

10.5. Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra. By taking into account the definition of the Hilbert algebras, $\mathfrak{A}^2$ is dense in $\mathfrak{A}$. More precisely, for any $\xi \in \mathfrak{A}$ we have $\xi \in \overline{\mathfrak{A}\xi}$. Indeed, if $\{\xi_i\} \subset \mathfrak{A}$ is a net, such that $L_{\xi_i} \xrightarrow{so} 1$, then

$$\mathfrak{A}\xi \ni L_{\xi_i}(\xi) \to \xi.$$

Actually, for any $\xi \in \mathfrak{A}$ we have $\xi \in \overline{\xi\mathfrak{A}\xi}$, as a consequence of the following lemma.

Lemma 1. *If $\xi \in \mathfrak{A}$, $\xi \neq 0$, then $L_{\xi\xi^\#} \neq 0$, and*

$$p_n(\|L_{\xi\xi^\#}\|^{-1}\xi\xi^\#)\xi \to \xi,$$

where p_n is the polynomial defined by

$$p_n(\lambda) = 1 - (1 - \lambda)^n.$$

Proof. Let $\{\eta_i\} \subset \mathfrak{A}'$ be a net, such that $R_{\eta_i} \xrightarrow{so} 1$. Then

$$L_\xi(\eta_i) = R_{\eta_i}(\xi) \to \xi;$$

hence

$$\xi \in \overline{L_\xi(\mathscr{H})} = \mathrm{l}(L_\xi)\mathscr{H} = \mathrm{s}(L_{\xi\xi^\#})\mathscr{H}.$$

In particular, $L_{\xi\xi^\#} \neq 0$.

On the other hand, according to a remark we made in Section 2.22, we have

$$p_n(\|L_{\xi\xi^\#}\|^{-1}L_{\xi\xi^\#}) \xrightarrow{so} \mathrm{s}(\|L_{\xi\xi^\#}\|^{-1}L_{\xi\xi^\#}) = \mathrm{s}(L_{\xi\xi^\#});$$

hence

$$p_n(\|L_{\xi\xi^\#}\|^{-1}\xi\xi^\#)\xi = p_n(\|L_{\xi\xi^\#}\|^{-1}L_{\xi\xi^\#})\xi \to \mathrm{s}(L_{\xi\xi^\#})\xi = \xi.$$

Q.E.D.

In particular, from what we have just proved, it follows that the $*$-representation $L\colon \mathfrak{A} \to \mathscr{B}(\mathscr{H})$ is injective.

In order to avoid any confusion, in what follows we shall mark the symbols $L, S, \ldots$, which correspond to $\mathfrak{A}$, in the following manner: $L^{\mathfrak{A}}, S^{\mathfrak{A}}, \ldots$

Lemma 2. *Let $\mathfrak{A}_1 \subset \mathscr{H}$ be a left Hilbert algebra and $\mathfrak{A}_2$ an involutive subalgebra of $\mathfrak{A}_1$, which is dense in $\mathscr{H}$. Then $\mathfrak{A}_2$ is a left Hilbert subalgebra of $\mathfrak{A}_1$.*

Proof. If is sufficient to prove that $(\mathfrak{A}_2)^2$ is dense in $\mathfrak{A}_2$. In accordance with Lemma 1, for any $\xi \in \mathfrak{A}_2$, $\xi \neq 0$, we have

$$(\mathfrak{A}_2)^2 \ni p_n(\|L^{\mathfrak{A}_1}_{\xi\xi^{\#}}\|^{-1}\xi\xi^{\#})\xi \to \xi,$$

where p_n is the polynomial from the statement of Lemma 1.

Q.E.D.

Lemma 3. *Let $\mathfrak{A}_2$ be a left Hilbert subalgebra of a left Hilbert algebra $\mathfrak{A}_1 \subset \mathscr{H}$. Then the following assertions are equivalent*

(i) $S^{\mathfrak{A}_1} = S^{\mathfrak{A}_2}$.

(ii) $(\mathfrak{A}_1)' = (\mathfrak{A}_2)'$;

(iii) $(\mathfrak{A}_1)'' = (\mathfrak{A}_2)''$.

Proof. The proofs of the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are immediate, on the basis of the corresponding definitions; the implication (iii) $\Rightarrow$ (i) follows from the considerations we have just made in Section 10.4.

Q.E.D.

In particular, if $\mathfrak{A}$ is a left Hilbert algebra, then $\mathfrak{A}^2$ is a left Hilbert subalgebra of $\mathfrak{A}$, and $(\mathfrak{A}^2)'' = \mathfrak{A}''$.

10.6. In this section we consider an important example of a left Hilbert algebra.

Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and $\xi_0 \in \mathscr{H}$ a separating cyclic vector for $\mathscr{M}$ (see 3.8). Then

$$\mathfrak{A} = \mathscr{M}\xi_0 = \{x\xi_0;\ x \in \mathscr{M}\},$$

endowed with the operations

$$(x\xi_0)(y\xi_0) = xy\xi_0,$$

$$(x\xi_0)^{\#} \doteq x^*\xi_0,$$

and with the scalar product of $\mathscr{H}$, becomes a left Hilbert algebra. Indeed, the first two conditions are immediate, the third one is trivially satisfied because $1 \in \mathscr{M}$, whereas the fourth one follows from the relation

$$(1) \qquad (x\xi_0 | x'\xi_0) = (x'^*\xi_0 | (x\xi_0)^{\#}), \quad x \in \mathscr{M}, \quad x' \in \mathscr{M}',$$

and by taking into account the fact that $\overline{\mathscr{M}'\xi_0} = \mathscr{H}$.

For $\xi = x\xi_0 \in \mathfrak{A}$ we obviously have $L_\xi = x$, hence

$$\mathfrak{L}(\mathfrak{A}) = \{L_\xi;\ \xi \in \mathfrak{A}\} = \mathscr{M}.$$

If $\eta \in \mathfrak{A}'$, then $R_\eta \in \mathfrak{L}(\mathfrak{A})' = \mathcal{M}'$, and

$$R_\eta(\xi_0) = L_{\xi_0}(\eta) = \eta.$$

Thus,

$$\mathfrak{A}' \subset \mathcal{M}'\xi_0 = \{x'\xi_0;\quad x' \in \mathcal{M}'\}.$$

With the help of relation (1) it is easy to see that, for any $x' \in \mathcal{M}'$, we have

$$x'\xi_0 \in \mathfrak{A}', \quad \text{and} \quad S^*(x'\xi_0) = x'^*\xi_0, \quad R_{x'\xi_0} = x'.$$

Consequently, we have

$$\mathfrak{A}' = \mathcal{M}'\xi_0,$$

and the operations of a right Hilbert algebra in $\mathfrak{A}'$ are the following ones

$$(x'\xi_0)(y'\xi_0) = R_{y'\xi_0}(x'\xi_0) = y'x'\xi_0,$$

$$(x'\xi_0)^\flat = S^*(x'\xi_0) = x'^*\xi_0.$$

By analogy with the preceding argument, it follows that

$$\mathfrak{A}'' = \mathcal{M}''\xi_0 = \mathcal{M}\xi_0 = \mathfrak{A}.$$

Because of their relevance to the following sections, we restate the following facts, which have already been proved:

The closure S of the antilinear operator

$$\mathcal{H} \supset \mathcal{M}\xi_0 \ni x\xi_0 \mapsto x^*\xi_0 \in \mathcal{H}$$

has as its adjoint S^* the closure of the antilinear operator

$$\mathcal{H} \supset \mathcal{M}'\xi_0 \ni x'\xi_0 \mapsto x'^*\xi_0 \in \mathcal{H}.$$

Also, if $\eta \in \mathcal{D}_{S^*}$, then the operator

$$\mathcal{H} \supset \mathcal{M}\xi_0 \ni x\xi_0 \mapsto x\eta \in \mathcal{H},$$

is preclosed and its closure is affiliated to $\mathcal{M}'$.

10.7. This section contains an important application of the theory already developed, to the commutation theorem for tensor products.

Lemma 1. *Let $\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{M}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be von Neumann algebras with separating cyclic vectors $\xi_1 \in \mathcal{H}_1$ and $\xi_2 \in \mathcal{H}_2$, respectively. Then*

$$(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)' = \mathcal{M}_1' \overline{\otimes} \mathcal{M}_2'.$$

Proof. Let

$$\xi = \xi_1 \otimes \xi_2 \in \mathcal{H}_1 \overline{\otimes} \mathcal{H}_2 = \mathcal{H}.$$

Then ξ is a cyclic and separating vector for $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$.

We now consider the operators

$S_1 =$ the closure of $[\mathcal{M}_1\xi_1 \ni x_1\xi_1 \mapsto x_1^*\xi_1 \in \mathcal{H}_1]$,

$S_2 =$ the closure of $[\mathcal{M}_2\xi_2 \ni x_2\xi_2 \mapsto x_2^*\xi_2 \in \mathcal{H}_2]$,

$S\ =$ the closure of $[(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)\xi \ni x\xi \mapsto x^*\xi \in \mathcal{H}]$.

By using the fact that $\mathcal{M}_1 \otimes \mathcal{M}_2$ is w-dense in $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ it is easy to verify that

$$S = S_1 \overline{\otimes} S_2.$$

In accordance with Proposition 9.33, we now infer that

$$S^* = S_1^* \overline{\otimes} S_2^*.$$

According to the last statement made in the preceding section, we get

$S^* =$ the closure of $[(\overline{\mathcal{M}}_1 \otimes \mathcal{M}_2)'\xi \ni x'\xi \mapsto x'^*\xi \in \mathcal{H}]$,

$S_1^* =$ the closure of $[\mathcal{M}_1'\xi_1 \ni x_1'\xi_1 \mapsto x_1'^*\xi_1 \in \mathcal{H}_1]$,

$S_2^* =$ the closure of $[\mathcal{M}_2'\xi_2 \ni x_2'\xi_2 \mapsto x_2'^*\xi_2 \in \mathcal{H}_2]$,

$S_1^* \overline{\otimes} S_2^* =$ the closure of $[(\mathcal{M}_1' \overline{\otimes} \mathcal{M}_2')\xi \ni y'\xi \mapsto y'^*\xi \in \mathcal{H}]$.

Therefore, we have

$$S^{(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)'} = S^* = S_1^* \overline{\otimes} S_2^* = S^{(\mathcal{M}_1' \overline{\otimes} \mathcal{M}_2')}.$$

From the inclusion

$$(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)'\xi \supset (\mathcal{M}_1' \overline{\otimes} \mathcal{M}_2')\xi,$$

with the help of Lemma 3 from Section 10.5, we obtain

$$(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)'\xi = (\mathcal{M}_1' \overline{\otimes} \mathcal{M}_2')\xi,$$

and this equality implies that

$$(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)' = \mathcal{M}_1' \overline{\otimes} \mathcal{M}_2'.$$

Q.E.D.

Lemma 2. *Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2$ be von Neumann algebras, $\mathcal{M}_1$ being $*$-isomorphic to $\mathcal{N}_1$ and $\mathcal{M}_2$ being $*$-isomorphic to $\mathcal{N}_2$. If*

$$(\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2)' = \mathcal{N}_1' \overline{\otimes} \mathcal{N}_2',$$

then

$$(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)' = \mathcal{M}_1' \overline{\otimes} \mathcal{M}_2'.$$

Proof. According to exercise E.8.8, we can separately consider the cases of the spatial isomorphism, of the induction and of the amplification.

(I) The case of the spatial isomorphism is trivial.

(II) The case of the induction: if

$$\mathcal{M}_1 = (\mathcal{N}_1)_{e_1'},\ e_1' \in \mathcal{P}_{\mathcal{N}_1'},\ \mathcal{M}_2 = (\mathcal{N}_2)_{e_2'},\ e_2' \in \mathcal{P}_{\mathcal{N}_2'},$$

then, by taking into account exercise E.3.15 and Theorem 3.13, we have

$$
\begin{aligned}
(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)' &= ((\mathcal{N}_1)_{e_1'} \overline{\otimes} (\mathcal{N}_2)_{e_2'})' = ((\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2)_{e_1' \overline{\otimes} e_2'})' \\
&= (\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2)'_{e_1' \overline{\otimes} e_2'} = (\mathcal{N}_1' \overline{\otimes} \mathcal{N}_2')_{e_1' \overline{\otimes} e_2'} \\
&= (\mathcal{N}_1')_{e_1'} \overline{\otimes} (\mathcal{N}_2')_{e_2'} = (\mathcal{N}_1)'_{e_1'} \overline{\otimes} (\mathcal{N}_2)'_{e_2'} = \mathcal{M}_1' \overline{\otimes} \mathcal{M}_2'.
\end{aligned}
$$

(III) The case of the amplification: let

$$\mathcal{N}_1 \subset \mathcal{B}(\mathcal{H}_1), \quad \mathcal{N}_2 \subset \mathcal{B}(\mathcal{H}_2)$$

$$\mathcal{M}_1 = \mathcal{N}_1 \overline{\otimes} \mathcal{C}(\mathcal{K}_1), \quad \mathcal{M}_2 = \mathcal{N}_2 \overline{\otimes} \mathcal{C}(\mathcal{K}_2),$$

where $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2$ are Hilbert spaces. We define a unitary operator

$$u: \mathcal{H}_1 \overline{\otimes} \mathcal{K}_1 \overline{\otimes} \mathcal{H}_2 \overline{\otimes} \mathcal{K}_2 \to \mathcal{H}_1 \overline{\otimes} \mathcal{H}_2 \overline{\otimes} \mathcal{K}_1 \overline{\otimes} \mathcal{K}_2$$

by the formula

$$u(\xi_1 \otimes \eta_1 \otimes \xi_2 \otimes \eta_2) = \xi_1 \otimes \xi_2 \otimes \eta_1 \otimes \eta_2.$$

Then, with the help of Proposition 3.17, we infer that

$$
\begin{aligned}
(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)' &= (\mathcal{N}_1 \overline{\otimes} \mathcal{C}(\mathcal{K}_1) \overline{\otimes} \mathcal{N}_2 \overline{\otimes} \mathcal{C}(\mathcal{K}_2))' \\
&= (u^*(\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2 \overline{\otimes} \mathcal{C}(\mathcal{K}_1) \overline{\otimes} \mathcal{C}(\mathcal{K}_2))u)' \\
&= u^*((\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2) \overline{\otimes} \mathcal{C}(\mathcal{K}_1) \overline{\otimes} \mathcal{C}(\mathcal{K}_2))'u \\
&= u^*((\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2)' \overline{\otimes} \mathcal{B}(\mathcal{K}_1) \overline{\otimes} \mathcal{B}(\mathcal{K}_2))u \\
&= u^*(\mathcal{N}_1' \overline{\otimes} \mathcal{N}_2' \overline{\otimes} \mathcal{B}(\mathcal{K}_1) \overline{\otimes} \mathcal{B}(\mathcal{K}_2))u \\
&= \mathcal{N}_1' \overline{\otimes} \mathcal{B}(\mathcal{K}_1) \overline{\otimes} \mathcal{N}_2' \overline{\otimes} \mathcal{B}(\mathcal{K}_2) \\
&= (\mathcal{N}_1 \overline{\otimes} \mathcal{C}(\mathcal{K}_1))' \overline{\otimes} (\mathcal{N}_2 \overline{\otimes} \mathcal{C}(\mathcal{K}_2))' = \mathcal{M}_1' \overline{\otimes} \mathcal{M}_2'.
\end{aligned}
$$

Q.E.D.

We now prove the commutation theorem for tensor products:

Theorem. *For any pair of von Neumann algebras* $\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1)$, $\mathcal{M}_2 \subset \mathcal{B}(\mathcal{H}_2)$ *the following relation holds*:

$$(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)' = \mathcal{M}_1' \overline{\otimes} \mathcal{M}_2'.$$

Proof. In accordance with Lemma 1, the assertion is true if $\mathcal{M}_1$ and $\mathcal{M}_2$ have separating cyclic vectors.

With the help of exercise E.5.6, Proposition 5.18, and Lemma 2, the assertion of the theorem extends to the case in which $\mathcal{M}_1$ and $\mathcal{M}_2$ are of countable type.

Then the assertion of the theorem trivially extends to the case in which the unit projections in $\mathscr{M}_1$ and $\mathscr{M}_2$ are piecewise of countable type (7.2).

By taking into account Lemma 7.2, Proposition 3.17 and Theorem 4.22, the assertion of the theorem easily obtains for the case in which $\mathscr{M}_1$ is finite, or uniform (8.4), and $\mathscr{M}_2$ is finite, or uniform, too.

Finally, with the help of Theorem 4.17 and of Proposition 8.4, the assertion of the theorem obtains in the general case.

Q.E.D.

Corollary. *Let $\mathscr{M}_1, \mathscr{M}_2$ be von Neumann algebras, $\mathscr{Z}_1, \mathscr{Z}_2$ their centers, respectively. Then the center of $\mathscr{M}_1 \overline{\otimes} \mathscr{M}_2$ is $\mathscr{Z}_1 \overline{\otimes} \mathscr{Z}_2$.*

Proof. We denote by $\mathscr{Z}$ the center of $\mathscr{M}_1 \overline{\otimes} \mathscr{M}_2$. Obviously,

$$\mathscr{Z}_1 \overline{\otimes} \mathscr{Z}_2 \subset \mathscr{Z}.$$

On the other hand, the inclusions

$$\mathscr{M}_1 \overline{\otimes} \mathscr{M}_2 \subset \mathscr{Z}',$$

$$\mathscr{M}_1' \overline{\otimes} \mathscr{M}_2' \subset (\mathscr{M}_1 \overline{\otimes} \mathscr{M}_2)' \subset \mathscr{Z}'$$

imply that

$$\mathscr{Z}_1' \overline{\otimes} \mathscr{Z}_2' = \mathscr{R}(\mathscr{M}_1, \mathscr{M}_1') \overline{\otimes} \mathscr{R}(\mathscr{M}_2, \mathscr{M}_2') \subset \mathscr{Z}'.$$

With the help of the preceding theorem, we obtain

$$\mathscr{Z} \subset (\mathscr{Z}_1' \overline{\otimes} \mathscr{Z}_2')' = \mathscr{Z}_1 \overline{\otimes} \mathscr{Z}_2.$$

Q.E.D.

10.8. In this section we exhibit a class of "positive" elements in $\mathscr{D}_S$ (respectively, in $\mathscr{D}_{S^*}$).

Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra. For any vector $\xi \in \mathscr{H}$ we define the linear operator L_ξ^0 by

$$L_\xi^0(\eta) = R_\eta(\xi), \quad \eta \in \mathscr{D}_{(L_\xi^0)} = \mathfrak{A}'.$$

We observe that

$$\xi \in \overline{L_\xi^0(\mathfrak{A}')}.$$

Indeed, if $\{\eta_i\} \subset \mathfrak{A}'$ and $R_{\eta_i} \xrightarrow{so} 1$, then

$$L_\xi^0(\mathfrak{A}') \ni L_\xi^0 \eta_i = R_{\eta_i}\xi \to \xi.$$

If L_ξ^0 is preclosed, we denote its closure by L_ξ. By analogy with Lemma 1 from Section 10.2, one can show that if L_ξ^0 is preclosed, then L_ξ is affiliated to $\mathscr{R}(\mathfrak{A}')' = \mathfrak{L}(\mathfrak{A})$. In Section 10.4 we saw that if $\xi \in \mathscr{D}_S$, then L_ξ^0 is preclosed. Also, if L_ξ^0 is symmetric, then L_ξ^0 is preclosed (9.4).

We write

$$\mathfrak{P}_S = \{\xi \in \mathscr{H};\ L_\xi^0 \text{ is positive}\}.$$

Let $\xi \in \mathfrak{P}_S$ and let A be the Friedrichs extension of L_ξ^0 (9.6). In accordance with exercise E.9.27, A is affiliated to $\mathfrak{L}(\mathfrak{A})$. With the help of Proposition 10.4, it immediately follows that, for any function $f \in \mathscr{B}([0, +\infty))$, such that $\sup \{\lambda^2 |f(\lambda)|;\ \lambda \in [0, +\infty)\} < +\infty$, we have

$$Af(A)\xi \in \mathfrak{A}'' \text{ and}$$

$$S(Af(A)\xi) = A\bar{f}(A)\xi, \quad L_{Af(A)\xi} = A^2 f(A).$$

Analogously, we denote

$$\mathfrak{P}_{S^*} = \{\eta \in \mathscr{H};\ R_\eta^0 \text{ is positive}\}.$$

Proposition. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra.*

The following assertions regarding a vector $\xi_0 \in \mathscr{H}$ are equivalent

(i) $\xi_0 \in \mathfrak{P}_S$;

(ii) $\xi_0 \in \mathscr{D}_S$, $S\xi_0 = \xi_0$ *and* L_{ξ_0} *is positive*;

(iii) ξ_0 *belongs to the norm closure of the set* $\{L_\xi S\xi;\ \xi \in \mathfrak{A}''\}$;

(iv) $(\xi_0 | \eta) \geqslant 0$, *for any* $\eta \in \mathfrak{P}_{S^*}$.

The following assertions regarding a vector $\eta_0 \in \mathscr{H}$ are equivalent

(j) $\eta_0 \in \mathfrak{P}_{S^*}$;

(jj) $\eta_0 \in \mathscr{D}_{S^*}$, $S^*\eta_0 = \eta_0$ *and* R_{η_0} *is positive*;

(jjj) η_0 *belongs to the norm closure of the set* $\{R_\eta S^*\eta;\ \eta \in \mathfrak{A}'\}$;

(jw) $(\xi | \eta_0) \geqslant 0$, *for any* $\xi \in \mathfrak{P}_S$.

Proof. We first observe that, for any $\eta \in \mathfrak{A}'$ we have

$$(1) \qquad (\xi_0 | R_{S^*\eta}\eta) = (R_\eta \xi_0 | \eta) = (L_{\xi_0}^0 \eta | \eta).$$

We prove that (i) $\Rightarrow$ (ii). From condition (i) and relation (1) we get, for any $\eta \in \mathfrak{A}'$,

$$(\xi_0 | R_{S^*\eta}\eta) \geqslant 0.$$

Hence, for any $\eta_1, \eta_2 \in \mathfrak{A}'$, we have

$$(\xi_0 | R_{S^*\eta_1}\eta_2) = \frac{1}{4}\sum_{k=0}^{3} i^k (\xi_0 | R_{S^*(\eta_1 + i^k \eta_2)}(\eta_1 + i^k \eta_2))$$

$$= \frac{1}{4}\sum_{k=0}^{3} i^k (R_{S^*(\eta_1 + i^k \eta_2)}(\eta_1 + i^k \eta_2) | \xi_0)$$

$$= (R_{S^*\eta_2}\eta_1 | \xi_0) = (S^* R_{S^*\eta_1}(\eta_2) | \xi_0).$$

Since $\overline{S^*|(\mathfrak{A}')^2} = S^*$ (see 10.4), it follows that

$$\xi_0 \in \mathscr{D}_S \text{ and } S\xi_0 = \xi_0,$$

thereby proving assertion (ii).

We now show that (ii) $\Rightarrow$ (iii). Let A be the Friedrichs extension of L_{ξ_0}. For any natural number n we consider the function $f_n \in \mathscr{B}([0, +\infty))$, defined by

$$f_n(\lambda) = \lambda^{-3/2}\chi_{(1/n, n)}(\lambda).$$

Since

$$\lim_{n\to\infty} \lambda^3 f_n^2(\lambda) = \chi_{(0, \infty)}(\lambda),$$

from Theorem 9.11 (vi) and Corollary 9.13 (iii) it follows that

$$A^3 f_n^2(A) \xrightarrow{so} s(A).$$

On the other hand, we have

$$\xi_n = Af_n(A)\xi_0 \in \mathfrak{A}'' \text{ and}$$

$$S\xi_n = Af_n(A)\xi_0, \quad L_{\xi_n} = A^2 f_n(A) \geqslant 0.$$

Since

$$\xi_0 \in \overline{L_{\xi_0}(\mathfrak{A}')} \subset s(A)\mathscr{H},$$

we have

$$L_{\xi_n} S\xi_n = A^3 f_n^2(A)\xi_0 \to \xi_0,$$

and assertion (iii) is thus proved.

The implications (j) $\Rightarrow$ (jj) $\Rightarrow$ (jjj) can be proved similary.

Let $\xi \in \mathfrak{A}''$, $\eta \in \mathfrak{A}'$. Then

$$\begin{aligned}(L_\xi S\xi \mid R_\eta S^*\eta) &= (R_{S^*\eta} L_\xi S\xi \mid S^*\eta) = (L_\xi R_{S^*\eta} S\xi \mid S^*\eta) \\ &= (L_\xi L_\xi^* S^*\eta \mid S^*\eta) = \|L_\xi^* S^*\eta\|^2 \geqslant 0.\end{aligned}$$

The implication (iii) $\Rightarrow$ (iv) now follows by tending to the limit, and by taking into account the implication (j) $\Rightarrow$ (jjj).

Since $\{R_{S^*\eta}\eta;\ \eta \in \mathfrak{A}'\} \subset \mathfrak{P}_{S^*}$, the implication (iv) $\Rightarrow$ (i) obviously follows from relation (1).

The implications (jjj) $\Rightarrow$ (jw) $\Rightarrow$ (j) can be proved similarly.

Q.E.D.

Corollary. *Let $\mathfrak{A}$ be a left Hilbert algebra. Then*

(1) *$\mathfrak{P}_S$ is a closed convex cone, included in $\mathscr{D}_S$; $\mathscr{D}_S$ is the linear hull of $\mathfrak{P}_S$.*

(2) *$\mathfrak{P}_{S^*}$ is a closed convex cone, included in $\mathscr{D}_{S^*}$; $\mathscr{D}_{S^*}$ is the linear hull of $\mathfrak{P}_{S^*}$.*

Proof. The fact that $\mathfrak{P}_S$ is a closed convex cone, included in $\mathscr{D}_S$, obviously follows from the preceding proposition.

Let $\xi \in \mathscr{D}_S$. We must prove that ξ is a linear combination of elements in $\mathfrak{P}_S$. We can assume that $S\xi = \xi$. Then, by taking into account the fact that $\overline{S|\mathfrak{A}''} = S$, it follows that for any n there exists

$$\xi_n \in \mathfrak{A}'', \quad S\xi_n = \xi_n, \quad \|\xi_n\| \leqslant 1/2^n,$$

such that

$$\sum_{n=1}^{\infty} \xi_n = \xi.$$

The operators

$$a_n = L_{\xi_n} \in \mathfrak{L}(\mathfrak{A})$$

are self-adjoint. In accordance with Corollary 2.10, we consider the decompositions

$$a_n = a_n^+ - a_n^-;$$

then

$$a_n^+, a_n^- \in \mathfrak{L}(\mathfrak{A}), \quad a_n^+, \ a_n^- \geqslant 0,$$

$$|a_n| = a_n^+ + a_n^-, \ \mathrm{s}(a_n) = \mathrm{s}(a_n^+) + \mathrm{s}(a_n^-), \ \mathrm{s}(a_n^+)\mathrm{s}(a_n^-) = 0.$$

We define

$$\xi_n^+ = \mathrm{s}(a_n^+)\xi_n, \quad \xi_n^- = \mathrm{s}(a_n^-)\xi_n.$$

We obviously have

$$\|\xi_n^+\| \leqslant 1/2^n, \quad \|\xi_n^-\| \leqslant 1/2^n,$$

$$\xi_n = \mathrm{s}(a_n)\xi_n = \mathrm{s}(a_n^+)\xi_n + \mathrm{s}(a_n^-)\xi_n = \xi_n^+ - \xi_n^-.$$

It is easy to verify that

$$L_{\xi_n^+} \subset \mathrm{s}(a_n^+)a_n = \mathrm{s}(a_n^+)|a_n| \geqslant 0,$$

$$L_{\xi_n^-} \subset -\mathrm{s}(a_n^-)a_n = \mathrm{s}(a_n^-)|a_n| \geqslant 0;$$

hence

$$\xi_n^+ \in \mathfrak{P}_S, \quad \xi_n^- \in \mathfrak{P}_S.$$

We denote

$$\xi^+ = \sum_{n=1}^{\infty} \xi_n^+, \ \xi^- = \sum_{n=1}^{\infty} \xi_n^-.$$

Since $\mathfrak{P}_S$ is closed, we have

$$\xi^+ \in \mathfrak{P}_S, \quad \xi^- \in \mathfrak{P}_S,$$

and, obviously,

$$\xi = \xi^+ - \xi^-.$$

We have thus proved assertion (1). Assertion (2) can be proved analogously.

Q.E.D.

The assertions (iv) and (jw) from the statement of the proposition show that $\mathfrak{P}_S$ and $\mathfrak{P}_{S^*}$ are cones "polar" to one another.

10.9. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra and ξ_0 a separating cyclic vector for $\mathscr{M}$.

In accordance with Section 10.6,

$$\mathfrak{A} = \mathscr{M}\xi_0 \subset \mathscr{H}$$

is canonically endowed with a structure of a left Hilbert algebra and

$$\mathfrak{A}' = \mathscr{M}'\xi_0, \quad \mathfrak{A}'' = \mathfrak{A}.$$

For $\xi, \eta \in \mathscr{H}$ we consider the forms $\omega_{\xi,\eta} \in \mathscr{M}_*$, $\omega'_{\xi,\eta} \in (\mathscr{M}')_*$ given by

$$\omega_{\xi,\eta}(x) = (x\xi \mid \eta), \quad x \in \mathscr{M},$$
$$\omega'_{\xi,\eta}(x') = (x'\xi \mid \eta), \quad x' \in \mathscr{M}'.$$

In particular, we have (see 5.22)

$$\omega_{\xi,\xi} = \omega_\xi, \quad \omega'_{\xi,\xi} = \omega'_\xi, \quad \xi \in \mathscr{H}.$$

It is obvious that, for any $\xi, \eta \in \mathscr{H}$, we have

$$R_a\omega_{\xi,\eta} = \omega_{a\xi,\eta}, \quad a \in \mathscr{M},$$
$$R_{a'}\omega'_{\xi,\eta} = \omega'_{a'\xi,\eta}, \quad a' \in \mathscr{M}'.$$

For any $\xi \in \mathscr{H}$, the operator L^0_ξ is given by

$$L^0_\xi(x'\xi_0) = x'\xi, \quad x'\xi_0 \in \mathscr{D}_{(L^0_\xi)} = \mathscr{M}'\xi_0.$$

This operator is positive iff

$$\omega'_{\xi,\xi_0} \geq 0.$$

Consequently, with the notations from Section 10.8, we have

$$\mathfrak{P}_S = \{\xi \in \mathscr{H};\ \omega'_{\xi,\xi_0} \geq 0\}.$$

If $\xi \in \mathfrak{P}_S$ and if we denote by A the Friedrichs extension of L^0_ξ, then A is a positive self-adjoint operator in $\mathscr{H}$, affiliated to $\mathscr{M}$, and

$$\xi_0 \in \mathscr{D}_A, \quad A\xi_0 = \xi.$$

Conversely, if A is a positive self-adjoint operator in $\mathscr{H}$, affiliated to $\mathscr{M}$, and if $\xi_0 \in \mathscr{D}_A$, then

$$A\xi_0 \in \mathfrak{P}_S, \quad L_{A\xi_0} \subset A.$$

Thus,

$$\mathfrak{P}_S = \{A\xi_0;\ A \text{ positive self-adjoint, affiliated to } \mathscr{M}, \xi_0 \in \mathscr{D}_A\}.$$

Lemma. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra, with the separating cyclic vector $\xi_0 \in \mathscr{H}$. For any normal form φ on $\mathscr{M}$ there exists a unique vector $\xi \in \mathfrak{P}_S$, such that*

$$\varphi = \omega_\xi.$$

In particular, there exists a positive self-adjoint operator A in $\mathscr{H}$, affiliated to $\mathscr{M}$, such that

$$\xi_0 \in \mathscr{D}_A, \quad \varphi = \omega_{A\xi_0}.$$

Proof. In order to prove the uniqueness of the vector ξ, let $\xi_1, \xi_2 \in \mathfrak{P}_S$, be such that $\omega_{\xi_1} = \omega_{\xi_2}$. Then

$$\|x\xi_1\| = \|x\xi_2\|, \quad x \in \mathscr{M};$$

hence, there exists a partial isometry $v' \in \mathscr{M}'$, such that

$$v'x\xi_1 = x\xi_2, \quad x \in \mathscr{M},$$

$$v'([\mathscr{M}\xi_1]^\perp) = \{0\}.$$

In particular,

$$v'\xi_1 = \xi_2, \quad v'^*\xi_2 = \xi_1,$$

and therefore

$$\omega'_{\xi_2,\xi_0} = \omega'_{v'\xi_1,\xi_0} = R_{v'}\,\omega'_{\xi_1,\xi_0},$$

$$\omega'_{\xi_1,\xi_0} = \omega'_{v'^*\xi_2,\xi_0} = R_{v'^*}\omega'_{\xi_2,\xi_0}.$$

Since $\xi_1, \xi_2 \in \mathfrak{P}_S$, we have $\omega'_{\xi_1,\xi_0} \geqslant 0$, $\omega'_{\xi_2,\xi_0} \geqslant 0$, and, by taking into account the uniqueness of the polar decomposition (5.16), it follows that

$$\omega'_{\xi_1,\xi_0} = \omega'_{\xi_2,\xi_0}.$$

Since $\overline{\mathscr{M}'\xi_0} = \mathscr{H}$, we infer that

$$\xi_1 = \xi_2.$$

We now prove the existence of the vector ξ. In accordance with Theorem 5.23, there exists a vector $\zeta \in \mathscr{H}$, such that

$$\varphi = \omega_\zeta.$$

Let now, in accordance with Theorem 5.16,

$$\omega'_{\zeta,\xi_0} = R_{v'}\psi'$$

be the polar decomposition of the form $\omega'_{\zeta,\zeta_0} \in (\mathcal{M}')_*$. We define

$$\xi = v'^*\zeta.$$

Then

$$\omega'_{\xi,\zeta_0} = \omega'_{v'^*\zeta,\zeta_0} = R_{v'^*}\omega'_{\zeta,\zeta_0} = \psi' \geq 0$$

and, consequently, we have

$$\xi \in \mathfrak{P}_S.$$

On the other hand,

$$\omega'_{\zeta,\zeta_0} = R_{v'}\psi' = R_{v'}\omega'_{\xi,\zeta_0} = \omega'_{v'\xi,\zeta_0},$$

whence

$$\zeta = v'\xi.$$

Thus, for any $x \in \mathcal{M}$, we have

$$\begin{aligned}\varphi(x) = \omega_\zeta(x) &= (x\zeta|\zeta) = (x\zeta|v'\xi) = (v'^*x\zeta|\xi) \\ &= (xv'^*\zeta|\xi) = (x\xi|\xi) = \omega_\xi(x).\end{aligned}$$

Consequently, we have

$$\varphi = \omega_\xi.$$

Q.E.D.

10.10. Another important application of the theory developed so far is a general Radon-Nikodym type theorem, which we shall present in this section.

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and φ a normal form on $\mathcal{M}$. Let A be a positive self-adjoint operator in $\mathcal{H}$, affiliated to $\mathcal{M}$. We write (cf. 9.9):

$$e_n = \chi_{[0,n)}(A) \in \mathcal{M}.$$

One says that A is *of summable square with respect to* φ if

$$c = \sup_n \varphi(A^2e_n) < +\infty;$$

then

$$\varphi(A^2e_n) \uparrow c.$$

In this case, for any $x \in \mathcal{M}$ and any $m > n$, we have

$$\begin{aligned}|\varphi(Ae_mxAe_m) - \varphi(Ae_nxAe_n)| &\leq |\varphi(Ae_mxA(e_m - e_n))| + |\varphi(A(e_m - e_n)\,xAe_n)| \\ &\leq \varphi(Ae_mxx^*Ae_m)^{1/2}\varphi(A^2(e_m - e_n))^{1/2} + \varphi(A^2(e_m - e_n))^{1/2}\varphi(Ae_nx^*xAe_n)^{1/2} \\ &\leq 2\|x\|c^{1/2}[\varphi(A^2e_m) - \varphi(A^2e_n)]^{1/2}.\end{aligned}$$

Consequently, the sequence $\{\varphi(Ae_n x Ae_n)\}$ is fundamental. We write

$$L_A R_A \varphi(x) = \lim_{n\to\infty} \varphi(Ae_n x Ae_n), \quad x \in \mathscr{M}.$$

Therefore, we get a linear form $L_A R_A \varphi$ on $\mathscr{M}$. Tending to the limit for $m \to \infty$, in the preceding inequality, we get

$$|(L_A R_A \varphi)(x) - (L_{Ae_n} R_{Ae_n} \varphi)(x)| \leqslant 2c^{1/2}[c - \varphi(A^2 e_n)]^{1/2}\|x\|.$$

Since $L_{Ae_n} R_{Ae_n} \varphi \in (\mathscr{M}_*)^+$, it follows that

$$L_A R_A \varphi \in (\mathscr{M}_*)^+.$$

By taking into account Theorem 9.11 (ii), it is obvious that if

$$\varphi = \omega_\xi, \quad \xi \in \mathscr{H},$$

then the operator A is of summable square with respect to φ iff $\xi \in \mathscr{D}_A$. In this case

$$L_A R_A \varphi = \omega_{A\xi}.$$

Theorem. *Let φ and ψ be normal forms on the von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ such that*

$$s(\varphi) \leqslant s(\psi).$$

Then there exists a positive self-adjoint operator A in $\mathscr{H}$, affiliated to $\mathscr{M}$, such that

$$s(A) \leqslant s(\psi),$$

$$\varphi = L_A R_A \psi.$$

Moreover, if the projection $s(\psi)$ is finite in $\mathscr{M}$, then the operator A is uniquely determined by the required properties.

Proof. Without any loss of generality, we can assume that $s(\psi) = 1$. From Proposition 5.18, we infer that there exists a $*$-isomorphism π of $\mathscr{M}$ onto a von Neumann algebra $\mathscr{N} \subset \mathscr{B}(\mathscr{K})$, $\pi(1) = 1$, and a separating cyclic vector $\zeta_0 \in \mathscr{K}$ for $\mathscr{N}$, such that

$$\psi = \omega_{\zeta_0} \circ \pi.$$

Then

$$\varphi = \theta \circ \pi,$$

where θ is a normal form on the von Neumann algebra $\mathscr{N} \subset \mathscr{B}(\mathscr{K})$, uniquely determined by this relation.

From Lemma 10.9 we infer that there exists a positive self-adjoint operator B in $\mathscr{K}$, affiliated to $\mathscr{N}$, such that

$$\theta = \omega_{B\zeta_0}.$$

Let $A = \pi^{-1}(B)$ be the positive self-adjoint operator in $\mathscr{H}$, affiliated to $\mathscr{M}$, which is obtained by transporting the operator B by the $*$-isomorphism π^{-1} (see Section 9.25). If we denote

$$e_n = \chi_{[0,n]}(A), \quad f_n = \chi_{[0,n]}(B),$$

then we have

$$\psi(A^2e_n) = \psi(\pi^{-1}(B^2f_n)) = \omega_{\zeta_0}(B^2f_n) = \|f_nB\zeta_0\|^2 \leqslant \|B\zeta_0\|^2,$$

hence A is of summable square with respect to ψ. Then

$$(L_AR_A\psi)(x) = \lim_{n\to\infty}(Ae_nxAe_n) = \lim_{n\to\infty}\psi(\pi^{-1}(Be_n)x\pi^{-1}(Be_n))$$

$$= \lim_{n\to\infty}\omega_{\zeta_0}(Be_n\pi(x)Be_n) = \omega_{B\zeta_0}(\pi(x)) = \theta(\pi(x)) = \varphi(x), \quad x \in \mathscr{M},$$

hence

$$L_AR_A\psi = \varphi.$$

We have proved the first part of the theorem.

Let us now assume that $s(\psi)$ is finite. We can assume again that $s(\psi) = 1$, hence $\mathscr{M}$ is finite. Moreover, by considering the $*$-isomorphism π and the possibility of transporting by π the positive self-adjoint operators which are affiliated to $\mathscr{M}$ (see 9.25), we can assume that $\mathscr{M}$ has a separating cyclic vector $\zeta_0 \in \mathscr{H}$ and $\psi = \omega_{\zeta_0}$.

Therefore let A be a positive self-adjoint operator in $\mathscr{H}$, which is affiliated to $\mathscr{M}$ and such that

$$\varphi = L_AR_A\psi = \omega_{A\zeta_0}.$$

Then the vector $A\zeta_0$ is uniquely determined by φ (see 10.9). On the other hand, A is an extension of the positive operator $L_{A\zeta_0}$ (see 10.9). Since $\mathscr{M}$ is finite, Corollary 9.8 implies that A is, indeed, determined in a unique manner by φ.

Q.E.D.

Another case in which the uniqueness of the operator A holds is that in which the normal form φ is dominated (see E.9.33) by the normal form ψ. Indeed, in this case it is easy to verify that the operator A is bounded and one applies Theorem 5.21.

10.11. This section contains the main technical result for the subsequent development of the theory; with its help a bridge between $\mathfrak{A}''$ and $\mathfrak{A}'$ is established.

Proposition. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra and $\lambda \in \mathbb{C}\setminus\mathbb{R}^+$. Then, for any $\xi \in \mathfrak{A}''$, we have $(\lambda - \Delta^{-1})^{-1}\xi \in \mathfrak{A}'$ and*

$$\|R_{(\lambda-\Delta^{-1})^{-1}\xi}\| \leqslant 2^{-1/2}(|\lambda| - \mathrm{Re}\,\lambda)^{-1/2}\|L_\xi\|.$$

Similarly, for any $\eta \in \mathfrak{A}'$, we have $(\lambda - \Delta)^{-1}\eta \in \mathfrak{A}''$ and

$$\|L_{(\lambda-\Delta)^{-1}\eta}\| \leqslant 2^{-1/2}(|\lambda| - \mathrm{Re}\,\lambda)^{-1/2}\|R_\eta\|.$$

Proof. We shall prove only the first assertion, the proof of the second one being similar.

We denote

$$\eta = (\lambda - \Delta^{-1})^{-1}\xi.$$

Then

$$\eta \in \mathscr{D}_{(\Delta^{-1})} \subset \mathscr{D}_{(\Delta^{-1/2})} = \mathscr{D}_{S^*}.$$

We consider the polar decompositions

$$R_\eta = uA = Bu.$$

With Lemma 1 from 10.2, we infer that A and B are affiliated to $\mathfrak{L}(\mathfrak{A})'$.

By taking into account Corollary 5 from 10.3, we infer that, for any function $f \in \mathscr{B}([0, +\infty))$, assumed only to be positive and with compact support, we have the following sequence of relations

$$\begin{aligned}
\|L_\xi\|^2\|f(A)S^*\eta\|^2 &\geqslant \|L_\xi f(A)S^*\eta\|^2 = \|f(A)L_\xi S^*\eta\|^2 = \|f(A)(R_\eta)^*\xi\|^2\\
&= \|f(A)Au^*\xi\|^2 = \|Af(A)u^*(\lambda - \Delta^{-1})\eta\|^2\\
&= |\lambda|^2\|Af(A)u^*\eta\|^2 + \|Af(A)u^*\Delta^{-1}\eta\|^2 - 2\,\mathrm{Re}[\lambda(Af(A)u^*\eta|Af(A)u^*\Delta^{-1}\eta)]\\
&\geqslant 2|\lambda|\,\|Af(A)u^*\eta\|\,\|Af(A)u^*\Delta^{-1}\eta\| - 2\,\mathrm{Re}[\lambda(Af(A)u^*\eta|Af(A)u^*\Delta^{-1}\eta)]\\
&\geqslant 2|\lambda|\,|(Af(A)u^*\eta|Af(A)u^*\Delta^{-1}\eta)| - 2\,\mathrm{Re}[\lambda(Af(A)u^*\eta|Af(A)u^*\Delta^{-1}\eta)]\\
&= 2|\lambda|\,|(B^2f^2(B)\eta|SS^*\eta)| - 2\,\mathrm{Re}[\lambda(B^2f^2(B)\eta|SS^*\eta)]\\
&= 2|\lambda|\,|(A^2f^2(A)S^*\eta|S^*\eta)| - 2\,\mathrm{Re}\,[\lambda(A^2f^2(A)S^*\eta|S^*\eta)]\\
&= 2|\lambda|\,\|Af(A)S^*\eta\|^2 - 2\,\mathrm{Re}\,[\lambda\|Af(A)S^*\eta\|^2]\\
&= 2(|\lambda| - \mathrm{Re}\,\lambda)\,\|Af(A)S^*\eta\|^2.
\end{aligned}$$

Consequently,

$$2^{-1/2}(|\lambda| - \mathrm{Re}\,\lambda)^{-1/2}\|L_\xi\|\,\|f(A)S^*\eta\| \geqslant \|Af(A)S^*\eta\|.$$

We denote $c = 2^{-1/2}(|\lambda| - \mathrm{Re}\,\lambda)^{-1/2}\|L_\xi\|$ and $f_n = \chi_{(c+n^{-1},\, n)}$. From the above inequality we obtain successively

$$c\|f_n(A)S^*\eta\| \geqslant \|Af_n(A)S^*\eta\| \geqslant (c + n^{-1})\|f_n(A)S^*\eta\|,$$

$$f_n(A)S^*\eta = 0,$$

$$Af_n(A)u^*\xi = f_n(A)\,(R_\eta)^*\xi = L_\xi f_n(A)S^*\eta = 0, \quad \xi \in \mathfrak{A},$$

$$Af_n(A)u^* = 0,$$

$$Af_n(A) = 0,$$

$$f_n(A) = 0.$$

Thus

$$\chi_{(c,+\infty)}(A) = so\text{-}\lim_{n\to\infty} f_n(A) = 0;$$

hence A is bounded and $\|A\| \leqslant c$.

It follows that R_η is bounded; hence $\eta \in \mathfrak{A}'$, and $\|R_\eta\| \leqslant c$.

Q.E.D.

10.12. In this section we present the fundamental theorem of Tomita.

Lemma 1. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra, $\xi \in \mathfrak{A}''$, $\lambda > 0$ and $\eta = (\lambda + \Delta^{-1})^{-1}\xi$. Then $\eta \in \mathfrak{A}'$ and the relation*

$$(L_\xi(\zeta_1)|\zeta_2) = \lambda(J(R_\eta)^* J\Delta^{-1/2}\zeta_1|\Delta^{1/2}\zeta_2) + (J(R_\eta)^* J\Delta^{1/2}\zeta_1|\Delta^{-1/2}\zeta_2)$$

holds for any $\zeta_1, \zeta_2 \in \mathscr{D}_{(\Delta^{1/2})} \cap \mathscr{D}_{(\Delta^{-1/2})}$.

Proof. Proposition 10.11 obviously implies that $\eta \in \mathfrak{A}'$.

We shall first assume that

$$\zeta_1, \zeta_2 \in (1 + \Delta^{-1})^{-1}\mathfrak{A}''.$$

Since $\mathfrak{A}'' \subset \mathscr{D}_S = \mathscr{D}_{(\Delta^{1/2})}$, we have

$$\zeta_1, \zeta_2 \in \Delta^{1/2}(\Delta + 1)^{-1}(\Delta^{1/2}\mathfrak{A}'') \subset \mathscr{D}_{(\Delta^{1/2})}.$$

On the other hand, from Proposition 10.11 we infer that

$$\zeta_1, \zeta_2 \in \mathfrak{A}' \subset \mathscr{D}_{S^*} = \mathscr{D}_{(\Delta^{-1/2})}.$$

With the help of Corollary 2 from 10.3 we infer that the following sequence of equalities holds:

$$\begin{aligned}
(L_\xi(\zeta_1)|\zeta_2) &= (R_{\zeta_1}(\xi)|\zeta_2) = (\xi|(R_{\zeta_1})^*\zeta_2) = ((\lambda + \Delta^{-1})\eta|(R_{\zeta_1})^*\zeta_2)\\
&= \lambda(\eta|(R_{\zeta_1})^*\zeta_2) + (SS^*\eta|R_{S^*\zeta_1}(\zeta_2)) = \lambda(\eta|(R_{\zeta_1})^*\zeta_2) + (S^*R_{S^*\zeta_1}(\zeta_2)|S^*\eta)\\
&= \lambda(\eta|(R_{\zeta_1})^*\zeta_2) + (R_{S^*\zeta_2}(\zeta_1)|S^*\eta) = \lambda(R_{\zeta_1}(\eta)|\zeta_2) + (\zeta_1|R_{\zeta_2}S^*\eta)\\
&= \lambda(R_{\zeta_1}(\eta)|SS\zeta_2) + (SS\zeta_1|R_{\zeta_2}S^*\eta) = \lambda(S\zeta_2|S^*R_{\zeta_1}(\eta)) + (S^*R_{\zeta_2}S^*\eta|S\zeta_1)\\
&= \lambda(S\zeta_2|R_{S^*\eta}S^*\zeta_1) + (R_\eta S^*\zeta_2|S\zeta_1) = \lambda(S\zeta_2|(R_\eta)^*S^*\zeta_1) + (S^*\zeta_2|(R_\eta)^*S\zeta_1)\\
&= \lambda(J\Delta^{1/2}\zeta_2|(R_\eta)^*J\Delta^{-1/2}\zeta_1) + (J\Delta^{-1/2}\zeta_2|(R_\eta)^*J\Delta^{1/2}\zeta_1)\\
&= \lambda(J(R_\eta)^*J\Delta^{-1/2}\zeta_1|\Delta^{1/2}\zeta_2) + (J(R_\eta)^*J\Delta^{1/2}\zeta_1|\Delta^{-1/2}\zeta_2).
\end{aligned}$$

If we can prove that, for any $\zeta \in \mathscr{D}_{(\Delta^{1/2})} \cap \mathscr{D}_{(\Delta^{-1/2})}$ there exists a sequence $\{\zeta_n\} \subset (1 + \Delta^{-1})^{-1}\mathfrak{A}''$, such that

$$\zeta_n \to \zeta, \ \Delta^{1/2}\zeta_n \to \Delta^{1/2}\zeta, \ \Delta^{-1/2}\zeta_n \to \Delta^{-1/2}\zeta,$$

then the assertion immediately follows from the preceding equalities.

Let then $\zeta \in \mathscr{D}_{(\Delta^{1/2})} \cap \mathscr{D}_{(\Delta^{-1/2})}$. Since the set $\Delta^{1/2}\mathfrak{A}'' = JS\mathfrak{A}'' = J\mathfrak{A}''$ is dense in $\mathscr{H}$, there exists a sequence $\{\xi_n\} \subset \mathfrak{A}''$, such that

$$\Delta^{1/2}\xi_n \to \Delta^{1/2}\zeta + \Delta^{-1/2}\zeta.$$

If we write

$$\zeta_n = (1 + \Delta^{-1})^{-1}\xi_n \in (1 + \Delta^{-1})^{-1}\mathfrak{A}'',$$

we have

$$\zeta_n = \Delta^{-1/2}(1 + \Delta^{-1})^{-1}(\Delta^{1/2}\xi_n) \to \Delta^{-1/2}(1 + \Delta^{-1})^{-1}(\Delta^{1/2}\zeta + \Delta^{-1/2}\zeta) = \zeta,$$

$$\Delta^{1/2}\zeta_n = (1 + \Delta^{-1})^{-1}(\Delta^{1/2}\xi_n) \to (1 + \Delta^{-1})^{-1}(\Delta^{1/2}\zeta + \Delta^{-1/2}\zeta) = \Delta^{1/2}\zeta,$$

$$\Delta^{-1/2}\zeta_n = \Delta^{-1}(1 + \Delta^{-1})^{-1}(\Delta^{1/2}\xi_n) \to \Delta^{-1}(1 + \Delta^{-1})^{-1}(\Delta^{1/2}\zeta + \Delta^{-1/2}\zeta) = \Delta^{-1/2}\zeta.$$

Q.E.D.

Lemma 2. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra and $\xi \in \mathfrak{A}''$. Then, for any $t \in \mathbb{R}$, we have*

$$J\Delta^{it}\xi \in \mathfrak{A}' \text{ and}$$

$$S^*J\Delta^{it}\xi = J\Delta^{it}S\xi, \quad R_{J\Delta^{it}\xi} = J\Delta^{it}L_\xi\Delta^{-it}J.$$

Proof. By taking into account Lemma 1 and Proposition 9.23, in which we make $A = B = \Delta$, we infer that, for any $\lambda > 0$, we have

$$(\lambda + \Delta^{-1})^{-1}\xi \in \mathfrak{A}' \text{ and}$$

$$J(R_{(\lambda+\Delta^{-1})^{-1}\xi})^*J = \int_{-\infty}^{+\infty} \frac{\lambda^{it-\frac{1}{2}}}{e^{\pi t} + e^{-\pi t}} \Delta^{it}L_\xi\Delta^{-it}\,dt,$$

i.e.,

$$(R_{(\lambda+\Delta^{-1})^{-1}\xi})^* = \int_{-\infty}^{+\infty} \frac{\lambda^{-it-\frac{1}{2}}}{e^{\pi t} + e^{-\pi t}} J\Delta^{it}L_\xi\Delta^{-it}J\,dt.$$

On the other hand, from Corollary 9.23, we infer that for any $\lambda > 0$ and any $\zeta \in \mathfrak{A}$, we have the equalities

$$(R_{(\lambda+\Delta^{-1})^{-1}\xi})^*\zeta = L_\zeta S^*(\lambda + \Delta^{-1})^{-1}\xi$$

$$= L_\zeta J\Delta^{-1/2}(\lambda + \Delta^{-1})^{-1}\xi = \int_{-\infty}^{+\infty} \frac{\lambda^{-it-\frac{1}{2}}}{e^{\pi t} + e^{-\pi t}} L_\zeta J\,\Delta^{it}\xi\,dt.$$

Let $\zeta \in \mathfrak{A}$. From the preceding formulas we infer that the equality

$$\int_{-\infty}^{+\infty} \lambda^{-it} \frac{1}{e^{\pi t} + e^{-\pi t}} (J\Delta^{it}L_\xi\Delta^{-it}J\zeta - L_\zeta J\Delta^{it}\xi)\,dt = 0$$

holds for any $\lambda > 0$, i.e., the Fourier transform of the mapping

$$t \mapsto \frac{1}{e^{\pi t} + e^{-\pi t}} (J\Delta^{it}L_\xi\Delta^{-it}J\zeta - L_\zeta J\Delta^{it}\xi)$$

vanishes identically. Since the Fourier transform is injective, it follows that

$$L_\zeta J\Delta^{it}\xi = J\Delta^{it}L_\xi\Delta^{-it}J\zeta, \quad t \in \mathbb{R}.$$

Thus, for any $t \in \mathbb{R}$ and any $\zeta \in \mathfrak{A}$, we have

$$L_\zeta(J\Delta^{it}\xi) = (J\Delta^{it}L_\xi\Delta^{-it}J)\zeta;$$

substituting $S\xi$ for ξ, we find that

$$L_\zeta(J\Delta^{it}S\xi) = (J\Delta^{it}L_\xi\Delta^{-it}J)^*\zeta.$$

If we now apply Proposition 10.3, we infer that, for any $t \in \mathbb{R}$, the following relations hold:

$$J\Delta^{it}\xi \in \mathfrak{A}' \text{ and}$$

$$S^*J\Delta^{it}\xi = J\Delta^{it}S\xi, \quad R_{J\Delta^{it}\xi} = J\Delta^{it}L_\xi\Delta^{-it}J.$$

Q.E.D.

We now prove the fundamental theorem of Tomita:

Theorem. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra. Then*

(1) $J\mathfrak{A}'' = \mathfrak{A}'$ *and, for any $\xi \in \mathfrak{A}''$, we have*

$$S^*J\xi = JS\xi, \quad R_{J\xi} = JL_\xi J;$$

(2) $\Delta^{it}\mathfrak{A}'' = \mathfrak{A}''$, $t \in \mathbb{R}$, *and, for any $\xi \in \mathfrak{A}''$, we have*

$$S\Delta^{it}\xi = \Delta^{it}S\xi, \quad L_{\Delta^{it}\xi} = \Delta^{it}L_\xi\Delta^{-it}.$$

Similarly,

(1') $J\mathfrak{A}' = \mathfrak{A}''$ *and, for any $\eta \in \mathfrak{A}'$, we have*

$$SJ\eta = JS^*\eta, \quad L_{J\eta} = JR_\eta J;$$

(2') $\Delta^{it}\mathfrak{A}' = \mathfrak{A}'$, $t \in \mathbb{R}$, *and, for any $\eta \in \mathfrak{A}'$, we have*

$$S^*\Delta^{it}\eta = \Delta^{it}S^*\eta, \quad R_{\Delta^{it}\eta} = \Delta^{it}R_\eta\Delta^{-it}.$$

Proof. If we apply Lemma 2 for $t = 0$, it follows that $J\mathfrak{A}'' \subset \mathfrak{A}'$ and, for any $\xi \in \mathfrak{A}''$, we have

$$S^*J\xi = JS\xi, \quad R_{J\xi} = JL_\xi J.$$

Let $\eta \in \mathfrak{A}'$. Then, for any $\zeta \in \mathfrak{A}'$ and $\xi \in \mathfrak{A}''$, we have

$$\begin{aligned}(R_\zeta J\eta|\xi) &= (J\eta|R_{S^*\zeta}(\xi)) = (J\eta|L_\xi S^*\zeta) = (L_{S\xi}J\eta|S^*\zeta)\\ &= (JR_{JS\xi}(\eta)|S^*\zeta) = (JR_{S^*J\xi}(\eta)|S^*\zeta) = (JS^*R_{S^*\eta}J\xi|J\Delta^{-1/2}\zeta)\\ &= (\Delta^{-1/2}\zeta|\Delta^{1/2}J(R_\eta)^*J\xi) = (\zeta|J(R_\eta)^*J\xi) = (JR_\eta J\zeta|\xi).\end{aligned}$$

Thus, for any $\zeta \in \mathfrak{A}'$, we have

$$R_\zeta(J\eta) = (JR_\eta J)\zeta,$$

and, substituting $S^*\eta$ for η, we get

$$R_\zeta(JS^*\eta) = (JR_\eta J)^*\zeta.$$

If we now apply Proposition 10.4, we infer that $J\eta \in \mathfrak{A}''$ and

$$SJ\eta = JS^*\eta, \quad L_{J\eta} = JR_\eta J.$$

Consequently,

$$J\mathfrak{A}'' \subset \mathfrak{A}' = J(J\mathfrak{A}') \subset J\mathfrak{A}'',$$

i.e.,

$$J\mathfrak{A}'' = \mathfrak{A}'.$$

Now let $\xi \in \mathfrak{A}''$ and $t \in \mathbb{R}$. From Lemma 2 we infer that $\eta = J\Delta^{it}\xi \in \mathfrak{A}'$ and

$$S^*J\Delta^{it}\xi = J\Delta^{it}S\xi, \quad R_{J\Delta^{it}\xi} = J\Delta^{it}L_\xi\Delta^{-it}J.$$

From the first part of the proof we now infer that $\Delta^{it}\xi = J\eta \in \mathfrak{A}''$ and

$$S\Delta^{it}\xi = SJ\eta = JS^*\eta = \Delta^{it}S\xi,$$

$$L_{\Delta^{it}\xi} = L_{J\eta} = JR_\eta J = \Delta^{it}L_\xi\Delta^{-it}.$$

Obviously,

$$\Delta^{it}\mathfrak{A}'' \subset \mathfrak{A}'' = \Delta^{it}(\Delta^{-it}\mathfrak{A}'') \subset \Delta^{it}\mathfrak{A}'',$$

i.e.,

$$\Delta^{it}\mathfrak{A}'' = \mathfrak{A}''.$$

We have thus proved assertions (1) and (2). Assertions (1′) and (2′) readily follow from these. On the other hand, it is clear that assertions (1′) and (2′) have direct proofs, analogous to those given for assertions (1) and (2).

Q.E.D.

According to the theorem, J is the natural bridge linking $\mathfrak{A}''$ to $\mathfrak{A}'$. Another such bridge is $\Delta^{1/2}$:

$$\Delta^{1/2}\mathfrak{A}'' = JJ\Delta^{1/2}\mathfrak{A}'' = JS\mathfrak{A}'' = J\mathfrak{A}'' = \mathfrak{A}'.$$

The following corollary allows the construction of useful elements in $\mathfrak{A}'$ and $\mathfrak{A}''$.

Corollary. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra and $f \in \mathscr{L}^1(\mathbb{R})$. Then, for any $\xi \in \mathfrak{A}''$, we have $\xi_f = \int_{-\infty}^{+\infty} f(t)\Delta^{it}\xi \, dt \in \mathfrak{A}''$ and*

$$S\xi_f = \int_{-\infty}^{+\infty} \bar{f}(t)\Delta^{it}S\xi \, dt, \quad L_{\xi_f} = \int_{-\infty}^{+\infty} f(t)\Delta^{it}L_\xi\Delta^{-it}dt.$$

Similarly, for any $\eta \in \mathfrak{A}'$ we have $\eta_f = \int_{-\infty}^{+\infty} f(t)\, \Delta^{it}\eta \, dt \in \mathfrak{A}'$ and

$$S^*\eta_f = \int_{-\infty}^{+\infty} \bar{f}(t)\Delta^{it}S^*\eta \, dt, \quad R_{\eta_f} = \int_{-\infty}^{+\infty} f(t)\Delta^{it}R_\eta\Delta^{-it} \, dt.$$

Proof. Let

$$\zeta = \int_{-\infty}^{+\infty} \bar{f}(t)\Delta^{it}S\xi \, dt.$$

From the preceding theorem we infer that, for any $\theta \in \mathfrak{A}'$, we have

$$R_\theta(\xi_f) = \int_{-\infty}^{+\infty} f(t)R_\theta \Delta^{it}\xi \, dt = \int_{-\infty}^{+\infty} f(t)L_{\Delta^{it}\xi}(\theta) \, dt = \left[\int_{-\infty}^{+\infty} f(t)\Delta^{it}L_\xi\Delta^{-it} \, dt\right](\theta)$$

and

$$R_\theta(\zeta) = \int_{-\infty}^{+\infty} \bar{f}(t)R_\theta S\Delta^{it}\xi \, dt = \int_{-\infty}^{+\infty} \bar{f}(t) \, (L_{\Delta^{it}\xi})^*(\theta) \, dt = \left[\int_{-\infty}^{+\infty} f(t)\Delta^{it}L_\xi\Delta^{-it} \, dt\right]^*(\theta).$$

If we now apply Proposition 10.4, the first assertion follows.

The second assertion can be proved similarly.

Q.E.D.

10.13. We now consider a left Hilbert algebra $\mathfrak{A} \subset \mathscr{H}$.

Let $z \in \mathfrak{L}(\mathfrak{A}) \cap \mathfrak{L}(\mathfrak{A})'$ and $\xi \in \mathscr{D}_S$. With the help of Corollary 3 from 10.3, we infer that, for any $\eta_1, \eta_2 \in \mathfrak{A}'$, we have

$$\begin{aligned}(z\xi | R_{\eta_1}(\eta_2)) &= (\xi | R_{\eta_1}z^*(\eta_2)) = (SS\xi | R_{\eta_1}z^*(\eta_2)) \\ &= (S^*R_{\eta_1}z^*(\eta_2) | S\xi) = (R_{S^*\eta_2} zS^*\eta_1 | S\xi) \\ &= (zR_{S^*\eta_2}S^*\eta_1 \mid S\xi) = (S^*R_{\eta_1}(\eta_2) | z^*S\xi).\end{aligned}$$

If we now take into account relation 10.4 (1), we infer that

$$(z\xi|\eta) = (S^*\eta|z^*S\xi), \quad \eta \in \mathscr{D}_{S^*};$$

consequently, we have

$$z\xi \in \mathscr{D}_S \text{ and } Sz\xi = z^*S\xi.$$

Let $u \in \mathfrak{L}(\mathfrak{A}) \cap \mathfrak{L}(\mathfrak{A})'$ be a unitary element and $\xi \in \mathscr{D}_\Delta \subset \mathscr{D}_{(\Delta^{1/2})} = \mathscr{D}_S$. From the preceding results we infer that $u\xi \in \mathscr{D}_S = \mathscr{D}_{(\Delta^{1/2})}$ and

$$\Delta^{1/2}u\xi = JSu\xi = Ju^*S\xi = Ju^*J\Delta^{1/2}\xi.$$

Thus, for any $\zeta \in \mathscr{D}_{(\Delta^{1/2}u)} = \mathscr{D}_{(\Delta^{1/2})}$, we have

$$(\Delta\xi|\zeta) = (\Delta^{1/2}\xi|\Delta^{1/2}\zeta) = (Ju^*J\Delta^{1/2}\xi|Ju^*J\Delta^{1/2}\zeta) = (\Delta^{1/2}u\xi|\Delta^{1/2}u\zeta);$$

hence

$$\Delta^{1/2}u\xi \in \mathscr{D}_{(\Delta^{1/2}u)^*} = \mathscr{D}_{(u^*\Delta^{1/2})} \text{ and } u^*\Delta u\xi = \Delta\xi.$$

Consequently, *the modular operator Δ is affiliated to the von Neumann algebra* $(\mathfrak{L}(\mathfrak{A}) \cap \mathfrak{L}(\mathfrak{A})')' = \mathscr{R}(\mathfrak{L}(\mathfrak{A}), \mathfrak{L}(\mathfrak{A})')$.

Corollary 1. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra. Then*

$$\mathfrak{L}(\mathfrak{A}) \ni x \mapsto Jx^*J \in \mathscr{B}(\mathscr{H})$$

is a $*$-antiisomorphism of $\mathfrak{L}(\mathfrak{A})$ onto $\mathfrak{L}(\mathfrak{A})'$, which acts identically on the center.

Proof. From Theorem 10.12 we infer that, for any $\xi \in \mathfrak{A}''$, we have $JS\xi \in \mathfrak{A}'$ and

$$J(L_\xi)^*J = JL_{S\xi}J = R_{JS\xi} \in \mathfrak{R}(\mathfrak{A}').$$

On the other hand, from the same theorem we infer that for any $\eta \in \mathfrak{A}'$, we have $SJ\eta \in \mathfrak{A}''$ and

$$J(L_{SJ\eta})^*J = JL_{J\eta}J = R_\eta.$$

Thus, the mapping in the statement of the corollary is a $*$-antiisomorphism of $\mathfrak{L}(\mathfrak{A}) = \mathfrak{L}(\mathfrak{A}'')$ onto $\mathfrak{R}(\mathfrak{A}') = \mathfrak{L}(\mathfrak{A})'$.

From the discussion at the beginning of this section we infer that, for any $z \in \mathfrak{L}(\mathfrak{A}) \cap \mathfrak{L}(\mathfrak{A})'$ and any $\xi \in \mathscr{D}_S$, we have

$$Jz^*J\xi = Jz^*\Delta^{1/2}S\xi = J\Delta^{1/2}z^*S\xi = Sz^*S\xi = z\xi,$$

and this shows that the considered mapping acts identically on the center.

Q.E.D.

Corollary 2. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra. Then the formula*

$$\sigma_t(x) = \Delta^{it}x\Delta^{-it}$$

yields a so-continuous group of $*$-automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ of $\mathfrak{L}(\mathfrak{A})$, which acts identically on the center.

Proof. According to Theorem 10.12, for any $\xi \in \mathfrak{A}''$ and any $t \in \mathbb{R}$, we have $\Delta^{it}\xi \in \mathfrak{A}''$ and

$$\sigma_t(L_\xi) = \Delta^{it}L_\xi\Delta^{-it} = L_{\Delta^{it}\xi} \in \mathfrak{L}(\mathfrak{A}).$$

Thus, $\{\sigma_t\}_{t\in\mathbb{R}}$ is a *so*-continuous group of $*$-automorphisms of $\mathfrak{L}(\mathfrak{A})$.

At the beginning of this section we saw that Δ is affiliated to the commutant of the center of $\mathfrak{L}(\mathfrak{A})$, whence it obviously follows that any $*$-automorphism σ_t acts identically on the center.

Q.E.D.

The *so*-continuous group $\{\sigma_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{L}(\mathfrak{A})$ is called the *group of the modular automorphisms* of $\mathfrak{L}(\mathfrak{A})$, associated to the left Hilbert algebra $\mathfrak{A}$.

10.14. In Section 10.6 to any von Neumann algebra $\mathscr{M}$, with a separating cyclic vector, we associated a left Hilbert algebra, such that $\mathscr{M} = \mathfrak{L}(\mathfrak{A})$. By taking into account Section 5.18, this association can be described in the following equivalent manner: to any von Neumann algebra $\mathscr{M}$, of countable type, and to any faithful normal form φ on $\mathscr{M}$, we associated a left Hilbert algebra $\mathfrak{A}_\varphi \subset \mathscr{H}_\varphi$, such that $\pi_\varphi(\mathscr{M}) = \mathfrak{L}(\mathfrak{A}_\varphi)$.

In this section we extend the above association to the case of arbitrary von Neumann algebras, and to some "unbounded forms", called weights.

Let $\mathcal{M}$ be a von Neumann algebra. A mapping

$$\varphi : \mathcal{M}^+ \to [0, +\infty] = \mathbb{R}^+ \cup \{+\infty\}$$

is called a *weight* if

(1) $$\varphi(a + b) = \varphi(a) + \varphi(b), \quad a, b \in \mathcal{M}^+,$$

(2) $$\varphi(\lambda a) = \lambda\varphi(a), \quad a \in \mathcal{M}^+, \quad \lambda \geqslant 0.\text{*)}$$

From condition (1) it follows that

$$a, b \in \mathcal{M}^+, \quad a \leqslant b \Rightarrow \varphi(a) \leqslant \varphi(b).$$

For any weight φ on $\mathcal{M}^+$ one defines

$$\mathfrak{F}_\varphi = \{a \in \mathcal{M}^+; \ \varphi(a) < +\infty\}.$$

It is obvious that $\mathfrak{F}_\varphi$ is a face (see 3.21) and, therefore, by taking into account Proposition 3.21, it follows that

$$\mathfrak{N}_\varphi = \{x \in \mathcal{M}; \ \varphi(x^*x) < +\infty\}$$

is a left ideal of $\mathcal{M}$, and

$$\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^* \mathfrak{N}_\varphi \subset \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$$

is a $*$-subalgebra of $\mathcal{M}$, such that

$$\mathfrak{M}_\varphi^+ = \mathfrak{F}_\varphi,$$

$$\mathfrak{M}_\varphi = \text{the linear hull of } \mathfrak{F}_\varphi.$$

From the latter property it easily follows that φ uniquely extends to a positive linear form on $\mathfrak{M}_\varphi$, which is also denoted by φ.

The weight φ is said to be *semifinite* if

(3) $$\mathfrak{M}_\varphi \text{ is } w\text{-dense in } \mathcal{M}.$$

The weight φ is said to be *faithful* if

(4) $$a \in \mathcal{M}^+, \ \varphi(a) = 0 \Rightarrow a = 0.$$

We shall say that the weight φ is *normal* if

(5) there exists a family $\{\varphi_i\}$ of normal forms on $\mathcal{M}$, such that

$$\varphi(a) = \sum_i \varphi_i(a), \quad a \in \mathcal{M}^+.$$

It is easy to see that if φ is normal, then φ is lower w-semicontinuous. In particular, for any family $\{a_\alpha\} \subset \mathcal{M}^+$, which is increasingly directed and bounded, we have

$$\varphi\,(\sup_\alpha a_\alpha) = \sup_\alpha \varphi(a_\alpha).$$

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*) With the convention that $0 \cdot (+\infty) = 0$.

The construction by which to any positive linear form one can associate a $*$-representation (see 5.18) can be extended to weights.

Let φ be a faithful semifinite normal weight on $\mathscr{M}^+$. Since φ is faithful, the positive sesquilinear form

$$(x|y)_\varphi = \varphi(y^*x), \quad x, y \in \mathfrak{N}_\varphi,$$

is a scalar product on $\mathfrak{N}_\varphi$. We shall denote by $\mathscr{H}_\varphi$ the Hilbert space obtained by the completion of $\mathfrak{N}_\varphi$ and, for any $x \in \mathfrak{N}_\varphi$, we shall denote by $x_\varphi \in \mathscr{H}_\varphi$ the image of x through the canonical embedding of $\mathfrak{N}_\varphi$ in $\mathscr{H}_\varphi$. Any element $x \in \mathscr{M}$ determines an operator $\pi_\varphi(x) \in \mathscr{B}(\mathscr{H}_\varphi)$, given by the relations

$$\pi_\varphi(x)y_\varphi = (xy)_\varphi, \quad y \in \mathfrak{N}_\varphi.$$

It is easy to verify that $\pi_\varphi: \mathscr{M} \to \mathscr{B}(\mathscr{H}_\varphi)$ is a $*$-representation.

Since φ is normal, the same argument as that used in the proof of Proposition 5.18 shows that the $*$-representation π_φ is w-continuous. It is clear that $\pi_\varphi(1)=1$; hence $\pi_\varphi(\mathscr{M})$ is a von Neumann algebra.

From the faithfulness and the semifiniteness of the weight φ it follows that π_φ is injective, hence π_φ is a $*$-isomorphism of the von Neumann algebra $\mathscr{M}$ onto the von Neumann algebra $\pi_\varphi(\mathscr{M})$.

Since φ is semifinite, Proposition 3.21 shows that there exists a family $\{u_\alpha\} \subset \mathfrak{M}_\varphi$, such that $u_\alpha \uparrow 1$. Then, for any $x \in \mathfrak{N}_\varphi$, we have

$$(*) \quad \begin{aligned} \|x_\varphi - \pi_\varphi(u_\alpha)\, x_\varphi\|_\varphi^2 &= \|x_\varphi - (u_\alpha x)_\varphi\|_\varphi^2 = \varphi((x - u_\alpha x)^*(x - u_\alpha x)) \\ &\leqslant 2[\varphi(x^*x) - \varphi(x^*u_\alpha x)] \to 0. \end{aligned}$$

We hence infer that $\pi_\varphi(u_\alpha) \uparrow 1$.

On the other hand, since $x \in \mathfrak{N}_\varphi$ and $u_\alpha \in \mathfrak{M}_\varphi^+$ imply that $u_\alpha x \in \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi = \mathfrak{M}_\varphi$, from relation $(*)$ we infer that $\mathfrak{M}_\varphi$ is densely embedded in $\mathscr{H}_\varphi$. Similarly, since $x \in \mathfrak{M}_\varphi$ and $u_\alpha \in \mathfrak{M}_\varphi$ imply that $u_\alpha x \in \mathfrak{M}_\varphi^2$, from the same relation we infer that $\mathfrak{M}_\varphi^2$ is dense in $\mathfrak{M}_\varphi$, with respect to the topology corresponding to the scalar product we have just defined. Consequently

$$\mathfrak{M}_\varphi^2 \text{ is densely embedded in } \mathscr{H}_\varphi.$$

Theorem. *Let $\mathscr{M}$ be a von Neumann algebra and φ a faithful, semifinite, normal weight on $\mathscr{M}^+$. Then $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, endowed with the structure of $*$-algebra induced by $\mathscr{M}$, and with the scalar product induced by that of $\mathscr{H}_\varphi$, is a left Hilbert algebra $\mathfrak{A}_\varphi \subset \mathscr{H}_\varphi$, and the following relations hold*

$$\mathfrak{A}_\varphi = \mathfrak{A}_\varphi'',$$

$$\pi_\varphi(\mathscr{M}) = \mathfrak{L}(\mathfrak{A}_\varphi),$$

$$\varphi(a) = \begin{cases} \|\xi\|^2, \textit{ if there exists a } \xi \in \mathfrak{A}, \textit{ such that} \\ \qquad \pi_\varphi(a)^{1/2} = L_\xi. \\ +\infty, \textit{ in the contrary case.} \end{cases} \quad ; a \in \mathscr{M}^+$$

Proof. Conditions (i) and (ii) from 10.1 are easy to verify.

Since $\mathfrak{M}_\varphi \subset \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, and since $\mathfrak{M}_\varphi^2$ is densely embedded in $\mathscr{H}_\varphi$, it follows that condition (iii) from 10.1 is also satisfied.

In order to verify condition (iv) from 10.1, we must prove that for any net $\{x_\alpha\} \subset \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ the following implication holds:

$$\left.\begin{aligned} &\varphi(x_\alpha^* x_\alpha) \xrightarrow[\alpha]{} 0 \\ &\varphi((x_\alpha - x_\beta)(x_\alpha - x_\beta)^*) \xrightarrow[\alpha,\beta]{} 0 \end{aligned}\right\} \Rightarrow \varphi(x_\alpha x_\alpha^*) \xrightarrow[\alpha]{} 0.$$

From Sections 5.18 and 10.6 we infer that such an implication is true for faithful normal forms. Since φ is normal, there exists an increasingly directed family $\{\varphi_\nu\}$ of normal forms on $\mathscr{M}$, such that

$$\varphi(a) = \sup_\nu \varphi_\nu(a), \quad a \in \mathscr{M}^+.$$

If we denote by $e_\nu \in \mathscr{M}$ the support of φ_ν, the restriction of φ_ν to $e_\nu \mathscr{M} e_\nu$ is faithful. From the hypotheses of the implication we must prove, it follows that

$$\varphi_\nu((e_\nu x_\alpha e_\nu)^*(e_\nu x_\alpha e_\nu)) \xrightarrow[\alpha]{} 0,$$

$$\varphi_\nu((e_\nu x_\alpha e_\nu - e_\nu x_\beta e_\nu)(e_\nu x_\alpha e_\nu - e_\nu x_\beta e_\nu)^*) \xrightarrow[\alpha,\beta]{} 0,$$

and, therefore,

$$\varphi_\nu(x_\alpha e_\nu x_\alpha^*) = \varphi_\nu((e_\nu x_\alpha e_\nu)(e_\nu x_\alpha e_\nu)^*) \xrightarrow[\alpha]{} 0.$$

Since φ is faithful, we have

$$e_\nu \mathop{\uparrow}_\nu 1.$$

For any α, β and $\nu \leqslant \mu$ we have

$$\begin{aligned} \varphi_\nu(x_\alpha e_\mu x_\alpha^*)^{1/2} &\leqslant \varphi_\mu(x_\alpha e_\mu x_\alpha^*)^{1/2} \\ &\leqslant \varphi_\mu((x_\alpha - x_\beta)e_\mu(x_\alpha - x_\beta)^*)^{1/2} + \varphi_\mu(x_\beta e_\mu x_\beta^*)^{1/2} \\ &\leqslant \varphi((x_\alpha - x_\beta)(x_\alpha - x_\beta)^*)^{1/2} + \varphi_\mu(x_\beta e_\mu x_\beta^*)^{1/2}. \end{aligned}$$

Let $\varepsilon > 0$. Then there exists an α_ε, such that for any $\alpha, \beta \geqslant \alpha_\varepsilon$ we have

$$\varphi((x_\alpha - x_\beta)(x_\alpha - x_\beta)^*)^{1/2} \leqslant \varepsilon.$$

Then

$$\varphi_\nu(x_\alpha e_\mu x_\alpha^*)^{1/2} \leqslant \varepsilon + \varphi_\mu(x_\beta e_\mu x_\beta^*)^{1/2}, \ \alpha, \beta \geqslant \alpha_\varepsilon, \quad \nu \leqslant \mu.$$

By tending to the limit with respect to β, from this relation we get

$$\varphi_\nu(x_\alpha e_\mu x_\alpha^*)^{1/2} \leqslant \varepsilon, \quad \alpha \geqslant \alpha_\varepsilon, \ \nu \leqslant \mu,$$

and, by tending to the limit with respect to μ, we obtain

$$\varphi_\nu(x_\alpha x_\alpha^*)^{1/2} \leqslant \varepsilon, \quad \alpha \geqslant \alpha_\varepsilon, \text{ any } \nu.$$

Finally, if we now compute the l.u.b., from this inequality we get

$$\varphi(x_\alpha x_\alpha^*)^{1/2} \leqslant \varepsilon, \quad \alpha \geqslant \alpha_\varepsilon.$$

Consequently, condition (iv) from 10.1 is verified.

Since $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^* \supset \mathfrak{M}_\varphi$ is w-dense in $\mathscr{M}$, $\pi_\varphi(\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*) = \{L_{x_\varphi}; x_\varphi \in \mathfrak{A}_\varphi\}$ is w-dense in $\pi_\varphi(\mathscr{M})$. Therefore,

$$\pi_\varphi(\mathscr{M}) = \mathfrak{L}(\mathfrak{A}_\varphi).$$

By considering on $\mathfrak{N}_\varphi$ the positive sesquilinear form

$$(x_\varphi, y_\varphi) \mapsto \varphi_\nu(y^*x)$$

and by observing that it is bounded from above by the scalar product in $\mathscr{H}_\varphi$, it follows that there exists an $a'_\nu \in \mathscr{B}(\mathscr{H}_\varphi)$, $0 \leqslant a'_\nu \leqslant 1$, such that

$$(a'_\nu x_\varphi \mid y_\varphi)_\varphi = \varphi_\nu(y^*x), \quad x, y \in \mathfrak{N}_\varphi.$$

It is easy to verify that $a'_\nu \in (\pi_\varphi(\mathscr{M}))'$.

Let $\{u_\alpha\} \subset \mathfrak{M}_\varphi$ be such that $u_\alpha \xrightarrow{so} 1$ and $\|u_\alpha\| \leqslant 1$. Then

$$\|(a'_\nu)^{1/2}(u_\alpha)_\varphi - (a'_\nu)^{1/2}(u_\beta)_\varphi\|_\varphi^2 = \varphi_\nu((u_\alpha - u_\beta)^*(u_\alpha - u_\beta)) \xrightarrow[\alpha,\beta]{} 0;$$

hence, there exists a vector $\eta_\nu \in \mathscr{H}$, such that

$$(a'_\nu)^{1/2}(u_\alpha)_\varphi \to \eta_\nu.$$

For any $x, y \in \mathfrak{N}_\varphi$ we have

$$\begin{aligned}(\pi_\varphi(x)\eta_\nu \mid (a'_\nu)^{1/2}y_\varphi)_\varphi &= \lim_\alpha (\pi_\varphi(x)\,(a'_\nu)^{1/2}(u_\alpha)_\varphi \mid (a'_\nu)^{1/2}y_\varphi)_\varphi \\ &= \lim_\alpha (a'_\nu((xu_\alpha)_\varphi) \mid y_\varphi)_\varphi = \lim_\alpha \varphi_\nu(y^*xu_\alpha) \\ &= \varphi_\nu(y^*x) = ((a'_\nu)^{1/2}x_\varphi \mid (a'_\nu)^{1/2}y_\varphi)_\varphi.\end{aligned}$$

Since $\pi_\varphi(x)\eta_\nu = \lim_\alpha (a'_\nu)^{1/2}((xu_\alpha)_\varphi) \in \overline{(a'_\nu)^{1/2}\mathscr{H}_\varphi}$, we infer that

$$\pi_\varphi(x)\eta_\nu = (a'_\nu)^{1/2}(x_\varphi), \quad x \in \mathfrak{N}_\varphi.$$

With the help of Proposition 10.3, one shows that $\eta_\nu \in (\mathfrak{A}_\varphi)'$, $S^*\eta_\nu = \eta_\nu$ and $R_{\eta_\nu} = (a'_\nu)^{1/2}$.

Let $\xi \in (\mathfrak{A}_\varphi)''$. Since $L_\xi \in \mathfrak{L}(\mathfrak{A}_\varphi) = \pi_\varphi(\mathscr{M})$, there exists an $x \in \mathscr{M}$, such that $L_\xi = \pi_\varphi(x)$. For any ν we have

$$\varphi_\nu(x^*x) = \|\pi_\varphi(x)\eta_\nu\|^2 = \|L_\xi(\eta_\nu)\|^2 = \|R_{\eta_\nu}(\xi)\|^2 \leqslant \|\xi\|^2;$$

hence,

$$\varphi(x^*x) \leqslant \|\xi\|^2 < +\infty.$$

Similarly, one can prove that

$$\varphi(xx^*) \leqslant \|S\xi\|^2 < +\infty.$$

Consequently, $x \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, and, therefore, $x_\varphi \in \mathfrak{A}_\varphi$. Since $L_\xi = \pi_\varphi(x) = L_{x_\varphi}$, we infer that $\xi = x_\varphi \in \mathfrak{A}_\varphi$.

The proof of the formula, given in the statement of the theorem, for φ, offers no difficulties.

Q.E.D.

We remark that *any von Neumann algebra $\mathscr{M}$ has a faithful, semifinite normal weight*. Indeed, if $\{\varphi_i\}$ is a maximal family of normal forms on $\mathscr{M}$, whose supports are mutually orthogonal, then the formula

$$\varphi(a) = \sum_i \varphi_i(a), \quad a \in \mathscr{M}^+,$$

yields a faithful, semifinite normal weight on $\mathscr{M}^+$.

Thus, *any von Neumann algebra is $*$-isomorphic to a von Neumann algebra of the form $\mathfrak{L}(\mathfrak{A})$, where $\mathfrak{A}$ is a left Hilbert algebra.*

10.15. A von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is said to be *standard* if there exists a conjugation $J : \mathscr{H} \to \mathscr{H}$, such that the mapping $x \mapsto Jx^*J$ be a $*$-antiisomorphism of $\mathscr{M}$ onto $\mathscr{M}'$, which acts identically on the center.

For particular cases of standard von Neumann algebras, the reader is referred to exercises E.7.15, E.7.16, E.7.17, E.7.18, E.7.19 and E.8.5. To the same end, exercises E.3.9, E.3.10, E.6.9 are also useful.

In accordance with Section 10.6 and Corollary 1 from Section 10.13, any von Neumann algebra with a separating cyclic vector is standard. Conversely, let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a standard von Neumann algebra of countable type and J the corresponding conjugation. In accordance with Lemma 7.18, there exist $\xi, \eta \in \mathscr{H}$, such that the projections p_ξ and p'_η be central, and $p_\xi + p'_\eta = 1$. From exercise E. 6.9 we infer that

$$p'_{J\xi} = Jp_\xi J = p_\xi, \quad p_{J\eta} = Jp'_\eta J = p'_\eta;$$

hence $J\xi + \eta$ is a cyclic vector, whereas $\xi + J\eta$ is a separating vector for $\mathscr{M}$. In accordance with exercise E.6.3, it follows that $\mathscr{M}$ has a separating cyclic vector.

Thus, *the standard von Neumann algebras of countable type are precisely the von Neumann algebras with a separating cyclic vector*. In accordance with Proposition 5.18, any von Neumann algebra of countable type is $*$-isomorphic to a standard von Neumann algebra, whereas, in accordance with Corollary 5.25, any $*$-isomorphism between two standard von Neumann algebras of countable type is spatial.

The following result extends these statements to the general case:

Corollary. *Any von Neumann algebra is $*$-isomorphic to a standard von Neumann algebra, and any $*$-isomorphism between two standard von Neumann algebras is spatial.*

Proof. In accordance with the remark at the end of section 10.14, and with Corollary 1 from Section 10.13, any von Neumann algebra is $*$-isomorphic to a standard von Neumann algebra.

By taking into account Theorem 4.17 and Proposition 8.5, the second assertion of the statement can be considered separately, for the finite, respectively the uniform, von Neumann algebras. In the first case, by using Lemma 7.2, the assertion is reduced to the case of the von Neumann algebras of countable type, whereas in the second case one applies Theorem 8.6, and one uses the fact that the uniformity (8.6) of a standard von Neumann algebra is equal to the uniformity of its commutant.

Q.E.D.

10.16. In this section we begin a construction inverse to that developed in Section 10.14. More precisely, to any left Hilbert algebra $\mathfrak{A}$ we associate a function $\varphi_{\mathfrak{A}}: \mathfrak{L}(\mathfrak{A})^+ \to \mathbb{R}^+ \cup \{+\infty\}$, which measures the "weight" of $\mathfrak{A}''$ in the operators belonging to $\mathfrak{L}(\mathfrak{A})$:

$$\varphi_{\mathfrak{A}}(a) = \begin{cases} \|\xi\|^2 \text{ if there exists a } \xi \in \mathfrak{A}'', \text{ such that} \\ \qquad a^{1/2} = L_\xi \\ +\infty \text{ in the contrary case} \end{cases} \quad ; \ a \in \mathfrak{L}(\mathfrak{A})^+$$

We first prove that $\varphi_{\mathfrak{A}}$ is increasing:

$$(1) \qquad [a, b \in \mathfrak{L}(\mathfrak{A})^+, \quad a \leqslant b \Rightarrow \varphi_{\mathfrak{A}}(a) \leqslant \varphi_{\mathfrak{A}}(b)].$$

If $\varphi_{\mathfrak{A}}(b) = +\infty$, then the implication is trivially true. We now assume that $b^{1/2} = L_\zeta$, $\zeta \in \mathfrak{A}''$. It is easy to see that the relations

$$x(b^{1/2}\eta) = a^{1/2}\eta, \quad \eta \in \mathscr{H},$$

$$x(\theta) = 0, \quad \theta \in [b^{1/2}\mathscr{H}]^\perp,$$

determine an operator $x \in \mathfrak{L}(\mathfrak{A})$, $\|x\| \leqslant 1$, such that $a^{1/2} = xb^{1/2}$. We denote $\xi = x\zeta$. Since, for any $\eta \in \mathfrak{A}'$, we have

$$R_\eta(\xi) = R_\eta x(\zeta) = xR_\eta(\zeta) = xL_\zeta(\eta) = xb^{1/2}(\eta) = a^{1/2}(\eta),$$

Proposition 10.4 implies that $\xi = \mathfrak{A}''$ and $L_\xi = a^{1/2}$. Thus

$$\varphi_{\mathfrak{A}}(a) = \|\xi\|^2 \leqslant \|x\|^2 \|\zeta\|^2 \leqslant \|\zeta\|^2 = \varphi_{\mathfrak{A}}(b).$$

We now prove that $\varphi_{\mathfrak{A}}$ is additive

$$(2) \qquad [\varphi_{\mathfrak{A}}(a + b) = \varphi_{\mathfrak{A}}(a) + \varphi_{\mathfrak{A}}(b), \quad a, b \in \mathfrak{L}(\mathfrak{A})^+].$$

Since $\varphi_{\mathfrak{A}}$ is increasing, if $\varphi_{\mathfrak{A}}(a) = +\infty$ or $\varphi_{\mathfrak{A}}(b) = +\infty$, then relation (2) is obviously satisfied. Thus we can assume that

$$a^{1/2} = L_{\gamma}, \quad b^{1/2} = L_{\delta}, \quad \gamma, \delta \in \mathfrak{A}''.$$

The equalities

$$x((a+b)^{1/2}\eta) = a^{1/2}\eta, \quad \eta \in \mathscr{H},$$

$$x(\theta) = 0, \quad \theta \in [(a+b)^{1/2}\mathscr{H}]^{\perp},$$

$$y((a+b)^{1/2}\eta) = b^{1/2}\eta, \quad \eta \in \mathscr{H},$$

$$y(\theta) = 0, \quad \theta \in [(a+b)^{1/2}\mathscr{H}],$$

determine the operators $x, y \in \mathfrak{L}(\mathfrak{A})$, $\|x\|$, $\|y\| \leqslant 1$, such that $a^{1/2} = x(a+b)^{1/2}$ and $b^{1/2} = y(a+b)^{1/2}$. Since

$$(a+b)^{1/2}(x^*x + y^*y)\,(a+b)^{1/2} = a + b,$$

the positive operator $(x^*x + y^*y)^{1/2}$ is isometric on $[(a+b)^{1/2}\mathscr{H}]$ and, obviously, vanishes on $[(a+b)^{1/2}\mathscr{H}]^{\perp}$. Thus

$$x^*x + y^*y = \mathrm{s}(a+b).$$

We denote

$$\xi = x^*\gamma + y^*\delta.$$

For any $\eta \in \mathfrak{A}'$ we have

$$\begin{aligned} R_{\eta}(\xi) &= x^*R_{\eta}(\gamma) + y^*R_{\eta}(\delta) = (x^*L_{\gamma} + y^*L_{\delta})\,(\eta) \\ &= (x^*a^{1/2} + y^*b^{1/2})\,(\eta) = (x^*x + y^*y)(a+b)^{1/2}(\eta) \\ &= (a+b)^{1/2}(\eta). \end{aligned}$$

With the help of Proposition 10.4 we infer that

$$\xi \in \mathfrak{A}'', \quad S\xi = \xi, \quad L_{\xi} = (a+b)^{1/2}.$$

For any $\eta \in \mathfrak{A}'$ we have

$$R_{\eta}(x\xi) = xR_{\eta}(\xi) = xL_{\xi}(\eta) = x(a+b)^{1/2}(\eta) = a^{1/2}(\eta) = L_{\gamma}(\eta) = R_{\eta}(\gamma);$$

hence, making $R_{\eta} \xrightarrow{so} 1$, it follows that

$$x\xi = \gamma.$$

Similarly,

$$y\xi = \delta.$$

Since $\xi \in \overline{L_\xi \mathscr{H}} = s(a+b)\mathscr{H}$ (see 10.5), we have

$$\varphi_{\mathfrak{A}}(a+b) = \|\xi\|^2 = ((x^*x + y^*y)\xi \mid \xi) = \|x\xi\|^2 + \|y\xi\|^2$$
$$= \|\gamma\|^2 + \|\delta\|^2 = \varphi_{\mathfrak{A}}(a) + \varphi_{\mathfrak{A}}(b).$$

It is obvious that

$$(3) \qquad [\varphi_{\mathfrak{A}}(\lambda a) = \lambda \varphi_{\mathfrak{A}}(a), \quad a \in \mathfrak{L}(\mathfrak{A})^+, \quad \lambda > 0].$$

From relations (2) and (3) we infer that $\varphi_{\mathfrak{A}}$ is a weight on $\mathfrak{L}(\mathfrak{A})^+$, which is called the *weight associated to the left Hilbert algebra* $\mathfrak{A}$.

With the help of Theorem 10.14 it is easy to verify that if $\mathscr{M}$ is a von Neumann algebra and φ is a faithful, semifinite, normal weight on $\mathscr{M}^+$, and if we consider the weight $\varphi_{\mathfrak{A}_\varphi}$, associated to the left Hilbert algebra $\mathfrak{A}_\varphi$, then

$$\varphi = \varphi_{\mathfrak{A}_\varphi} \circ \pi_\varphi .$$

For the von Neumann algebra $\mathfrak{L}(\mathfrak{A})$ and the weight $\varphi_{\mathfrak{A}}$ we shall denote briefly (see Section 10.14):

$$\mathfrak{F}_{\mathfrak{A}} = \mathfrak{F}_{\varphi_{\mathfrak{A}}}, \quad \mathfrak{N}_{\mathfrak{A}} = \mathfrak{N}_{\varphi_{\mathfrak{A}}}, \quad \mathfrak{M}_{\mathfrak{A}} = \mathfrak{M}_{\varphi_{\mathfrak{A}}}.$$

We now show that $\varphi_{\mathfrak{A}}$ measures indeed the "weight" of $\mathfrak{A}''$ in the operators belonging to $\mathfrak{L}(\mathfrak{A})$. More precisely:

(4) $\Bigg[$ *for any operator* $x \in \mathfrak{L}(\mathfrak{A})$ *we have the equivalence: there exists a* $\xi \in \mathfrak{A}''$ *such that* $x = L_\xi \Leftrightarrow x \in \mathfrak{N} \cap \mathfrak{N}^*$; *moreover, if* $\xi, \zeta \in \mathfrak{A}''$, *then* $(L_\zeta)^* L_\xi \in \mathfrak{M}_{\mathfrak{A}}$ *and*

$$\varphi_{\mathfrak{A}}((L_\zeta)^* L_\xi) = (\xi \mid \zeta). \Bigg]$$

Indeed, let $x = v|x|$, $x^* = v^*|x^*|$ be the polar decompositions. Assuming that $x = L_\xi$, $\xi \in \mathfrak{A}''$, for any $\eta \in \mathfrak{A}'$ we have

$$R_\eta(v^*\xi) = v^* R_\eta(\xi) = v^* L_\xi(\eta) = |x|(\eta),$$
$$R_\eta(vS\xi) = vR_\eta(S\xi) = v(L_\xi)^*(\eta) = |x^*|(\eta),$$

and, with the help of Proposition 10.4, we infer that

$$v^*\xi \in \mathfrak{A}'', \quad S(v^*\xi) = v^*\xi, \quad L_{v^*\xi} = |x|,$$
$$vS\xi \in \mathfrak{A}'', \quad S(vS\xi) = vS\xi, \quad L_{vS\xi} = |x^*|.$$

Thus

$$\varphi_{\mathfrak{A}}(x^*x) = \varphi_{\mathfrak{A}}(|x|^2) = \|v^*\xi\|^2 = \|\xi\|^2 < +\infty,$$
$$\varphi_{\mathfrak{A}}(xx^*) = \varphi_{\mathfrak{A}}(|x^*|^2) = \|vS\xi\|^2 = \|S\xi\|^2 < +\infty,$$

i.e.,

$$x \in \mathfrak{N}_{\mathfrak{A}} \cap \mathfrak{N}_{\mathfrak{A}}^* .$$

Conversely, if $x \in \mathfrak{N}_{\mathfrak{A}} \cap \mathfrak{N}_{\mathfrak{A}}^*$, then there exist $\xi, \zeta \in \mathfrak{A}''$, such that $|x| = L_\xi$, $|x^*| = L_\zeta$. For any $\eta \in \mathfrak{A}'$ we have

$$R_\eta(v\xi) = vR_\eta(\xi) = v|x|(\eta) = x(\eta),$$

$$R_\eta(v^*\zeta) = v^*R_\eta(\zeta) = v^*|x^*|(\eta) = x^*(\eta),$$

and, with the help of Proposition 10.4, we infer that

$$v\xi \in \mathfrak{A}'', \quad S(v\xi) = v^*\zeta, \quad L_{v\xi} = x.$$

Let $\xi, \zeta \in \mathfrak{A}''$. From the first part of the proof it follows that L_ξ and L_ζ belong to $\mathfrak{N}_{\mathfrak{A}} \cap \mathfrak{N}_{\mathfrak{A}}^*$, hence $(L_\zeta)^*L_\xi \in \mathfrak{M}_{\mathfrak{A}}$. With the help of the polarization identity

$$(L_\zeta)^*L_\xi = 4^{-1}\sum_{k=0}^{3} i^k (L_{\zeta+i^k\xi})^* L_{\zeta+i^k\xi},$$

one easily obtains the equality

$$\varphi_{\mathfrak{A}}((L_\zeta)^*L_\xi) = (\xi \mid \zeta).$$

From assertion (4) one easily infers that $\varphi_{\mathfrak{A}}$ is semifinite, i.e.,

(5) $[\mathfrak{M}_{\mathfrak{A}}^+ = \{a \in \mathfrak{L}(\mathfrak{A})^+;\ \varphi_{\mathfrak{A}}(a) < +\infty\}$ *is w-dense in* $\mathfrak{L}(\mathfrak{A})^+]$.

It is immediately verified that $\varphi_{\mathfrak{A}}$ is faithful, i.e.,

(6) $[a \in \mathfrak{L}(\mathfrak{A})^+,\ \varphi_{\mathfrak{A}}(a) = 0 \Rightarrow a = 0]$.

The normality of $\varphi_{\mathfrak{A}}$ is a more difficult problem and it will be proved later (see 10.18). Here we shall prove only that $\varphi_{\mathfrak{A}}$ *is lower w-semicontinuous.* To this end it is sufficient to prove that the set

$$\{a \in \mathfrak{L}(\mathfrak{A})^+;\ \|a\| \leqslant c,\ \varphi_{\mathfrak{A}}(a) \leqslant \lambda\}$$

is *so*-closed, where c and λ are arbitrary positive constants (one uses the Krein-Šmulian theorem, see C.1.1, and Corollary 1.5). Let $\{a_i\}$ be a net in the above set, which is *so*-convergent to $a \in \mathfrak{L}(\mathfrak{A})^+$. For any i there exists a $\xi_i \in \mathfrak{A}''$, such that $a_i^{1/2} = L_{\xi_i}$ and $\|\xi_i\|^2 \leqslant \lambda$. Since $L_{S\xi_i} = (L_{\xi_i})^* = L_{\xi_i}$ and, since the representation L is injective (10.5), it follows that $S\xi_i = \xi_i$. The set

$$\{f \in \mathscr{C}([0, c]);\ f(a_i) \xrightarrow{so} f(a)\}$$

is closed in the norm topology and contains all polynomials; hence, it coincides with $\mathscr{C}([0, c])$. Consequently,

$$L_{\xi_i} = a_i^{1/2} \xrightarrow{so} a^{1/2}.$$

With the help of Corollary 3 from Section 10.4, it follows that there exists a $\xi \in \mathfrak{A}''$, such that

$$a^{1/2} = L_\xi \text{ and } \xi_i \to \xi, \text{ weakly.}$$

Thus,

$$\varphi_{\mathfrak{A}}(a) = \|\xi\|^2 \leqslant \underline{\lim}_i \|\xi_i\|^2 \leqslant \lambda.$$

In particular, we have proved that

(7) $$\left[\begin{array}{l} \textit{for any bounded, increasingly directed family } \{a_i\} \subset \mathfrak{L}(\mathfrak{A})^+ \\ \textit{one has} \\ \qquad\qquad \varphi_{\mathfrak{A}}(\sup_i a_i) = \sup_i \varphi_{\mathfrak{A}}(a_i). \end{array}\right].$$

With the help of Theorem 10.12 one easily infers that $\varphi_{\mathfrak{A}}$ is invariant with respect to the modular automorphisms $\sigma_t(\cdot) = \Delta^{it} \cdot \Delta^{-it}$, $t \in \mathbb{R}$:

(8) $$[\varphi_{\mathfrak{A}}(\sigma_t(a)) = \varphi_{\mathfrak{A}}(a), \quad a \in \mathfrak{L}(\mathfrak{A})^+, \quad t \in \mathbb{R}].$$

At the end of this section we shall prove two other useful properties of the pair $(\mathfrak{L}(\mathfrak{A}), \varphi_{\mathfrak{A}})$, which are important for their own sake. In accordance with a remark we have made above, these will be properties shared by any pair $(\mathcal{M}, \varphi)$, consisting of a von Neumann algebra $\mathcal{M}$, and a faithful, semifinite, normal weight φ on $\mathcal{M}^+$.

We denote by $\mathfrak{L}(\mathfrak{A})_\infty$ the set of all elements $x \in \mathfrak{L}(\mathfrak{A})$, such that the mapping $it \mapsto \sigma_t(x) = \Delta^{it} x \Delta^{-it}$ has an entire analytic continuation. It is easy to see that this continuation takes values in $\mathfrak{L}(\mathfrak{A})$. Obviously, *any fixed element for the group of modular automorphisms belongs to* $\mathfrak{L}(\mathfrak{A})_\infty$. In particular, $\mathfrak{L}(\mathfrak{A})_\infty$ *contains the center of* $\mathfrak{L}(\mathfrak{A})$. It is easy to verify that $\mathfrak{L}(\mathfrak{A})_\infty$ is a subalgebra of $\mathfrak{L}(\mathfrak{A})$. If F is the entire analytic continuation of $it \mapsto \sigma_t(x)$, then $\alpha \mapsto F(-\bar{\alpha})^*$ is an entire analytic continuation of $it \mapsto \sigma_t(x^*)$. It follows that $\mathfrak{L}(\mathfrak{A})_\infty$ is self-adjoint. On the other hand, if $x \in \mathfrak{L}(\mathfrak{A})$ and if we define

$$x_n = \sqrt{n/\pi} \int_{-\infty}^{+\infty} e^{-nt^2} \sigma_t(x)\, dt, \qquad n \in \mathbb{N},$$

then the mapping

$$\alpha \mapsto \sqrt{n/\pi} \int_{-\infty}^{+\infty} e^{-n(t+i\alpha)^2} \sigma_t(x)\, dt, \qquad n \in \mathbb{N},$$

is an entire analytic continuation of the mapping $is \mapsto \sigma_s(x_n)$; hence

$$x_n \in \mathfrak{L}(\mathfrak{A})_\infty, \qquad n \in \mathbb{N}.$$

By taking into account the *so*-continuity of the mapping $t \mapsto \sigma_t(x)$ and by applying the Lebesgue dominated convergence theorem, one easily infers that

$$x_n \xrightarrow{so} x.$$

Consequently,

$$\mathfrak{L}(\mathfrak{A})_\infty \textit{ is a so-dense } *\textit{-subalgebra of } \mathfrak{L}(\mathfrak{A}).$$

We observe that the structure of the entire analytic continuation of the mapping $it \mapsto \sigma_t(x)$, $x \in \mathfrak{L}(\mathfrak{A})_\infty$, is given by Proposition 9.24. We now prove that

$$\xi \in \mathfrak{A}'', \quad x \in \mathfrak{L}(\mathfrak{A})_\infty \Rightarrow x\xi \in \mathfrak{A}'', \quad L_{x\xi} = xL_\xi.$$

Indeed, since the mapping $it \mapsto \Delta^{it}x\Delta^{-it}$ has an entire analytic continuation, whereas the mapping $it \mapsto \Delta^{it}\xi$ has a continuous extension to the set $\{\alpha \in \mathbb{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant 1/2\}$, which is analytic in $\{\alpha \in \mathbb{C}; 0 < \operatorname{Re}\alpha < 1/2\}$ (in accordance with Corollary 9.21, because $\xi \in \mathscr{D}_{(\Delta^{1/2})}$), it follows that the mapping

$$it \mapsto \Delta^{it}x\xi = (\Delta^{it}x\Delta^{-it})\,\Delta^{it}\xi$$

has a continuous extension to $\{\alpha \in \mathbb{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant 1/2\}$, which is analytic in $\{\alpha \in \mathbb{C}; 0 < \operatorname{Re}\alpha < 1/2\}$. With the help of Corollary 9.21, we infer that $x\xi \in \mathscr{D}_{(\Delta^{1/2})} = \mathscr{D}_S$. On the other hand, for any $\eta \in \mathfrak{A}'$ we have

$$\|L_{x\xi}(\eta)\| = \|R_\eta x\xi\| = \|xR_\eta(\xi)\| = \|xL_\xi(\eta)\| \leqslant \|xL_\xi\|\,\|\eta\|,$$

and, therefore, $x\xi \in \mathfrak{A}''$. Obviously, $L_{x\xi} = xL_\xi$. We are now able to prove that

(9) $$[\mathfrak{L}(\mathfrak{A})_\infty \mathfrak{M}_\mathfrak{A} \mathfrak{L}(\mathfrak{A})_\infty = \mathfrak{M}_\mathfrak{A}].$$

To this end, it is sufficient to verify the inclusion $\mathfrak{L}(\mathfrak{A})_\infty \mathfrak{F}_\mathfrak{A} \subset \mathfrak{M}_\mathfrak{A}$. Let then $x \in \mathfrak{L}(\mathfrak{A})_\infty$ and $a \in \mathfrak{F}_\mathfrak{A}$. There exists a $\xi \in \mathfrak{A}''$, such that $a^{1/2} = L_\xi$. From the implication we have proved above, we infer that $x\xi \in \mathfrak{A}''$ and

$$xa^{1/2} = xL_\xi = L_{x\xi} \in \mathfrak{N}^*_\mathfrak{A}.$$

Since

$$a^{1/2} = L_\xi \in \mathfrak{N}_\mathfrak{A},$$

it follows that

$$xa = (xa^{1/2})\,a^{1/2} \in \mathfrak{N}^*_\mathfrak{A}\mathfrak{N}_\mathfrak{A} = \mathfrak{M}_\mathfrak{A}.$$

The invariance property expressed by relation (9) allows the construction of an increasingly directed family of elements in $\mathfrak{M}^+_\mathfrak{A}$, whose supremum is 1, of a very particular nature:

(10) *there exists a family $\{e_i\}_{i\in I} \subset \mathfrak{L}(\mathfrak{A})$ of mutually orthogonal projections of countable type, such that $\sigma_t(e_i) = e_i$, $i \in I$, $t \in \mathbb{R}$, and such that, for any $i \in I$,*

(*) *there exists a sequence $\{a_{i,n}\}_{n\geqslant 1} \subset \mathfrak{M}^+_\mathfrak{A}$, such that $a_{i,n} \uparrow e_i$ and, for any $n \geqslant 1$ and any rational r, there exists an integer $m(n,r) \geqslant 1$, such that $\sigma_r(a_{i,n}) \leqslant a_{i,m(n,r)}$.*

Indeed, let $\{e_\iota\}_{\iota\in I}\subset \mathcal{L}(\mathfrak{A})$ be a maximal family with the above property. We assume that $e=1-\sum_\iota e_\iota\neq 0$. From assertion (5) we infer that there exists a $b\in\mathfrak{M}_{\mathfrak{A}}^+$, such that $a=ebe\neq 0$. Since $\sigma_t(e)=e$, $t\in\mathbb{R}$, we have $e\in\mathcal{L}(\mathfrak{A})_\infty$ and, from assertion (9), we infer that $a\in\mathfrak{M}_{\mathfrak{A}}^+$. Let $\{r_n\}_{n\geqslant 1}$ be an enumeration of the set of rational numbers. We define

$$a_{0,n}=\left(n^{-1}+\sum_{j=1}^{n}\sigma_{r_j}(a)\right)^{-1}\sum_{j=1}^{n}\sigma_{r_j}(a),\qquad n\geqslant 1.$$

It is easy to verify that the sequence $\{a_{0,n}\}$ is increasing and *so*-convergent to the projection $e_0=\bigvee_{j=1}^{\infty}s(\sigma_{r_j}(a))$. If $n\geqslant 1$ and r is rational, then we choose $m(n,r)\geqslant 1$, such that

$$\{r_j\}_{1\leqslant j\leqslant m(n,r)}\supset\{r+r_j\}_{1\leqslant j\leqslant n}.$$

Then, for any rational r and any $n\geqslant 1$ we have

$$\sigma_r(a_{0,n})\leqslant a_{0,m(n,r)}.$$

It follows that for any rational r we have

$$\sigma_r(e_0)\leqslant e_0.$$

Consequently, for any $t\in\mathbb{R}$,

$$\sigma_t(e_0)=e_0.$$

Let $\{f_\alpha\}_{\alpha\in\Gamma}$ be a family of mutually orthogonal projections in $\mathcal{L}(\mathfrak{A})$, such that $\sum_\alpha f_\alpha\leqslant e_0$. For any $n\geqslant 1$ we have, in accordance with assertion (7),

$$\sum_\alpha\varphi_{\mathfrak{A}}((a_{0,n})^{1/2}f_\alpha(a_{0,n})^{1/2})=\varphi_{\mathfrak{A}}(\sum_\alpha(a_{0,n})^{1/2}f_\alpha(a_{0,n})^{1/2})=\varphi_{\mathfrak{A}}(a_{0,n})<+\infty.$$

With the help of assertion (6) we infer that there exists an at most countable subset $\Gamma_n\subset\Gamma$, such that

$$\alpha\notin\Gamma_n\Rightarrow(a_{0,n})^{1/2}f_\alpha(a_{0,n})^{1/2}=0.$$

If $\alpha\notin\bigcup_n\Gamma_n$, then

$$f_\alpha=wo\text{-}\lim_n(a_{0,n})^{1/2}f_\alpha(a_{0,n})^{1/2}=0.$$

We have thus proved that

e_0 is a projection of countable type.

Consequently, $0\neq e_0\leqslant 1-\sum_\iota e_\iota$ has property (*), thus contradicting the maximality of the family $\{e_\iota\}_{\iota\in I}$. It follows that

$$\sum_{\iota\in I}e_\iota=1.$$

In the last part of this section we shall prove the following assertion

$$(11)\quad \left[\begin{array}{l} \textit{there exists a faithful, semifinite normal weight } \varphi \textit{ on } \mathfrak{L}(\mathfrak{A})^+, \textit{ such that} \\ \qquad \text{(i) } \varphi \leqslant \varphi_{\mathfrak{A}}; \\ \qquad \text{(ii) } \varphi(a) = \varphi_{\mathfrak{A}}(a), \textit{ for any } a \in \mathfrak{M}_{\mathfrak{A}}^+; \\ \qquad \text{(iii) } \varphi(\sigma_t(a)) = \varphi(a), \textit{ for any } a \in \mathfrak{L}(\mathfrak{A})^+,\ t \in \mathbb{R}. \end{array}\right]$$

The proof of the fact that $\varphi_{\mathfrak{A}}$ is normal will consist in showing that conditions (i)—(iii) from (11) imply that $\varphi = \varphi_{\mathfrak{A}}$.

In order to prove assertion (11) we shall first observe that to $\mathfrak{A}'$ one can also associate a weight $\varphi_{\mathfrak{A}'}$ on $\mathfrak{R}(\mathfrak{A}')^+$, namely

$$\varphi_{\mathfrak{A}'}(a') = \begin{cases} \|\eta\|^2, \text{ if there exists an } \eta \in \mathfrak{A}', \text{ such that} \\ \qquad (a')^{1/2} = R_\eta, \\ +\infty, \text{ in the contrary case,} \end{cases} \quad ;\ a' \in \mathfrak{R}(\mathfrak{A}')^+.$$

It is easy to verify that $\varphi_{\mathfrak{A}'}$ has properties analogous to properties (1)—(10), already proved for $\varphi_{\mathfrak{A}}$. In what follows we shall actually prove that the pair $(\mathfrak{R}(\mathfrak{A}'), \varphi_{\mathfrak{A}'})$ has the property analogous to property (11).

We shall denote by $\{\sigma'_t\}$ *the group of the modular automorphisms of* $\mathfrak{R}(\mathfrak{A}')$:

$$\sigma'_t(x') = \Delta^{-it} x' \Delta^{it}, \quad x' \in \mathfrak{R}(\mathfrak{A}'), \quad t \in \mathbb{R}.$$

Let $\{a_{i,n}\} \subset \mathfrak{M}_{\mathfrak{A}}^+$ be the elements from (10). For any i and n there exists a $\xi_{i,n} \in \mathfrak{A}''$, such that

$$(a_{i,1})^{1/2} = L_{\xi_{i,1}}; \quad (a_{i,n} - a_{i,n-1})^{1/2} = L_{\xi_{i,n}}, \quad n \geqslant 2.$$

Obviously,

$$\sum_i \sum_n (L_{\xi_{i,n}})^2 = 1.$$

We define a weight φ' on $\mathfrak{R}(\mathfrak{A}')^+$ by the formula

$$\varphi'(a') = \sum_i \sum_n \omega_{\xi_{i,n}}(a'), \qquad a' \in \mathfrak{R}(\mathfrak{A}')^+.$$

If $a' \in \mathfrak{R}(\mathfrak{A}')^+$ and $\varphi_{\mathfrak{A}'}(a') < +\infty$, then there exists an $\eta \in \mathfrak{A}'$, such that $(a')^{1/2} = R_\eta$; hence

$$\omega_{\xi_{i,n}}(a') = \|R_\eta(\xi_{i,n})\|^2 = \|L_{\xi_{i,n}}(\eta)\|^2 = \omega_\eta((L_{\xi_{i,n}})^2),$$

and therefore

$$\varphi'(a') = \sum_i \sum_n \omega_{\xi_{i,n}}(a') = \omega_\eta(1) = \|\eta\|^2 = \varphi_{\mathfrak{A}'}(a').$$

Consequently, φ' is a semifinite, normal weight on $\mathfrak{R}(\mathfrak{A}')^+$, which coincides with $\varphi_{\mathfrak{A}'}$ on $\{a' \in \mathfrak{R}(\mathfrak{A}')^+;\ \varphi_{\mathfrak{A}'}(a') < +\infty\}$.

Let $n \geqslant 1$, r be rational and $m(n, r) \geqslant 1$, as in (10). For any $\eta \in \mathfrak{A}'$, such that $R_\eta \geqslant 0$, we have

$$\sum_{j=1}^{n} \omega_{\xi_{i,j}}(\sigma'_r((R_\eta)^2)) = \sum_{j=1}^{n} \omega_{\xi_{i,j}}((R_{\Delta^{-ir}\eta})^2) = \sum_{j=1}^{n} \omega_{\Delta^{-ir}\eta}((L_{\xi_{i,j}})^2)$$

$$= \omega_{\Delta^{-ir}\eta}(a_{i,n}) = \omega_\eta(\sigma_r(a_{i,n})) \leqslant \omega_\eta(a_{i,m(n,r)}) = \sum_{j=1}^{m(n,r)} \omega_{\xi_{i,j}}((R_\eta)^2).$$

Consequently, for any $a' \in \mathscr{R}(\mathfrak{A}')^+$ we have

$$\sum_{j=1}^{n} \omega_{\xi_{i,j}}(\sigma'_r(a')) \leqslant \sum_{j=1}^{m(n,r)} \omega_{\xi_{i,j}}(a') \leqslant \varphi'(a').$$

Hence we infer that $\varphi'(\sigma'_r(a')) \leqslant \varphi'(a')$, for any $a' \in \mathscr{R}(\mathfrak{A}')^+$ and any rational r. With the help of the lower w-semicontinuity of φ', it is easy to infer that

$$\varphi'(\sigma'_t(a')) = \varphi'(a'), \quad a' \in \mathscr{R}(\mathfrak{A}')^+, \quad t \in \mathbb{R}.$$

We have still to prove that φ' is faithful. As for normal forms (see 5.15), by a similar argument one easily shows that the set

$$\{a' \in \mathscr{R}(\mathfrak{A}'); \ 0 \leqslant a' \leqslant 1, \ \varphi'(a') = 0\}$$

has a greatest element e', which is a projection. Since φ' is invariant with respect to the group of modular automorphisms $\{\sigma'_t\}_t$, it follows that

$$\sigma'_t(e') = e', \quad t \in \mathbb{R}.$$

Let $a' \in \mathscr{R}(\mathfrak{A}')^+$ be such that $\varphi_{\mathfrak{A}'}(a') < +\infty$. With the help of the assertion similar to assertion (9), it follows that $\varphi_{\mathfrak{A}'}(e'a'e') < +\infty$. Thus

$$\varphi_{\mathfrak{A}'}(e'a'e') = \varphi'(e'a'e') \leqslant \|a'\|\varphi'(e') = 0;$$

hence $e'a'e' = 0$. Since the set

$$\{a' \in \mathscr{R}(\mathfrak{A}')^+; \ \varphi_{\mathfrak{A}'}(a') < +\infty\}$$

is w-dense in $\mathscr{R}(\mathfrak{A}')^+$, we infer that $e' = 0$.

10.17. In this section we show that the weight $\varphi_{\mathfrak{A}}$ determines the group of the modular automorphisms $\{\sigma_t\}$ only in terms of the von Neumann algebra $\mathfrak{L}(\mathfrak{A})$.

Let $\mathscr{M}$ be a von Neumann algebra, φ a weight on $\mathscr{M}^+$ and $\{\pi_t\}_{t \in \mathbb{R}}$ a group of $*$-automorphisms of $\mathscr{M}$. We assume that the group $\{\pi_t\}$ *leaves invariant the weight* φ, i.e.,

$$\varphi(\pi_t(a)) = \varphi(a), \ a \in \mathscr{M}^+, \ t \in \mathbb{R}.$$

One says that φ *satisfies the Kubo-Martin-Schwinger condition* (briefly, the KMS-condition) *for the elements* $x, y \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, *with respect to* $\{\pi_t\}$, if there exists a bounded continuous function

$$\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\ \alpha \leqslant 1\} \ni \alpha \mapsto f_{x,y}(\alpha) \in \mathbb{C},$$

which is analytic in $\{\alpha \in \mathbb{C}; 0 < \operatorname{Re} \alpha < 1\}$, and such that

$$f_{x,y}(\mathrm{i}t) = \varphi(x\pi_t(y)), \quad t \in \mathbb{R},$$
$$f_{x,y}(1 + \mathrm{i}t) = \varphi(\pi_t(y)x), \quad t \in \mathbb{R}.$$

Theorem. *Let* $\mathfrak{A} \subset \mathscr{H}$ *be a left Hilbert algebra,* $\mathscr{M} = \mathfrak{L}(\mathfrak{A})$, $\{\sigma_t\}_{t\in\mathbb{R}}$ *the group of modular automorphisms and* $\varphi = \varphi_{\mathfrak{A}}$.

Then φ *satisfies the KMS-condition with respect to* $\{\sigma_t\}$ *for any pair of elements in* $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$.

Conversely, if $\{\pi_t\}_{t\in\mathbb{R}}$ *is a group of* $*$*-automorphisms of* $\mathscr{M}$, *which leaves invariant the weight* φ *and with respect to which* φ *satisfies the KMS-condition, for any pair of elements in* $(\mathfrak{M}_\varphi)^2$, *then*

$$\pi_t = \sigma_t, \quad t \in \mathbb{R}.$$

Proof. Let $x, y \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$. In accordance with assertion (4) from Section 10.16, there exist $\xi, \zeta \in \mathfrak{A}''$, such that

$$x = L_\xi, \quad y = L_\zeta.$$

Then the function

$$\alpha \mapsto (\Delta^\alpha\zeta \mid S\xi)$$

is bounded and continuous on $\{\alpha \in \mathbb{C}; 0 \leqslant \operatorname{Re} \alpha \leqslant 1/2\}$ and it is analytic in $\{\alpha \in \mathbb{C}; 0 < \operatorname{Re} \alpha < 1/2\}$; also, the function

$$\alpha \mapsto (\xi \mid J\Delta^{\alpha-\frac{1}{2}}\zeta)$$

is bounded and continuous on $\{\alpha \in \mathbb{C};\ 1/2 \leqslant \operatorname{Re} \alpha \leqslant 1\}$, and analytic in $\{\alpha \in \mathbb{C}; 1/2 < \operatorname{Re} \alpha < 1\}$. Since, for any $t \in \mathbb{R}$, we have

$$(\Delta^{\frac{1}{2}+\mathrm{i}t}\zeta \mid S\xi) = (\Delta^{\frac{1}{2}+\mathrm{i}t}\zeta \mid \Delta^{-\frac{1}{2}}J\xi) = (\Delta^{\mathrm{i}t}\zeta \mid J\xi)$$
$$= (\xi \mid J\Delta^{\mathrm{i}t}\zeta) = (\xi \mid J\Delta^{\left(\frac{1}{2}+\mathrm{i}t\right)-\frac{1}{2}}\zeta),$$

the two functions coincide on the line $\alpha = \dfrac{1}{2} + \mathrm{i}t$, $t \in \mathbb{R}$.

Thus, we can define

$$f_{x,y}(\alpha) = \begin{cases} (\Delta^\alpha\zeta \mid S\xi), & \text{if} \quad 0 \leqslant \operatorname{Re}\alpha \leqslant 1/2, \\ (\xi \mid J\Delta^{\alpha-\frac{1}{2}}\zeta), & \text{if} \quad 1/2 \leqslant \operatorname{Re}\alpha \leqslant 1. \end{cases}$$

The function $f_{x,y}$ thus defined is bounded and continuous on $\{\alpha \in \mathbb{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$ and analytic in $\{\alpha \in \mathbb{C}; 0 < \operatorname{Re}\alpha < 1\}$. With the help of assertion (4) from Section 10.16, we infer that for any $t \in \mathbb{R}$ we have

$$f_{x,y}(it) = (\Delta^{it}\zeta \mid S\xi) = \varphi(L_\xi L_{\Delta^{it}\zeta}) = \varphi(x\sigma_t(y)),$$

$$f_{x,y}(1+it) = (\xi \mid J\Delta^{\frac{1}{2}+it}\zeta) = (\xi \mid S\Delta^{it}\zeta) = \varphi(L_{\Delta^{it}\zeta}L_\xi) = \varphi(\sigma_t(y)x).$$

We have thus proved the first part of the theorem.

Let now $\{\pi_t\}$ be a group of $*$-automorphisms of $\mathscr{M}$, which leaves invariant the weight φ and with respect to which φ satisfies the KMS-condition, for any pair of elements in $(\mathfrak{M}_\varphi)^2$. We denote

$$\mathfrak{A}_0 = \{\xi \in \mathfrak{A}''; \ L_\xi \in (\mathfrak{M}_\varphi)^2\}.$$

Then $\mathfrak{A}_0$ is a $*$-subalgebra of $\mathfrak{A}''$. With the help of assertion (4) from Section 10.16 it is easy to see that $\mathfrak{A}_0 \supset (\mathfrak{A}'')^4$ and, therefore, in accordance with the last remark in Section 10.5, $\mathfrak{A}_0$ is a left Hilbert subalgebra of $\mathfrak{A}''$ and $(\mathfrak{A}_0)'' = \mathfrak{A}''$. In particular, $\mathfrak{L}(\mathfrak{A}_0) = \mathscr{M}$. From the invariance of φ with respect to $\{\pi_t\}$ it is easy to infer tha for any $\xi \in \mathfrak{A}_0$ and any $t \in \mathbb{R}$ there exists a unique element in $\mathfrak{A}_0$, denoted by $u_t\xi$ such that

$$L_{u_t\xi} = \pi_t(L_\xi);$$

also, for any $t \in \mathbb{R}$, the mapping $\mathfrak{A}_0 \ni \xi \mapsto u_t\xi$ extends to a unitary operator $u_t \in \mathscr{B}(\mathscr{H})$. Obviously, $\{u_t\}$ is a one-parameter group. From the KMS-condition we infer that, for any $\xi, \zeta \in \mathfrak{A}_0$, the mapping

$$t \mapsto \varphi(L_{S\xi}\pi_t(L_\zeta)) = (u_t\zeta \mid \xi)$$

is continuous. From this result it is easy to infer that the group $\{u_t\}$ is *wo*-continuous.

For any $\xi \in \mathfrak{A}_0$ we have

$$L_{Su_t\xi} = (\pi_t(L_\xi))^* = \pi_t(L_\xi^*) = L_{u_tS\xi}, \quad t \in \mathbb{R};$$

hence

$$Su_t\xi = u_tS\xi, \quad t \in \mathbb{R}.$$

Since $\overline{S \mid \mathfrak{A}_0} = S$, the preceding equality is true for any $\xi \in \mathscr{D}_S$. Thus, for any $\xi \in \mathscr{D}_{(\Delta^{1/2})} = \mathscr{D}_S$, we have

$$\|\Delta^{1/2}u_t\xi\| = \|Su_t\xi\| = \|u_tS\xi\| = \|S\xi\| = \|\Delta^{1/2}\xi\|, \quad t \in \mathbb{R};$$

hence, for any $\xi \in \mathscr{D}_\Delta$, $\zeta \in \mathscr{D}_{(\Delta^{1/2})}$, we have:

$$(\Delta^{1/2}u_t\xi \mid \Delta^{1/2}u_t\zeta) = (\Delta^{1/2}\xi \mid \Delta^{1/2}\zeta) = (\Delta\xi \mid \zeta), \quad t \in \mathbb{R}.$$

From these equalities it is easy to see that, for any $\xi \in \mathscr{D}_\Delta$, $t \in \mathbb{R}$, we have

$$\Delta^{1/2}u_t\xi \in \mathscr{D}_{(\Delta^{1/2})^*} = \mathscr{D}_{(\Delta^{1/2})},$$

i.e.,

$$u_t\xi \in \mathscr{D}_\Delta,$$

and

$$\Delta u_t\xi = u_t\Delta\xi.$$

From the KMS-condition we infer that, for any $\xi, \zeta \in \mathfrak{A}_0$, there exists a bounded and continuous function $f_{\xi,\eta}$ on $\{\alpha \in \mathbf{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$, which is analytic in $\{\alpha \in \mathbf{C}; 0 < \operatorname{Re}\alpha < 1\}$, such that, for any $t \in \mathbf{R}$,

$$f_{\xi,\zeta}(\mathrm{i}t) = \varphi(L_{S\xi}\pi_t(L_\zeta)) = (u_t\zeta \mid \xi),$$

$$f_{\xi,\zeta}(1 + \mathrm{i}t) = \varphi(\pi_t(L_\zeta)\, L_{S\xi}) = (S\xi \mid Su_t\zeta) = (S\xi \mid u_tS\zeta).$$

From the equality $\overline{S \mid \mathfrak{A}_0} = S$, and with the help of the Phragmen-Lindelöf principle, it is easy to see that the preceding assertion extends for any $\xi, \zeta \in \mathscr{D}_S$.

If $\zeta \in \mathscr{D}_\Delta$, then, for any $\xi \in \mathscr{D}_S$ and any $t \in \mathbf{R}$, we have

$$f_{\xi,\zeta}(1 + \mathrm{i}t) = (J\Delta^{1/2}\xi \mid J\Delta^{1/2}u_t\zeta) = (\Delta^{1/2}u_t\zeta \mid \Delta^{1/2}\xi) = (u_t\Delta\zeta \mid \xi).$$

From the fact that $\overline{\mathscr{D}}_S = \mathscr{H}$, and with the help of the Phragmen-Lindelöf principle, it is easy to see that, for any $\zeta \in \mathscr{D}_\Delta$ and any $\xi \in \mathscr{H}$, there exists a bounded continuous function $f_{\xi,\zeta}$ on $\{\alpha \in \mathbf{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$, which is analytic in $\{\alpha \in \mathbf{C}; 0 < \operatorname{Re}\alpha < 1\}$, and such that, for any $t \in \mathbf{R}$,

$$f_{\xi,\zeta}(\mathrm{i}t) = (u_t\zeta \mid \xi),$$

$$f_{\xi,\zeta}(1 + \mathrm{i}t) = (u_t\Delta\zeta \mid \xi).$$

Moreover, for any $\alpha \in \mathbf{C}$, $0 < \operatorname{Re}\alpha \leqslant 1$, we have

$$|f_{\xi,\zeta}(\alpha)| \leqslant \max\{\|\zeta\|, \|\Delta\zeta\|\}\, \|\xi\|.$$

From this inequality it is easy to infer that, for any $\zeta \in \mathscr{D}_\Delta$, the mapping

$$\mathrm{i}t \mapsto u_t\zeta$$

has a weakly continuous extension to $\{\alpha \in \mathbf{C}; 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$ which is weakly analytic in $\{\alpha \in \mathbf{C}; 0 < \operatorname{Re}\alpha < 1\}$. Moreover, the value of this extension at 1 is $\Delta\zeta$. As a result of the Stone representation theorem (9.20), there exists a positive self-adjoint operator A in $\mathscr{H}$, such that

$$u_t = A^{\mathrm{i}t}, \quad t \in \mathbf{R}.$$

From the preceding results we infer that, for any $\zeta \in \mathscr{D}_\Delta$, we have

$$\Delta\zeta = A\zeta.$$

Consequently, we have the following relations

$$\Delta \subset A$$

$$A = A^* \subset \Delta^* = \Delta,$$

which imply that:

$$A = \Delta.$$

Therefore, for any $\xi \in \mathfrak{A}_0$ and any $t \in \mathbb{R}$, we have

$$\pi_t(L_\xi) = L_{u_t \xi} = L_{\Delta^{it}\xi} = \Delta^{it} L_\xi \Delta^{-it} = \sigma_t(L_\xi),$$

whence we infer that, for any $x \in \mathfrak{L}(\mathfrak{A}_0) = \mathscr{M}$ we have

$$\pi_t(x) = \sigma_t(x), \quad t \in \mathbb{R}.$$

Q.E.D.

Let φ be a faithful, semifinite, normal weight on a von Neumann algebra $\mathscr{M}$, and let $\mathfrak{A}_\varphi$ be the left Hilbert algebra we have constructed in Section 10.15. If $\{\sigma_t\}$ is the group of the modular automorphism of $\mathfrak{L}(\mathfrak{A}_\varphi)$, which is associated to $\mathfrak{A}_\varphi$, we shall denote

$$\sigma_t^\varphi = \pi_\varphi^{-1} \circ \sigma_t \circ \pi_\varphi, \quad t \in \mathbb{R},$$

and we shall say that $\{\sigma_t^\varphi\}$ is *the group of the modular automorphisms of* $\mathscr{M}$, *which is associated to* φ.

Obviously, the assertions made in the preceding theorem are true for $\mathscr{M}$, φ and $\{\sigma_t^\varphi\}$.

10.18. Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra, and $\varphi_\mathfrak{A}$ the associated weight on $\mathfrak{L}(\mathfrak{A})^+$. In this section we shall prove that $\varphi_\mathfrak{A}$ is *normal.*

In accordance with assertion (11) from Section 10.16, there exists a faithful, semifinite, normal weight $\varphi \leqslant \varphi_\mathfrak{A}$ on $\mathfrak{L}(\mathfrak{A})^+$, which is invariant with respect to the group $\{\sigma_t\}$ and which coincides with $\varphi_\mathfrak{A}$ on $\mathfrak{M}_\mathfrak{A}^+$.

Let $a \in \mathfrak{M}_\varphi^+$. We define

$$a_n = \sqrt{n/\pi} \int_{-\infty}^{+\infty} e^{-nt^2} \sigma_t(a) \, dt, \quad n \in \mathbb{N}.$$

From Proposition 3.21 and Corollary 3.20, we infer that there exists a net $\{u_\nu\} \subset \mathfrak{M}_\mathfrak{A}^+$, such that $u_\nu \uparrow 1$. From assertion (9) from Section 10.16, and the discussion preceding it, we have $a_n u_\nu a_n \in \mathfrak{M}_\mathfrak{A}^+$; hence

$$\varphi_\mathfrak{A}(a_n u_\nu a_n) = \varphi(a_n u_\nu a_n) \leqslant \varphi((a_n)^2) \leqslant \|a_n\| \, \varphi(a_n).$$

By taking into account the normality of φ and the fact that $\{\sigma_t\}$ leaves invariant the weight φ, we have

$$\varphi(a_n) = \sqrt{n/\pi} \int_{-\infty}^{+\infty} e^{-nt^2} \varphi(\sigma_t(a)) \, dt = \varphi(a), \quad n \in \mathbb{N};$$

hence, for any ν, we have $\varphi_{\mathfrak{N}}(a_n u_\nu a_n) \leqslant \varphi(a)$, $n \in \mathbb{N}$; hence

$$\varphi_{\mathfrak{N}}((a_n)^2) \leqslant \varphi(a), \quad n \in \mathbb{N}.$$

Since $(a_n)^2 \xrightarrow{w} a^2$, with the help of the lower w-semicontinuity of the weight $\varphi_{\mathfrak{N}}$, we infer that

$$\varphi_{\mathfrak{N}}(a^2) \leqslant \varphi(a) + \infty,$$

i.e., $a^2 \in \mathfrak{M}_{\mathfrak{N}}^+$.

From what we have just proved, it follows that

$$(\mathfrak{M}_\varphi)^2 \subset \mathfrak{M}_{\mathfrak{N}}.$$

In accordance with Theorem 10.17, $\varphi_{\mathfrak{N}}$ satisfies the KMS-condition with respect to $\{\sigma_t\}$, for any pair of elements in $(\mathfrak{M}_\varphi)^2 \subset \mathfrak{M}_{\mathfrak{N}}$. Since on $(\mathfrak{M}_\varphi)^2 \subset \mathfrak{M}_{\mathfrak{N}}$ the weights φ and $\varphi_{\mathfrak{N}}$ coincide, it follows that φ satisfies the KMS-condition, for any pair of elements in $(\mathfrak{M}_\varphi)^2$. If we now apply the uniqueness part of Theorem 10.17, we infer that

$$\sigma_t^\varphi = \sigma_t, \quad t \in \mathbb{R}.$$

We also consider the left Hilbert algebra $\mathfrak{A}_\varphi$ and we denote by S_φ, Δ_φ, J_φ, the corresponding operators. Let

$$v_\nu = 1/\sqrt{\pi} \int_{-\infty}^{+\infty} e^{-t^2} \sigma_t(u_\nu) \, dt.$$

Since the mapping $is \mapsto \sigma_s(v_\nu)$ has an entire analytic continuation, from Proposition 9.24 we infer that, for any $\alpha \in \mathbb{C}$, we have

$$\mathscr{D}_{((\Delta_\varphi)^\alpha \pi_\varphi(v_\nu)(\Delta_\varphi)^{-\alpha})} = \mathscr{D}_{(\Delta_\varphi)^{-\alpha}}$$

and the operator $(\Delta_\varphi)^\alpha \pi_\varphi(v_\nu)(\Delta_\varphi)^{-\alpha}$ is bounded. We denote

$$F_{v_\nu}(\alpha) = \overline{(\Delta_\varphi)^\alpha \pi_\varphi(v_\nu)(\Delta_\varphi)^{-\alpha}} = 1/\sqrt{\pi} \int_{-\infty}^{+\infty} e^{-(t+i\alpha)^2} \sigma_t(u_\nu) \, dt.$$

It is easy to verify that, for any $\alpha \in \mathbb{C}$,

$$F_{v_\nu}(\alpha) \xrightarrow{so} 1,$$

$$\|F_{v_\nu}(\alpha)\| \leqslant e^{(\mathrm{Re}\,\alpha)^2}.$$

For any $a \in \mathfrak{M}_\varphi^+$ and any ν we have

$$\varphi_{\mathfrak{N}}(v_\nu a v_\nu) \leqslant \varphi_{\mathfrak{N}}((v_\nu)^2) < +\infty;$$

hence

$$\varphi_{\mathfrak{N}}(v_\nu a v_\nu) = \varphi(v_\nu a v_\nu).$$

With the help of the lower w-semicontinuity of $\varphi_{\mathfrak{A}}$, we get

$$\begin{aligned}\varphi_{\mathfrak{A}}(a) &\leqslant \sup_{\nu} \varphi_{\mathfrak{A}}(v_\nu a v_\nu) = \sup_{\nu} \varphi(v_\nu a v_\nu) = \sup_{\nu} \|(a^{1/2}v_\nu)_\varphi\|_\varphi^2 \\ &= \sup_{\nu} \|S_\varphi \pi_\varphi(v_\nu)\, S_\varphi((a^{1/2})_\varphi)\|_\varphi^2 \\ &= \sup_{\nu} \|J_\varphi(\Delta_\varphi)^{1/2}\pi_\varphi(v_\nu)\, (\Delta_\varphi)^{-1/2}J_\varphi((a^{1/2})_\varphi)\|_\varphi^2 \\ &= \sup_{\nu} \|\pi_\varphi(F_{v_\nu}(1/2))\, J_\varphi((a^{1/2})_\varphi)\|_\varphi^2 \\ &\leqslant \mathrm{e}^{1/4}\|(a^{1/2})_\varphi\|_\varphi^2 \leqslant \mathrm{e}^{1/4}\varphi(a) < +\infty;\end{aligned}$$

hence, $a \in \mathfrak{M}_{\mathfrak{A}}^+$.

Consequently, we have

$$\mathfrak{M}_\varphi^+ = \mathfrak{M}_{\mathfrak{A}}^+,$$

whence

$$\varphi = \varphi_{\mathfrak{A}}.$$

From the results in Section 10.16 and from those we have just obtained we infer the following

Theorem. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra. We define the mapping $\varphi_{\mathfrak{A}} : \mathfrak{L}(\mathfrak{A})^+ \to \mathbb{R}^+ \cup \{+\infty\}$ by the formula*

$$\varphi_{\mathfrak{A}}(a) = \begin{cases} \|\xi\|^2, & \text{if there exists a } \xi \in \mathfrak{A}'', \text{ such that } a^{1/2} = L_\xi, \\ +\infty, & \text{in the contrary case,} \end{cases} \quad ; \quad a \in \mathfrak{L}(\mathfrak{A})^+.$$

Then $\varphi_{\mathfrak{A}}$ us a faithful, semifinite, normal weight on $\mathfrak{L}(\mathfrak{A})^+$ and the mapping $\xi \mapsto L_\xi$ is a $*$-isomorphism of $\mathfrak{A}''$ onto $\mathfrak{N}_{\varphi_{\mathfrak{A}}} \cap \mathfrak{N}_{\varphi_{\mathfrak{A}}}^*$, such that

$$(\xi \mid \zeta) = \varphi_{\mathfrak{A}}((L_\zeta)^* L_\xi), \qquad \xi, \zeta \in \mathfrak{A}''.$$

Two pairs $(\mathscr{M}_j, \varphi_j)$, $j = 1, 2$, where φ_j is a faithful, semifinite, normal weight on the von Neumann algebra $\mathscr{M}_j$, are said to be *equivalent* if there exists a $*$-isomorphism $\pi : \mathscr{M}_1 \to \mathscr{M}_2$ such that $\varphi_1 = \varphi_2 \circ \pi$. Two left Hilbert algebras $\mathfrak{A}_j \subset \mathscr{H}_j$, $j = 1, 2$, are said to be *equivalent* if there exists a unitary operator $u : \mathscr{H}_1 \to \mathscr{H}_2$, such that the restriction of u to $\mathfrak{A}_1$ be a $*$-isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$.

From the above theorem and from Theorem 10.14 we infer that *the associations*

$$(\mathscr{M}, \varphi) \mapsto \mathfrak{A}_\varphi,$$

$$\mathfrak{A} \mapsto (\mathfrak{L}(\mathfrak{A}),\ \varphi_{\mathfrak{A}}),$$

establish bijections, inverse to one another, between the classes of equivalence of pairs $(\mathscr{M}, \varphi)$, where φ is a faithful, semifinite normal weight on the von Neumann algebra $\mathscr{M}$, and the classes of equivalence of left Hilbert algebras $\mathfrak{A}$, such that $\mathfrak{A} = \mathfrak{A}''$.

10.19. The aim of the following sections is to exhibit some "suitable" elements in $\mathfrak{A}'' \cap \mathfrak{A}'$ and to prove their "abundance".

Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra. We consider the vector space $\mathfrak{T}_0$, generated by the set

$$\{\Delta^n \exp(-r\Delta)\exp(-s\Delta^{-1})\xi;\ \xi \in \mathfrak{A}'',\ r, s > 0,\ n \in \mathbb{Z}\}.$$

Lemma 1. $\mathfrak{T}_0$ *is contained in* $\mathfrak{A}'' \cap \mathfrak{A}'$.

Proof. Let $\xi \in \mathfrak{A}''$, $r, s > 0$ and $n \in \mathbb{Z}$. We consider the curve $\Gamma: \mathbb{R} \to \mathbb{C}$, given by the formula

$$\Gamma(t) = \begin{cases} -t-1+\mathrm{i}, & \text{if } t \leqslant -1, \\ -\mathrm{e}^{\mathrm{i}\frac{\pi}{2}t}, & \text{if } -1 \leqslant t \leqslant 1, \\ t-1-\mathrm{i}, & \text{if } t \geqslant 1. \end{cases}$$

We shall assume that $n \geqslant 0$. From Proposition 9.27 we infer that

$$\exp(-s\Delta^{-1})\xi = (2\pi\mathrm{i})^{-1}\int_\Gamma \exp(-s\lambda)(\lambda - \Delta^{-1})^{-1}\xi\, \mathrm{d}\lambda.$$

With the help of Proposition 10.11, it is easy to verify that

$$\exp(-s\Delta^{-1})\xi \in \mathfrak{A}'.$$

A similar argument now shows that

$$\Delta^n \exp(-r\Delta)\exp(-s\Delta^{-1})\xi \in \mathfrak{A}''.$$

Assuming that $n \leqslant 0$ and by repeating the preceding argument, one obtains successively

$$\Delta^n \exp(-s\Delta^{-1})\xi = (\Delta^{-1})^{-n}\exp(-s\Delta^{-1})\xi \in \mathfrak{A}',$$

$$\Delta^n \exp(-r\Delta)\exp(-s\Delta^{-1})\xi = \exp(-r\Delta)(\Delta^n \exp(-s\Delta^{-1})\xi) \in \mathfrak{A}''.$$

Hence, $\mathfrak{T}_0 \subset \mathfrak{A}''$.

Let $\xi \in \mathfrak{A}''$, $r, s \geqslant 0$, $n \in \mathbb{Z}$ again. From the preceding argument we infer that

$$\Delta^n \exp(-r\Delta)\exp\left(-\frac{s}{2}\Delta^{-1}\right)\xi \in \mathfrak{A}''.$$

The argument used in the first part of the proof shows that

$$\Delta^n \exp(-r\Delta)\exp(-s\Delta^{-1})\xi = \exp\left(-\frac{s}{2}\Delta^{-1}\right)\left(\Delta^n \exp(-r\Delta)\exp\left(-\frac{s}{2}\Delta^{-1}\right)\xi\right) \in \mathfrak{A}'.$$

Consequently, $\mathfrak{T}_0 \subset \mathfrak{A}'$.

Q.E.D.

It is obvious that $\Delta\mathfrak{T}_0 = \mathfrak{T}_0$ and $\Delta^{-1}\mathfrak{T}_0 = \mathfrak{T}_0$. With the help of Lemma 1 we get

$$\xi \in \mathfrak{T}_0 \Rightarrow S\xi = S(SS^*)\Delta\xi = S^*\Delta\xi \in S^*\mathfrak{A}' \subset \mathfrak{A}',$$

$$\xi \in \mathfrak{T}_0 \Rightarrow S^*\xi = S^*(S^*S)\Delta^{-1}\xi = S\Delta^{-1}\xi \in S\mathfrak{A}'' \subset \mathfrak{A}''.$$

Lemma 2. *$\mathfrak{T}_0$ is contained in $\bigcap_{\alpha\in\mathbb{C}} \mathcal{D}_{\Delta^\alpha}$ and, for any $\alpha \in \mathbb{C}$,*

$$\overline{\Delta^\alpha | \mathfrak{T}_0} = \Delta^\alpha.$$

Proof. Since $\mathfrak{T}_0 \subset \bigcap_{n\in\mathbb{Z}} \mathcal{D}_{\Delta^n}$, with Corollary 9.21 we infer that

$$\mathfrak{T}_0 \subset \bigcap_{\alpha\in\mathbb{C}} \mathcal{D}_{\Delta^\alpha}.$$

Let $\alpha \in \mathbb{C}$ and assume that $(\xi, \Delta^\alpha\xi) \in \mathcal{G}_{\Delta^\alpha}$ is orthogonal to the graph of the operator $\Delta^\alpha | \mathfrak{T}_0$. Then, for any $\zeta \in \mathfrak{A}''$ we have

$$(\xi \,|\, \exp(-\Delta)\exp(-\Delta^{-1})\zeta) + (\Delta^\alpha\xi \,|\, \Delta^\alpha \exp(-\Delta)\exp(-\Delta^{-1})\zeta) = 0,$$

$$(\xi \,|\, (1 + \Delta^{2\mathrm{Re}\alpha})\exp(-\Delta)\exp(-\Delta^{-1})\zeta) = 0.$$

Since the operator $(1 + \Delta^{2\mathrm{Re}\alpha})\exp(-\Delta)\exp(-\Delta^{-1}) \in \mathscr{B}(\mathscr{H})$ is positive and injective, its range is dense in $\mathscr{H}$. Since $\mathfrak{A}''$ is dense in $\mathscr{H}$, it follows that

$$(1 + \Delta^{2\mathrm{Re}\alpha})\exp(-\Delta)\exp(-\Delta^{-1})\mathfrak{A}''$$

is dense in $\mathscr{H}$. Consequently, $\xi = 0$.

We have thus proved that

$$\overline{\Delta^\alpha | \mathfrak{T}_0} = \Delta^\alpha.$$

Q.E.D.

Lemma 3. *For any $\xi \in \mathfrak{T}_0$ and any $n \in \mathbb{Z}$, we have*

$$\mathcal{D}_{(\Delta^n L_\xi \Delta^{-n})} = \mathcal{D}_{(\Delta^{-n})} \text{ and } \Delta^n L_\xi \Delta^{-n} \subset L_{\Delta^n\xi},$$

$$\mathcal{D}_{(\Delta^n R_\xi \Delta^{-n})} = \mathcal{D}_{(\Delta^{-n})} \text{ and } \Delta^n R_\xi \Delta^{-n} \subset R_{\Delta^n\xi}.$$

Proof. It is obvious that it is sufficient to prove the assertions in the lemma only for $n = 1$ and $n = -1$.

We consider the case $n = 1$. From Lemma 2 and Proposition 9.24, it is sufficient to prove that, for any $\zeta \in \mathfrak{T}_0$, we have

$$\zeta \in \mathcal{D}_{(\Delta L_\xi \Delta^{-1})} \text{ and } \Delta L_\xi \Delta^{-1}(\zeta) = L_{\Delta\xi}(\zeta),$$

$$\zeta \in \mathcal{D}_{(\Delta R_\xi \Delta^{-1})} \text{ and } \Delta R_\xi \Delta^{-1}(\zeta) = R_{\Delta\xi}(\zeta).$$

Indeed, by taking into account the remark made just after Lemma 1, we have

$$L_\xi(\Delta^{-1}(\zeta) = SL_{S\Delta^{-1}\zeta}S\xi = SL_{S^*\zeta}S\xi = SR_{S\xi}S^*\zeta = SS^*R_\zeta S^*S\xi = \Delta^{-1}L_{\Delta\xi}(\zeta);$$

hence

$$L_\xi\Delta^{-1}(\zeta) \in \mathcal{D}_\Delta \text{ and } \Delta L_\xi \Delta^{-1}(\zeta) = L_{\Delta\xi}(\zeta).$$

Similarly

$$R_\xi\Delta^{-1}(\zeta) = L_{\Delta^{-1}\zeta}(\xi) = SL_{S\xi}S\Delta^{-1}\zeta = SL_{S\xi}S^*\zeta = SR_{S^*\zeta}S\xi$$
$$= SS^*R_{S^*S\xi}(\zeta) = \Delta^{-1}R_{\Delta\xi}(\zeta);$$

hence

$$R_\xi \Delta^{-1}(\zeta) \in \mathscr{D}_\Delta \text{ and } \Delta R_\xi \Delta^{-1}(\zeta) = R_{\Delta\xi}(\zeta).$$

The case $n = -1$ can be treated similarly.

Q.E.D.

We have not as yet used the fundamental theorem of Tomita (10.12). In what follows we shall use it, in order to extend Lemma 3, by replacing, in its statement, $n \in \mathbb{Z}$, by $\alpha \in \mathbb{C}$.

Lemma 4. *For any $\xi \in \mathfrak{T}_0$ and any $\alpha \in \mathbb{C}$ we have*

$$\Delta^\alpha \xi \in \mathfrak{A}'' \cap \mathfrak{A}',$$

$$\mathscr{D}_{(\Delta^\alpha L_\xi \Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{-\alpha})} \text{ and } \Delta^\alpha L_\xi \Delta^{-\alpha} \subset L_{\Delta^\alpha \xi},$$

$$\mathscr{D}_{(\Delta^\alpha R_\xi \Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{-\alpha})} \text{ and } \Delta^\alpha R_\xi \Delta^{-\alpha} \subset R_{\Delta^\alpha \xi}.$$

Proof. Let $\xi \in \mathfrak{T}_0$. In accordance with Lemma 2 and Corollary 9.21, for any $\alpha \in \mathbb{C}$ we have $\xi \in \mathscr{D}_{\Delta^\alpha}$, whereas the mapping

$$\alpha \mapsto \Delta^\alpha \xi$$

is entire analytic.

On the other hand, in accordance with Lemma 2 and Proposition 9.24, for any $\alpha \in \mathbb{C}$ we have $\mathscr{D}_{(\Delta^\alpha L_\xi \Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{-\alpha})}$, the operator $\Delta^\alpha L_\xi \Delta^{-\alpha}$ is bounded and the mapping

$$\alpha \mapsto F(\alpha) = \overline{\Delta^\alpha L_\xi \Delta^{-\alpha}} \in \mathscr{B}(\mathscr{H})$$

is entire analytic.

For any $\alpha \in \mathbb{C}$, we have $\Delta^\alpha \xi \in \mathscr{D}_{(\Delta^{1/2})} = \mathscr{D}_S$; hence the closed operator $L_{\Delta^\alpha \xi}$ is defined. Theorem 10.12 shows that, for any $t \in \mathbb{R}$, we have

$$\Delta^{it} \xi \in \mathfrak{A}'' \text{ and } L_{\Delta^{it}\xi} = \Delta^{it} L_\xi \Delta^{-it} = F(it).$$

Thus, for any $\eta \in \mathfrak{A}'$ the entire analytic functions

$$\alpha \mapsto R_\eta(\Delta^\alpha \xi) = L_{\Delta^\alpha \xi}(\eta),$$
$$\alpha \mapsto F(\alpha)(\eta)$$

coincide on the imaginary axis and, therefore, they coincide on the entire complex plane $\mathbb{C}$.

Therefore, for any $\alpha \in \mathbb{C}$, we have

$$\|L_{\Delta^\alpha \xi}(\eta)\| = \|F(\alpha)(\eta)\| \leqslant \|F(\alpha)\| \, \|\eta\|, \quad \eta \in \mathfrak{A}';$$

hence

$$\Delta^\alpha \xi \in \mathfrak{A}'' \text{ and } L_{\Delta^\alpha \xi} = F(\alpha) \supset \Delta^\alpha L_\xi \Delta^{-\alpha}.$$

Similarly one can show that for any $\alpha \in \mathbb{C}$, we have $\Delta^\alpha \xi \in \mathfrak{A}'$, and

$$\mathscr{D}_{(\Delta^\alpha R_\xi \Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{-\alpha})} \text{ and } \Delta^\alpha R_\xi \Delta^{-\alpha} \subset R_{\Delta^\alpha \xi}.$$

Q.E.D.

10.20. Let $\mathfrak{A}\subset\mathscr{H}$ be a left Hilbert algebra. We observe that, for $\xi,\eta\in\mathfrak{A}''\cap\mathfrak{A}'$, we have

$$L_\xi(\eta)=R_\eta(\xi).$$

We can, therefore, use the notation $\xi\eta$ both in the sense of the product in $\mathfrak{A}''$ and in the sense of the product in $\mathfrak{A}'$. Obviously, $\mathfrak{A}'\cap\mathfrak{A}''$ is an algebra.

We now consider the vector space

$$\mathfrak{T}=\left\{\xi\in\bigcap_{\alpha\in\mathbb{C}}\mathscr{D}_{\Delta^\alpha}\;\middle|\;\begin{array}{l}\text{for any } \alpha\in\mathbb{C} \text{ we have}\\ \Delta^\alpha\xi\in\mathfrak{A}''\cap\mathfrak{A}',\\ \mathscr{D}_{(\Delta^\alpha L_\xi\Delta^{-\alpha})}=\mathscr{D}_{(\Delta^{-\alpha})} \text{ and } \Delta^\alpha L_\xi\Delta^{-\alpha}\subset L_{\Delta^\alpha\xi},\\ \mathscr{D}_{(\Delta^\alpha R_\xi\Delta^{-\alpha})}=\mathscr{D}_{(\Delta^{-\alpha})} \text{ and } \Delta^\alpha R_\xi\Delta^{-\alpha}\subset R_{\Delta^\alpha\xi}.\end{array}\right\}$$

The following theorem of "calculus in $\mathfrak{T}$" is the culminating point of Tomita's theory.

Theorem. *Let $\mathfrak{A}\subset\mathscr{H}$ be a left Hilbert algebra. Then $\mathfrak{T}$ is a left Hilbert subalgebra of $\mathfrak{A}''$ and*

$$\mathfrak{T}'=\mathfrak{A}',\quad \mathfrak{T}''=\mathfrak{A}''.$$

Moreover,

(1) $$\mathfrak{T}\subset\mathscr{D}_{\Delta^\alpha},\ \Delta^\alpha\mathfrak{T}=\mathfrak{T} \text{ and } \overline{\Delta^\alpha|\mathfrak{T}}=\Delta^\alpha,\quad \alpha\in\mathbb{C};$$

(2) $$J\mathfrak{T}=\mathfrak{T};$$

(3) $$\Delta^\alpha J\xi=J\Delta^{-\bar{\alpha}}\xi,\quad \xi\in\mathfrak{T},\quad \alpha\in\mathbb{C};$$

(4) $$\Delta^\alpha(\xi\eta)=(\Delta^\alpha\xi)(\Delta^\alpha\eta),\quad \xi,\ \eta\in\mathfrak{T},\quad \alpha\in\mathbb{C};$$

(5) $$J(\xi\eta)=J(\eta)\,J(\xi),\quad \xi,\eta\in\mathfrak{T}.$$

Proof. We first prove assertion (1). Let $\alpha\in\mathbb{C}$. From the definition of $\mathfrak{T}$ we have $\mathfrak{T}\subset\mathscr{D}_{\Delta^\alpha}$.

Let $\xi\in\mathfrak{T}$ and $\beta\in\mathbb{C}$. It is easy to see that if $\eta\in\bigcap_{\gamma\in\mathbb{C}}\mathscr{D}_{\Delta^\gamma}$, then η belongs to the domains of definition of the operators $\Delta^{\alpha+\beta}L_\xi\Delta^{-\alpha-\beta}$ and $\Delta^\beta L_{\Delta^\alpha\xi}\Delta^{-\beta}$ and the following equality holds

$$\Delta^\beta L_{\Delta^\alpha\xi}\Delta^{-\beta}(\eta)=\Delta^{\alpha+\beta}L_\xi\Delta^{-\alpha-\beta}(\eta)=L_{\Delta^{\alpha+\beta}\xi}(\eta).$$

Since

$$\overline{\Delta^{-\beta}|\bigcap_{\gamma\in\mathbb{C}}\mathscr{D}_{\Delta^\gamma}}=\Delta^{-\beta},$$

from the preceding equality and from Proposition 9.24, we infer that

$$\mathscr{D}_{(\Delta^\beta L_{\Delta^\alpha\xi}\Delta^{-\beta})}=\mathscr{D}_{(\Delta^{-\beta})} \text{ and } \Delta^\beta L_{\Delta^\alpha\xi}\Delta^{-\beta}\subset L_{\Delta^\beta(\Delta^\alpha\xi)}.$$

Similarly

$$\mathscr{D}_{(\Delta^\beta R_{\Delta^\alpha\xi}\Delta^{-\beta})}=\mathscr{D}_{(\Delta^{-\beta})} \text{ and } \Delta^\beta R_{\Delta^\alpha\xi}\Delta^{-\beta}\subset R_{\Delta^\beta(\Delta^\alpha\xi)}.$$

Consequently,

$$\Delta^{\alpha}\mathfrak{T} = \mathfrak{T}.$$

In accordance with Lemma 4 from Section 10.19, the vector space $\mathfrak{T}_0$ is contained in $\mathfrak{T}$. Thus, Lemma 2 from Section 10.19 shows that

$$\overline{\Delta^{\alpha}|\mathfrak{T}} = \Delta^{\alpha}.$$

We now prove assertion (3). With the help of the formula $f(\Delta)J = J\bar{f}(\Delta^{-1})$ from Section 10.1 we infer that, for any $\xi \in \mathfrak{T}$ and any $\alpha \in \mathbf{C}$, we have $J\xi \in \mathscr{D}_{\Delta^{\alpha}}$ and

$$\Delta^{\alpha}J\xi = J\Delta^{-\bar{\alpha}}\xi.$$

We now prove assertion (2). Let $\xi \in \mathfrak{T}$. Then, as we have seen, $J\xi \in \bigcap_{\alpha \in \mathbf{C}} \mathscr{D}_{\Delta^{\alpha}}$, and, from Theorem 10.12, we infer that

$$\Delta^{\alpha}J\xi = J(\Delta^{-\bar{\alpha}}\xi) \in J(\mathfrak{A}'' \cap \mathfrak{A}') \subset \mathfrak{A}'' \cap \mathfrak{A}', \qquad \alpha \in \mathbf{C}.$$

By using again Theorem 10.12, we infer that for any $\alpha \in \mathbf{C}$, we have the relations

$$\Delta^{\alpha}L_{J\xi}\Delta^{-\alpha} = \Delta^{\alpha}JR_{\xi}J\Delta^{-\alpha} = J\Delta^{-\bar{\alpha}}R_{\xi}\Delta^{\bar{\alpha}}J \subset JR_{\Delta^{-\bar{\alpha}}\xi}J = L_{J\Delta^{-\bar{\alpha}}\xi} = L_{\Delta^{\alpha}J\xi}$$

and

$$\mathscr{D}_{(\Delta^{\alpha}L_{J\xi}\Delta^{-\alpha})} = \mathscr{D}_{(J\Delta^{-\bar{\alpha}}R_{\xi}\Delta^{\alpha}J)} = \mathscr{D}_{(\Delta^{\bar{\alpha}}J)} = \mathscr{D}_{(J\Delta^{-\bar{\alpha}})} = \mathscr{D}_{(\Delta^{-\alpha})}.$$

Similarly, one can show that, for any $\alpha \in \mathbf{C}$, we have

$$\mathscr{D}_{(\Delta^{\alpha}R_{J\xi}\Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{-\alpha})} \text{ and } \Delta^{\alpha}R_{J\xi}\Delta^{-\alpha} \subset R_{\Delta^{\alpha}J\xi}.$$

We now prove assertion (4). Let $\xi, \eta \in \mathfrak{T}$ and $\alpha \in \mathbf{C}$. Then

$$\Delta^{\alpha}\eta \in \mathscr{D}_{(\Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{\alpha}L_{\xi}\Delta^{-\alpha})};$$

hence

$$\xi\eta = L_{\xi}(\eta) = L_{\xi}\Delta^{-\alpha}(\Delta^{\alpha}\eta) \in \mathscr{D}_{(\Delta^{\alpha})}$$

and

$$\Delta^{\alpha}(\xi\eta) = \Delta^{\alpha}L_{\xi}\Delta^{-\alpha}(\Delta^{\alpha}\eta) = L_{\Delta^{\alpha}\xi}(\Delta^{\alpha}\eta) = (\Delta^{\alpha}\xi)(\Delta^{\alpha}\eta).$$

We also have

$$\mathscr{D}_{(\Delta^{\alpha}L_{\xi\eta}\Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{\alpha}L_{\xi}\Delta^{-\alpha})(\Delta^{\alpha}L_{\eta}\Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{-\alpha})}$$

and

$$\Delta^{\alpha}L_{\xi\eta}\Delta^{-\alpha} = (\Delta^{\alpha}L_{\xi}\Delta^{-\alpha})(\Delta^{\alpha}L_{\eta}\Delta^{-\alpha}) \subset L_{\Delta^{\alpha}\xi}L_{\Delta^{\alpha}\eta} = L_{\Delta^{\alpha}(\xi\eta)}.$$

Similarly, one proves the following relations

$$\mathscr{D}_{(\Delta^{\alpha}R_{\xi\eta}\Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{-\alpha})} \text{ and } \Delta^{\alpha}R_{\xi\eta}\Delta^{-\alpha} \subset R_{\Delta^{\alpha}(\xi\eta)}.$$

We have thus proved that, if $\xi, \eta \in \mathfrak{T}$, then

$$\xi\eta \in \mathfrak{T},$$

whereas from assertions (1) and (2) we infer that, if $\xi \in \mathfrak{T}$, then

$$S\xi = J\Delta^{1/2}\xi \in \mathfrak{T}.$$

Since $\mathfrak{T} \supset \mathfrak{T}_0$, we infer that $\mathfrak{T}$ is dense in $\mathscr{H}$. From what we have already proved and from Lemma 2, Section 10.5, it follows that $\mathfrak{T}$ is a left Hilbert subalgebra of $\mathfrak{A}''$.

From assertion (1) we infer that $\overline{\Delta^{1/2}|\mathfrak{T}} = \Delta^{1/2}$ and, therefore,

$$\overline{S|\mathfrak{T}} = S.$$

With the help of Lemma 3, from Section 10.5, we deduce the equalities

$$\mathfrak{T}' = \mathfrak{A}', \qquad \mathfrak{T}'' = \mathfrak{A}''.$$

Q.E.D.

We shall call $\mathfrak{T}$ the *Tomita algebra* associated to the left Hilbert algebra $\mathfrak{A}$.

10.21. We shall now prove a criterion with the help of which we can establish that an element belongs to $\mathfrak{T}$.

Corollary 1. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra, $\mathfrak{T}$ the associated Tomita algebra, $\xi \in \bigcap_{\alpha \in \mathbb{C}} \mathscr{D}_{\Delta^{\alpha}}$ and $\{\varepsilon_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ a family of real numbers, such that*

$$\lim_{n \to -\infty} \varepsilon_n = -\infty, \quad \lim_{n \to +\infty} \varepsilon_n = +\infty.$$

Then the following assertions are equivalent:

(i) $\xi \in \mathfrak{T}$;

(ii) *for any $n \in \mathbb{Z}$ we have $\Delta^{\varepsilon_n}\xi \in \mathfrak{A}''$;*

(iii) *for any $n \in \mathbb{Z}$ we have $\Delta^{\varepsilon_n}\xi \in \mathfrak{A}'$;*

(iv) *for any $n \in \mathbb{Z}$ we have $\mathfrak{T} \subset \mathscr{D}_{(\Delta^{\varepsilon_n}L_{\xi}\Delta^{-\varepsilon_n})}$ and the operator $\Delta^{\varepsilon_n}L_{\xi}\Delta^{-\varepsilon_n}|\mathfrak{T}$ is bounded;*

(v) *for any $n \in \mathbb{Z}$ we have $\mathfrak{T} \subset \mathscr{D}_{(\Delta^{\varepsilon_n}R_{\xi}\Delta^{-\varepsilon_n})}$ and the operator $\Delta^{\varepsilon_n}R_{\xi}\Delta^{-\varepsilon_n}|\mathfrak{T}$ is bounded.*

Proof. We shall prove the implications

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iv)} \Rightarrow \text{(i)};$$

the proofs of the implications

$$\text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(v)} \Rightarrow \text{(i)}$$

are completely similar.

The implication (i) $\Rightarrow$ (ii) is obvious.

We now assume that $\Delta^{\varepsilon_n}\xi \in \mathfrak{A}''$, for any $n \in \mathbb{Z}$. Then, for any $\eta \in \mathfrak{T}$ and any $n \in \mathbb{Z}$, we have

$$L_\xi \Delta^{-\varepsilon_n}(\eta) = R_{\Delta^{-\varepsilon_n}(\eta)}(\xi) = \Delta^{-\varepsilon_n} R_\eta \Delta^{\varepsilon_n}\xi = \Delta^{-\varepsilon_n} L_{\Delta^{\varepsilon_n}\xi}(\eta);$$

hence

$$L_\xi \Delta^{-\varepsilon_n}(\eta) \in \mathscr{D}_{(\Delta^{\varepsilon_n})} \text{ and } \Delta^{\varepsilon_n} L_\xi \Delta^{-\varepsilon_n}(\eta) = L_{\Delta^{\varepsilon_n}\xi}(\eta).$$

Thus,

$$\mathfrak{T} \subset \mathscr{D}_{(\Delta^{\varepsilon_n} L_\xi \Delta^{-\varepsilon_n})} \text{ and } \Delta^{\varepsilon_n} L_\xi \Delta^{-\varepsilon_n} | \mathfrak{T} \subset L_{\Delta^{\varepsilon_n}\xi} \in \mathscr{B}(\mathscr{H}).$$

Finally, let us assume that $\mathfrak{T} \subset \mathscr{D}_{(\Delta^{\varepsilon_n} L_\xi \Delta^{-\varepsilon_n})}$ and also that the operator $\Delta^{\varepsilon_n} L_\xi \Delta^{-\varepsilon_n} | \mathfrak{T}$ is bounded, for any $n \in \mathbb{Z}$. We denote

$$x_0 = \overline{\Delta^{\varepsilon_0} L_\xi \Delta^{-\varepsilon_0} | \mathfrak{T}} \in \mathscr{B}(\mathscr{H}).$$

Then, for any $n \in \mathbb{Z}$, we have $\mathfrak{T} \subset \mathscr{D}_{(\Delta^{\varepsilon_n - \varepsilon_0} x_0 \Delta^{\varepsilon_0 - \varepsilon_n})}$ and the operator $\Delta^{\varepsilon_n - \varepsilon_0} x_0 \Delta^{\varepsilon_0 - \varepsilon_n} | \mathfrak{T} = \Delta^{\varepsilon_n} L_\xi \Delta^{-\varepsilon_n} | \mathfrak{T}$ is bounded. With the help of Theorem 10.20(1), from Proposition 9.24 we infer that, for any $\alpha \in \mathbb{C}$, we have $\mathscr{D}_{(\Delta^\alpha x_0 \Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{-\alpha})}$ and the operator $\Delta^\alpha x_0 \Delta^{-\alpha}$ is bounded. In particular, the operator $\Delta^{-\varepsilon_0} x_0 \Delta^{\varepsilon_0}$ is bounded.

On the other hand, if $\eta \in \mathfrak{T}$, then $\Delta^{\varepsilon_0}\eta \in \mathfrak{T} \subset \mathscr{D}_{(\Delta^{\varepsilon_0} L_\xi \Delta^{-\varepsilon_0})}$; hence

$$L_\xi(\eta) = \Delta^{-\varepsilon_0}(\Delta^{\varepsilon_0} L_\xi \Delta^{-\varepsilon_0})\Delta^{\varepsilon_0}(\eta) = \Delta^{-\varepsilon_0} x_0 \Delta^{\varepsilon_0}(\eta).$$

Thus, $L_\xi | \mathfrak{T} \subset \Delta^{-\varepsilon_0} x_0 \Delta^{\varepsilon_0}$ is bounded. It follows that the closed operator L_ξ is bounded, hence $\xi \in \mathfrak{A}''$.

If we now again apply Proposition 9.24, we infer that, for any $\alpha \in \mathbb{C}$, we have $\mathscr{D}_{(\Delta^\alpha L_\xi \Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{-\alpha})}$ and the operator $\Delta^\alpha L_\xi \Delta^{-\alpha}$ is bounded, whereas the mapping

$$\alpha \mapsto F(\alpha) = \overline{\Delta^\alpha L_\xi \Delta^{-\alpha}} \in \mathscr{B}(\mathscr{H})$$

is entire analytic.

Theorem 10.12 now shows that, for any $t \in \mathbb{R}$, we have

$$\Delta^{it}\xi \in \mathfrak{A}'' \text{ and } L_{\Delta^{it}\xi} = \Delta^{it} L_\xi \Delta^{-it} = F(it).$$

Thus, for any $\eta \in \mathfrak{A}'$, the entire analytic functions

$$\alpha \mapsto R_\eta(\Delta^\alpha \xi) = L_{\Delta^\alpha \xi}(\eta),$$

$$\alpha \mapsto F(\alpha)(\eta),$$

coincide on the imaginary axis; hence, everywhere.

Consequently, for any $\alpha \in \mathbb{C}$, we have

$$\|L_{\Delta^\alpha\xi}(\eta)\| = \|F(\alpha)(\eta)\| \leqslant \|F(\alpha)\| \, \|\eta\|, \quad \eta \in \mathfrak{A}';$$

therefore

$$\Delta^\alpha \xi \in \mathfrak{A}'' \text{ and } L_{\Delta^\alpha \xi} = F(\alpha) \supset \Delta^\alpha L_\xi \Delta^{-\alpha}.$$

For any $\alpha \in \mathbb{C}$ we have $\Delta^{\alpha-(1/2)}\xi \in \mathfrak{A}''$; hence

$$\Delta^{\alpha}\xi = \Delta^{1/2}(\Delta^{\alpha-(1/2)}\xi) \in \Delta^{1/2}\mathfrak{A}'' \subset \mathfrak{A}'.$$

Finally, for any $\alpha \in \mathbb{C}$ and any $\zeta \in \mathfrak{T}$ we have

$$R_{\xi}\Delta^{-\alpha}(\zeta) = L_{\Delta^{-\alpha}\zeta}(\xi) = \Delta^{-\alpha}L_{\zeta}\Delta^{\alpha}\xi = \Delta^{-\alpha}R_{\Delta^{\alpha}\xi}(\zeta) \in \mathscr{D}_{\Delta^{\alpha}}$$

and

$$\Delta^{\alpha}R_{\xi}\Delta^{-\alpha}(\zeta) = R_{\Delta^{\alpha}\xi}(\zeta).$$

If we now apply Proposition 9.24, it follows that, for any $\alpha \in \mathbb{C}$, we have

$$\mathscr{D}_{(\Delta^{\alpha}R_{\xi}\Delta^{-\alpha})} = \mathscr{D}_{(\Delta^{-\alpha})} \text{ and } \Delta^{\alpha}R_{\xi}\Delta^{-\alpha} \subset R_{\Delta^{\alpha}\xi}.$$

Q.E.D.

Let us now consider a vector $\xi \in \mathscr{H}$. For any $\varepsilon > 0$ we shall define

$$\xi_{\varepsilon} = \sqrt{\varepsilon/\pi}\int_{-\infty}^{+\infty} e^{-\varepsilon t^2}\Delta^{it}\xi \, dt \in \mathscr{H}.$$

Then the mapping

$$\alpha \mapsto \sqrt{\varepsilon/\pi}\int_{-\infty}^{+\infty} e^{-\varepsilon(t+i\alpha)^2}\Delta^{it}\,\xi \, dt$$

is an entire analytic continuation of the mapping

$$is \mapsto \Delta^{is}\xi_{\varepsilon};$$

hence, in accordance with Corollary 9.21, we have

$$\xi_{\varepsilon} \in \bigcap_{\alpha \in \mathbb{C}} \mathscr{D}_{\Delta^{\alpha}},$$

$$\Delta^{\alpha}\xi_{\varepsilon} = \sqrt{\varepsilon/\pi}\int_{-\infty}^{+\infty} e^{-\varepsilon(t+i\alpha)^2}\Delta^{it}\xi \, dt, \qquad \alpha \in \mathbb{C}.$$

Therefore, by taking into account Corollary 10.12 and the above corollary, it follows that if ξ belongs to $\mathfrak{A}''$, or to $\mathfrak{A}'$, then, for any $\varepsilon > 0$, the element ξ_{ε} belongs to $\mathfrak{T}$.

We observe that

$$\xi_{\varepsilon} \xrightarrow[\varepsilon \to +\infty]{} \xi, \qquad \xi \in \mathscr{H},$$

a fact which is easy to establish, by taking into account the continuity of the mapping $t \mapsto \Delta^{it}\xi$ at 0 and by using the Lebesgue dominated convergence theorem.

One can similarly prove that

$$L_{\xi_\varepsilon} = \sqrt{\varepsilon/\pi}\int_{-\infty}^{+\infty} e^{-\varepsilon t^2}\Delta^{it}L_\xi\Delta^{-it}\,dt \xrightarrow[\varepsilon\to+\infty]{so} L_\xi, \qquad \xi \in \mathfrak{A}'',$$

$$R_{\xi_\varepsilon} = \sqrt{\varepsilon/\pi}\int_{-\infty}^{+\infty} e^{-\varepsilon t^2}\Delta^{it}R_\xi\Delta^{-it}\,dt \xrightarrow[\varepsilon\to+\infty]{so} R_\xi, \qquad \xi \in \mathfrak{A}'.$$

From the preceding arguments we retain the following

Corollary 2. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra and $\mathfrak{T}$ the associated Tomita algebra. Then, for any $\xi \in \mathfrak{A}''$, there exists a sequence $\{\xi_n\} \subset \mathfrak{T}$, such that*

(1) $\xi_n \to \xi, \quad S\xi_n \to S\xi$;

(2) $L_{\xi_n} \xrightarrow{so} L_\xi, \quad (L_{\xi_n})^* \xrightarrow{so} (L_\xi)^*$;

(3) $\|L_{\xi_n}\| \leqslant \|L_\xi\|$, *for any* $n \in \mathbb{N}$.

Similarly, for any $\eta \in \mathfrak{A}'$ there exists a sequence $\{\eta_n\} \subset \mathfrak{T}$, such that

(1′) $\eta_n \to \eta, \quad S^*\eta_n \to S^*\eta$;

(2′) $R_{\eta_n} \xrightarrow{so} R_\eta, \quad (R_{\eta_n})^* \xrightarrow{so} (R_\eta)^*$;

(3′) $\|R_{\eta_n}\| \leqslant \|R_\eta\|$, *for any* $n \in \mathbb{N}$.

10.22. In this section we show that $\mathfrak{T}$ contains sufficiently many elements ξ, such that $\|\Delta^\alpha\xi\|$, $\|L_{\Delta^\alpha\xi}\|$ and $\|R_{\Delta^\alpha\xi}\|$ have exponential upper bounds depending on $\operatorname{Re}\alpha$.

Let $\mathfrak{A}\subset\mathscr{H}$ be a left Hilbert algebra and $\mathfrak{T}$ the associated Tomita algebra. We define the set $\mathfrak{S}$, consisting of all elements $\xi \in \mathfrak{T}$, such that there exist $\lambda_1, \lambda_2 \in (0, +\infty)$, $\lambda_1 \leqslant \lambda_2$, such that

$$\|\Delta^\alpha\xi\| \leqslant \lambda_2^{\operatorname{Re}\alpha},\ \|L_{\Delta^\alpha\xi}\| \leqslant \lambda_2^{\operatorname{Re}\alpha}\|L_\xi\|,\ \|R_{\Delta^\alpha\xi}\| \leqslant \lambda_2^{\operatorname{Re}\alpha}\|R_\xi\|, \text{ if } \operatorname{Re}\alpha \geqslant 0,$$

$$\|\Delta^\alpha\xi\| \leqslant \lambda_1^{\operatorname{Re}\alpha},\ \|L_{\Delta^\alpha\xi}\| \leqslant \lambda_1^{\operatorname{Re}\alpha}\|L_\xi\|,\ \|R_{\Delta^\alpha\xi}\| \leqslant \lambda_1^{\operatorname{Re}\alpha}\|R_\xi\|, \text{ if } \operatorname{Re}\alpha \leqslant 0.$$

We recall that if $f \in \mathscr{L}^1(\mathbb{R})$ and if its inverse Fourier transform $\hat{f}$, defined by

$$\hat{f}(s) = \int_{-\infty}^{+\infty} f(t)\,e^{its}\,dt, \qquad s \in \mathbb{R},$$

belongs to $\mathscr{L}^1(\mathbb{R})$, then the following inversion formula holds:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(s)\,e^{-its}\,ds, \qquad t \in \mathbb{R}.$$

If $f \in \mathscr{L}^1(\mathbb{R})$, $\hat{f} \in \mathscr{C}^2(\mathbb{R})$ and $\operatorname{supp}\hat{f} \subset [c_1, c_2]$, then f has an entire analytic continuation

$$\alpha \mapsto (2\pi)^{-1}\int_{c_1}^{c_2} \hat{f}(s)\,e^{-i\alpha s}\,ds,$$

which is also denoted by f, and the following equality holds

$$\alpha^2 f(\alpha) = -(2\pi)^{-1}\int_{c_1}^{c_2} \hat{f}''(s)\, e^{-i\alpha s}\, ds.$$

Consequently, if $\operatorname{Im}\alpha \geqslant 0$, we have

$$|f(\alpha)| \leqslant \frac{c_2 - c_1}{2\pi}(\sup_{s\in\mathbb{R}} |\hat{f}(s)| + \sup_{s\in\mathbb{R}} |\hat{f}''(s)|)\frac{1}{1+|\alpha|^2} e^{c_2 \operatorname{Im}\alpha},$$

whereas if $\operatorname{Im}\alpha \leqslant 0$, we have

$$|f(\alpha)| \leqslant \frac{c_2 - c_1}{2\pi}(\sup_{s\in\mathbb{R}} |\hat{f}(s)| + \sup_{s\in\mathbb{R}} |\hat{f}''(s)|)\frac{1}{1+|\alpha|^2} e^{c_1 \operatorname{Im}\alpha}.$$

Corollary. *Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra and $\mathfrak{T}$ the associated Tomita algebra. Then $\mathfrak{S}$ is a left Hilbert subalgebra of $\mathfrak{A}''$ and*

$$\mathfrak{S}' = \mathfrak{A}', \qquad \mathfrak{S}'' = \mathfrak{A}''.$$

Moreover,

(1) $\qquad \mathfrak{S} = \mathfrak{T} \cap \mathscr{S}_\Delta \cap \mathscr{S}_{\Delta^{-1}}$

$$= \left\{\int_{-\infty}^{+\infty} f(t)\,\Delta^{it}\xi\, dt;\ \xi \in \mathfrak{T},\ f \in \mathscr{L}^1(\mathbb{R}),\ \operatorname{supp}\hat{f} \textit{ is compact}\right\};$$

(2) $\qquad \Delta^\alpha \mathfrak{S} = \mathfrak{S}$ *and* $\overline{\Delta^\alpha | \mathfrak{S}} = \Delta^\alpha$, *for any* $\alpha \in \mathbb{C}$;

(3) $\qquad J\mathfrak{S} = \mathfrak{S}$.

Proof. Let $\xi \in \mathfrak{S}$. Then there exist $\lambda_1, \lambda_2 \in (0, +\infty)$, $\lambda_1 \leqslant \lambda_2$, such that

$$\|\Delta^\alpha \xi\| \leqslant \lambda_2^{\operatorname{Re}\alpha}\|\xi\|,\ \operatorname{Re}\alpha \geqslant 0;\ \|\Delta^\alpha\xi\| \leqslant \lambda_1^{\operatorname{Re}\alpha}\|\xi\|,\ \operatorname{Re}\alpha \leqslant 0.$$

Then, for any $\varepsilon > 0$, we have

$$(\lambda_2 + \varepsilon)^n \|\chi_{(\lambda_2+\varepsilon, +\infty)}(\Delta)\xi\| \leqslant \|\Delta^n \xi\| \leqslant \lambda_2^n \|\xi\|, \qquad n \geqslant 0,$$

whence

$$\chi_{(\lambda_2+\varepsilon, +\infty)}(\Delta)\,\xi = 0.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\xi \in \chi_{[0, \lambda_2]}(\Delta)\,\mathscr{H} \subset \mathscr{S}_\Delta.$$

One can similarly prove that

$$\xi \in \chi_{[\lambda_1, +\infty]}(\Delta)\,\mathscr{H} = \chi_{[0, \lambda_1^{-1}]}(\Delta^{-1})\,\mathscr{H} \subset \mathscr{S}_{\Delta^{-1}}.$$

Consequently, we have proved that

(*) $$\mathfrak{S} \subset \mathfrak{T} \cap \mathscr{S}_\Delta \cap \mathscr{S}_{\Delta^{-1}}.$$

Let $\xi \in \mathfrak{T} \cap \mathscr{S}_\Delta \cap \mathscr{S}_{\Delta^{-1}}$. There exist $\lambda_1, \lambda_2 \in (0, +\infty)$, $\lambda_1 \leqslant \lambda_2$, such that

$$\xi \in \chi_{[\lambda_1, \lambda_2]}(\Delta)\mathscr{H}.$$

We now consider a function $f \in \mathscr{L}^1(\mathbb{R})$, such that $\hat{f} \in \mathscr{C}^2(\mathbb{R})$, $\hat{f}(s) = 1$, for any $s \in [\ln \lambda_1, \ln \lambda_2]$, and such that $\operatorname{supp} \hat{f}$ be compact. Then

$$\int_{-\infty}^{+\infty} f(t)\, \lambda^{it} \chi_{[\lambda_1, \lambda_2]}(\lambda)\, \mathrm{d}t = \chi_{[\lambda_1, \lambda_2]}(\lambda), \quad \lambda \in (0, +\infty).$$

With the help of Theorem 9.11 (vi), it is easy to prove that

$$\int_{-\infty}^{+\infty} f(t)\, \Delta^{it} \chi_{[\lambda_1, \lambda_2]}(\Delta)\, \mathrm{d}t = \chi_{[\lambda_1, \lambda_2]}(\Delta);$$

hence

$$\int_{-\infty}^{+\infty} f(t)\, \Delta^{it}\xi\, \mathrm{d}t = \int_{-\infty}^{+\infty} f(t)\, \Delta^{it} \chi_{[\lambda_1, \lambda_2]}(\Delta)\, \xi\, \mathrm{d}t = \chi_{[\lambda_1, \lambda_2]}(\Delta)\, \xi = \xi.$$

Consequently,

$$(**) \quad \mathfrak{T} \cap \mathscr{S}_\Delta \cap \mathscr{S}_{\Delta^{-1}} \subset \left\{\int_{-\infty}^{+\infty} f(t)\, \Delta^{it}\xi\, \mathrm{d}t;\ \xi \in \mathfrak{T},\ f \in \mathscr{L}^1(\mathbb{R}),\ \operatorname{supp} \hat{f} \text{ is compact}\right\}.$$

Finally, let us consider an element

$$\zeta = \int_{-\infty}^{+\infty} f(t)\, \Delta^{it}\xi\, \mathrm{d}t,$$

where $\xi \in \mathfrak{T}$ and $f \in \mathscr{L}^1(\mathbb{R})$, $\operatorname{supp} \hat{f} \subset [c_1, c_2]$, $c_1, c_2 \in \mathbb{R}$, $c_1 \leqslant c_2$. Then

$$\alpha \mapsto \int_{-\infty}^{+\infty} f(t)\, \Delta^{it}(\Delta^{\alpha}\xi)\, \mathrm{d}t$$

is an entire analytic continuation of the mapping

$$is \mapsto \int_{-\infty}^{+\infty} f(t)\, \Delta^{it}(\Delta^{is}\xi)\, \mathrm{d}t = \Delta^{is}\zeta.$$

If we now apply Corollary 9.21, we get

$$\zeta \in \bigcap_{\alpha \in \mathbb{C}} \mathscr{D}_{\Delta^\alpha} \quad \text{and} \quad \Delta^{\alpha}\zeta = \int_{-\infty}^{+\infty} f(t)\, \Delta^{it}(\Delta^{\alpha}\xi)\, \mathrm{d}t, \quad \alpha \in \mathbb{C}.$$

In accordance with Corollary 10.12 and Corollary 1 from Section 10.21, we infer that

$$\zeta \in \mathfrak{T}.$$

Let $g \in \mathscr{L}^1(\mathbb{R})$, be such that $\hat{g} \in \mathscr{C}^2(\mathbb{R})$, $\hat{g}(s) = 1$ for any $s \in [c_1, c_2]$, and $\operatorname{supp} \hat{g} \subset [c_1 - \varepsilon, c_2 + \varepsilon]$, $\varepsilon > 0$. Then, if we denote by "$*$" the convolution product in $\mathscr{L}^1(\mathbb{R})$, we have

$$\int_{-\infty}^{+\infty} g(t)\, \Delta^{it}\zeta \,\mathrm{d}t = \int_{-\infty}^{+\infty} (g * f)(t)\, \Delta^{it}\xi \,\mathrm{d}t.$$

Since $(g * f)^\wedge = \hat{g}\hat{f} = \hat{f}$, we infer that $g * f = f$ and, therefore,

$$\int_{-\infty}^{+\infty} g(t)\, \Delta^{it}\zeta \,\mathrm{d}t = \int_{-\infty}^{+\infty} f(t)\, \Delta^{it}\xi \,\mathrm{d}t = \zeta.$$

With the help of Corollary 9.21, as above we can prove that

$$\Delta^{\alpha}\zeta = \int_{-\infty}^{+\infty} g(t)\, \Delta^{it}(\Delta^{\alpha}\zeta) \,\mathrm{d}t, \qquad \alpha \in \mathbb{C}.$$

We denote

$$c = \frac{c_2 - c_1 + 2\varepsilon}{2\pi} (\sup_{s \in \mathbb{R}} |\hat{g}(s)| + \sup_{s \in \mathbb{R}} |\hat{g}''(s)|).$$

With the help of the Cauchy integral formula and the remarks at the beginning of the present section, we infer that, for any $n \geqslant 0$,

$$\|\Delta^n\zeta\| = \left\| \int_{-\infty}^{+\infty} g(t)\, \Delta^{it+n}\zeta \,\mathrm{d}t \right\| = \left\| \int_{-\infty}^{+\infty} g(t + in)\, \Delta^{it}\zeta \,\mathrm{d}t \right\|$$

$$\leqslant c\|\zeta\| \int_{-\infty}^{+\infty} \frac{1}{1+t^2} \mathrm{d}t \cdot \mathrm{e}^{c_2 n} = \pi c \|\zeta\|\, \mathrm{e}^{c_2 n}.$$

If we now apply the "three lines" theorem (see N. Dunford and J. Schwartz [1], VI.10.3), we obtain, for any $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha \geqslant 0$, and any $n \geqslant \operatorname{Re}\alpha$,

$$\|\Delta^{\alpha}\zeta\| \leqslant \|\zeta\|^{1 - \frac{\operatorname{Re}\alpha}{n}} \cdot \|\Delta^n\zeta\|^{\frac{\operatorname{Re}\alpha}{n}} \leqslant \|\zeta\|^{1 - \frac{\operatorname{Re}\alpha}{n}} \cdot (\pi c \|\zeta\|)^{\frac{\operatorname{Re}\alpha}{n}} \cdot \mathrm{e}^{c_2 \operatorname{Re}\alpha}.$$

Tending to the limit for $n \to +\infty$, we get

$$\|\Delta^{\alpha}\zeta\| \leqslant \|\zeta\|\, \mathrm{e}^{c_2 \operatorname{Re}\alpha}, \qquad \operatorname{Re}\alpha \geqslant 0.$$

Similarly, one can prove that

$$\|\Delta^{\alpha}\zeta\| \leqslant \|\zeta\|\, \mathrm{e}^{c_1 \operatorname{Re}\alpha}, \qquad \operatorname{Re}\alpha \leqslant 0.$$

On the other hand, in accordance with Corollary 10.12,

$$L_{\zeta} = \int_{-\infty}^{+\infty} g(t)\, L_{\Delta^{it}\zeta} \,\mathrm{d}t.$$

With the help of Proposition 9.24, it is easy to verify that

$$L_{\Delta^{\alpha}\zeta} = \int_{-\infty}^{+\infty} g(t)\, L_{\Delta^{it}(\Delta^{\alpha}\zeta)} \,\mathrm{d}t, \qquad \alpha \in \mathbb{C}.$$

If we argue as above, we first obtain

$$\|L_{\Delta^n\zeta}\| \leqslant \pi c \|L_\zeta\| \, e^{c_2 n}, \quad n \geqslant 0,$$

and then,

$$\|L_{\Delta^\alpha\zeta}\| \leqslant \|L_\zeta\| \, e^{c_2 \operatorname{Re} \alpha}, \quad \operatorname{Re} \alpha \geqslant 0;$$

similarly, one obtains

$$\|L_{\Delta^\alpha\zeta}\| \leqslant \|L_\zeta\| \, e^{c_1 \operatorname{Re} \alpha}, \quad \operatorname{Re} \alpha \leqslant 0.$$

If we now repeat the above arguments for R, instead of L, we obtain

$$\|R_{\Delta^\alpha\zeta}\| \leqslant \|R_\zeta\| \, e^{c_2 \operatorname{Re} \alpha}, \quad \operatorname{Re} \alpha \geqslant 0,$$
$$\|R_{\Delta^\alpha\zeta}\| \leqslant \|R_\zeta\| \, e^{c_1 \operatorname{Re} \alpha}, \quad \operatorname{Re} \alpha \leqslant 0.$$

Thus, $\zeta \in \mathfrak{S}$.

Consequently, we have just proved the inclusion

$$(***) \qquad \left\{\int_{-\infty}^{+\infty} f(t)\, \Delta^{it}\xi \, dt;\ \xi \in \mathfrak{T},\ f \in \mathscr{L}^1(\mathbb{R}),\ \operatorname{supp} \hat{f} \text{ is compact}\right\} \subset \mathfrak{S}.$$

From relations (*), (**), (***) we infer that assertion (1) is true.

If $\xi \in \mathfrak{T}$ and $f \in \mathscr{L}^1(\mathbb{R})$ is such that $\operatorname{supp} \hat{f}$ is compact, then, by taking into account Theorem 10.20, we get

$$\Delta^\alpha \int_{-\infty}^{+\infty} f(t)\, \Delta^{it}\xi \, dt = \int_{-\infty}^{+\infty} f(t)\, \Delta^{it}(\Delta^\alpha \xi) \, dt \in \mathfrak{S}, \quad \alpha \in \mathbb{C},$$

$$J \int_{-\infty}^{+\infty} f(t)\, \Delta^{it}\xi \, dt = \int_{-\infty}^{+\infty} \overline{f}(t)\, \Delta^{it}(J\xi) \, dt \in \mathfrak{S}.$$

Thus, we have proved that

$$\Delta^\alpha \mathfrak{S} = \mathfrak{S}, \quad \alpha \in \mathbb{C}; \quad J\mathfrak{S} = \mathfrak{S}.$$

We consider a function $f_0 \in \mathscr{L}^1(\mathbb{R})$, such that $\hat{f}_0(s) = 1$ for any $s \in [-1, 1]$, and such that $\operatorname{supp} \hat{f}_0$ be compact. We write

$$f_n(t) = n f_0(nt), \quad n \geqslant 1.$$

Then $\hat{f}_n(s) = \hat{f}_0(s/n)$, $s \in \mathbb{R}$; hence

$$\left(\int_{-\infty}^{+\infty} f_n(t)\, \lambda^{it} \, dt\right) \chi_{[e^{-n}, e^n]}(\lambda) = \chi_{[e^{-n}, e^n]}(\lambda).$$

With the help of Theorem 9.11 (vi), it is easy to prove that

$$\left(\int_{-\infty}^{+\infty} f_n(t)\, \Delta^{it} \, dt\right) \chi_{[e^{-n}, e^n]}(\Delta) = \chi_{[e^{-n}, e^n]}(\Delta).$$

Since, for any n, we have

$$\left\| \int_{-\infty}^{+\infty} f_n(t)\, \Delta^{it}\, dt \right\| \leqslant \int_{-\infty}^{+\infty} |f_n(t)|\, dt = \int_{-\infty}^{+\infty} |f_0(t)|\, dt,$$

from the preceding equality we infer that

$$\int_{-\infty}^{+\infty} f_n(t)\, \Delta^{it}\, dt \xrightarrow{so} 1.$$

Thus, for any $\xi \in \mathfrak{T}$ and any $\alpha \in \mathbf{C}$, we get

$$\Delta^{\alpha} \int_{-\infty}^{+\infty} f_n(t)\, \Delta^{it}\xi\, dt = \int_{-\infty}^{+\infty} f_n(t)\, \Delta^{it}(\Delta^{\alpha}\xi)\, dt \to \Delta^{\alpha}\xi.$$

With the help of assertion (1) and of Theorem 10.20(1), it follows that

$$\overline{\Delta^{\alpha}|\mathfrak{S}} = \overline{\Delta^{\alpha}|\mathfrak{T}} = \Delta^{\alpha}, \qquad \alpha \in \mathbf{C}.$$

Thus, assertions (2) and (3) are also true.

Finally, if we now use the definition of $\mathfrak{S}$, it is easy to see that

$$\xi_1, \xi_2 \in \mathfrak{S} \Rightarrow L_{\xi_1}(\xi_2) \in \mathfrak{S}.$$

With the help of assertions (2) and (3) we get

$$\xi \in \mathfrak{S} \Rightarrow S\xi = J\Delta^{1/2}\xi \in \mathfrak{S}.$$

From assertion (2) we infer that $\overline{\Delta^{1/2}|\mathfrak{S}} = \Delta^{1/2}$; hence

$$\overline{S|\mathfrak{S}} = S.$$

If we apply Lemmas 2 and 3 from Section 10.5, we infer that $\mathfrak{S}$ is a left Hilbert subalgebra of $\mathfrak{A}''$, and

$$\mathfrak{S}' = \mathfrak{A}', \qquad \mathfrak{S}'' = \mathfrak{A}''.$$

Q.E.D.

10.23. Let $\mathfrak{A} \subset \mathscr{H}$ be a left Hilbert algebra and $\mathfrak{T}$ the Tomita algebra associated to $\mathfrak{A}$ (see Section 10.20). In Section 10.8 we introduced two cones, polar to one another:

$$\mathfrak{P}_S \subset \mathscr{D}_S = \mathscr{D}_{(\Delta^{1/2})}, \qquad \mathfrak{P}_{S^*} \subset \mathscr{D}_{S^*} = \mathscr{D}_{(\Delta^{-1/2})}.$$

With the help of Proposition 10.8 and of Theorem 10.12, it is easy to verify that

$$J\mathfrak{P}_S = \mathfrak{P}_{S^*}.$$

Since $S\xi = \xi$ for any $\xi \in \mathfrak{P}_S$, we infer that

$$\Delta^{1/2}\mathfrak{P}_S = \mathfrak{P}_{S^*}.$$

Consequently,

$$\Delta^{1/4}\mathfrak{P}_S = \Delta^{-1/4}\mathfrak{P}_{S^*}.$$

In the present section we shall study the set

$$\mathfrak{P} = \overline{\Delta^{1/4}\mathfrak{P}_S} = \overline{\Delta^{-1/4}\mathfrak{P}_{S^*}}.$$

Since $\mathfrak{P}_S$ and $\mathfrak{P}_{S^*}$ are convex cones, it follows that $\mathfrak{P}$ is a closed convex cone.

In order to make the notations as simple and as expressive as possible, we shall use the following abbreviations, already introduced above (see Section 10.4):

$$\xi\eta = L_\xi(\eta), \text{ for } \xi, \eta \in \mathfrak{A}''; \ \xi\eta = R_\eta(\xi), \text{ for } \xi, \eta \in \mathfrak{A}'.$$

We recall that, if $\xi, \eta \in \mathfrak{A}' \cap \mathfrak{A}''$, then $L_\xi(\eta) = R_\eta(\xi)$. With these notations, we have the equalities

$$\mathfrak{P}_S = \{\overline{\xi(S\xi);\ \xi \in \mathfrak{A}''}\}, \qquad \mathfrak{P}_{S^*} = \{\overline{\eta(S^*\eta);\ \eta \in \mathfrak{A}'}\}.$$

Let $\xi \in \mathfrak{A}''$ and let $\{\xi_n\} \subset \mathfrak{T}$ be a sequence having the properties from Corollary 2, Section 10.21. Then it is easy to prove that

$$\xi_n(S\xi_n) = L_{\xi_n}(S\xi_n) \to L_\xi(S\xi) = \xi(S\xi).$$

Consequently,

(1) $$\mathfrak{P}_S = \overline{\mathfrak{P}_S^0}, \text{ where } \mathfrak{P}_S^0 = \{\xi(S\xi);\ \xi \in \mathfrak{T}\}.$$

Similarly,

(1') $$\mathfrak{P}_{S^*} = \overline{\mathfrak{P}_{S^*}^0}, \text{ where } \mathfrak{P}_{S^*}^0 = \{\eta(S^*\eta);\ \eta \in \mathfrak{T}\}.$$

By taking into account Theorem 10.20, it follows that for any $\xi \in \mathfrak{T}$ we have

$$\Delta^{1/4}(\xi(S\xi)) = \Delta^{1/4}(\xi)\, \Delta^{1/4}(S\xi) = (\Delta^{1/4}(\xi))\, (J(\Delta^{1/4}(\xi))).$$

Since $\Delta^{1/4}\mathfrak{T} = \mathfrak{T}$, we hence infer that, if we denote

$$\mathfrak{P}^0 = \{\zeta(J\zeta);\ \zeta \in \mathfrak{T}\},$$

then we have

$$\Delta^{1/4}\mathfrak{P}_S^0 = \mathfrak{P}^0.$$

Similarly,

$$\Delta^{-1/4}\mathfrak{P}_{S^*}^0 = \mathfrak{P}^0.$$

If $\xi \in \mathfrak{P}_S$, then there exists a sequence $\{\xi_n\} \subset \mathfrak{P}_S^0$, such that $\xi_n \to \xi$. Since $S\xi_n = \xi_n$, $S\xi = \xi$ and $\Delta^{1/2} = JS$, it follows that $\Delta^{1/2}\xi_n \to \Delta^{1/2}\xi$. Therefore,

$$\|\Delta^{1/4}\xi - \Delta^{1/4}\xi_n\|^2 = (\Delta^{1/2}(\xi - \xi_n)|\xi - \xi_n) \to 0,$$

whence $\Delta^{1/4}\xi \in \overline{\Delta^{1/4}\mathfrak{P}_S^0} = \overline{\mathfrak{P}^0}$. Consequently, $\mathfrak{P} \subset \overline{\mathfrak{P}^0}$. On the other hand, it is obvious that $\mathfrak{P}^0 = \Delta^{1/4}\mathfrak{P}_S^0 \subset \Delta^{1/4}\mathfrak{P}_S \subset \mathfrak{P}$. Therefore,

(2) $$\mathfrak{P} = \overline{\mathfrak{P}^0} = \overline{\{\zeta(J\zeta);\ \zeta \in \mathfrak{T}\}}.$$

With the help of relation (2) and of Corollary 2 from Section 10.21, it is easy to verify that

$$\mathfrak{P} = \overline{\{\xi(J\xi);\ \xi \in \mathfrak{A}''\}} = \overline{\{(J\eta)\eta;\ \eta \in \mathfrak{A}'\}}. \tag{3}$$

From relation (2) it obviously follows that

$$\zeta \in \mathfrak{P} \Rightarrow J\zeta = \zeta. \tag{4}$$

For any $t \in \mathbb{R}$ and any $\zeta \in \mathfrak{T}$, in accordance with Theorem 10.20, we have

$$\Delta^{it}(\zeta(J\zeta)) = \Delta^{it}(\zeta)\,\Delta^{it}(J\zeta) = (\Delta^{it}(\zeta))\,(J(\Delta^{it}(\zeta))).$$

Thus, $\Delta^{it}\mathfrak{P}^0 \subset \mathfrak{P}^0$. By taking into account relation (2), we get

$$\Delta^{it}\mathfrak{P} = \mathfrak{P}, \qquad t \in \mathbb{R}. \tag{5}$$

Since $\mathfrak{P}$ is a closed convex cone, from (5) we infer that

$$\zeta \in \mathfrak{P},\ \ f \in \mathscr{L}^1(\mathbb{R}),\ \ f \geqslant 0 \Rightarrow \int_{-\infty}^{+\infty} f(t)\,\Delta^{it}\zeta\,\mathrm{d}t \in \mathfrak{P}. \tag{6}$$

If $\xi \in \mathfrak{A}''$ and $\zeta \in \mathfrak{T}$, then, by taking into account Corollary 1 from Section 10.13, we get

$$[L_\xi(JL_\xi J)]\,(\zeta(J\zeta)) = L_\xi JL_\xi(JL_\zeta J)\,(\zeta) = L_\xi J(JL_\zeta J)\,L_\xi(\zeta) = L_\xi L_\zeta JL_\xi(\zeta) = (\xi\zeta)\,(J(\xi\zeta));$$

hence $[L_\xi(JL_\xi J)]\,\mathfrak{P}^0 \subset \mathfrak{P}$, and, therefore, $[L_\xi(JL_\xi J)]\,\mathfrak{P} \subset \mathfrak{P}$. With the help of Kaplansky's density theorem (3.10), we now infer that

$$x \in \mathfrak{L}(\mathfrak{A}) \Rightarrow [x(JxJ)]\,\mathfrak{P} \subset \mathfrak{P}. \tag{7}$$

For any $\xi \in \mathfrak{P}_S$ and any $\eta \in \mathfrak{P}_{S^\bullet}$ we have

$$(\Delta^{1/4}\xi | \Delta^{-1/4}\eta) = (\xi|\eta) \geqslant 0,$$

because the cones $\mathfrak{P}_S$ and $\mathfrak{P}_{S^\bullet}$ are polar to one another (10.8). From this result and from the definition of $\mathfrak{P}$, it follows that

$$\zeta, \theta \in \mathfrak{P} \Rightarrow (\zeta|\theta) \geqslant 0.$$

On the other hand, let $\zeta \in \mathscr{H}$ be such that $(\zeta|\theta) \geqslant 0$, for any $\theta \in \mathfrak{P}$. For any $n \in \mathbb{N}$, we denote

$$\zeta_n = \sqrt{n/\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-nt^2}\,\Delta^{it}\zeta\,\mathrm{d}t, \qquad \theta_n = \sqrt{n/\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-nt^2}\,\Delta^{it}\theta\,\mathrm{d}t.$$

If $\theta \in \mathfrak{P}$, then, in accordance with relation (6), we have $\theta_n \in \mathfrak{P}$ and, therefore,

$$(\zeta_n|\theta) = (\zeta|\theta_n) \geqslant 0, \qquad n \in \mathbb{N}.$$

In particular, if $\xi \in \mathfrak{P}_S$, then $\Delta^{1/4}\xi \in \mathfrak{P}$ and

$$(\Delta^{1/4}\zeta_n|\xi) = (\zeta_n|\Delta^{1/4}\xi) \geqslant 0, \quad n \in \mathbb{N};$$

hence, in accordance with Proposition 10.8, we infer that $\Delta^{1/4}\zeta_n \in \mathfrak{P}_{S^*}$. Thus,

$$\zeta_n \in \Delta^{-1/4}\mathfrak{P}_{S^*} \subset \mathfrak{P}.$$

Since $\zeta_n \to \zeta$ and since $\mathfrak{P}$ is closed, it follows that $\zeta \in \mathfrak{P}$.

Consequently, the cone $\mathfrak{P}$ is *selfpolar*, i.e., for any $\zeta \in \mathscr{H}$ we have the equivalence

(8) $$\zeta \in \mathfrak{P} \Leftrightarrow (\zeta|\theta) \geqslant 0, \quad \text{for any } \theta \in \mathfrak{P}.$$

If $\zeta \in \mathfrak{P} \cap (-\mathfrak{P})$, then, from relation (8), we infer that $(\zeta|-\zeta) \geqslant 0$, whence $\zeta = 0$. Consequently,

(*) $$\mathfrak{P} \cap (-\mathfrak{P}) = \{0\}.$$

Since $\mathfrak{P}$ is a convex cone, having property (*), we infer that $\mathfrak{P}$ determines an order relation "$\leqslant$" in the set $\{\zeta \in \mathscr{H};\ J\zeta = \zeta\}$:

$$\zeta \leqslant \theta \Leftrightarrow \theta - \zeta \in \mathfrak{P}.$$

We shall now prove that *for any $\zeta \in \mathscr{H}$, such that $J\zeta = \zeta$, there exist $\zeta^+, \zeta^- \in \mathfrak{P}$, such that*

(**) $$\zeta = \zeta^+ - \zeta^-, \quad \zeta^+ \perp \zeta^-,$$

and these elements are uniquely determined by these conditions.

Indeed, let $\zeta \in \mathscr{H}$ be such that $J\zeta = \zeta$. Since $\mathfrak{P}$ is a closed convex subset of the Hilbert space $\mathscr{H}$, there exists a unique element $\zeta^+ \in \mathfrak{P}$, such that

$$\|\zeta^+ - \zeta\| = \inf\{\|\theta - \zeta\|;\ \theta \in \mathfrak{P}\}.$$

We denote $\zeta^- = \zeta^+ - \zeta$. For any $\theta \in \mathfrak{P}$ and any $t \geqslant 0$ we have

$$0 \leqslant \|(\zeta^+ + t\theta) - \zeta\|^2 - \|\zeta^+ - \zeta\|^2 = t^2\|\theta\|^2 + 2t\,\mathrm{Re}\,(\zeta^-|\theta),$$

whence we infer that $\mathrm{Re}\,(\zeta^-|\theta) \geqslant 0$. Since $J\zeta^- = \zeta^-$ and $J\theta = \theta$, we have $(\zeta^-|\theta) = \mathrm{Re}\,(\zeta^-|\theta) \geqslant 0$. From relation (8) we infer that $\zeta^- \in \mathfrak{P}$. On the other hand, for any $t \in (0, 1)$ we have

$$0 \leqslant \|(1-t)\,\zeta^+ - \zeta\|^2 - \|\zeta^+ - \zeta\|^2 = t^2\|\zeta^+\|^2 - 2t\,\mathrm{Re}\,(\zeta^+|\zeta^-),$$

whence we infer that $\mathrm{Re}\,(\zeta^+|\zeta^-) \leqslant 0$. Since $\zeta^+, \zeta^- \in \mathfrak{P}$, from relation (8) we infer that $(\zeta^+|\zeta^-) \geqslant 0$. Consequently, we have $\zeta^+ \perp \zeta^-$.

Let us now assume that $\zeta = \xi - \eta$, $\xi, \eta \in \mathfrak{P}$, $\xi \perp \eta$. Then $\zeta^+ - \xi = \zeta^- - \eta$ and, therefore,

$$\|\zeta^+ - \xi\|^2 = (\zeta^+ - \xi|\zeta^- - \eta) = -(\zeta^+|\eta) - (\xi|\zeta^-) \leqslant 0,$$

whence $\xi = \zeta^+$, $\eta = \zeta^-$.

In particular, from the proved assertion, we infer that $\mathcal{H}$ coincides with the linear hull of $\mathfrak{P}$:

$$\mathcal{H} = (\mathfrak{P} - \mathfrak{P}) + i(\mathfrak{P} - \mathfrak{P}).$$

We observe that any selfpolar convex cone $\mathfrak{P}$ in a Hilbert space $\mathcal{H}$ determines a unique conjugation J in $\mathcal{H}$, such that $J\zeta = \zeta$, for any $\zeta \in \mathfrak{P}$. Indeed, if $\zeta \in \mathcal{H}$ and $\zeta \perp \mathfrak{P}$, then $\zeta \in \mathfrak{P}$, because $\mathfrak{P}$ is selfpolar and, therefore, $(\zeta|\zeta) = 0$, whence $\zeta = 0$. Thus $\mathfrak{P}$ is a total subset of $\mathcal{H}$; hence $(\mathfrak{P} - \mathfrak{P}) + i(\mathfrak{P} - \mathfrak{P})$ is a dense subset of $\mathcal{H}$. Consequently, the conjugation J is uniquely determined by the formula

$$J(\xi + i\eta) = \xi - i\eta, \quad \xi, \eta \in \mathfrak{P} - \mathfrak{P}.$$

The decomposition (**) is valid in this more general situation, with the same proof. In particular, in this general situation $\mathcal{H}$ coincides with the linear hull of $\mathfrak{P}$, too.

We shall say that a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is *hyperstandard* if there exists a conjugation $J: \mathcal{H} \to \mathcal{H}$ and a selfpolar convex cone $\mathfrak{P} \subset \mathcal{H}$, such that

1) the mapping $x \mapsto Jx^*J$ is a $*$-antiisomorphism of $\mathcal{M}$ onto $\mathcal{M}'$, which acts identically on the center;

2) $\zeta \in \mathfrak{P} \Rightarrow J\zeta = \zeta$;

3) $x \in \mathcal{M} \Rightarrow [x(JxJ)]\mathfrak{P} \subset \mathfrak{P}$.

From the above results we infer that, *for any left Hilbert algebra $\mathfrak{A} \subset \mathcal{H}$, the von Neumann algebra $\mathfrak{L}(\mathfrak{A}) \subset \mathcal{B}(\mathcal{H})$ is a hyperstandard von Neumann algebra.*

10.24. We now consider a hyperstandard von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, with the conjugation J and the selfpolar cone $\mathfrak{P}$. We denote by $\mathcal{Z}$ the center of $\mathcal{M}$.

Let e be a projection in $\mathcal{M}$. Then JeJ is a projection in $\mathcal{M}'$ and $z(JeJ) = z(e)$. From Corollary 3.9, we easily infer that

$$e \neq 0 \Leftrightarrow e(JeJ) \neq 0.$$

We denote by q the projection $e(JeJ) = (JeJ)e \in \mathcal{B}(\mathcal{H})$. By taking into account Sections 3.13, 3.14 and 3.15, we infer that $\mathcal{M}_q \subset \mathcal{B}(q\mathcal{H})$ is a von Neumann algebra, whose commutant is $(\mathcal{M}')_q$ and whose center is $\mathcal{Z}_q$; also, the mapping

$$x_e \mapsto x_q$$

is a $*$-isomorphism of the reduced von Neumann algebra $\mathcal{M}_e$ onto the von Neumann algebra $\mathcal{M}_q$. Since $Jq = qJ$, and $q\mathfrak{P} = [e(JeJ)]\mathfrak{P} \subset \mathfrak{P}$, it is easy to prove that $\mathcal{M}_q \subset \mathcal{B}(q\mathcal{H})$ is a hyperstandard von Neumann algebra, whose conjugation is qJq and whose selfpolar cone is $q\mathfrak{P}$.

This remark will enable us to reduce some of the problems to the case of the hyperstandard von Neumann algebras of countable type.

Lemma 1. *For any projection $e \in \mathcal{M}$ there exists a family $\{\xi_n\}_{n \in I} \subset \mathfrak{P}$, such that the projections p_{ξ_n} be non-zero, mutually orthogonal and such that*

$$e = \sum_{n \in I} p_{\xi_n}.$$

If e is of countable type, then there exists a $\xi_0 \in \mathfrak{P}$, such that $e = p_{\xi_0}$.

In particular, if $\mathcal{M}$ is of countable type, then $\mathcal{M}$ has a separating cyclic vector $\xi_0 \in \mathfrak{P}$.

Proof. If $e \neq 0$, then $e(JeJ) \neq 0$ and, since $\mathcal{H}$ is the linear hull of $\mathfrak{P}$, there exists a non-zero vector $\xi \in [e(JeJ)]\mathfrak{P} \subset \mathfrak{P}$. Obviously, $p_\xi \leqslant e$. Thus, the first assertion follows with a familiar argument based on the Zorn lemma.

If e is of countable type, then the set I is at most countable and we can assume that $\|\xi_n\| = 1$. If we define $\xi_0 = \sum_n \frac{1}{2^n}\xi_n$, it follows that $\xi_0 \in \mathfrak{P}$ and $e = p_{\xi_0}$.

The last assertion of the lemma follows from the remark that $J\xi_0 = \xi_0$, for any $\xi_0 \in \mathfrak{P}$, and from the fact that $Jp_{\xi_0}J = p'_{J\xi_0}$(see E.6.9).

Q.E.D.

If the von Neumann algebra $\mathcal{M}$ is of countable type and if $\xi_0 \in \mathfrak{P}$ is a separating cyclic vector, then we can consider the left Hilbert algebra $\mathfrak{A} = \mathcal{M}\xi_0 \subset \mathcal{H}$ and we have $\mathcal{M} = \mathfrak{L}(\mathfrak{A})$ (see Section 10.6). On the one hand, by hypothesis, $\mathcal{M}$ is a hyperstandard von Neumann algebra. On the other hand, as we have seen in Section 10.23, $\mathfrak{L}(\mathfrak{A})$ is endowed with a natural structure of a hyperstandard von Neumann algebra. We shall denote by $S^{\mathfrak{A}}$, $J^{\mathfrak{A}}$ the operators which are associated to the left Hilbert algebra $\mathfrak{A} = \mathcal{M}\xi_0$, and by $\mathfrak{P}^{\mathfrak{A}}$ the selfpolar convex cone, which is associated to the left Hilbert algebra $\mathfrak{A} = \mathcal{M}\xi_0$; it is easy to verify that

$$\mathfrak{P}^{\mathfrak{A}} = \overline{\{[x(J^{\mathfrak{A}}xJ^{\mathfrak{A}})]\,\xi_0;\quad x \in \mathcal{M}\}}.$$

Under these assumptions, we have the following

Lemma 2. *$J^{\mathfrak{A}} = J$ and $\mathfrak{P}^{\mathfrak{A}} = \mathfrak{P}$.*

Proof. For any $x \in \mathcal{M}$ we have $JxJ \in \mathcal{M}'$. Consequently,

$$[J(S^{\mathfrak{A}})^*J]\,x\xi_0 = J(S^{\mathfrak{A}})^*(JxJ)\,\xi_0 = J(JxJ)^*\xi_0 = x^*\xi_0.$$

From these equalities it is easy to infer that $S^{\mathfrak{A}} \subset J(S^{\mathfrak{A}})^*J$, whereas a similar argument shows that $(S^{\mathfrak{A}})^* \subset JS^{\mathfrak{A}}J$. Thus, $JS^{\mathfrak{A}} = (S^{\mathfrak{A}})^*J$. By taking account Proposition 9.2, we infer that

$$(JS^{\mathfrak{A}})^* = (S^{\mathfrak{A}})^*J = JS^{\mathfrak{A}},$$

hence the linear operator $JS^{\mathfrak{A}}$ is self-adjoint. For any $x \in \mathcal{M}$ we have

$$((JS^{\mathfrak{A}})(x\xi_0)\,|\,x\xi_0) = (Jx^*\xi_0|x\xi_0) = (xJx\xi_0|\xi_0) = ([x(JxJ)]\,\xi_0|\xi_0) \geqslant 0,$$

because ξ_0 and $[x(JxJ)]\,\xi_0$ belong to $\mathfrak{P}$. Since $JS^{\mathfrak{A}} = \overline{JS^{\mathfrak{A}}|\mathcal{M}\xi_0}$, it follows that the operator $JS^{\mathfrak{A}}$ is positive. Since $S^{\mathfrak{A}} = J(JS^{\mathfrak{A}})$, by taking into account Section 10.1 and the uniqueness of the polar decomposition, we get

$$J = J^{\mathfrak{A}}.$$

Then

$$\mathfrak{P}^{\mathfrak{A}} = \overline{\{[x(J^{\mathfrak{A}}xJ^{\mathfrak{A}})]\,\xi_0;\ x \in \mathcal{M}\}} = \overline{\{[x(JxJ)]\xi_0;\ x \in \mathcal{M}\}} \subset \mathfrak{P},$$

and, since $\mathfrak{P}^{\mathfrak{A}}$ and $\mathfrak{P}$ are, both, selfpolar, we infer that

$$\mathfrak{P}^{\mathfrak{A}} = \mathfrak{P}.$$

Q.E.D.

Thus, if the hyperstandard von Neumann algebra $\mathscr{M}$ is of countable type, then J and $\mathfrak{P}$ are naturally derived from a left Hilbert algebra $\mathfrak{A} = \mathscr{M}\xi_0$, where $\xi_0 \in \mathfrak{P}$ is a separating cyclic vector for $\mathscr{M}$. Consequently, in this case, we can avail ourselves of the powerful instrument of the left Hilbert algebras.

We shall continue to assume that $\mathscr{M}$ is of countable type and we choose a separating cyclic vector $\xi_0 \in \mathfrak{P}$. We shall denote by Δ the modular operator corresponding to $\mathfrak{A} = \mathscr{M}\xi_0$.

We obviously have $\Delta\xi_0 = S^*S\xi_0 = \xi_0$, whence $(1 + \Delta)^{-1}\xi_0 = \dfrac{1}{2}\xi_0$. It is easy to verify, first for polynomials and then, by tending to the limit, that for any function $F \in \mathscr{B}([0, 1])$ we have

$$F((1 + \Delta)^{-1})\,\xi_0 = F(1/2)\,\xi_0.$$

By taking into account Section 9.10, we infer that, for any bounded $f \in \mathscr{B}([0, +\infty))$, we have

$$f(\Delta)\,\xi_0 = F_f((1 + \Delta)^{-1})\,\xi_0 = F_f(1/2)\xi_0 = f(1)\,\xi_0.$$

In particular,

$$\Delta^{it}\xi_0 = \xi_0, \qquad t \in \mathbb{R},$$

and, by analytic continuation, we obtain

$$\Delta^{\alpha}\xi_0 = \xi_0, \qquad \alpha \in \mathbf{C}.$$

Lemma 3. *The mapping $\Phi: a \mapsto \Delta^{1/4}a\xi_0$ is an order isomorphism of the set of all self-adjoint operators $a \in \mathscr{M}$ onto the set of all vectors $\xi \in \mathscr{H}$, such that $J\xi = \xi$ and having the property that there exists a $\lambda > 0$, for which $-\lambda\xi_0 \leqslant \xi \leqslant \lambda\xi_0$.*

Proof. Since $\Delta^{1/4}$ is injective, and since ξ_0 is separating for $\mathscr{M}$, the mapping Φ is injective. In accordance with Section 10.9, and with the definition of $\mathfrak{P}$ (Section 10.23) we infer that for $a \in \mathscr{M}$, $a = a^*$, we have

$$a \geqslant 0 \Leftrightarrow a\xi_0 \in \mathfrak{P}_S \Leftrightarrow \Delta^{1/4}\,a\xi_0 \in \mathfrak{P}.$$

Thus, the mapping Φ is an order isomorphism and we have still to show that Φ is surjective.

For $a \in \mathscr{M}$, $a = a^*$, we have $(1 + \Delta^{1/2})(a\xi_0) = a\xi_0 + Ja\xi_0$, whence

$$\Delta^{1/4}(a\xi_0) = (\Delta^{1/4} + \Delta^{-1/4})^{-1}\,(a\xi_0 + Ja\xi_0).$$

Since the operator $(\Delta^{1/4} + \Delta^{-1/4})^{-1}$ is bounded, it follows that

$$a_\iota \xrightarrow{w} a \Rightarrow \Phi(a_\iota) \to \Phi(a) \text{ weakly.}$$

Since the set $\{a \in \mathscr{M};\ 0 \leqslant a \leqslant 1\}$ is w-compact, it follows that the set $\Phi(\{a \in \mathscr{M};$ $0 \leqslant a \leqslant 1\})$ is weakly compact in $\mathscr{H}$.

Let now $\xi \in \mathscr{H}$, $J\xi = \xi$, $0 \leqslant \xi \leqslant \xi_0$. With the help of relation (6) from Section 10.23, we obtain

$$0 \leqslant \xi_n = \sqrt{n/\pi} \int_{-\infty}^{+\infty} e^{-nt^2} \Delta^{it} \xi \, dt \leqslant \sqrt{n/\pi} \int_{-\infty}^{+\infty} e^{-nt^2} \Delta^{it} \xi_0 \, dt = \xi_0.$$

For any $\eta \in \mathfrak{P}_S$, we have $\Delta^{-1/4}\eta \in \mathfrak{P}$; hence

$$(\Delta^{-1/4}\xi_n \mid \eta) = (\xi_n \mid \Delta^{-1/4}\eta) \geqslant 0.$$

Thus, $\Delta^{-1/4}\xi_n \in \mathfrak{P}_S$ (see Section 10.8). Similarly, $\xi_0 - \Delta^{-1/4}\xi_n = \Delta^{-1/4}(\xi_0 - \xi_n) \in \mathfrak{P}_S$. From Proposition 10.8, we infer that the operators $L^0_{\Delta^{-1/4}\xi_n}$ and $1 - L^0_{\Delta^{-1/4}\xi_n} = L^0_{\xi_0 - \Delta^{-1/4}\xi_n}$ are positive. Consequently, the operator $a_n = L_{\Delta^{-1/4}\xi_n}$ is bounded and $0 \leqslant a_n \leqslant 1$. Obviously, $\xi_n = \Phi(a_n)$. Since $\Phi(a_n) = \xi_n \to \xi$ and since the set $\Phi(\{a \in \mathscr{M};\ 0 \leqslant a \leqslant 1\})$ is closed, it follows that there exists an $a \in \mathscr{M}$, $0 \leqslant a \leqslant 1$, such that $\xi = \Phi(a)$.

Hence we easily infer that Φ is surjective.

Q.E.D.

We now return to the general situation in which $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a hyperstandard von Neumann algebra, with the conjugation J and the selfpolar convex cone $\mathfrak{P}$. The remarks we made so far enable us to prove the following important result.

Proposition. *For any* $\xi, \eta \in \mathfrak{P}$ *we have*

$$\|\xi - \eta\|^2 \leqslant \|\omega_\xi - \omega_\eta\| \leqslant \|\xi - \eta\| \, \|\xi + \eta\|.$$

Proof. The second inequality follows from the relation

$$(\omega_\xi - \omega_\eta)(x) = \frac{1}{2} [(x(\xi + \eta) \mid \xi - \eta) + (x(\xi - \eta) \mid \xi + \eta)], \quad x \in \mathscr{M},$$

and holds for any $\xi, \eta \in \mathscr{H}$.

Let $e = s(\omega_\xi) \vee s(\omega_\eta)$ and $q = e(JeJ)$. Then e is of countable type and $\xi, \eta \in q\mathfrak{P}$. In accordance with the remark we made at the beginning of the present section, $\mathscr{M}_q$ is hyperstandard and $*$-isomorphic with $\mathscr{M}_e$. Consequently, in order to prove the first inequality, we can assume that $\mathscr{M}$ is of countable type.

We shall first consider the case in which the vector $\xi + \eta$ is separating. Since $J(\xi + \eta) = \xi + \eta$, it follows that $\xi + \eta$ is also cyclic (see. E.6.9). We shall denote by Δ the modular operator associated to the left Hilbert algebra $\mathfrak{A} = \mathscr{M}(\xi + \eta)$. In accordance with Lemma 2, $\mathfrak{P}$ is the selfpolar convex cone associated to $\mathfrak{A}$. Since

$$-(\xi + \eta) \leqslant \xi - \eta \leqslant \xi + \eta,$$

from Lemma 3 we infer that there exists an $a \in \mathscr{M}$, $-1 \leqslant a \leqslant 1$, such that

$$\xi - \eta = \Delta^{1/4} a(\xi + \eta).$$

Then

$$\|\omega_\xi - \omega_\eta\| \geqslant (\omega_\xi - \omega_\eta)(a) = \mathrm{Re}(a(\xi + \eta) \mid \xi - \eta) = (\Delta^{-1/4}(\xi - \eta) \mid \xi - \eta).$$

Since $J\Delta^{1/4} = \Delta^{-1/4}J$, it follows that

$$(\Delta^{-1/4}(\xi - \eta) \mid \xi - \eta) = (\Delta^{1/4}(\xi - \eta) \mid \xi - \eta);$$

hence

$$\|\omega_\xi - \omega_\eta\| \geqslant \left(\frac{1}{2}(\Delta^{1/4} + \Delta^{-1/4})(\xi - \eta) \mid \xi - \eta\right) \geqslant \|\xi - \eta\|^2.$$

If the vector $\xi + \eta$ is not separating, then $1 - p_{(\xi+\eta)} \neq 0$. In accordance with Lemma 1, there exists a $\zeta \in \mathfrak{P}$, such that

$$p_\zeta = 1 - p_{(\xi+\eta)}.$$

We now consider the vectors

$$\xi_n = \xi + \frac{1}{n}\zeta \in \mathfrak{P}, \quad \eta_n = \eta + \frac{1}{n}\zeta \in \mathfrak{P}, \quad n \in \mathbb{N}.$$

In accordance with the remark just made after the Corollary 3.8, it follows that

$$p'_{(\xi_n+\eta_n)} = p'_{\left(\xi+\eta+\frac{2}{n}\zeta\right)} = 1,$$

because $p_{(\xi+\eta)}$ and $p_{\frac{2}{n}\zeta}$ are orthogonal. Consequently, we have $p_{(\xi_n+\eta_n)} = 1$.

In accordance with the first case, just considered, we have

$$\|\omega_{\xi_n} - \omega_{\eta_n}\| \geqslant \|\xi_n - \eta_n\|^2, \quad n \in \mathbb{N}.$$

If we tend to the limit, in this inequality, for $n \to \infty$, we obtain

$$\|\omega_\xi - \omega_\eta\| \geqslant \|\xi - \eta\|^2.$$

Q.E.D.

Corollary 1. *For $\xi, \eta \in \mathfrak{P}$ we have*

(i) $\xi \perp \eta \Leftrightarrow p_\xi \perp p_\eta$;

(ii) *if $\zeta \in \mathfrak{P}$ and $\zeta \perp \eta$ imply that $\zeta \perp \xi$, then $p_\xi \leqslant p_\eta$;*

(iii) $\xi \leqslant \eta \Rightarrow p_\xi \leqslant p_\eta$.

Proof. (i) If $\xi \perp \eta$, then

$$\|\omega_\xi - \omega_\eta\| \geqslant \|\xi - \eta\|^2 = \|\xi\|^2 + \|\eta\|^2 = \|\omega_\xi\| + \|\omega_\eta\|;$$

hence, in accordance with exercise E.5.15, $p_\xi = s(\omega_\xi) \perp s(\omega_\eta) = p_\eta$. The converse is obvious.

(ii) In accordance with Lemma 1, there exists a family $\{\zeta_n\}_{n\in I} \subset \mathfrak{P}$, such that

$$1 - p_\eta = \sum_{n\in I} p_{\zeta_n}.$$

Now the assertion (ii) is easy to prove, by using assertion (i).

(iii) if $0 \leqslant \xi \leqslant \eta$, then, for any $\zeta \in \mathfrak{P}$ we have

$$0 \leqslant (\xi|\zeta) \leqslant (\eta|\zeta),$$

and assertion (iii) follows from assertion (ii).

Q.E.D.

Corollary 2. *The mapping $\xi \mapsto \omega_\xi$ is a homeomorphism of $\mathfrak{P}$ onto a closed subset of $(\mathcal{M}_*)^+ = \{\varphi \in \mathcal{M}_*;\ \varphi \geqslant 0\}$, with respect to the norm topologies.*

Proof. From the preceding proposition it obviously follows that the mapping in the statement of the proposition is a homeomorphism of $\mathfrak{P}$ onto $\{\omega_\xi;\ \xi \in \mathfrak{P}\} \subset \subset (\mathcal{M}_*)^+$. If $\{\omega_{\xi_n}\}$, $\xi_n \in \mathfrak{P}$, is a Cauchy sequence, then the same proposition shows that $\{\xi_n\}$ is a Cauchy sequence in $\mathfrak{P}$; hence, there exists a $\xi \in \mathfrak{P}$ such that $\omega_{\xi_n} \to \omega_\xi$.

Q.E.D.

10.25. In this section we shall present a Radon-Nikodym type theorem, which is similar to Theorem 5.23, for normal forms on hyperstandard von Neumann algebras.

We consider a hyperstandard von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, whose conjugation is J and whose selfpolar cone is $\mathfrak{P}$.

Lemma 1. *Let $\xi_0 \in \mathfrak{P}$. For any normal form φ on $\mathcal{M}$, such that $\varphi \leqslant \omega_{\xi_0}$, there exists an $\eta \in \mathfrak{P}$, such that*

$$\varphi = \omega_{\xi_0,\eta} + \omega_{\eta,\xi_0} \text{ and } \eta \leqslant \frac{1}{2}\xi_0.$$

Proof. Let $e = s(\omega_{\xi_0})$ and $q = e(JeJ)$. Then $\xi_0 \in q\mathfrak{P}$ is a separating cyclic vector for the hyperstandard von Neumann algebra $\mathcal{M}_q \subset \mathcal{B}(q\mathcal{H})$, which is $*$-isomorphic to $\mathcal{M}_e$ (see Section 10.24). Consequently, we can assume that ξ_0 is a separating cyclic vector for $\mathcal{M}$. In this case we shall denote by S and Δ the operators which are associated to the left Hilbert algebra $\mathfrak{A} = \mathcal{M}\xi_0$. In accordance with Lemma 2 from Section 10.24, J and $\mathfrak{P}$ are also associated to $\mathfrak{A}$.

By taking into account Lemma 5.19, it follows that there exists an operator $a' \in \mathcal{M}'$, $0 \leqslant a' \leqslant 1$, such that

$$\varphi(x) = (x\xi_0 | a'\xi_0), \qquad x \in \mathcal{M}.$$

From Section 10.9, we infer that $a'\xi_0, (1 - a')\xi_0 \in \mathfrak{P}_{S^*}$. Thus,

$$\zeta = \Delta^{-1/4}(a'\xi_0) \in \mathfrak{P}, \quad \xi_0 - \zeta = \Delta^{-1/4}((1 - a')\xi_0) \in \mathfrak{P}.$$

We define

$$\eta = (1 + \Delta^{1/2})^{-1} a'\xi_0 = (1 + \Delta^{1/2})^{-1} \Delta^{1/4}\zeta.$$

If we denote by f the function $t \mapsto 2/(e^{2\pi t} + e^{-2\pi t})$, then, by applying Corollary 9.23, for $A = \Delta^{-1/2}$, we infer that

$$\eta = \int_{-\infty}^{+\infty} f(t)\, \Delta^{it}\, \zeta \, dt.$$

10.26. From Sections 10.14 and 10.23 we infer that *any von Neumann algebra is $*$-isomorphic to a hyperstandard von Neumann algebra.*

It is obvious that any hyperstandard von Neumann algebra is standard. Consequently, in accordance with Corollary 10.15, any $*$-isomorphism between two hyperstandard von Neumann algebras is spatial. Theorem 10.25 implies the following much more precise result.

Corollary. *Let $\mathcal{M}_k \subset \mathcal{B}(\mathcal{H}_k)$ be a hyperstandard von Neumann algebra, J_k its conjugation and $\mathfrak{P}_k$ its selfpolar cone, $k = 1, 2$. If*

$$\pi : \mathcal{M}_1 \to \mathcal{M}_2$$

is a $*$-isomorphism, then there exists a unitary operator

$$u : \mathcal{H}_1 \to \mathcal{H}_2$$

uniquely determined by the conditions

(1) $\pi(x_1) = u \circ x_1 \circ u^*$, *for any* $x_1 \in \mathcal{M}_1$;

(2) $J_2 = u \circ J_1 \circ u^*$;

(3) $\mathfrak{P}_2 = u(\mathfrak{P}_1)$.

Proof. If the unitary operator u has the required properties, then, for any vector $\xi_1 \in \mathfrak{P}_1$, we have $\omega_{u\xi_1} = \omega_{\xi_1} \circ \pi^{-1}$ and $u\xi_1 \in \mathfrak{P}_2$. From Theorem 10.25 we infer that $u\xi_1$ is uniquely determined. Since $\mathcal{H}_1$ is the linear hull of $\mathfrak{P}_1$, we infer that the unitary operator u is uniquely determined by the stated properties.

In order to prove the existence of u, we first consider the case in which $\mathcal{M}_1$ is of countable type. Then $\mathcal{M}_1$ has a separating cyclic vector $\xi_1 \in \mathfrak{P}_1$ (see Lemma 1 from Section 10.24). In accordance with Theorem 10.25, there exists a vector $\xi_2 \in \mathfrak{P}_2$, such that $\omega_{\xi_2} = \omega_{\xi_1} \circ \pi^{-1}$. It follows that ξ_2 is separating for $\mathcal{M}_2$; since $J_2\xi_2 = \xi_2$, we infer that ξ_2 is also cyclic.

As in the proof of Corollary 5.25, it is possible to prove that the relations

$$u(x_1\xi_1) = \pi(x_1)\, \xi_2, \quad x_1 \in \mathcal{M}_1,$$

one determines a unitary operator $u: \mathcal{H}_1 \to \mathcal{H}_2$, which satisfies condition (1) from the statement of the theorem.

If S_k is the operator associated to the left Hilbert algebra $\mathfrak{A}_k = \mathcal{M}_k\xi_k$, $k = 1, 2$, then it is easy to prove that

$$S_2 = u \circ S_1 \circ u^*.$$

By taking into account Lemma 2 from Section 10.24, we infer that u satisfies condition (2) from the statement of the theorem.

Finally, if we now use again Lemma 2 from Section 10.24, we obtain

$$\begin{aligned}\mathfrak{P}_2 &= \overline{\{[x_2(J_2x_2J_2)]\, \xi_2\, ;\ x_2 \in \mathcal{M}_2\}} \\ &= \overline{\{[(ux_1u^*)(uJ_1u^*)(ux_1u^*)(uJ_1u^*)]\xi_2;\ x_1 \in \mathcal{M}_1\}} \\ &= u(\overline{\{[x_1(J_1x_1J_1)]\, \xi_1;\ x_1 \in \mathcal{M}_1\}}) = u(\mathfrak{P}_1).\end{aligned}$$

In the general case, there exists an increasingly directed family $\{e_{1,i}\}_{i\in I}$, consisting of projections of countable type in $\mathcal{M}_1$ whose l.u.b. is equal to 1. For any $i\in I$ we denote

$$e_{2,i}=\pi(e_{1,i}),\quad q_{1,i}=e_{1,i}(J_1e_{1,i}J_1),\quad q_{2,i}=e_{2,i}(J_2e_{2,i}J_2).$$

By taking into account the remark made at the beginning of Section 10.24, it follows that, for any $i\in I$, there exists a uniquely determined $*$-isomorphism

$$\pi_i\colon(\mathcal{M}_1)_{q_{1,i}}\to(\mathcal{M}_2)_{q_{2,i}},$$

such that

$$\pi_i((x_1)_{q_{1,i}})=(\pi(x_1))_{q_{2,i}},\qquad x_1\in\mathcal{M}_1.$$

In accordance with the first part of the proof, for any $i\in I$, there exists a unitary operator

$$u_i:q_{1,i}(\mathcal{H}_1)\to q_{2,i}(\mathcal{H}_2),$$

which is uniquely determined by the properties similar to (1), (2), (3), from the statement of the theorem, but applied to the hyperstandard von Neumann algebras $(\mathcal{M}_1)_{q_{1,i}}$, $(\mathcal{M}_2)_{q_{2,i}}$ and to the $*$-isomorphism π_i.

It follows that if $i\leqslant k$, then $u_i\subset u_k$. Consequently, there exists a uniquely determined unitary operator $u:\mathcal{H}_1\to\mathcal{H}_2$, which is an extension of all the operators u_i, $i\in I$, and it is easy to verify that it satisfies conditions (1), (2), (3) in the statement of the theorem.

Q.E.D.

Let $\mathcal{M}\subset\mathcal{B}(\mathcal{H})$ be a hyperstandard von Neumann algebra, J its conjugation and $\mathfrak{P}$ its selfpolar cone. By taking into account Sections 10.14 and 10.23, from the above corollary we infer that there exists a left Hilbert algebra $\mathfrak{A}\subset\mathcal{H}$, such that $J=J^{\mathfrak{A}}$ and $\mathfrak{P}=\mathfrak{P}^{\mathfrak{A}}$. Consequently, in any hyperstandard von Neumann algebra we have at our disposal the tool of the left Hilbert algebras.

We remark that there exist standard von Neumann algebras, which are not hyperstandard (see, e.g., U. Haagerup, preprint of [2], Proposition 5.3).

10.27. We now return to the study of the faithful semifinite, normal weights on von Neumann algebras.

Let $\mathcal{M}$ be a von Neumann algebra, $\mathcal{Z}$ its center and φ a faithful, semifinite, normal weight on $\mathcal{M}^+$. We write

$$\mathcal{M}^{\varphi}_{\infty}=\{x\in\mathcal{M};\ it\mapsto\sigma^{\varphi}_t(x)\text{ has an entire analytic continuation}\}.$$

In Section 10.16, we showed that $\mathcal{M}^{\varphi}_{\infty}$ is a *so*-dense $*$-subalgebra and

$$\mathcal{M}^{\varphi}_{\infty}\cdot\mathfrak{M}_{\varphi}\cdot\mathcal{M}^{\varphi}_{\infty}=\mathfrak{M}_{\varphi}.$$

We now write

$$\mathcal{M}^{\varphi}_{0}=\{x\in\mathcal{M};\quad\sigma^{\varphi}_t(x)=x,\quad t\in\mathbb{R}\}.$$

Obviously,

$$\mathcal{Z}\subset\mathcal{M}^{\varphi}_{0}\subset\mathcal{M}^{\varphi}_{\infty}.$$

The set $\mathcal{M}^{\varphi}_{0}$ is a von Neumann algebra and it is called the *centralizer* of φ.

Theorem. *Let $\mathcal{M}$ be a von Neumann algebra, φ a faithful, semifinite normal weight on $\mathcal{M}^+$ and $x \in \mathcal{M}$. Then the following assertions are equivalent*

(i) $x \in \mathcal{M}_0^\varphi$;

(ii) $x\mathfrak{M}_\varphi \subset \mathfrak{M}_\varphi$, $\mathfrak{M}_\varphi x \subset \mathfrak{M}_\varphi$ *and* $\varphi(xy) = \varphi(yx)$, $\quad y \in \mathfrak{M}_\varphi$.

Proof. In accordance with Theorems 10.15 and 10.18, we can assume that $\mathcal{M} = \mathfrak{L}(\mathfrak{A})$, where $\mathfrak{A}$ is a left Hilbert algebra and $\varphi = \varphi_{\mathfrak{A}}$.

Let $x \in \mathcal{M}_0^\varphi$. By taking into account the remarks made at the beginning of the section, we have

$$x\mathfrak{M}_\varphi \subset \mathfrak{M}_\varphi, \quad \mathfrak{M}_\varphi x \subset \mathfrak{M}_\varphi.$$

Thus, if $a \in \mathfrak{M}_\varphi^+$ and $a^{1/2} = L_\xi$, $\xi \in \mathfrak{A}''$, then

$$x\xi \in \mathfrak{A}'', \ L_{x\xi} = xL_\xi \text{ and } x^*\xi \in \mathfrak{A}'', \ L_{x^*\xi} = x^*L_\xi.$$

By taking into account Proposition 9.24, we have

$$\begin{aligned}\varphi(xa) &= \varphi(L_{x\xi}L_{S\xi}) = (S\xi \mid Sx\xi) = (J\Delta^{1/2}\xi \mid J\Delta^{1/2}x\xi)\\ &= ((\Delta^{1/2}x\Delta^{-1/2})\,\Delta^{1/2}\xi \mid \Delta^{1/2}\xi) = (x\Delta^{1/2}\xi \mid \Delta^{1/2}\xi)\\ &= (\Delta^{1/2}\xi \mid (\Delta^{1/2}x^*\Delta^{-1/2})\,\Delta^{1/2}\xi) = (J\Delta^{1/2}x^*\xi \mid J\Delta^{1/2}\xi)\\ &= (Sx^*\xi \mid S\xi) = \varphi(L_\xi(L_{x^*\xi})^*) = \varphi(L_\xi(x^*L_\xi)^*) = \varphi(ax).\end{aligned}$$

Thus, we have shown that (i) $\Rightarrow$ (ii).

Conversely, let us now assume that x satisfies condition (ii). Then, obviously, the elements $\sigma_t(x)$, $t \in \mathbb{R}$, satisfy condition (ii), too.

For any $\xi, \eta \in \mathfrak{T}$ let us consider the function $f_{\xi,\eta}$, which is bounded and continuous on $\{\alpha \in \mathbb{C};\ 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$, and analytic in $\{\alpha \in \mathbb{C};\ 0 < \operatorname{Re}\alpha < 1\}$, and given by the formula

$$f_{\xi,\eta}(\alpha) = (x\Delta^{-\alpha+1}\eta \mid \Delta^{\bar{\alpha}}\xi).$$

If we use the first part of condition (ii), then, for any $t \in \mathbb{R}$, we obtain

$$\sigma_t(x)\eta \in \mathfrak{A}'', \ L_{\sigma_t(x)\eta} = \sigma_t(x)\,L_\eta \text{ and } \sigma_t(x^*)\,\xi \in \mathfrak{A}'', \ L_{\sigma_t(x^*)\xi} = \sigma_t(x^*)\,L_\xi.$$

With the help of the second part of condition (ii), we obtain

$$\begin{aligned}f_{\xi,\eta}(1+it) &= (x\Delta^{-it}\eta \mid \Delta^{-it}S^*S\xi) = (\sigma_t(x)\eta \mid S^*S\xi)\\ &= (S\xi \mid S\sigma_t(x)\eta) = \varphi(\sigma_t(x)\,L_\eta L_{S\xi}) = \varphi(L_\eta L_{S\xi}\,\sigma_t(x))\\ &= \varphi(L_\eta(\sigma_t(x^*)\,L_\xi)^*) = (S\sigma_t(x^*)\xi \mid S\eta) = (S^*S\eta|\Delta^{it}x^*\Delta^{-it}\xi)\\ &= (x\Delta^{-it+1}\eta \mid \Delta^{-it}\xi) = f_{\xi,\eta}(it).\end{aligned}$$

Thus, $f_{\xi,\eta}$ can be extended, by periodicity, to a bounded entire analytic function. Liouville's theorem now implies that $f_{\xi,\eta}$ is a constant. In particular,

$$(\sigma_t(x)\,\eta \mid \Delta\xi) = f_{\xi,\eta}(it) = f_{\xi,\eta}(1) = (x\eta \mid \Delta\xi), \qquad t \in \mathbb{R}.$$

Since the vectors $\xi, \eta \in \mathfrak{X}$ are arbitrary, we hence infer that

$$\sigma_t(x) = x, \qquad t \in \mathbb{R}.$$

We have thus proved the implication (ii) $\Rightarrow$ (i).

Q.E.D.

Corollary. *Let $\mathcal{M}$ be a von Neumann algebra, φ a faithful, semifinite, normal weight on $\mathcal{M}^+$ and $u \in \mathcal{M}$ a unitary element. Then the following assertions are equivalent*

(i) $u \in \mathcal{M}_0^\varphi$;

(ii) $\varphi(u^*au) = \varphi(uau^*) = \varphi(a), \; a \in \mathfrak{M}_\varphi^+$.

Proof. According to the theorem, we have the implication (i) $\Rightarrow$ (ii).

Let us now assume that u satisfies condition (ii). For any $a \in \mathfrak{M}_\varphi^+$ we have $u^*au \in \mathfrak{M}_\varphi^+$, hence $a^{1/2}u \in \mathfrak{N}_\varphi$. Since we obviously have $a^{1/2} \in \mathfrak{N}_\varphi^*$, it follows that

$$au = a^{1/2}(a^{1/2}u) \in \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi = \mathfrak{M}_\varphi.$$

Consequently, we have $\mathfrak{M}_\varphi u \subset \mathfrak{M}_\varphi$. Similarly, one can show that $u\mathfrak{M}_\varphi \subset \mathfrak{M}_\varphi$. Now one can easily prove that u satisfies condition (ii) in the statement of the theorem, hence $u \in \mathcal{M}_0^\varphi$.

Q.E.D.

In particular, it follows that φ is a trace iff the centralizer of φ coincides with $\mathcal{M}$, i.e., iff the group $\{\sigma_t^\varphi\}$ acts identically on $\mathcal{M}$.

10.28. We now prove a remarkable Radon-Nikodym type property, due to A. Connes, which establishes a link between the groups of modular automorphisms, associated to any pair of faithful, semifinite, normal weights on a von Neumann algebra.

Theorem. *Let $\mathcal{M}$ be a von Neumann algebra and φ, ψ two faithful, semifinite, normal weights on $\mathcal{M}^+$. Then, there exists a so-continuous mapping*

$$\mathbb{R} \ni t \mapsto u_t \in \mathcal{M},$$

such that

(1) u_t *is unitary*, $t \in \mathbb{R}$;

(2) $u_{t+s} = u_t\sigma_t^\varphi(u_s), \quad s, t \in \mathbb{R}$;

(3) $\sigma_t^\psi(x) = u_t\sigma_t^\varphi(x)u_t^*, \quad x \in \mathcal{M}, \quad t \in \mathbb{R}$.

Proof. We denote $\mathcal{N} = \mathrm{Mat}_2(\mathcal{M})$ (see 2.32 and 3.16). For any

$$a = (a_{ij}) \in \mathcal{N}, \quad a \geqslant 0,$$

we define

$$\theta(a) = \varphi(a_{11}) + \psi(a_{22}) \in \mathbb{R}^+ \cup \{+\infty\}.$$

It is easy to see that θ is a faithful, normal weight on $\mathcal{N}^+$. If $x_{11}, x_{21} \in \mathfrak{N}_\varphi$ and $x_{12}, x_{22} \in \mathfrak{N}_\psi$, then $(x_{ij}) \in \mathfrak{N}_\theta$. We hence infer that $\mathfrak{N}_\theta$ is *so*-dense in $\mathcal{N}$. Consequently, θ is also semifinite.

We denote by e_{ij}, $i = 1, 2$; $j = 1, 2$, the "matrix units" of $\mathcal{N}$:

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we denote

$$u = e_{11} - e_{22},$$

then u is a self-adjoint unitary operator and

$$u \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} u = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix}, \quad (a_{ij}) \in \mathcal{N}^+$$

It follows that, for any $a \in \mathcal{N}^+$, we have

$$\theta(uau) = \theta(a).$$

If we now apply Corollary 10.27, we infer that $u \in \mathcal{N}_0^\theta$. Since we obviously have $1 \in \mathcal{N}_0^\theta$, it follows that

$$e_{11} = \frac{1}{2}(1 + u) \in \mathcal{N}_0^\theta,$$

$$e_{22} = \frac{1}{2}(1 - u) \in \mathcal{N}_0^\theta.$$

Since $e_{11} \in \mathcal{N}_0^\theta$, for any $x \in \mathcal{M}$ and any $t \in \mathbb{R}$, we have

$$e_{11}\sigma_t^\theta\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right)e_{11} = \sigma_t^\theta\left(e_{11}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} e_{11}\right) = \sigma_t^\theta\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right);$$

hence $\sigma_t^\theta\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right)$ is of the form

$$\sigma_t^\theta\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} \pi_t(x) & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to verify that the above relation determines a group $\{\pi_t\}$ of $*$-automorphisms of $\mathcal{M}$, which leaves φ invariant. Since, for any $x, y \in \mathfrak{M}_\varphi$ and any $t \in \mathbb{R}$, we have

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{M}_\theta,$$

$$\theta\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \sigma_t^\theta\left(\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}\right)\right) = \varphi(x\pi_t(y)),$$

$$\theta\left(\sigma_t^\theta\left(\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}\right) \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) = \varphi(\pi_t(y)x),$$

from Theorem 10.17 we infer that φ satisfies the KMS-condition, with respect to $\{\pi_t\}$, for any pair of elements in $\mathfrak{M}_\varphi$. From the uniqueness part of Theorem 10.17, we infer that

$$\pi_t = \sigma_t^\varphi, \quad t \in \mathbb{R}.$$

Consequently, we have

$$\sigma_t^\theta\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} \sigma_t^\varphi(x) & 0 \\ 0 & 0 \end{pmatrix}, \ x \in \mathcal{M}.$$

One can similarly prove that

$$\sigma_t^\theta\left(\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_t^\psi(y) \end{pmatrix}, \ y \in \mathcal{M}.$$

By taking into account the fact that $e_{11}, e_{22} \in \mathcal{N}_0^\theta$, it is easy to see that for any $t \in \mathbb{R}$, we have

$$e_{11}\sigma_t^\theta(e_{21}) = \sigma_t^\theta(e_{21})e_{22} = 0;$$

hence, $\sigma_t^\theta(e_{21})$ is of the form

$$\sigma_t^\theta(e_{21}) = \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix}, \quad u_t \in \mathcal{M}.$$

Since the group $\{\sigma_t^\theta\}$ is *so*-continuous, the mapping

$$\mathbb{R} \ni t \mapsto u_t \in \mathcal{M}$$

is *so*-continuous. Then, since

$$\begin{aligned} \sigma_t^\theta(e_{21})^* \sigma_t^\theta(e_{21}) &= \sigma_t^\theta(e_{12}e_{21}) = \sigma_t^\theta(e_{11}) = e_{11}, \\ \sigma_t^\theta(e_{21})\, \sigma_t^\theta(e_{21})^* &= \sigma_t^\theta(e_{21}e_{12}) = \sigma_t^\theta(e_{22}) = e_{22}, \end{aligned}$$

it follows that the operators u_t are unitary. For any $x \in \mathcal{M}$ and any $t \in \mathbb{R}$, we have

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & \sigma_t^\psi(x) \end{pmatrix} &= \sigma_t^\theta\left(\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right) = \sigma_t^\theta\left(e_{21}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} e_{12}\right) = \sigma_t^\theta(e_{21})\sigma_t^\theta\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right)\sigma_t^\theta(e_{12}) \\ &= \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix}\begin{pmatrix} \sigma_t^\varphi(x) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & u_t^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & u_t\sigma_t^\varphi(x)u_t^* \end{pmatrix}; \end{aligned}$$

hence

$$\sigma_t^\psi(x) = u_t\sigma_t^\varphi(x)u_t^*.$$

Finally, for any $t, s \in \mathbb{R}$, we have

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ u_{t+s} & 0 \end{pmatrix} &= \sigma_{t+s}^\theta(e_{21}) = \sigma_t^\theta\left(\begin{pmatrix} 0 & 0 \\ u_s & 0 \end{pmatrix}\right) = \sigma_t^\theta\left(e_{21}\begin{pmatrix} u_s & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix}\begin{pmatrix} \sigma_t^\varphi(u_s) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u_t\sigma_t^\varphi(u_s) & 0 \end{pmatrix}; \end{aligned}$$

hence
$$u_{t+s} = u_t\sigma_t^\varphi(u_s).$$
Q.E.D.

The preceding theorem enables us to define the notion of "commutation", relatively to weights.

Corollary. *Let $\mathscr{M}$ be a von Neumann algebra and φ, ψ faithful, semifinite, normal weights on $\mathscr{M}^+$. Then the following assertions are equivalent*

(i) $\{\sigma_t^\psi\}_{t\in\mathbb{R}}$ *leaves invariant the weight* φ;

(ii) $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ *leaves invariant the weight* ψ;

(iii) *there exists a so-continuous group* $\{u_t\}_{t\in\mathbb{R}}$ *of unitary operators in* $\mathscr{M}_0^\varphi \cap \mathscr{M}_0^\psi$, *such that*
$$\sigma_t^\psi(x) = u_t\sigma_t^\varphi(x)u_t^*, \quad x \in \mathscr{M}, \quad t \in \mathbb{R}.$$

Proof. The implication (iii) $\Rightarrow$ (i) obviously follows from Corollary 10.27.

Let us now assume that $\{\sigma_t^\psi\}_{t\in\mathbb{R}}$ leaves invariant the weight φ and let us consider a *so*-continuous mapping $t \mapsto u_t$, as in the statement of Theorem 10.28. For any $t \in \mathbb{R}$, we have
$$\varphi(a) = \varphi(\sigma_{-t}^\varphi(a)) = \varphi(\sigma_t^\psi(\sigma_{-t}^\varphi(a))) = \varphi(u_tau_t^*), \quad a \in \mathscr{M}^+;$$
hence, in accordance with Corollary 10.27,
$$u_t \in \mathscr{M}_0^\varphi.$$
Then, for any t, $s \in \mathbb{R}$, we have
$$u_{t+s} = u_t\sigma_t^\varphi(u_s) = u_tu_s;$$
hence, $\{u_t\}$ is a one-parameter group. Finally, for any $t \in \mathbb{R}$, we have
$$\sigma_s^\psi(u_t) = u_s\sigma_s^\varphi(u_t)u_s^* = u_su_tu_s^* = u_t, \quad s \in \mathbb{R};$$
hence
$$u_t \in \mathscr{M}_0^\psi.$$
Consequently, (i) $\Leftrightarrow$ (iii).

Similarly, one proves that (ii) $\Leftrightarrow$ (iii).

Q.E.D.

If the faithful, semifinite, normal weights φ and ψ satisfy the equivalent conditions from the statement of the preceding corollary, we shall say that φ and ψ *commute*.

10.29. At the end of this chapter we present a criterion of semifiniteness for the von Neumann algebras, expressed in terms of the group of modular automorphisms.

Theorem. *For any von Neumann algebra $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ the following assertions are equivalent*

(i) $\mathscr{M}$ *is semifinite*;

(ii) *there exists a faithful, semifinite, normal weight* φ *on* $\mathscr{M}^+$, *and a so-continuous group* $\{u_t\}_{t\in\mathbb{R}}$ *of unitary operators in* $\mathscr{M}$, *such that*
$$\sigma_t^\varphi(x) = u_txu_t^*, \quad x \in \mathscr{M}, \quad t \in \mathbb{R};$$

(iii) *for any faithful, semifinite, normal weight* φ *on* $\mathscr{M}$ *there exists a so-continuous group* $\{u_t\}_{t\in\mathbb{R}}$ *of unitary operators in* $\mathscr{M}$, *such that*

$$\sigma_t^\varphi(x) = u_t x u_t^*, \quad x \in \mathscr{M}, \quad t \in \mathbb{R}.$$

Proof. If $\mathscr{M}$ is semifinite, then, in accordance with Corollary 7.15, there exists a faithful, semifinite, normal trace μ on $\mathscr{M}^+$. With the help of Corollary 10.27, we have

$$\sigma_t^\mu(x) = x, \quad x \in \mathscr{M}, \quad t \in \mathbb{R}.$$

If φ is any faithful, semifinite, normal weight on $\mathscr{M}^+$, then $\{\sigma_t^\mu\}$ leaves the weight φ invariant. Thus, if we now apply Corollary 10.28, it follows that there exists a *so*-continuous group $\{u_t\}_{t\in\mathbb{R}}$ of unitary operators in $\mathscr{M}$, such that

$$\sigma_t^\varphi(x) = u_t \sigma_t^\mu(x) u_t^* = u_t x u_t^*, \quad x \in \mathscr{M}, \quad t \in \mathbb{R}.$$

Since on $\mathscr{M}^+$ there exists a faithul, semifinite, normal weight (see Section 10.14), the implication (iii) $\Rightarrow$ (ii) is trivial.

Finally, let us assume that assertion (ii) is true. According to the Stone representation theorem (9.20), there exists a positive self-adjoint operator A in $\mathscr{H}$, such that $s(A) = 1$, which is affiliated to $\mathscr{M}$, and such that

$$u_t = A^{it}, \quad t \in \mathbb{R}.$$

For any natural n we denote

$$e_n = \chi_{\left(\frac{1}{n}, n\right)}(A) \in \mathscr{M}.$$

Then

$$e_n \in \mathscr{M}_0^\varphi, \quad n \in \mathbb{N},$$

and

$$e_n \uparrow s(A) = 1.$$

In order to prove the semifiniteness of $\mathscr{M}$, it is sufficient to show that the reduced von Neumann algebras $\mathscr{M}_{e_n}$ are semifinite.

Let us choose a natural number n. According to Theorem 10.27 we have

$$e_n \mathfrak{M}_\varphi e_n \subset \mathfrak{M}_\varphi;$$

hence, the weight φ_n, defined on $(\mathscr{M}_{e_n})^+$ by the formula

$$\varphi_n(e_n a \mid e_n \mathscr{H}) = \varphi(e_n a e_n), \quad a \in \mathscr{M}^+,$$

is semifinite. It is easy to see that φ_n is normal and faithful. If we denote $a_n = Ae_n \in \mathscr{M}_{e_n}$, we have

$$\sigma_t^{\varphi_n}(x) = a_n^{it} x a_n^{-it}, \quad x \in (\mathscr{M}_{e_n})^+, \quad t \in \mathbb{R}.$$

Thus, our problem reduced to the following one: to show that, if $\mathfrak{A} \subset \mathscr{H}$ is a left Hilbert algebra and if there exists an invertible $a \in \mathfrak{L}(\mathfrak{A})^+$, such that

$$\sigma_t(x) = a^{it} x a^{-it}, \quad x \in \mathfrak{L}(\mathfrak{A}), \quad t \in \mathbb{R},$$

then $\mathfrak{L}(\mathfrak{A})$ is semifinite.

Indeed, let us define on $\mathfrak{L}(\mathfrak{A})^+$ the weight

$$\mu(b) = \varphi_{\mathfrak{A}}(a^{-1/2}ba^{-1/2}), \quad b \in \mathfrak{L}(\mathfrak{A})^+.$$

It is easy to see that μ is normal and faithful. Since $a \in \mathfrak{L}(\mathfrak{A})_0^{\varphi_{\mathfrak{A}}}$, it follows that μ is semifinite and $\mathcal{M}_\mu = \mathcal{M}_{\varphi_{\mathfrak{A}}}$.

For any $\xi, \zeta \in \mathfrak{A}''$ and any $t \in \mathbb{R}$, with the help of Proposition 9.24, we infer that

$$\begin{aligned}
&(a^{-\left(\frac{1}{2}+it\right)}\Delta^{\frac{1}{2}+it}\xi \,|\, Ja^{\left(-\frac{1}{2}+it\right)}\Delta^{\frac{1}{2}}\zeta) \\
&= (\Delta^{\frac{1}{2}}[\Delta^{-\frac{1}{2}}a^{-\left(\frac{1}{2}+it\right)}\Delta^{\frac{1}{2}}]\Delta^{it}\xi | J\Delta^{\frac{1}{2}}[\Delta^{-\frac{1}{2}}a^{\left(-\frac{1}{2}+it\right)}\Delta^{\frac{1}{2}}]\zeta) \\
&= (\Delta^{\frac{1}{2}}a^{-\left(\frac{1}{2}+it\right)}\Delta^{it}\xi \,|\, \Delta^{-\frac{1}{2}}Ja^{\left(-\frac{1}{2}+it\right)}\zeta) \\
&= (a^{-\left(\frac{1}{2}+it\right)}\Delta^{it}\xi | J a^{\left(-\frac{1}{2}+it\right)}\zeta).
\end{aligned}$$

Thus, by the formula

$$f_{\xi,\zeta}(\alpha) = \begin{cases} (a^{-\alpha}\Delta^{\alpha}\xi \,|\, Ja^{\alpha-1}\Delta^{\frac{1}{2}}\zeta), & \text{if } 0 \leqslant \operatorname{Re}\alpha \leqslant \dfrac{1}{2}, \\[2ex] (a^{-\alpha}\Delta^{\alpha-\frac{1}{2}}\xi \,|\, Ja^{\alpha-1}\zeta), & \text{if } \dfrac{1}{2} \leqslant \operatorname{Re}\alpha \leqslant 1, \end{cases}$$

we define a bounded and continuous function $f_{\xi,\zeta}$ on $\{\alpha \in \mathbb{C};\, 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$, which is analytic in $\{\alpha \in \mathbb{C};\, 0 < \operatorname{Re}\alpha < 1\}$. For any $t \in \mathbb{R}$, we have

$$\begin{aligned}
f_{\xi,\zeta}(it) &= (a^{-it}\Delta^{it}\xi \,|\, J\Delta^{\frac{1}{2}}[\Delta^{-\frac{1}{2}}a^{it-1}\Delta^{\frac{1}{2}}]\zeta) \\
&= (a^{-it}\Delta^{it}\xi \,|\, Sa^{it-1}\zeta) = \varphi_{\mathfrak{A}}(a^{-1}a^{it}L_\zeta a^{-it}L_{\Delta^{it}\xi}) \\
&= \varphi_{\mathfrak{A}}(a^{-1}\sigma_t(L_\zeta)\sigma_t(L_\xi)) = \mu(L_\zeta L_\xi),
\end{aligned}$$

$$\begin{aligned}
f_{\xi,\zeta}(1+it) &= (a^{-1-it}\Delta^{\frac{1}{2}+it}\xi \,|\, Ja^{it}\zeta) \\
&= (\Delta^{\frac{1}{2}}[\Delta^{-\frac{1}{2}}a^{-1-it}\Delta^{\frac{1}{2}}]\Delta^{it}\xi \,|\, Ja^{it}\zeta) \\
&= (a^{it}\zeta \,|\, Sa^{-1-it}\Delta^{it}\xi) = \varphi_{\mathfrak{A}}(a^{-1}a^{-it}L_{\Delta^{it}\xi}a^{it}L_\zeta) \\
&= \varphi_{\mathfrak{A}}(a^{-1}\sigma_{-t}(\sigma_t(L_\xi))L_\zeta) = \mu(L_\xi L_\zeta).
\end{aligned}$$

Consequently, $f_{\xi,\zeta}$ is constant on the imaginary axis; hence, everywhere.

We hence infer that

$$\mu(L_\zeta L_\xi) = \mu(L_\xi L_\zeta), \quad \xi, \zeta \in A''.$$

In particular, we have

$$\mu(xy) = \mu(yx), \quad x, y \in \mathfrak{M}_\mu = \mathfrak{M}_{\varphi_\mathfrak{A}};$$

consequently, μ satisfies the KMS-condition with respect to the identity group for any pair of elements in $\mathfrak{M}_\mu$. With the help of Theorem 10.17, we infer that

$$\sigma_t^\mu(x) = x, \quad x \in \mathfrak{L}(\mathfrak{A}), \quad t \in \mathbb{R};$$

hence, according to Corollary 10.27, μ is a trace. Thus, with the help of Corollary 7.15, we infer that $\mathfrak{L}(\mathfrak{A})$ is semifinite.

Q.E.D.

Exercises

E.10.1. Let $\mathfrak{A}$ be a complex algebra endowed with an involution $\#$ and a scalar product $(\cdot|\cdot)$. We suppose that $\mathfrak{A}$ satisfies conditions (i), (ii), (iii) from 10.1, and that

$$\mathscr{H} \supset \mathfrak{A}^2 \ni \xi_1\xi_2 \mapsto (\xi_1\xi_2)^\# \in \mathscr{H}$$

is a preclosed antilinear operator. Show that the operator

$$\mathscr{H} \supset \mathfrak{A} \ni \xi \mapsto \xi^\# \in \mathscr{H}$$

is preclosed, hence $\mathfrak{A}$ is a left Hilbert algebra.

E.10.2. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra with a separating cyclic vector $\xi_0 \in \mathscr{H}$. Give direct proofs to the following assertions (see Section 10.6):

(1) The adjoint S^* of the closure S of the antilinear operator

$$\mathscr{H} \supset \mathscr{M}\xi_0 \ni x\xi_0 \mapsto x^*\xi_0 \in \mathscr{H}$$

is the closure of the antilinear operator

$$\mathscr{H} \supset \mathscr{M}'\xi_0 \ni x'\xi_0 \mapsto x'^*\xi_0 \in \mathscr{H}.$$

(2) If $\eta \in \mathscr{D}_{S^*}$, then the operator

$$\mathscr{H} \supset \mathscr{M}\xi_0 \ni x\xi_0 \mapsto x\eta \in \mathscr{H}$$

is preclosed and its closure is affiliated to $\mathscr{M}'$.

E.10.3. Let G be a locally compact topological group, dg a left invariant Haar measure on G and $\vartheta: G \to \mathbb{R}^+$, the modular function. Show that the set $\mathfrak{A}_G$ of all continuous complex functions, which are defined on G and whose supports are compact, is a left Hilbert algebra with respect to the operations:

$$(\xi\eta)(g) = \int \xi(h)\eta(h^{-1}g)\mathrm{d}h,$$

$$\xi^\#(g) = \vartheta(g)^{-1}\overline{\xi(g^{-1})},$$

and the scalar product

$$(\xi|\eta) = \int \xi(g)\overline{\eta(g)}\mathrm{d}g.$$

Determine, in this case, the objects $\mathscr{H}, \Delta, J, \mathfrak{S}$.

E.10.4. Let $\mathscr{H}$ be a Hilbert space. Show that the set $\mathfrak{A} = \mathscr{F}(\mathscr{H})$ of all operators in $\mathscr{B}(\mathscr{H})$, whose ranges are finitely dimensional, is a left Hilbert algebra with respect to the $*$-algebra operations induced by those of $\mathscr{B}(\mathscr{H})$ and with the scalar product

$$(x|y) = \mathrm{tr}(y^*x).$$

Determine, in this case, the objects $\mathscr{H}, \Delta, J, \mathfrak{A}''$.

E.10.5. A left Hilbert algebra $\mathfrak{A} \subset \mathscr{H}$ is said to be *unimodular* if $\Delta = 1$.

Show that for a left Hilbert algebra $\mathfrak{A} \subset \mathscr{H}$, the following assertions are equivalent

(i) $\mathfrak{A}$ is unimodular;
(ii) S is isometric;
(iii) $\varphi_{\mathfrak{A}}$ is a trace.

Show that if G is a locally compact topological group, then the left Hilbert algebra $\mathfrak{A}_G$ (E.10.3) is unimodular iff the group G is unimodular.

E.10.6. Prove that any von Neumann algebra is $*$-isomorphic to a standard von Neumann algebra along the lines of the proof for Theorem 10.7.

E.10.7. Let $\mathscr{M}$ be a standard von Neumann algebra and J its conjugation. Show that $\mathscr{R}(\mathscr{M}, \mathscr{M}')$ is the w-closed linear hull of the set

$$\{x(JxJ);\quad x \in \mathscr{M}\}.$$

(Hint: use a polarization relation).

In the following three exercises, $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra with the separating cyclic vector $\xi_0 \in \mathscr{H}$, whereas J_{ξ_0} is the canonical conjugation associated to $\mathfrak{A} = \mathscr{M}\xi_0$.

E.10.8. For $\eta \in \mathscr{H}$ the following assertions are equivalent:

(i) $\eta \in \mathfrak{A}'$;
(ii) R^0_η is bounded;
(iii) the form ω_η is dominated (E.9.33) by the form ω_{ξ_0}.

E.10.9. For $x \in \mathscr{M}$, the following assertions are equivalent:

(i) $\omega_{x\xi_0} \leqslant \omega_{\xi_0}$;
(ii) $\|\Delta^{1/2}x\Delta^{-1/2}\| \leqslant 1$.

E.10.10. Show that if J is a conjugation in $\mathscr{H}$, with the properties

(1) the mapping $x \mapsto Jx^*J$ is a $*$-antiisomorphism of $\mathscr{M}$ onto $\mathscr{M}'$, which acts identically on the center;
(2) $J\xi_0 = \xi_0$;
(3) $(\xi_0|[x(JxJ)]\xi_0) \geqslant 0$, for any $x \in \mathscr{M}$;

then $J = J_{\xi_0}$.
(Hint: prove that $JS = JJ_{\xi_0}\Delta^{1/2}$ is a positive self-adjoint operator).

In the following two exercises, $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra, $\mathscr{Z}$ is its center; $\xi_0, \zeta_0 \in \mathscr{H}$ are separating cyclic vectors, J_{ξ_0}, J_{ζ_0} are the corresponding canonical conjugations and $\mathfrak{P}_{\xi_0}$, $\mathfrak{P}_{\zeta_0}$ are the selfpolar convex cones associated respectively to the left Hilbert algebras $\mathscr{M}\xi_0$, $\mathscr{M}\zeta_0$.

E.10.11. Show that the following assertions are equivalent:

(i) $\zeta_0 \in \mathfrak{P}_{\xi_0}$;

(ii) $\xi_0 \in \mathfrak{P}_{\zeta_0}$;

(iii) $J_{\xi_0} = J_{\zeta_0}$ and $(z\xi_0 | z\zeta_0) \geqslant 0$, for any $z \in \mathscr{Z}$, $z \geqslant 0$.

(Hint for the proof of the implication (iii) $\Rightarrow$ (i): in accordance with Section 10.23, ζ_0 can be written $\zeta_0 = \zeta_0^+ - \zeta_0^-$, $p_{\zeta_0^+} \perp p_{\zeta_0^-}$, with respect to $\mathfrak{P}_{\xi_0}$; show that

$$(\zeta_0^+ | x(J_{\xi_0} x J_{\xi_0})\zeta_0^-) = 0, \quad x \in \mathscr{M};$$

from exercise E.10.7 we infer that $\zeta_0^+ \perp [\mathscr{M}\mathscr{M}'\zeta_0^-]$, whence

$$0 \leqslant (\xi_0 | [\mathscr{M}\mathscr{M}'\zeta_0^-]\zeta_0) = -(\xi_0 | \zeta_0^-) \leqslant 0;$$

hence, $(\xi_0 | \zeta_0^-) = 0$, $\zeta_0^- = 0$.)

E.10.12. Show that there exists a unitary $u' \in \mathscr{M}'$, such that

$$J_{\zeta_0} = u' \circ J_{\xi_0} \circ u'^*.$$

Infer that the $*$-automorphism

$$\mathscr{M} \ni x \mapsto J_{\xi_0} J_{\zeta_0} x J_{\zeta_0} J_{\xi_0} \in \mathscr{M}$$

is inner.

In the following six exercises, $\mathfrak{A} \subset \mathscr{H}$ is a left Hilbert algebra, the other notations corresponding to those introduced in the main text.

E.10.13. We define the set

$$\mathfrak{T}_1 = \left\{ \xi \in \bigcap_{n=-\infty}^{+\infty} \mathscr{D}_{\Delta^n} \;\middle|\; \begin{array}{l} \text{for any } n \in Z \text{ we have} \\ \qquad \Delta^n \xi \in \mathfrak{A}'' \cap \mathfrak{A}' \\ \mathscr{D}_{(\Delta^n L_\xi \Delta^{-n})} = \mathscr{D}_{(\Delta^{-n})} \text{ and } \Delta^n L_\xi \Delta^{-n} \subset L_{\Delta^n \xi} \\ \mathscr{D}_{(\Delta^n R_\xi \Delta^{-n})} = \mathscr{D}_{(\Delta^{-n})} \text{ and } \Delta^n R_\xi \Delta^{-n} \subset R_{\Delta^n \xi} \end{array} \right\}$$

With the help of Lemmas 1, 2, and 3 from 10.19, show that $\mathfrak{T}_1$ is a left Hilbert subalgebra of $\mathfrak{A}''$ and $\mathfrak{T}_1'' = \mathfrak{A}''$.

We now consider the operator T on $\mathscr{B}(\mathscr{H})$, defined as in E.9.36, with $A = B = \Delta$. Show that for any $\xi \in \mathfrak{T}_1$ and any $\lambda > 0$, we have

$$(\lambda + \Delta)^{-1} \xi \in \mathfrak{T}_1 \quad \text{and} \quad L_{(\lambda+\Delta)^{-1}\xi} = (\lambda + T)^{-1}(L_\xi).$$

With the help of exercises E.9.36 and E.9.37, infer from this result that

$$\mathfrak{T} = \mathfrak{T}_1.$$

Show that the preceding assertions imply Theorem 10.12.

E.10.14. For any $\xi \in \mathscr{H}$, the following assertions are equivalent:

(i) $\xi \in \mathfrak{S}$;

(ii) $\xi \in \mathfrak{A}'' \cap \mathscr{S}_\Delta \cap \mathscr{S}_{\Delta^{-1}}$;

(iii) $\xi \in \mathfrak{A}''$ and the mapping $it \mapsto \Delta^{it} L_\xi \Delta^{-it}$ has an entire analytic continuation F, such that

$$\overline{\lim_{n\to\infty}} \| F(n) \|^{1/n} < +\infty, \quad \overline{\lim_{n\to\infty}} \| F(-n) \|^{1/n} < +\infty.$$

E.10.15. Show that for any $\xi \in \mathfrak{A}''$ there exists a sequence $\{\xi_n\} \subset \mathfrak{S}$, such that

(1) $\xi_n \to \xi$, $S\xi_n \to S\xi$;

(2) $L_{\xi_n} \to L_\xi$, $(L\xi_n)^* \to (L_\xi)^*$;

(3) $\sup \|L_{\xi_n}\| < +\infty$.

E.10.16. Show that if $\xi \in \mathfrak{A}'' \cap \mathscr{D}_{(\Delta^{-\alpha})}$ and $\eta \in \mathfrak{A}' \cap \mathscr{D}_{(\Delta^\alpha)}$, then

$$R_\eta \Delta^{-\alpha}\xi \in \mathscr{D}_{(\Delta^\alpha)} \text{ and } \Delta^\alpha R_\eta \Delta^{-\alpha}\xi = L_\xi \Delta^\alpha \eta.$$

E.10.17. Show that the set

$$\{\xi \in \mathfrak{A}' \cap \mathscr{D}_\Delta; \quad \Delta\xi \in \mathfrak{A}'\}$$

is a left Hilbert subalgebra of $\mathfrak{A}''$ and that, for any ξ_1, ξ_2, belonging to this set, the following relation holds

$$\Delta(\xi_1\xi_2) = \Delta(\xi_1)\Delta(\xi_2).$$

E.10.18. For any $\lambda \in [0, 1/2]$, one defines the set

$$\mathfrak{P}_\lambda = \overline{\Delta^\lambda \mathfrak{P}_S}.$$

Show that

(1) $\Delta^{it}\mathfrak{P}_\lambda = \mathfrak{P}_\lambda$;

(2) $J\mathfrak{P}_\lambda = \mathfrak{P}_{\left(\frac{1}{2}-\lambda\right)}$;

(3) $\mathfrak{P}_\lambda$ is a convex cone, polar to $\mathfrak{P}_{\left(\frac{1}{2}-\lambda\right)}$;

(4) $\mathfrak{P}_\lambda = \overline{\{\xi(J\Delta^{\left(\frac{1}{2}-2\lambda\right)}\xi); \quad \xi \in \mathfrak{T}\}}$.

In the following four exercises, $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a hyperstandard von Neumann algebra, whose conjugation is J and whose selfpolar convex cone is $\mathfrak{P}$.

E.10.19. Let Aut $(\mathscr{M})$ be the group of the $*$-automorphisms of $\mathscr{M}$ and $U(\mathscr{M})$ the group of all unitary elements in $\mathscr{M}$. Show that there exists a group homomorphism

$$\text{Aut}(\mathscr{M}) \ni \pi \mapsto u_\pi \in U(\mathscr{M}),$$

which is uniquely determined by the following conditions

(1) $\pi(x) = u_\pi x u_\pi^*$, $\pi \in \text{Aut}(\mathscr{M})$, $x \in \mathscr{M}$;

(2) $u_\pi(\mathfrak{P}) = \mathfrak{P}$, $\pi \in \text{Aut}(\mathscr{M})$.

E.10.20. Show that the von Neumann algebra $\mathscr{M}' \subset \mathscr{B}(\mathscr{H})$ is also hyperstandard, with the same conjugation J and the same self polar convex cone $\mathfrak{P}$.

With the help of Theorem 10.25, infer from this result that for any vector $\zeta \in \mathscr{H}$ there exists a vector $\xi \in \mathfrak{P}$ and a partial isometry $v \in \mathscr{M}$, which are uniquely determined by the properties

$$\zeta = v\xi, \ v^*v = p_\xi.$$

The vector ξ is denoted by $|\zeta|$ and it is called the *modulus* of ζ, whereas the equalities $\zeta = v|\zeta|$, $v^*v = p_{|\zeta|}$ yield the *polar decomposition* of ζ.

E.10.21. Let $\zeta = v|\zeta|$ be the polar decomposition of a vector $\zeta \in \mathscr{H}$. Show that the polar decomposition of $J\zeta$ is

$$J\zeta = v^*([v(JvJ)]|\zeta|).$$

With the help of Corollary 1 from Section 10.24, infer from this result that if $J\zeta = \zeta$, then

$$|\zeta| = \zeta^+ + \zeta^-, \ v = p_{\zeta^+} - p_{\zeta^-},$$

where $\zeta = \zeta^+ - \zeta^-$ is the decomposition $(**)$ from Section 10.23.

E.10.22. Let φ, ψ be normal forms on $\mathscr{M}$. Show that

$$\varphi \leqslant \psi \Rightarrow \varphi^{1/2} \leqslant \psi^{1/2}.$$

E.10.23. Let φ be a faithful, semifinite weight on the von Neumann algebra $\mathscr{M}$. Show that if there exists an increasingly directed family $\{\varphi_\nu\}$ of normal forms on $\mathscr{M}$, such that

$$\varphi(a) = \sup_\nu \varphi_\nu(a), \ a \in \mathscr{M}^+,$$

then φ is normal.

E.10.24. Let $\mathscr{M}$ be a von Neumann algebra, φ a faithful, semifinite normal weight on $\mathscr{M}^+$ and π a $*$-automorphism of $\mathscr{M}$. Show that

$$\sigma_t^{\varphi\circ\pi} = \pi^{-1} \circ \sigma_t^\varphi \circ \pi, \quad t \in \mathbb{R}.$$

In particular, if π leaves invariant the weight φ, then π commutes with σ_t^φ for any $t \in \mathbb{R}$. The case of a trace shows that the converse is not true.

E.10.25. Let $\mathscr{M}$ be a von Neumann algebra and φ a faithful, semifinite, normal weight on $\mathscr{M}^+$, such that the restriction of φ to $(\mathscr{M}_0^\varphi)^+$ be semifinite. Then the von Neumann algebra $\mathscr{M}_0^\varphi$ is semifinite.

E.10.26. Let $\mathscr{M}$ be a von Neumann algebra, Aut($\mathscr{M}$) the group of all the $*$-automorphisms of $\mathscr{M}$ and Int($\mathscr{M}$) the group of all the inner $*$-automorphisms of $\mathscr{M}$. Show that Int($\mathscr{M}$) is an invariant subgroup of Aut($\mathscr{M}$). One denotes by Out($\mathscr{M}$) the quotient group Aut($\mathscr{M}$)/Int($\mathscr{M}$) and by ε the canonical homomorphism

$$\varepsilon : \mathrm{Aut}(\mathscr{M}) \to \mathrm{Out}(\mathscr{M}).$$

Show that if φ and ψ are faithful, semifinite, normal weights on $\mathscr{M}^+$, then

$$\mathfrak{c}(\sigma_t^\varphi) = \mathfrak{c}(\sigma_t^\psi), \quad t \in \mathbb{R}.$$

Show that the mapping

$$\mathbb{R} \ni t \mapsto \mathfrak{c}(\sigma_t^\varphi)$$

is a homomorphism of the additive group $\mathbb{R}$ into the center of the group Out $(\mathscr{M})$, which does not depend on the faithful, semifinite, normal weight φ on $\mathscr{M}^+$.

The kernel of the mapping $t \mapsto \mathfrak{c}(\sigma_t^\varphi)$ is denoted by $T(\mathscr{M})$. Show that if $\mathscr{M}$ is semifinite, then $T(\mathscr{M}) = \mathbb{R}$.

Comments

C.10.1. The theory of the left Hilbert algebras was devised by M. Tomita [10], [11] and it became known through M. Takesaki's lessons [18]. In M. Takesaki's book [18], the left Hilbert algebras appear as *generalized Hilbert algebras* whereas Tomita's algebra is introduced axiomatically as *modular Hilbert algebra* (see C.10.7). Although this terminology is still in use, the terminology we have introduced in our text is becoming more common in the literature. On the other hand, M. Takesaki's notations from [18], which differ from those used in our text, are currently used in the literature and, therefore, we indicate their correspondence with those introduced by us:

Our notations	M.Takesaki's notations (see [18])
$\mathfrak{A}, \mathfrak{A}', \mathfrak{A}''$	$\mathfrak{A}, \mathfrak{A}', \mathfrak{A}''$
$\mathfrak{L}(\mathfrak{A}), \mathfrak{R}(\mathfrak{A}')$	$\mathfrak{L}(\mathfrak{A}), \mathfrak{R}(\mathfrak{A}')$
$\mathscr{D}_S$	$\mathscr{D}^\#$
$S\xi$	$\xi^\#$
$\mathscr{D}_{S^*}$	$\mathscr{D}^\flat$
$S^*\eta$	$\eta^\flat$
L_ξ	$\pi(\xi)$
$L_\xi(\zeta)$	$\xi\zeta$
R_η	$\pi'(\eta)$
$R_\eta(\zeta)$	$\zeta\eta$
S^*	F
$\mathfrak{P}_S$	$\mathfrak{P}^\#$ (or $\mathscr{P}^\#$)
$\mathfrak{P}_{S^*}$	$\mathfrak{P}^\flat$ (or $\mathscr{P}^\flat$)

We give two examples of formulas which correspond to one another under these different notations

$L_\xi(\eta) = SL_{S\eta}S\xi$	$\xi\eta = (\eta^{\#}\xi^{\#})^{\#}$
$\mathfrak{P}_{S^*} = \{R_\eta S^*\eta;\ \eta \in \mathfrak{A}'\}$	$\mathfrak{P}^{\flat} = \{\eta^{\flat}\eta;\ \eta \in \mathfrak{A}'\}$

The left Hilbert algebras $\mathfrak{A} \subset \mathscr{H}$, such that $\mathfrak{A} = \mathfrak{A}''$, are called *maximal* (or "achieved", "full"; "achevée", in French).

For a left Hilbert algebras $\mathfrak{A} \subset \mathscr{H}$, F. Perdrizet [4] also introduced the sets

$$\mathscr{F}^{\#} = \{\xi \in \mathscr{H};\quad L_\xi^0 \text{ is preclosed}\},$$

$$\mathscr{B}^{\#} = \{\xi \in \mathscr{H};\quad L_\xi^0 \text{ is bounded}\},$$

and showed by examples that the inclusions, indicated in the following diagram by arrows

$$\begin{tikzcd} & \mathscr{B}^{\#} \arrow[dr] & & \\ \mathfrak{A}'' \arrow[ur] \arrow[dr] & & \mathscr{F}^{\#} \arrow[r] & \mathscr{H} \\ & \mathscr{D}^{\#} \arrow[ur] & & \end{tikzcd}$$

are, in general, strict inclusions. If, nevertheless, $\mathfrak{A}$ is of the form $\mathfrak{A} = \mathscr{M}\xi_0$ (see Section 10.6), then we obviously have $\mathfrak{A} = \mathscr{B}^{\#}$. F. Perdrizet [4] also introduced the sets

$$\mathfrak{P}_a^{\#} = \{\xi \in \mathfrak{P}^{\#};L_\xi \text{ is self-adjoint}\},$$

$$\mathfrak{A}^+ = \{\xi \in \mathfrak{A}'';\quad L_\xi \geqslant 0\},$$

$$\mathfrak{J}^+ = \{\xi\xi^{\#};\ \xi \in \mathfrak{A}''\},$$

and has shown, by examples, that the inclusions

$$\mathfrak{J}^+ \to \mathfrak{A}^+ \to \mathfrak{P}_a^{\#} \to \mathfrak{P}^{\#}$$

are, in general, strict. If $\mathfrak{A} = \mathscr{M}\xi_0$, then it is obvious that $\mathfrak{J}^+ = \mathfrak{A}^+$.

Similar considerations can be made for $\mathfrak{A}'$, endowed with the involution $\flat$.

C.10.2. The unimodular Hilbert algebras (E.10.5) have been known for a long time as *Hilbert algebras*, or *unitary algebras* and were the basis for obtaining the standard forms of the semifinite von Neumann algebras. Important contributions to this theory have been obtained by W. Ambrose [2], [3], J. Dixmier [19], H. A. Dye [1], R. Godement [6], [8], [11], F. J. Murray and J. von Neumann [1], [3], H. Nakano [2], R. Pallu de la Barrière [3], L. Pukánszky [3], V. Rokhlin [1],

I. E. Segal [1], [6], [11], O. Takenouchi [1], [2], [6], [7], M. Tomita [2], H. Umegaki [3], and others. We mention the fact that J. Dixmier [19] extended the notion of a (unimodular) Hilbert algebra to that of a *quasi-Hilbert algebra* (or *quasi-unitary algebra*) and showed that the set of all continuous complex functions with compact supports, defined on a locally compact topological group, can be canonically endowed with a quasi-Hilbert algebra structure (cf. E.10.3). The results concerning the (quasi-)Hilbert algebras and the standard forms for semifinite von Neumann algebras are set out in the book by J. Dixmier [26] (see also L. H. Loomis [1] and M. A. Rieffel [2]).

Let $\mathfrak{A} \subset \mathscr{H}$ be a unimodular Hilbert algebra. We shall use some notations from C.10.1.

From exercise E.10.5, we infer that $S = J$; hence

$$\mathscr{D}^{\#} = \mathscr{D}^{\flat} = \mathscr{H}$$

and, for any $\zeta \in \mathscr{H}$, we have

$$\zeta^{\#} = \zeta^{\flat} = J\zeta.$$

For any $\xi \in \mathscr{H} (= \mathscr{D}^{\#})$, the operator L_ξ^0 is preclosed and we have $(L_\xi)^* \supset L_{J\xi}$. In fact, in this particular case, we have the equality

$$(L_\xi)^* = L_{J\xi}.$$

Indeed, let $\eta \in \mathscr{D}_{(L_\xi)^*}$. Since $\eta \in \mathscr{H} = \mathscr{D}^{\flat}$, the closed operator R_η makes sense. With the help of Corollary 5 from Section 10.3, we infer the existence of a sequence $\{e_k\} \subset \mathfrak{L}(\mathfrak{A})'$ of projections, such that $e_k \uparrow 1$, and $e_k\eta \in \mathfrak{A}'$, $k=1, 2, \ldots$ Then $e_k\eta \in \mathscr{D}_{L_{J\xi}}$ and since $(L_\xi)^*$ is affiliated to $\mathfrak{L}(\mathfrak{A})$, we have

$$L_{J\xi}e_k\eta = (L_\xi)^*e_k\eta = e_k(L_\xi)^*\eta; \; k = 1, 2, \ldots$$

For $k \to \infty$ we obtain $e_k\eta \to \eta$ and $L_{J\xi}e_k\eta \to (L_\xi)^*\eta$; hence $\eta \in \mathscr{D}_{L_{J\xi}}$ and $L_{J\xi}\eta = (L_\xi)^*\eta$, thereby proving the asserted equality.

It is now easy to verify that

$$\mathfrak{P}_a^{\#} = \mathfrak{P}_a^{\flat} = \mathfrak{P}^{\#} = \mathfrak{P}^{\flat} = \mathfrak{P}.$$

On the other hand, from the equality $J = S$, and from Theorem 10.12, we infer that

$$\mathfrak{A}' = J\mathfrak{A}'' = S\mathfrak{A}'' = \mathfrak{A}'';$$

thus, by taking into account relation 10.4.2, we obtain the "commutation theorem":

$$\mathfrak{L}(\mathfrak{A})' = \mathfrak{R}(\mathfrak{A})$$

(for a simpler direct proof of this equality, see J. Dixmier [26] or M. A. Rieffel [2]).

Since $\mathscr{D}^{\#} = \mathscr{H}$ it is easy to see that

$$\xi \in \mathfrak{A}'', \; x \in \mathfrak{L}(\mathfrak{A}) \Rightarrow x\xi \in \mathfrak{A}'', \; L_{x\xi} = xL_{\xi}.$$

The weight $\varphi_{\mathfrak{A}}$, associated with the unimodular Hilbert algebra $\mathfrak{A}$, is a trace (E.10.5), which is called the *natural trace* associated to $\mathfrak{A}$ and it is usually denoted by $\mu_{\mathfrak{A}}$. Conversely, if μ is a faithful semifinite normal trace on the von Neumann algebra $\mathscr{M}$, then the left Hilbert algebra $\mathfrak{A}_\mu = \mathfrak{N}_\mu = \mathfrak{N}_\mu^*$ (see Section 10.14) is unimodular.

If $\mathscr{M} = \mathscr{B}(\mathscr{H})$ and $\mu = tr$ (see E.10.4) then the operators in $\mathfrak{A}_{tr} = \mathfrak{N}_{tr}$ are called the *Hilbert-Schmidt operators* in the Hilbert space $\mathscr{H}$. We mention that the unimodular Hilbert algebra $\mathfrak{A}_{tr}$ is complete with respect to the scalar product. More precisely, we have the following result, from T. Ogasawara and K. Yoshinaga [4], whose proof can be found in J. Dixmier [26], Prop. 6, Ch. I, § 8.5:

Proposition. *Let $\mathfrak{A}$ be a maximal unimodular Hilbert algebra such that $\mathfrak{L}(\mathfrak{A})$ is a factor. Then the following assertions are equivalent:*

(i) $\mathfrak{A}$ *is complete;*

(ii) $\mathfrak{L}(\mathfrak{A})$ *is of type I;*

(iii) *up to a multiplication of the norm in $\mathfrak{A}$ by a suitable constant, $\mathfrak{A}$ is isomorphic to the unimodular Hilbert algebra of all Hilbert-Schmidt operators on a Hilbert space.*

In accordance with Theorem 10.25, the mapping $\xi \mapsto \omega_\xi$ is a bijection of $\mathfrak{P}$ onto $(\mathfrak{L}(\mathfrak{A})_*)^+$ (this result also has a simpler direct proof; see F. Perdrizet [4], Prop. 3.3). Consequently, given a normal form φ on $\mathfrak{L}(\mathfrak{A})$, there exists a uniquely determined element $\xi \in \mathfrak{P} = \mathfrak{P}_a^{\#}$, such that $\varphi = \omega_\xi$. Then $A = L_\xi$ is a positive self-adjoint operator in $\mathscr{H}$, which is affiliated to $\mathfrak{L}(\mathfrak{A})$. If we denote $e_n = \chi_{(n-1,n)}(A)$, it is easy to infer that $e_n\xi \in \mathfrak{A}^+$, $L_{e_n\xi} = e_nA = Ae_n$ and $e_n\xi \to \xi$. We have

$$\mu_{\mathfrak{A}}(A^2e_n) = \mu_{\mathfrak{A}}((L_{e_n\xi})^*(L_{e_n\xi})) = \|e_n\xi\|^2 \leqslant \|\xi\|^2 < +\infty;$$

hence the operator A is *of summable square with respect to* $\mu_{\mathfrak{A}}$. Moreover, for any $x \in \mathfrak{L}(\mathfrak{A})^+$, we have

$$\begin{aligned} L_AR_A\mu_{\mathfrak{A}}(x) &= \lim_{n\to\infty} \mu_{\mathfrak{A}}(Ae_nxAe_n) = \lim_{n\to\infty} \mu_{\mathfrak{A}}((L_{(x^{1/2}e_n\xi)})^*(L_{(x^{1/2}e_n\xi)})) \\ &= \lim_{n\to\infty} \|x^{1/2}e_n\xi\|^2 = \|x^{1/2}\xi\|^2 = \varphi(x). \end{aligned}$$

Thus,

$$\varphi = L_AR_A\mu_{\mathfrak{A}}.$$

Conversely, let $\mathfrak{A}$ be a positive self-adjoint operator in $\mathscr{H}$, which is affiliated to $\mathfrak{L}(\mathfrak{A})$, of summable square with respect to $\mu_{\mathfrak{A}}$ and such that $\varphi = L_AR_A\mu_{\mathfrak{A}}$. Since $\mu_{\mathfrak{A}}(A^2e_n) < +\infty$, there exists a $\xi_n \in \mathfrak{A}^+$, such that $L_{\xi_n} = Ae_n$. For $n > m$ we have

$$\|\xi_n - \xi_m\|^2 = \mu_{\mathfrak{A}}((Ae_n - Ae_m)^2) = \mu_{\mathfrak{A}}(A(e_n - e_m)A) = \varphi(e_n - e_m);$$

hence $\{\xi_n\}$ is a Cauchy sequence. Let $\xi = \lim_{n\to\infty} \xi_n \in \mathfrak{P} = \mathfrak{P}_a^{\#}$. For any $\eta \in \mathfrak{A}'$ we have

$$L_\xi \eta = R_\eta \xi = \lim_{n\to\infty} R_\eta \xi_n = \lim_{n\to\infty} L_{\xi_n}\eta = \lim_{n\to\infty} Ae_n\eta = A_\eta.$$

Since the operators L_ξ and A are self-adjoint, it follows that $A = L_\xi$. Thus, for any $x \in \mathfrak{L}(\mathfrak{A})^+$, we have

$$\varphi(x) = L_A R_A \mu_{\mathfrak{A}}(x) = \mu_{\mathfrak{A}}((L_{x^{1/2}\xi})^*(L_{x^{1/2}\xi})) = \|x^{1/2}\xi\|^2 = \omega_\xi(x);$$

hence, $\varphi = \omega_\xi$. Consequently, ξ is uniquely determined by φ and $A = L_\xi$ is also uniquely determined by φ.

If we now take into account the possibility of using a $*$-isomorphism (see Section 9.26), from the preceding results we infer the following

Theorem. *Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a semifinite von Neumann algebra and μ a faithful, semifinite, normal trace on $\mathscr{M}^+$. For any normal form φ on $\mathscr{M}$ there exists a unique positive self-adjoint operator A in $\mathscr{H}$, which is affiliated to $\mathscr{M}$ and of summable square with respect to μ, such that*

$$\varphi = L_A L_A \mu.$$

This is the Radon-Nikodym type theorem, with respect to a semifinite, normal trace, obtained by I. E. Segal ([11], Th. 14) and L. Pukánszky ([2], Th. 1). The above proof belongs to F. Perdrizet ([4], Cor. 3.6). We mention the fact that if the trace μ is finite, then the theorem is a particular case of Theorem 10.10; in this case the theorem has been obtained by H. A. Dye ([1], Cor. 5.1).

In Section C.10.4, we shall present an extension of this theorem for weights.

C.10.3. For a weight φ on the von Neumann algebra $\mathscr{M}$ we consider the following properties

($N1$) $\{a_i\} \subset \mathscr{M}^+$, a w-summable family $\Rightarrow \varphi(\sum a_i) = \sum \varphi(a_i)$;

($N2$) $\{a_i\} \subset \mathscr{M}^+$, $a_i \uparrow a \Rightarrow \varphi(a_i) \uparrow \varphi(a)$;

($N3$) φ is lower w-semicontinuous;

($4N$) there exists a family $\{\varphi_\nu\}$ of normal forms on $\mathscr{M}$, such that

$$\varphi(a) = \sup_\nu \varphi_\nu(a), \quad a \in \mathscr{M}^+;$$

($N5$) there exists a family $\{\varphi_i\}$ of normal forms on $\mathscr{M}$, such that

$$\varphi(a) = \sum \varphi_i(a), \quad a \in \mathscr{M}^+.$$

In our text (see Section 10.14) we said that φ is normal if it satisfies property ($N5$). It is obvious that

$$(N5) \Rightarrow (N4) \Rightarrow (N3) \Rightarrow (N2) \Rightarrow (N1),$$

and it is natural to inquire about the equivalence of these properties (see J. Dixmier [26], p. 52—53, 2nd ed.)

Theorem 10.14 retains its validity for the faithful semifinite weights having property ($N4$). More precisely, F. Combes ([10], Th. 2.13) showed that if φ is a faithful semifinite weight on $\mathscr{M}^+$, which has property ($N3$), then $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, endowed with the structure of a $*$-algebra induced by that of $\mathscr{M}$ and with the scalar product induced by that of $\mathscr{H}_\varphi$, is a left Hilbert algebra $\mathfrak{A}_\varphi \subset \mathscr{H}_\varphi$, and $\pi_\varphi(\mathscr{M}) = \mathfrak{L}(\mathfrak{A}_\varphi)$; if φ has property ($N4$), then $\mathfrak{A}_\varphi = \mathfrak{A}_\varphi''$ and

$$\varphi = \varphi_{\mathfrak{A}} \circ \pi_\varphi.$$

A variant of proof for this fact can be found in M. Takesaki's course ([17], 13.5—13.12). At the basis of the proof lies a result about the "ε-filtration" of the normal forms, which are majorized by a weight, result which is due to F. Combes ([7], Lemma 1.9) (see, also, M. Takesaki [17], Th. 13.8, for a simpler form of this result, that which is actually used).

Since the weight which is associated with a left Hilbert algebra has property ($N5$) (in accordance with 10.18), it follows that

$$(N4) \Leftrightarrow (N5).$$

This equivalence has been established by G. K. Pedersen and M. Takesaki ([2], Th. 7.2); in our exposition of the results in Section 10.16.(9)—10.16.(11) and 10.18 we used the main arguments contained in this article.

U. Haagerup [1] completely solved the problem of the equivalence of the above properties, by showing that

$$(N1) \Leftrightarrow (N4).$$

We mention the fact that the elegant arguments in the article of U. Haagerup can be easily read and the equivalence $(N3) \Leftrightarrow (N4)$ is proved in a more general case. Also, U. Haagerup ([1], 1.12) shows by an example, in the commutative case, that the property

$$(N_0)\ \{e_i\} \subset \mathscr{M} \text{ family of orthogonal projections} \Rightarrow \varphi(\textstyle\sum e_i) = \sum \varphi(e_i),$$

is *not* equivalent to the normality (compare with Theorem 5.11) and the problem arises whether a result, analogous to Corollary 5.12, is true for weights.

We mention that the equivalence of the above properties, in the case of traces, is well known since a long time (see J. Dixmier [26], Cor. Prop. 2, Ch. I, § 6.1; see also E.8.10)

C.10.4. With the help of a technique similar to that used in Section 10.18 one can prove the following

Proposition. *Let φ and ψ be faithful, semifinite, normal weights on the von Neumann algebra $\mathscr{M}$. If φ and ψ commute and are equal on a σ_t^φ-invariant $*$-subalgebra of $\mathfrak{M}_\varphi$, which is w-dense in $\mathscr{M}$, then $\varphi = \psi$.*

For the details of the proof we refer to the article of G. K. Pedersen and M. Takesaki ([2], Lemma 5.2, Prop 5.9; see also Prop. 7.8, *loc. cit.*).

In what follows we choose two faithful, semifinite, normal weights φ and ψ on $\mathscr{M}^+$. We shall use the notations from the proof of Theorem 10.28, and we shall also denote (in accordance with A. Connes [6]) by

$$\frac{D\psi}{D\varphi} : t \mapsto \frac{D\psi}{D\varphi}(t)$$

the mapping $t \mapsto u_t$ that was obtained there. We shall show that

$$\textit{if } \frac{D\psi}{D\varphi}(t) = 1, \textit{ for any } t \in \mathbb{R}, \textit{ then } \varphi = \psi.$$

Indeed, from the hypothesis we infer that $\sigma_t^\theta(e_{21}) = e_{21}$, for any $t \in \mathbb{R}$, i.e., $e_{21} \in \mathscr{N}_0^\theta$. In accordance with Theorem 10.27, it follows that

$$x \in \mathfrak{M}_\theta \Rightarrow xe_{21},\ e_{21}x \in \mathfrak{M}_\theta \qquad \text{and } \theta(xe_{21}) = \theta(e_{21}x);$$

hence

$$(x_{ij}) \in \mathfrak{M}_\theta \Rightarrow x_{12} \in \mathfrak{M}_\varphi \cap \mathfrak{M}_\psi \quad \text{and} \quad \varphi(x_{12}) = \psi(x_{12}).$$

If $y_{11},\ y_{21} \in \mathfrak{N}_\varphi$ and $y_{12}, y_{22} \in \mathfrak{N}_\psi$, then $y = (y_{ij}) \in \mathfrak{N}_\theta$, whence

$$a = y_{11}y_{12} + y_{21}y_{22} \in \mathfrak{M}_\varphi \cap \mathfrak{M}_\psi \text{ and } \varphi(a) = \psi(a).$$

In particular, for $y_{11} = y_{12} = 0$, $y_{21} = u \in \mathfrak{N}_\varphi$, $y_{22} = v \in \mathfrak{N}_\psi$, we obtain

$$\text{(1)} \qquad u \in \mathfrak{N}_\varphi,\ v \in \mathfrak{N}_\psi \Rightarrow u^*v \in \mathfrak{M}_\varphi \cap \mathfrak{M}_\psi \text{ and } \varphi(u^*v) = \psi(u^*v).$$

If we make u run over an approximate unit fo $\mathfrak{N}_\varphi$, it follows that $\mathfrak{N}_\psi \subset \subset \overline{\mathfrak{M}_\varphi \cap \mathfrak{M}_\psi}^w$, hence $(\mathfrak{M}_\varphi \cap \mathfrak{M}_\psi)^+$ is a w-dense face of $\mathscr{M}^+$ (see Sections 3.20,3.21). If $a \in (\mathfrak{M}_\varphi \cap \mathfrak{M}_\psi)^+$, then $a^{1/2} \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\psi$ and, by applying the relation (1), in which we make $u = v = a^{1/2}$, we deduce that

$$a \in \mathfrak{M}_\varphi \cap \mathfrak{M}_\psi \Rightarrow \varphi(a) = \psi(a).$$

On the other hand, from the hypothesis it easily follows that the weights φ and ψ commute. Thus, if we now apply the above proposition, we obtain $\varphi = \psi$.

By using the fact $(e_{12})^* = e_{21}$, it is easy to prove that

$$\frac{D\varphi}{D\psi}(t) = \left[\frac{D\psi}{D\varphi}(t)\right]^{-1}, \quad t \in \mathbb{R}.$$

Also, if $\varphi_1, \varphi_2, \varphi_3$ are faithful, semifinite, normal weights on $\mathscr{M}^+$, then

$$\frac{D\varphi_3}{D\varphi_1}(t) = \frac{D\varphi_3}{D\varphi_2}(t) \cdot \frac{D\varphi_2}{D\varphi_1}(t), \quad t \in \mathbb{R}.$$

Indeed, let $\{e_{ij}, i, j = 1, 2, 3\}$ be the matrix units in $\mathrm{Mat}_3(\mathscr{M})$, and let ω be the weight on $\mathrm{Mat}_3(\mathscr{M})^+$, given by

$$\omega(x) = \varphi_1(x_{11}) + \varphi_2(x_{22}) + \varphi_3(x_{33}), \quad x = (x_{ij}).$$

The foregoing equality then follows by an argument similar to that used in the proof of Theorem 10.28, by taking into account the fact that $e_{31} = e_{32}e_{21}$.

Let φ be a faithful, semifinite, normal weight on $\mathscr{M}^+$, and let A be a positive self-adjoint operator in $\mathscr{H}$, which is affiliated to $\mathscr{M}_0^\varphi$; we denote $e_n = \chi_{(n-1,n)}(A)$.

One defines a semifinite, normal weight φ_A *on* $\mathscr{M}^+$ *by the relations*

$$\varphi_A(x) = \lim_{n\to\infty} \varphi((Ae_n)^{1/2} x (Ae_n)^{1/2}), \quad x \in \mathscr{M}^+.$$

If $s(A) = 1$, *then* φ_A *is faithful and*

$$\sigma_t^{\varphi_A} = A^{it}\sigma_t^\varphi(x)A^{-it}, \quad t \in \mathbb{R}, \quad x \in \mathscr{M}.$$

For the proofs we refer the reader to G. K. Pedersen and M. T. Takesaki ([2], § 4)

We define on $\mathrm{Mat}_2(\mathscr{M})^+$ two weights τ and ω by the relations

$$\tau(a) = \varphi(a_{11}) + \varphi_A(a_{22}),\ a = (a_{ij}),$$

$$\omega(a) = \varphi(a_{11}) + \varphi(a_{22}),\ a = (a_{ij}).$$

Then $\tau = \omega_B$, where $B = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$; hence

$$\sigma_t^\tau(x) = B^{it}\sigma_t^\omega(x)B^{-it}, \quad t \in \mathbb{R}, \quad x \in \mathrm{Mat}_2(\mathscr{M}).$$

In particular, if we make $x = e_{21}$, we obtain

$$\frac{D\varphi_A}{D\varphi}(t) = A^{it}, \quad t \in \mathbb{R}.$$

From the preceding results we infer the following Radon-Nikodym type theorem for weights, due to G. K. Pedersen and M. Takesaki ([2], Th. 5.12).

Theorem. *Let* φ *and* ψ *be two faithful, semifinite, normal weights on the von Neumann algebra* $\mathscr{M}$. *If* φ *and* ψ *commute, then there exists a uniquely determined positive self-adjoint operator* A *in* $\mathscr{H}$, *which is affiliated to* $\mathscr{M}_0^\varphi$, *such that* $s(A) = 1$ *and* $\psi = \varphi_A$.

Indeed, since φ and ψ commute, from the proof of Corollary 10.28 we infer that $\left\{\frac{D\psi}{D\varphi}(t)\right\}_{t\in\mathbb{R}}$ is a *so*-continuous group of unitary operators in $\mathscr{M}_0^\varphi$. From the Stone theorem (see 9.20) we infer that there exists a positive self-adjoint operator A in $\mathscr{H}$, such that $s(A) = 1$ and

$$\frac{D\psi}{D\varphi}(t) = A^{it}, \quad t \in \mathbb{R}.$$

In accordance with exercise E.9.25, A is affiliated to $\mathscr{M}_0^\varphi$; hence we can define the weight φ_A and we have

$$\frac{D\varphi_A}{D\varphi}(t) = A^{it}, \quad t \in \mathbb{R}.$$

Thus,

$$\frac{D\varphi_A}{D\psi}(t) = \frac{D\varphi_A}{D\varphi}(t)\frac{D\varphi}{D\psi}(t) = A^{it}(A^{it})^{-1} = 1, \quad t \in \mathbb{R};$$

Consequently, we have

$$\psi = \varphi_A.$$

The uniqueness of the operator A immediately follows from the uniqueness of the analytic generator in Stone's theorem*).

It is obvious that any weight commutes with any trace; hence, the theorem of I. E. Segal and L. Pukánszky, we have stated in section C.10.2, is a particular case of the theorem of G. K. Pedersen and M. Takesaki.

For a faithful, semifinite, normal weight φ on $\mathscr{M}^+$, $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ is the only group of $*$-automorphisms of $\mathscr{M}$, with respect to which φ satisfies the KMS-conditions (see Section 10.17). Consequently, if ψ is another faithful, semifinite, normal weight, which satisfies the KMS-conditions with respect to $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$, then $\sigma_t^\psi = \sigma_t^\varphi$, $t \in \mathbb{R}$. We hence infer that φ and ψ commute; hence, in accordance with the above theorem, there exists a positive self-adjoint operator A in $\mathscr{H}$, which is affiliated to $\mathscr{M}_0^\varphi$, such that $s(A) = 1$ and $\psi = \varphi_A$. Since

$$\sigma_t^\varphi(x) = \sigma_t^\psi(x) = \sigma_t^{\varphi_A}(x) = A^{it}\sigma_t^\varphi(x)A^{-it}, \quad x \in \mathscr{M},$$

it follows that A^{it} belongs to the center $\mathscr{Z}$ of $\mathscr{M}$, for any $t \in \mathbb{R}$; hence A is affiliated to $\mathscr{Z}$. We thus obtain the following

Corollary 1. *Let φ and ψ be faithful, semifinite, normal weights on $\mathscr{M}^+$. The following assertions are then equivalent:*

(i) *ψ satisfies the (KMS)-condition with respect to $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$;*

(ii) $\sigma_t^\psi = \sigma_t^\varphi$, $t \in \mathbb{R}$;

(iii) *there exists a positive self-adjoint operator A in $\mathscr{H}$, which is affiliated to $\mathscr{Z}$, such that $s(A) = 1$ and $\psi = \varphi_A$.*

The assertions in this corollary are true, for example, if φ and ψ are faithful semifinite normal traces on $\mathscr{M}^+$; if, moreover, $\psi \leqslant \varphi$, then $A \in \mathscr{Z}$, $0 \leqslant A \leqslant 1$. In particular, we have the following

Corollary 2. *Let $\mathscr{M}$ be a factor and φ, ψ two semifinite normal traces on $\mathscr{M}^+$. Then there exists a $\lambda \geqslant 0$ such that $\psi = \lambda\varphi$.*

Indeed, the corollary follows from the facts that the support of a trace is a central projection and $\varphi + \psi$ is also a semifinite normal trace, whereas $\varphi \leqslant \varphi + \psi$, $\psi \leqslant \varphi + \psi$, etc.

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*) This method if proving the theorem of G. K. Pedersen and M. Takesaki was communicated to us by Gr. Arsene. In a recent paper, G. A. Elliott [15] gives a yet simpler proof, and also indicates technical simplifications for the proofs of the results in Sections 10.18, 10.27 and 10.28.

The preceding results concerning the traces have direct and simpler proofs (see J. Dixmier [26], Ch. I., § 6.4). For finite traces they can easily be obtained from the Radon-Nikodym type theorem of Sakai (see E.7.14).

The problem now arises whether the theorem in this section can be extended for weights which do not commute. Theorem 10.10 is such an extension for the case in which φ and ψ are normal forms. In the article of G. K. Pedersen and M. Takesaki [2] another partial extension is given (*loc. cit.*, Prop. 7.6), as well as a negative result (*loc. cit.*, Prop. 7.7).

Another theorem of the Radon-Nikodym type, for weights, was obtained by A. van Daele [5], who generalized a theorem of S. Sakai for normal forms (see C.5.5).

C.10.5. Let φ and ψ be faithful normal forms on the von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$. We recall that by $|\varphi \pm i\psi|$ we denote the modulus of the w-continuous linear form $\varphi \pm i\psi$, in accordance with Theorem 5.16. M. Takesaki ([18], Th. 15.2) and R. H. Herman and M. Takesaki ([1], Th. 1, Th. 2) proved the following results:

Proposition 1. *The following assertions are equivalent:*

(i) *φ and ψ commute;*

(ii) *$\{\sigma_t^\varphi\}$ and $\{\sigma_s^\psi\}$ commute: $\sigma_t^\varphi \circ \sigma_s^\psi = \sigma_s^\psi \circ \sigma_t^\varphi$, $t, s \in \mathbb{R}$;*

(iii) $|\varphi + i\psi| = |\varphi - i\psi|$.

Proposition 2. *If π is a $*$-automorphism of $\mathcal{M}$, which acts identically on the center, then the following assertions are equivalent:*

(i) *φ is π-invariant: $\varphi \circ \pi = \varphi$;*

(ii) *π commutes with $\{\sigma_t^\varphi\}$: $\pi \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \pi$, $t \in \mathbb{R}$.*

The proofs of these propositions can also be found in the course of M. Takesaki ([17], 15.14—15.18).

Let now φ and ψ be faithful, semifinite, normal weights on $\mathcal{M}^+$. The problem now arises whether the equivalences (i) $\Leftrightarrow$ (ii) in the two propositions remain true. If π is an arbitrary $*$-automorphism of $\mathcal{M}$, then, with the help of the KMS-conditions, it is easy to prove that

$$\sigma_t^{\varphi\circ\pi} = \pi^{-1} \circ \sigma_t^\varphi \circ \pi, \quad t \in \mathbb{R}.$$

Thus it is obvious that the implications (i) $\Rightarrow$ (ii) in both propositions remain true for weights, too. Nevertheless, the converse implications are not true, in general, as G. K. Pedersen and M. Takesaki have shown ([2], Prop. 5.11). In the presence of some additional hypotheses, the equivalence (i) $\Leftrightarrow$ (ii) from Proposition 1 retains its validity for weights, too (see G. K. Pedersen and M. Takesaki [2], Lemma 5.8, Prop. 6.1, Cor. 6.4, Th. 6.6). We also mention that from condition (ii) (in Proposition 2), it follows that the weight $\varphi + \psi$ is semifinite (in accordance with *loc.cit.*, Prop. 5.10).

Let us now assume that φ and ψ are faithful forms on the hyperstandard von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and that $\psi \leqslant \varphi$. On the one hand we have the "derivative" $\dfrac{d\psi}{d\varphi}$, introduced in Section 10.25, on the other hand we have the "derivative" given by the theorem of Sakai (5.21). It is obvious that the two "derivatives" coincide iff $\frac{d\psi}{d\varphi} \geqslant 0$. One can prove the following result (see H. Araki [27], Th. 13):

Proposition 3. *The following assertions are equivalent:*

(i) $\left(\frac{d\psi}{d\varphi}\right)^* = \frac{d\psi}{d\varphi}$;

(ii) $\frac{d\varphi}{d\psi} \geqslant 0$;

(iii) $\sigma_t^\varphi \left(\frac{d\psi}{d\varphi}\right) = \frac{d\psi}{d\varphi}$, $t \in \mathbb{R}$;

(iv) ψ and φ *commute.*

C.10.6. The problem of the continuous dependence of the group $\{\sigma_t^\varphi\}$ of modular automorphisms with respect to the faithful normal form φ has been solved by A. Connes ([3], Th. 1) by the following

Theorem. *Let φ_n, φ be faithful normal forms on the von Neumann algebra $\mathcal{M}$ If $\|\varphi_n - \varphi\| \to 0$, then*

$$\sigma_t^{\varphi_n}(x) \xrightarrow{so} \sigma_t^\varphi(x), \quad t \in \mathbb{R}, \quad x \in \mathcal{M},$$

and the convergence is uniform with respect to t, for $|t| \leqslant t_0$.

A proof of this theorem, based on the methods developed in Sections 10.23—10.25, can be found in H. Araki ([27], Th. 10). See also A. Connes [26] for another result of this kind.

C.10.7. One calls a *modular Hilbert algebra* (or a *Tomita algebra*) a complex algebra $\mathfrak{A}$ with an involution $\xi \mapsto \xi^\#$, endowed also with a scalar product $(\cdot|\cdot)$ and with a group of algebra automorphisms $\{\Delta(\alpha)\}_{\alpha\in\mathbb{C}}$, depending on a complex parameter, which satisfies axioms (i), (ii), (iii) from Section 10.1, and also the axioms:

(IV) $(\Delta(\alpha)\xi)^\# = \Delta(-\overline{\alpha})\xi^\#$, for any $\xi \in \mathfrak{A}$, $\alpha \in \mathbb{C}$;

(V) $(\Delta(\alpha)\xi|\eta) = (\xi|\Delta(\overline{\alpha})\eta)$, for any $\xi, \eta \in \mathfrak{A}$, $\alpha \in \mathbb{C}$;

(VI) $(\Delta(1)\xi^\#|\eta^\#) = (\eta|\xi)$, for any $\xi, \eta \in \mathfrak{A}$;

(VII) $\mathbb{C} \ni \alpha \mapsto (\Delta(\alpha)\xi|\eta) \in \mathbb{C}$ is an entire analytic function, for any $\xi, \eta \in \mathfrak{A}$;

(VIII) $(1 + \Delta(t))\mathfrak{A}$ is dense in $\mathfrak{A}$, for any $t \in \mathbb{R}$.

It is easy to see that the Tomita algebra, associated to a left Hilbert algebra, is, in a natural manner, a modular Hilbert algebra, with

$$\Delta(\alpha) = \Delta^\alpha|\mathfrak{A}, \quad \alpha \in \mathbb{C}.$$

Let now $\mathfrak{A}$ be a modular Hilbert algebra and $\mathcal{H}$ the Hilbert space obtained by the completion of $\mathfrak{A}$. From axiom (V) one infers that $\Delta(it)$ is an isometric mapping; hence, by denoting by u_t the closure of $\Delta(it)$, we obtain a one-parameter group $\{u_t\}_{t\in\mathbb{R}}$ of unitary operators on $\mathcal{H}$. In accordance with the Stone theorem, there exists a positive self-adjoint operator Δ in $\mathscr{H}$, such that $s(\Delta) = 1$ and $u_t = \Delta^{it}$, $t \in \mathbb{R}$. One can then prove that

$$\Delta^{\alpha} \text{ is the closure of } \Delta(\alpha), \ \alpha \in \mathbb{C}.$$

The operator Δ is called the *modular operator* associated to the modular Hilbert algebra $\mathfrak{A}$. For any $t \in \mathbb{R}$, the mapping

$$\mathfrak{A} \ni \xi \mapsto \Delta^t \xi^{\#} \in \mathfrak{A}$$

is an involution in $\mathfrak{A}$, which is compatible with its algebra structure. It is easy to prove that the involution corresponding to $t = 1/2$ extends to a conjugation J of $\mathscr{H}$, called the *canonical conjugation* associated to the modular Hilbert algebra $\mathfrak{A}$. On the other hand, the involution corresponding to $t = 1$, i.e., $\eta \mapsto \eta^{\flat} = \Delta\eta^{\#}$, is called the *adjoint involution* and it has the property that

$$(\xi|\eta) = (\eta^{\flat}|\xi^{\#}), \quad \xi, \eta \in \mathfrak{A}.$$

Hence one can immediately infer that $\mathfrak{A}$ also satisfies axiom (iv) from Section 10.1, hence $\mathfrak{A}$ *is a left Hilbert algebra.* It is easy to prove that Δ and J are associated to the structure of a left Hilbert algebra of $\mathfrak{A}$, as in Section 10.1, i.e., $S = J\Delta^{1/2}$. For details we refer to M. Takesaki ([17], [18]).

The modular Hilbert algebras are a useful tool for the computations (see, for example, M. Takesaki [33]).

C.10.8. Bibliographical comments. In Sections 10.1—10.6, which contain the "elementary" part of the Tomita theory, we followed the lessons of M. Takesaki [18], but the systematic use of Proposition 10.3, exhibited by A. van Daele ([4], Lemma 2.6) allowed the simplification of the exposition given by M. Takesaki [18].

The commutation theorem for tensor products (10.7) has been known for a long time, for semifinite von Neumann algebras, and conjectured in general (J. Dixmier [26], Ch. I, § 2.4, § 6.9). S. Sakai [23] proved this theorem by assuming that only one of the two von Neumann algebras is semifinite. The general case was obtained by M. Tomita [10], [11], as a corollary of his main results. The direct proof we presented here was obtained by I. Cuculescu [5] and S. Sakai [32] (see, also, M. Takesaki [24], and L. Zsidó [1]). Recently, M. A. Rieffel and A. von Daele [1] obtained a simple proof of another nature, which does not use the theory of unbounded operators. See also R. Rousseau, A. van Daele and L. van Heeswijck [1].

The cones $\mathfrak{P}_S$ and $\mathfrak{P}_{S^*}$ (§ 10.9) were introduced by M. Takesaki [18], for the case $\mathfrak{A} = \mathscr{M}\xi_0$, and by F. Perdrizet [4] in the general case. Lemma 10.9, which is the main argument in the proof of the general Radon-Nikodym type theorem for normal forms (10.10), is due to M. Takesaki [18]. We mention that M. Takesaki [18] also gives a new proof to the Radon-Nikodym type theorem of Sakai, which is based on elementary results from the theory of Tomita.

Tomita's fundamental theorem (10.12) allows the conclusion that any von Neumann algebra is $*$-isomorphic to a standard von Neumann algebra (10.15), a result which concludes a long series of efforts in the development of the operator algebras theory, highlighted by the works of F. J. Murray and J. von Neumann [1],

[2], H. A. Dye [1], I. E. Segal [11], J. Dixmier [19], L. Pukánszky [3], M. Tomita [8], and others (see C.10.2). The proof of Theorem 10.12, given in the text, is due to A. van Daele [4] and it is different and simpler than the proofs given by M. Tomita, in [10], and by M. Takesaki, in [18]. Another proof, given by L. Zsidó [6], is indicated in exercise E.10.13*).

The culminating point in Tomita's theory, and the main technical instrument for handling the left Hilbert algebras, is the theorem on the existence of the Tomita algebra (10.20). For the exposition given in Sections 10.19—10.21 we developed the ideas from the article of L. Zsidó [6], the analytic continuation methods we use (Sections 9.15, 9.24) originating in the article by G. K. Pedersen and M. Takesaki [2], which also suggested to us a part of the criterion 10.21. The proof of Theorem 10.20, thus obtained, is different from that given by M. Takesaki in [18]. The algebra $\mathfrak{S}$ (Section 10.22) was first considered by L. Zsidó [6].

The weights were introduced by F. Combes [7] and G. K. Pedersen [1], whereas the link existing between the weights and the left Hilbert algebras was established by F. Combes [10] and M. Tomita [10]. For the exposition given by us in Sections 10.16 and 10.18 we used the articles by F. Combes [10], G. K. Pedersen and M. Takesaki [2], as well as the course by M. Takesaki [17].

The KMS-condition originates in theoretical physics, and it was framed into the theory of operator algebras by R. Haag, N. M. Hugenholtz and M. Winnink [1], who showed that, given a C^*-algebra, endowed with a one-parameter group of $*$-automorphisms, the cyclic representation associated to a positive form, which satisfies the KMS-condition, is standard. Another application of the KMS-conditions was given by N. M. Hugenholtz [1]. These papers appeared at the same time as M. Tomita's papers [10], [11] whereas M. Takesaki ([18], § 13) found the deep link between the Tomita theory and the KMS-condition, by proving Theorem 10.17 for the case of the faithful normal forms. Subsequently, F. Combes [12] and M. Takesaki ([17], Th. 14.6) proved a variant of Theorem 10.17 for the case of the weights. The KMS-condition for weights is similar to the condition $\varphi(xy) = \varphi(yx)$, which is characteristic of the traces. For various results on the KMS-condition, we refer to: H. Araki [8], H. Araki and H. Miyata [1], F. Combes [10], [12], N. M. Hugenholtz [2], O. Bratteli, A. Kishimoto and D. W. Robinson [1], D. W. Robinson [3], F. Rocca, M. Sirugue and D. Testard [1], [4], M. Sirugue and M. Winnink [1], [2], [3], M. Takesaki [17], [18], [19], [26], [27], M. Winnink [2]. We also mention the works which gave a name to the KMS-condition: R. Kubo, J. Phys. Soc. Japan, 12 (1957), p. 570, and P. C. Martin, J. Schwinger, Phys. Rev., 115 (1959), p. 1342.

The results in Section 10.27 are due to G. K. Pedersen and M. Takesaki ([2], § 3), whereas Theorem 10.28 is due to A. Connes [4], [6]. In our exposition we have used these sources.

The characterization of the semifinite von Neumann algebras in terms of the group of the modular automorphisms (Section 10.29) was obtained by M. Takesaki ([18], § 14) by a very intricate proof. A simpler proof was given by G. K. Pe-

*) In the case $\mathfrak{A} = \mathscr{M}\xi_0$, A. van Daele [3] gives for Proposition 10.11 a proof similar to the proof of Sakai's theorem, mentioned in C.5.5.

dersen and M. Takesaki ([2], Th. 7.4), on the basis of their theorem of the Radon-Nikodym type for weights (see C.10.4). The proof given in our text does not explicitly use this theorem, but the theorem of A. Connes.

The results in Sections 10.23–10.26 are from H. Araki [27], A. Connes [4], [7] and U. Haagerup [2]. In our exposition we followed the preprint of U. Haagerup [2]. The sets $\mathfrak{P}_\lambda$ (E.10.18) were introduced by H. Araki [27], [28], who also proved Radon-Nikodym type theorems relatively to these sets. A. Connes [7] found a characterization of the von Neumann algebras as ordered vector spaces, thus giving an answer to a problem posed by S. Sakai [4]. U. Haagerup [2] made a comprehensive study of the hyperstandard von Neumann algebras.

One of the most important applications of Tomita's theory concerns the classification of the factors of type III. In this direction we mention the wealth and depth of the results obtained by A. Connes [1—13], [19], [23], [24], [26]. Other important results were obtained by M. Takesaki [28], [29] and E. Störmer [15], [20].

A remarkable application of Tomita's theory to the structure of type III von Neumann algebras was found by M. Takesaki [30], [31], [32], [33], who showed that any type III von Neumann algebra is, in a unique manner, the cross-product of a type II_∞ von Neumann algebra by a one-parameter group of $*$-automorphisms. Particular cases of this theorem were previously proved by A. Connes [6] and M. Takesaki [28], [29].

For other results and applications concerning the Tomita theory we refer the reader to the Proceedings of some recent International Conferences on Operator Algebras such as: *C*-algebras and their applications to statistical mechanics and quantum field theory*, North-Holland, 1976; *Symposia Mathematica*, vol. XX, Academic Press, 1975; *Operator Algebras and their Applications to Mathematical Physics* (Conference held in Marseille, June, 1977; the book, edited by CNRS, is in preparation).

Appendix

Fixed point theorems

In this Appendix we shall prove Ryll-Nardzewski's fixed point theorem, which we used in Chapter 7.

A.1. Let $\mathfrak{X}$ be a vector space and $\mathscr{K} \subset \mathfrak{X}$ a convex set. A mapping $T : \mathscr{K} \to \mathscr{K}$ is said to be *affine* if for any $x_1, x_2 \in \mathscr{K}$ and any $\lambda \in [0, 1]$ we have

$$T(\lambda x_1 + (1-\lambda)x_2) = \lambda T(x_1) + (1-\lambda)T(x_2).$$

Lemma (Markov-Kakutani). *Let $\mathfrak{X}$ be a Hausdorff topological vector space, $\mathscr{K} \subset \mathfrak{X}$ a non-empty, compact, convex subset and $\mathfrak{T}$ a family of continuous, affine mappings of $\mathscr{K}$ into $\mathscr{K}$, which are mutually commuting. Then there exists an $x_0 \in \mathscr{K}$, such that*

$$Tx_0 = x_0, \quad T \in \mathfrak{T}.$$

Proof. For any $T \in \mathfrak{T}$ and any natural number n we write

$$T_n = \frac{1}{n}(I + T + \ldots + T^{n-1}).$$

The sets $T_n(\mathscr{K})$ are compact subsets of $\mathscr{K}$. For any $T^{(1)}, T^{(2)}, \ldots, T^{(k)} \in \mathfrak{T}$ and any natural numbers $n_1, \ldots, n_k$ we have

$$(T^{(1)})_{n_1} \ldots (T^{(k)})_{n_k} \mathscr{K} \subset \bigcap_{i=1}^{k} (T^{(i)})_{n_i}\mathscr{K};$$

hence

$$\bigcap_{i=1}^{k} (T^{(i)})_{n_i}\mathscr{K} \neq \emptyset.$$

Consequently, there exists an

$$x_0 \in \bigcap_{\substack{T \in \mathfrak{T} \\ n \in \mathbb{N}}} T_n(\mathscr{K}) \subset \mathscr{K}.$$

Let now $T \in \mathfrak{T}$ be arbitrary. Since the set $\mathscr{K} - \mathscr{K}$ is compact, for any neighbourhood $\mathscr{U}$ of the origin, there exists a natural number n, such that

$$\frac{1}{n}(\mathscr{K} - \mathscr{K}) \subset \mathscr{U}.$$

Since $x_0 \in T_n(\mathscr{K})$, there exists an $x \in \mathscr{K}$, such that $x_0 = T_n x$; consequently

$$Tx_0 - x_0 = \frac{1}{n}(T^n x - x) \in \frac{1}{n}(\mathscr{K} - \mathscr{K}) \subset \mathscr{U}.$$

Since $\mathscr{X}$ is separated and $\mathscr{U}$ was an arbitrary neighbourhood of the origin,

$$Tx_0 = x_0.$$

Q.E.D.

A.2. Let p be a seminorm on the vector space $\mathscr{X}$. For any subset $\mathscr{S} \subset \mathscr{X}$ let us denote

$$p\text{-diam}(\mathscr{S}) = \sup_{x, y \in \mathscr{S}} p(x - y).$$

Lemma (Namioka-Asplund [1]). *Let $\mathscr{X}$ be a Hausdorff locally convex vector space' $\mathscr{K} \subset \mathscr{X}$ a non-empty, separable, weakly compact, convex subset, p a continuous seminorm on $\mathscr{X}$ and $\varepsilon > 0$. Then there exists a closed convex subset $\mathscr{C} \subset \mathscr{X}$, such that*

$$\mathscr{C} \neq \mathscr{K} \textit{ and } p\text{-diam}(\mathscr{K} \setminus \mathscr{C}) \leqslant \varepsilon.$$

Proof. Let

$$\mathscr{U} = \{x;\ x \in \mathscr{X},\ p(x) \leqslant \varepsilon/4\}.$$

Since $\mathscr{K}$ is separable, there exists a sequence $\{x_n\} \subset \mathscr{K}$, such that

$$\mathscr{K} \subset \bigcup_{i=1}^{\infty} (x_i + \mathscr{U}).$$

We denote by $\mathscr{E}$ the weak closure of the set of all extreme points of $\mathscr{K}$. Since $\mathscr{E}$ is weakly compact, and the sets $x_i + \mathscr{U}$ are weakly closed, from

$$\mathscr{E} \subset \bigcup_{i=1}^{\infty} (x_i + \mathscr{U}),$$

and by taking into account the theorem of Baire, we infer that there exists an index i_0, and a weakly open subset $\mathscr{D} \subset \mathscr{X}$, such that

$$\varnothing \neq \mathscr{E} \cap \mathscr{D} \subset \mathscr{E} \cap (x_{i_0} + \mathscr{U}).$$

Let $\mathscr{K}_1$ be the closed convex hull of $\mathscr{E} \setminus \mathscr{D}$ and $\mathscr{K}_2$ the closed convex hull of $\mathscr{E} \cap \mathscr{D}$. From the Krein-Milman theorem, $\mathscr{K}$ is the convex hull of $\mathscr{K}_1 \cup \mathscr{K}_2$. Since the set $\mathscr{E} \setminus \mathscr{D}$ is weakly compact, it contains all the extreme points of $\mathscr{K}_1$ (by virtue of the converse Milman theorem; see R. R. Phelps [1], Ch. 1). Since $\mathscr{E} \cap \mathscr{D} \neq \varnothing$, it follows that $\mathscr{K}_1 \neq \mathscr{K}$. Obviously,

$$p\text{-diam}(\mathscr{K}_2) \leqslant p\text{-diam}(x_{i_0} + \mathscr{U}) \leqslant \frac{\varepsilon}{2}.$$

We denote $d = p\text{-diam}(\mathscr{K})$ and we consider a number δ, such that $0 < \delta < \min\{1, \varepsilon/4d\}$. The set

$$\mathscr{C} = \{\lambda y_1 + (1 - \lambda) y_2;\ y_1 \in \mathscr{K}_1,\ y_2 \in \mathscr{K}_2,\ \delta \leqslant \lambda \leqslant 1\},$$

is a closed convex subset of $\mathscr{K}$.

Let us assume that $\mathscr{C} = \mathscr{K}$. Let x be an extreme point of $\mathscr{K}$. Then there exist $y_1 \in \mathscr{K}_1$, $y_2 \in \mathscr{K}_2$, $\lambda \in [\delta, 1]$, such that

$$x = \lambda y_1 + (1 - \lambda) y_2.$$

If $\lambda = 1$, then $x = y_1$; if $\lambda < 1$, since x is extreme, it follows that $x = y_1 = y_2$. In both cases, $x = y_1 \in \mathscr{K}_1$. Since x is an arbitrary extreme point of $\mathscr{K}$, we obtain $\mathscr{K} \subset \mathscr{K}_1$, thus contradicting the fact that $\mathscr{K}_1 \neq \mathscr{K}$.

Consequently, $\mathscr{C} \neq \mathscr{K}$.

Since $\mathscr{K}$ is the convex hull of $\mathscr{K}_1 \cup \mathscr{K}_2$, for any $y \in \mathscr{K} \setminus \mathscr{C}$ there exist $y_1 \in \mathscr{K}_1$, $y_2 \in \mathscr{K}_2$, $\lambda \in [0, \delta]$, such that

$$y = \lambda y_1 + (1 - \lambda) y_2.$$

Thus,

$$p(y - y_2) = \lambda p(y_1 - y_2) \leqslant \delta d.$$

Since p-diam$(\mathscr{K}_2) \leqslant \varepsilon/2$, it follows that

$$p\text{-diam}(\mathscr{K} \setminus \mathscr{C}) \leqslant 2\delta d + \varepsilon/2 \leqslant \varepsilon.$$

Q.E.D.

A.3. Let $\mathfrak{X}$ be a locally convex vector space, $\mathscr{S} \subset \mathfrak{X}$ and $\mathfrak{S}$ a semigroup of mappings of $\mathscr{S}$ into $\mathscr{S}$. One says that $\mathfrak{S}$ is *non-contracting* if, for any $x, y \in \mathscr{S}$, $x \neq y$, there exists a continuous seminorm p on $\mathfrak{X}$, such that

$$\inf_{T \in \mathfrak{S}} p(Tx - Ty) > 0.$$

Theorem (Ryll-Nardzewski [1]). *Let $\mathfrak{X}$ be a Hausdorff locally convex vector space, $\mathscr{K} \subset \mathfrak{X}$ a non-empty, weakly compact, convex subset and $\mathfrak{S}$ a non-contracting semigroup of weakly continuous affine mappings of $\mathscr{K}$ into $\mathscr{K}$. Then there exists an $x_0 \in \mathscr{K}$, such that*

$$Tx_0 = x_0, \qquad T \in \mathfrak{S}.$$

Proof. Let $T_1, \ldots, T_n \in \mathfrak{S}$ and let

$$T_0 = \frac{1}{n}(T_1 + \ldots + T_n).$$

In accordance with Lemma A.1, there exists an $x_0 \in \mathscr{K}$, such that $T_0 x_0 = x_0$.

Let us assume that there exists an index i, $1 \leqslant i \leqslant n$, such that $T_i x_0 \neq x_0$. We can assume that

$$T_i x_0 \neq x_0, \qquad \text{for } 1 \leqslant i \leqslant m,$$

$$T_i x_0 = x_0, \qquad \text{for } i > m.$$

If we denote $T_0' = \dfrac{1}{m}(T_1 + \ldots + T_m)$, we have $T_0' x_0 = x_0$.

Since $\mathcal{S}$ is non-contracting, there exists a continuous seminorm p on $\mathscr{X}$ and an $\varepsilon > 0$, such that

$$(*) \qquad p(TT_i x_0 - Tx_0) \geqslant 2\varepsilon, \qquad T \in \mathcal{S}, \qquad 1 \leqslant i \leqslant m.$$

Let $\mathcal{S}_0$ be the subsemigroup of $\mathcal{S}$, generated by $T_1, \ldots, T_m$ and let $\mathscr{K}_0$ be the weakly closed convex hull of the set $\{Tx_0;\ T \in {}_0\}$. Obviously, $\mathscr{K}_0$ is a nonempty, separable, weakly compact, convex subset of $\mathscr{X}$. In accordance with Lemma A.2, there exists a closed convex subset $\mathscr{C}_0 \subset \mathscr{K}_0$, such that $\mathscr{C}_0 \neq \mathscr{K}_0$ and $p\text{-diam}(\mathscr{K}_0 \diagdown \mathscr{C}_0) \leqslant \varepsilon$.

Since $\mathscr{C}_0 \neq \mathscr{K}_0$, there exists a $S_0 \in \mathcal{S}_0$, such that $S_0 x_0 \in \mathscr{K}_0 \diagdown \mathscr{C}_0$. From the equality $T_0' x_0 = x_0$, we infer that

$$S_0 x_0 = \frac{1}{m}(S_0 T_1 x_0 + \ldots + S_0 T_m x_0).$$

Thus, there exists an index i, $1 \leqslant i \leqslant n$, such that $S_0 T_i x_0 \in \mathscr{K}_0 \diagdown \mathscr{C}_0$. Then

$$p(S_0 T_i x_0 - S_0 x_0) \leqslant p\text{-diam}(\mathscr{K}_0 \diagdown \mathscr{C}_0) \leqslant \varepsilon,$$

and this contradicts the relation $(*)$.

Consequently, any finite subset of $\mathcal{S}$ has a common fixed point. A familiar compactness argument shows that $\mathcal{S}$ has a fixed point.

Q.E.D.

A bibliography on operator algebras and related topics*)

The bibliography of the 1975 edition of this book has been updated by Grigore Arsene and Şerban Strătilă, using also the INCREST Preprints no. 6/1976, 5/1977. Thanks are due to the National Institute for Scientific and Technical Creation, Bucharest, for technical assistance.

The titles without references are preprints or works in preparation.

AARNES, J.

1. The Vitali-Hahn-Saks theorem for von Neumann algebras, *Math. Scand.*, 18 (1966), 87–92.
2. On the Mackey topology for von Neumann algebras, *Math. Scand.*, 22 (1968), 87–107.
3. On the continuity of automorphic representations of groups, *Comm. Math. Phys.*, 7 (1968), 332–336.
4. Physical states of C^*-algebras, *Acta Math.*, 122 (1969), 161–172.
5. Continuity of group representations with applications to C^*-algebras, *J. Fnal Analysis*, 5 (1970), 14–36.
6. Quasi-states on C^*-algebras, *Trans. AMS*, 149 (1970), 601–625.
7. Full sets of states on a C^*-algebra, *Math. Scand.*, 26 (1970), 141–148.
8. Problems (on operator algebras), *Proc. Fnal Analysis Week*, Aarhus Univ., 1969, 64–65.
9. Differentiable representations, I. Induced representations and Frobenius reciprocity, *Trans. AMS*, 220 (1976), 1–36.

AARNES, J.; EFFROS, E. C.; NIELSEN, O. A.

1. Locally compact spaces and two classes of C^*-algebras, *Pacific J. Math.*, 34 (1970), 1–16.

AARNES, J.; KADISON, R. V.

1. Pure states and approximate identities, *Proc. AMS*, 21 (1969), 749–752.

ABDALLA, J.; SZÜCS, J.

1. On an ergodic type theorem for von Neumann algebras, *Acta Sci. Math.*, 36 (1974), 167–172.

ABRAHAMSE, M.; DOUGLAS, R. G.

1. A class of subnormal operators related to multiply connected domains, *Adv. Math.*, 19 (1976), 106–148.

ACCARDI, L.

1. On the connection between the probabilistic and the Hilbert-space description of dynamical systems, *Lettere al Nuovo Cimento*, Serie 2, 8 (1973), 585–589.
2. On the non-commutative Markof property (Russian), *Funkt. Analiz i pril.*, 9: 1 (1975), 1–8.
3. Non-commutative Markof chains, *Proc. Summer School in Math. Phys.*, Camerino, 1974, 268–295.
4. Non-relativistic quantum mechanics as a non-commutative Markof process, in *Symposia Math.*, XX, 365–394, Academic Press, 1976; *Adv. Math.*, 20 (1976), 329–366.

---

*) A current bibliography of preprints on operator algebras is being prepared by the Centre de Physique Théorique, CNRS, Marseille, under the title "C^*-News".

5. *L'edificio matematico della meccanica quantistica nonrelativistica: situazione attuale*, Univ. Salerno.
6. Local perturbations of conditional expectations, Marseille, 1976.
7. Non-commutative Markof chains associated to a preassigned evolution: an application to the quantum theory of measurement.

AHIEZER, N. I.; GLAZMAN, I. M.
1. *The theory of linear operators in Hilbert spaces* (Russian), Moscow, 1976.

AIKEN, J.
1. A problem of Dyer, Porcelli and Rosenfeld, *Israel J. Math.*, **24** (1976), 191–198.
2. On perturbations of operators in the shift algebra.
3. *A perturbation property of W^*-algebras*, Dissertation, 1973.

AIZENMANN, M.; GALLAVOTTI, G.; GOLDSTEIN, S.; LEBOWITZ, J. L.
1. Stability and equilibrium states of infinite classical systems, *Comm. Math. Phys.*, **48** (1976), 1–14.

AKEMANN, C. A.
1. The dual space of an operator algebra, *Trans. AMS*, **126** (1967), 286–302.
2. Projections onto separable C^*-algebras of a W^*-algebra, *Bull. AMS*, **73** (1967), 925.
3. Sequential convergence in the dual of a W^*-algebra, *Comm. Math. Phys.*, **7** (1968), 222–224.
4. Interpolation in W^*-algebras, *Duke Math. J.*, **35** (1968), 525–533.
5. The general Stone-Weierstrass problem, *J. Fnal Analysis*, **4** (1969), 277–294.
6. Left ideal structure of C^*-algebras, *J. Fnal Analysis*, **6** (1970), 305–318.
7. Separable representations of a W^*-algebra, *Proc. AMS*, **24** (1970), 354–355.
8. Approximate units and maximal abelian C^*-subalgebras, *Pacific J. Math.*, **33** (1970), 543–550.
9. A Gelfand representation for arbitrary C^*-algebras, *Pacific J. Math.*, **39** (1971), 1–12.
10. A simpler proof that a type I group having a discrete dual space is compact.
11. Non-abelian Pontryagin duality, *Lect. Notes Math.*, No. 399, 1974, 18–21.

AKEMANN, C. A.; DOODS, P. G.; GAMLEN, J. L. B.
1. Weak compactness in the dual space of a C^*-algebra, *J. Fnal Analysis*, **10** (1972), 446–450.

AKEMANN, C. A.; ELLIOTT, G. A.; PEDERSEN, G. K.; TOMIYAMA, J.
1. Derivations and multipliers of C^*-algebras, *Amer. J. Math.*, **98** (1976), 679–708.

AKEMANN, C. A.; NEWBERGER, S. M.
1. Physical states on a C^*-algebra, *Proc. AMS*, **40** (1973), 500.

AKEMANN, C. A.; OSTRAND, P. A.
1. Computing norms in group C^*-algebras, *Amer. J. Math.*, **98** (1976), 1015–1047.
2. On a tensor product C^*-algebra associated with the free group of two generators, *J. Math. Soc. Japan*, **27** (1975), 589–599.
3. The spectrum of a derivation of a C^*-algebra, *J. London Math. Soc.*, **13** (1976), 525–530.

AKEMANN, C. A.; PEDERSEN, G. K.
1. Complications of semicontinuity in C^*-algebra, *Duke Math. J.*, **40** (1973), 785–796.
2. Ideal perturbations of elements in C^*-algebras, *Math. Scand.*, **41** (1977), 117–139.
3. Central sequences and inner derivations of separable C^*-algebras, Copenhagen, 1977.

AKEMANN, C. A.; PEDERSEN, G. K.; TOMIYAMA, J.
1. Multipliers of C^*-algebras, *J. Fnal Analysis*, **13** (1973), 277–301.

AKEMANN, C. A.; ROSENFELD, M.
1. Maximal one-sided ideals in operator algebras, *Amer. J. Math.*, **94** (1972), 723–728.

AKEMANN, C. A.; RUSSO, B.
1. Geometry of the unit sphere of a C^*-algebra and its dual, *Pacific J. Math.*, **32** (1970), 575–585.

AKEMANN, C. A.; WALTER, M. E.
1. Non-abelian Pontriagin duality, *Duke Math. J.*, 39 (1972), 451–464.
2. The Riemann-Lebesgue property for arbitrary locally compact groups, *Duke Math. J.*, 43 (1976), 225–236.

ALBERTI, P. M.
1. On states of a type II_1 factor, Leipzig, 1975.
2. A theorem on the comparability of projections infinite or countably decomposable infinite W^*-algebras, Leipzig, 1976.
3. On maximally unitarily mixed states on W^*-algebras; the central ideal in properly infinite W^*-algebras, Leipzig, 1976.
4. A Σ-property for positive linear forms on C^*-algebras, Leipzig, 1976.

ALBEVERIO, S.; HOEGH-KROHN, R.
1. Dirichlet forms and Markof semigroups on C^*-algebras, *Comm. Math. Phys.*, 56 (1977), 173–188.
2. Frobenius theory for positive maps of von Neumann algebras.
3. Ergodic actions by compact groups on von Neumann algebras, Marseille, 1978.
4. The method of Dirichlet forms, Marseille, 1978.
5. The structure of diffusion processes, Marseille, 1978.

ALEXANDER, F. E.; TOMIUK, B.
1. Complemented B^*-algebras, *Trans. AMS*, 137 (1969), 459–480.

ALFSEN, E. M.
1. *Convex compact sets and boundary integrals*, Springer Verlag, 1971.

ALFSEN, E. M.; EFFROS, E. G.
1. Structure in real Banach spaces, I, II, *Ann. Math.*, 96 (1976), 98–128; 129–174.

ALFSEN, E. M.; SCHULTZ, F. W.
1. On the geometry of non-commutative spectral theory, *Bull. AMS*, 81 (1975), 893–895.
2. *Non-commutative spectral theory for affine function space on convex sets*, Memoirs AMS, No. 172, 1976.
3. State spaces of Jordan algebras, Oslo, 1977.
4. On non-commutative spectral theory and Jordan algebras, Oslo, 1977.

ALFSEN, E. M.; SCHULTZ, F. W.; STÖRMER, E.
1. A Gelfand-Neumark theorem for Jordan algebras, *Adv. Math.*

ALLAN, G. R.
1. A note on B^*-algebras, *Proc. Cambridge Phil. Soc.*, 61 (1965), 29–32.

ALLAN, G. R.; SINCLAIR, A. M.
1. Power factorizations in Banach algebras with a bounded approximate identity, *Studia Math.*, 56 (1976), 31–38.

AMAR, E.
1. Représentations d'algèbres d'opérateurs, *C. R. Acad. Sc. Paris*, 280 (1975), 1507–1509.

AMBROSE, W.
1. Spectral resolution of unitary operators, *Duke Math. J.*, 2 (1944), 589–595.
2. Structure theorem for a special class of Banach algebras, *Trans. AMS*, 57 (1945), 364–386.
3. The L^2-system of a unimodular group, I, *Trans. AMS*, 65 (1949), 27–48.

AMEMIYA, I.; ARAKI, H.
1. A remark on Piron's paper, *Publ. RIMS, Kyoto Univ.*, 2 (1967), 423–427.

ANASTASIO, S.
1. Maximal abelian subalgebras in hyperfinite factors, *Amer. J. Math.*, 87 (1965), 955–971.
2. Non-normal abelian subalgebras, *Tohoku Math. J.*, 20 (1968), 26–37.
3. Some recent results in W^*-algebras, *Trans. New York Acad. Sc.*, 33 (1971), 405–410.

ANASTASIO, S.; WILLIG, P.
1. *The structure of factors*, Algorithmics Press, New York, 1974.

ANDERSEN, T. B.
1. On multipliers and order bounded operators in C^*-algebras, *Proc. AMS*, **25** (1970), 890–900.
2. Linear extensions, projections and split faces, *J. Fnal Analysis*, **17** (1974), 161–173.

ANDERSON, J.
1. Extreme points in sets of positive linear maps of $B(H)$.
2. On pure states of C^*-subalgebras, *Proc. AMS*, **55** (1976), 471.
3. Extensions, restrictions and representations of states on C^*-algebras.
4. A C^*-algebra for which Ext (A) is not a group.
5. On vector states and separable C^*-algebras, *Proc. AMS*, **65** (1977), 62–64.
6. A maximal abelian subalgebra of the Calkin algebra with the extension property.

ANDERSON, J.; BUNCE, J. W.
1. A type II_∞ factor representation of the Calkin algebra, *Amer. J. Math.*, **99** (1977), 515–521.

ANDERSON, J.; BUNCE, J. W.; DEDDENS, J. A.; WILLIAMS, J. P.
1. C^*-algebras and derivation ranges; D-symmetric operators.

ANDERSSON, S. I.
1. Properties of equilibrium states on quasilocal C^*-algebras, *Fortschr. Phys.*, **24**: 2 (1976), 55–83.

ANSHELEVIČ, V. V.
1. Central limit theorem for "non-commutative" stationary processes (Russian), *Uspekhi Mat. Nauk SSSR*, **28**: 5 (1973), 227–228.
2. Central limit theorem in "non-commutative" probability theory (Russian), *Dokl. Akad. Nauk SSSR*, **208**: 6 (1973), 1265–1267.

APOSTOL, C.
1. Spectral theory and functional calculus (Romanian), *Stud. Cerc. Mat.*, **20** (1968), 635–668.
2. Commutators on Hilbert space, *Rev. Roum. Math. Pures Appl.*, **18** (1973), 1013–1024.
3. Inner derivations with closed range, *Rev. Roum. Math. Pures Appl.*, **21** (1976), 249–265.

APOSTOL, C.; FOIAŞ, C.; VOICULESCU, D.
1. On strongly reductive algebras, *Rev. Roum. Math. Pures Appl.*, **21** (1976), 633–641.

APOSTOL, C.; ZSIDÓ, L.
1. Ideals in W^*-algebras and the function η of A. Brown and C. Pearcy, *Rev. Roum. Math. Pures Appl.*, **18** (1973), 1151–1170.

ARAKI, H.
1. Hamiltonian formalism and canonical commutation relations in quantum field theory, *J. Math. Phys.*, **1** (1960), 492–504.
2. *Einführung in die axiomatische Quantenfeldtheorie*, I, II, Lecture Notes, ETH Zürich, 1961/62.
3. A lattice of von Neumann algebras associated with quantum theory of free Boson field, *J. Math. Phys.*, **4** (1963), 1343–1362.
4. A generalization of Borchers' theorem, *Helv. Acta Phys.*, **36** (1963), 132–139.
5. Type of von Neumann algebras associated with free field, *Progr. Theor. Phys.*, **32** (1964), 956–965.
6. On the algebra of all local observables, *Progr. Theor. Phys.*, **32** (1964), 844–854.
7. Von Neumann algebras of local observables for free scalar field, *J. Math. Phys.*, **5** (1964), 1–13.
8. Multiple time analyticity of a quantum statistical state satisfying the KMS boundary conditions, *Publ. RIMS, Kyoto Univ.*, **4** (1968), 361–371.
9. On the diagonalization of a bilinear Hamiltonian by a Bogoliubov transformation, *Publ. RIMS, Kyoto Univ.*, **4** (1968), 387–413.
10. Local quantum theory, I, *Proc. Int. School Phys. "E. Fermi"*, **15** (1968), 65–96.
11. Systèmes à un nombre infini de degrés de liberté, *Colloques Int. CNRS*, **181** (1969), 75–86.

12. Gibbs states of a one-dimensional quantum lattice, *Comm. Math. Phys.*, 14 (1969), 120–157.
13. A classification of factors, II, *Publ. RIMS, Kyoto Univ.*, 4 (1969), 585–595.
14. Product states, *Cargese Lectures in Phys.*, 4 (1970), 1–30.
15. Factorizable representation of current algebras. Non-commutative extensions of Levi-Khinchin formula and cohomology of solvable groups with values in a Hilbert space, *Publ. RIMS, Kyoto Univ.*, 5 (1970), 361–422.
16. On the representations of the canonical commutation relations, *Comm. Math. Phys.*, 20 (1971), 9–24.
17. A remark on Bures distance function for normal states, *Publ. RIMS, Kyoto Univ.*, 6 (1971), 477–482.
18. On quasifree states of canonical anticommutation relations and Bogoliubiv automorphisms, *Publ. RIMS, Kyoto Univ.*, 6 (1971), 385–442.
19. Asymptotic ratio set and property L'_λ, *Publ. RIMS, Kyoto Univ.*, 6 (1971), 443–460.
20. On quasifree states of the canonical commutation relations II, *Publ. RIMS, Kyoto Univ.*, 7 (1971), 121–152.
21. Bures distance function and a generalization of Sakai's non-commutative Radon-Nikodym theorem, *Publ. RIMS, Kyoto Univ.*, 8 (1972), 335–362.
22. Remarks on spectra of modular operators of von Neumann algebras, *Comm. Math. Phys.*, 28 (1972), 267–278.
23. Normal positive linear mappings of norm 1 from a von Neumann algebra into its commutant and applications, *Publ. RIMS, Kyoto Univ.*, 8 (1973), 439–470.
24. Expansional in Banach algebras, *Ann. Éc. Norm. Sup.*, 6 (1973), 67–84.
25. Structure of some von Neumann algebras with isolated discrete modular spectrum, *Publ. RIMS, Kyoto Univ.*, 9 (1973), 1–44.
26. Relative Hamiltonian for faithful normal states of von Neumann algebras, *Publ. RIMS, Kyoto Univ.*, 9 (1973), 165–210.
27. Some properties of the modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain rule, *Pacific J. Math.*, 50 (1974), 309–354.
28. One-parameter family of Radon-Nikodym theorems for states of a von Neumann algebra, *Publ. RIMS, Kyoto Univ.*, 10 (1974), 1–10.
29. Golden-Thompson and Peierls-Bogoliubov inequalities for a general von Neumann algebra, *Comm. Math. Phys.*, 34 (1973), 167–178.
30. Introduction to operator algebras, in *Statistical Mechanics and Field Theory Lectures*, 1971, Haifa Summer School, New York-Jerusalem-London, 1972, 1–26.
31. Classifications of factors, in *Statistical Mechanics and Field Theory Lectures*, 1971, Haifa Summer School, New York-Jerusalem-London, 1972, 27–30.
32. Positive cone, Radon-Nikodym theorems, relative Hamiltonian and the Gibbs condition in statistical mechanics. An application of Tomita-Takesaki theory, in "*C^*-algebras and their applications to statistical mechanics and quantum field theory*", p. 64–100, North-Holland, 1976.
33. On the equivalence of the *KMS* conditions and the variational principle for quantum lattices systems, *Comm. Math. Phys.*, 38 (1974), 1–10.
34. Recent developments in the theory of operator algebras and their significance in theoretical physics, *Symposia Math.*, XX, 395–424, Academic Press, 1976.
35. On uniqueness of *KMS* states of one-dimensional quantum lattice systems, *Comm. Math. Phys.*, 44 (1975), 1–8.
36. Inequalities in von Neumann algebra, Kyoto, 1975.
37. Relative entropy and its applications, Coll. Int. CNRS, No. 248; *Les méthodes mathématiques de la théorie quantique des champs*, Marseille 1975, 61–79.
38. Relative entropy of states of von Neumann algebras, I, II, *Publ. RIMS, Kyoto Univ.*, 11 (1976), 809–834; 13 (1977), 173–192.
39. (Editor) *International Symposium on Mathematical Problems in Theoretical Physics*, Lecture Notes in Physics, 39 (1975), XII, 562 p., Springer Verlag.
40. Relative Hamiltonian for states of von Neumann algebras (Russian), *Trudy Mat. Inst. Akad. Nauk SSSR*, 135 (1975), 18–25.
41. Introduction to relative Hamiltonian and relative entropy, Lecture given at Marseille, 1975.

42. On *KMS* states of a C^*-dynamical systems, *Japan-US Seminary*, 1977.
43. Contact points of mathematics and physics, ZIF, Bielefeld, 1976.

ARAKI, H.; ELLIOTT, G. A.
1. On the definition of C^*-algebras, *Publ. RIMS, Kyoto Univ.*, **9** (1973), 93–112.

ARAKI, H.; HAAG, R.
1. Collision cross section in terms of local observables, *Comm. Math. Phys.*, **4** (1966).

ARAKI, H.; ION, P. D. F.
1. On the equivalence of KMS and Gibbs conditions for states of quantum lattice systems, *Comm. Math. Phys.*, **35** (1974), 1–12.

ARAKI, H.; KASTLER, D.; TAKESAKI, M.; HAAG, R.
1. Extension of *KMS* states and chemical potential, *Comm. Math. Phys.*, **53** (1977), 97–134.

ARAKI, H.; KISHIMOTO, A.
1. On clustering property, *Rep. Math. Phys.*, **10** (1976), 275–281.
2. Symmetry and equilibrium states, *Comm. Math. Phys.*, **52** (1977), 211–232.

ARAKI, H.; LIEB, E. H.
1. Entropy inequalities, *Comm. Math. Phys.*, **18** (1970), 160–170.

ARAKI, H.; MIYATA, H.
1. On *KMS* boundary condition, *Publ. RIMS, Kyoto Univ.*, **4** (1968), 373–385.

ARAKI, H.; NAKAGAMI, Y.
1. A remark on an infinite tensor product of von Neumann algebras, *Publ. RIMS, Kyoto Univ.*, **8** (1972), 363–374.

ARAKI, H.; SEWELL, G. L.
1. *KMS* conditions and local thermodynamical stability of quantum lattice systems, *Comm. Math. Phys.*, **52** (1977), 103–110.

ARAKI, H.; SHIRAISHI, M.
1. On quasifree states of the canonical commutation relations, I, *Publ. RIMS, Kyoto Univ.*, **7** (1971), 105–120.

ARAKI, H.; SMITH, M. S. B.; SMITH, L.
1. On the homotopical significance of the type of a von Neumann algebra factor, *Comm. Math. Phys.*, **22** (1971), 71–88.

ARAKI, H.; WOODS, E. J.
1. Representations of the canonical commutation relations describing a non-relativistic infinite free Bose gas, *J. Math. Phys.*, **4** (1963), 637–662.
2. Complete boolean algebras of type I factors, *Publ. RIMS, Kyoto Univ.*, **2** (1966), 157–242 (Corrections, ibidem, 451–452).
3. A classification of factors, *Publ. RIMS, Kyoto Univ.*, **3** (1968), 51–130.
4. Topologies induced by representations of the canonical commutation relations, *Rep. Math. Phys.*, **4** (1973), 227–254.

ARAKI, H.; WYSS, W.
1. Representations of canonical anticommutation relations, *Helv. Acta Phys.*, **37** (1964), 136–159.

ARCHBOLD, R. J.
1. Prime C^*-algebras and antilattices, *Proc. London Math. Soc.*, **24** (1972), 669–680.
2. Order and commutativity in C^*-algebras, *Proc. Cambridge Phil. Soc.*, **76** (1974), 153–156.
3. *Certain properties of operator algebras*, Thesis, Univ. of Newcastle upon Tyne, 1972.
4. Density theorems for the centre of a C^*-algebra, *J. London Math. Soc.*, **10** (1975), 189–197.
5. On the centre of a tensor product of C^*-algebras, *J. London Math. Soc.*, **10** (1975), 257–262.
6. On commuting C^*-algebras of operators, *Math. Scand.*, **38** (1976), 106–114.
7. A mean ergodic theorem associated with the free group on two generators, *J. London Math. Soc.*, **13** (1976), 339–345.
8. An averaging process for C^*-algebras related to weighted shifts, *Proc. London Math. Soc.*, **35** (1977), 541–554.

9. On the norm of an inner derivation of a C^*-algebra.
10. A note on the $*$-operator on an $[FIA]^-$ group.

ARENS, R.
1. On a theorem of Gelfand and Neumark, *Proc. Nat. Acad. Sci. USA*, 32 (1946), 237–239.
2. Representations of $*$-algebras, *Duke Math. J.*, 14 (1947), 269–282.
3. The adjoint of a bilinear operation, *Proc. AMS*, 2 (1951), 839–848.

ARENS, R.; KAPLANSKY, I.
1. Topological representations of algebras, *Trans. AMS*, 63 (1948), 457–481.

ARNAL, D.; JURZAK, J. P.
1. Topological aspects of algebras of unbounded operators, *J. Fnal Analysis*, 24 (1977), 397–425.

ARSENE, GR.
1. Non-commutative integration (Romanian), (unpublished, Bucharest, 1966).
2. Topologies on operator spaces (Romanian), in *Operator theory and operator algebras*, 81–92, Editura Academiei, Bucharest, 1973.

ARSENE, GR.; CEAUŞESCU, Z.
1. On intertwining dilations. IV, *Tohoku. Math. J.*, 30 (1978), 423–438.

ARSENE, GR.; ZSIDÓ, L.
1. Une propriété de type de Darboux dans les algèbres de von Neumann, *Acta Sci. Math.*, 30 (1969), 195–198.

ARVESON, W. B.
1. A theorem on the action of abelian unitary groups, *Pacific J. Math.*, 16 (1966), 205–212.
2. An algebraic conjugacy invariant for measure-preserving transformations, *Bull. AMS*, 73 (1967), 121–125.
3. Analyticity in operator algebras, *Amer. J. Math.*, 89 (1967), 578–642.
4. A density theorem for operator algebras, *Duke Math. J.*, 34 (1967), 635–647.
5. Operator algebras and measure preserving automorphisms, *Acta Math.*, 118 (1967), 95–109.
6. On subalgebras of C^*-algebras, *Bull. AMS*, 75 (1969), 790–794.
7. Subalgebras of C^*-algebras, I, II, *Acta Math.*, 123 (1969), 141–244; 128 (1972), 271–308.
8. Unitary invariants for compact operators, *Bull. AMS*, 76 (1970), 88–91.
9. Lattices of invariant subspaces, *Bull. AMS*, 78 (1972), 515–519.
10. On groups of automorphisms of operator algebras, *J. Fnal Analysis*, 15 (1974), 217–243.
11. A note on essentially normal operators, *Proc. Roy Irish. Acad.*, 74 (1974), 143–146.
12. A note on maximal subdiagonal algebras.
13. Operator algebras and invariant subspaces, *Ann. Math.*, 100 (1974), 433–532.
14. Representations of C^*-algebras, (unpublished).
15. Interpolation problems in nest algebras, *J. Fnal Analysis*, 20 (1975), 208–233.
16. Spectral theory for non-linear random processes, *Symposia Math.*, XX, 531–538, Academic Press, 1976.
17. Representations of unitary groups, *Japan-US Seminary*, 1977.
18. Notes on extensions of C^*-algebras, *Duke Math. J.*, 44 (1977), 329–357.
19. *An invitation to C^*-algebras*, Springer Verlag, 1976.

ARVESON, W. B.; FELDMAN, J.
1. A note on invariant subspaces, *Michigan Math. J.*, 15 (1968), 61–65.

ARVESON, W. B.; JOSEPHSON, K. B.
1. Operator algebras and measure preserving automorphisms, II, *J. Fnal Analysis*, 4 (1969), 100–135.

ARZUMANIAN, V. A.
1. Factor states on crossed products constructed from dynamical systems (Russian), *Izv. Akad. Nauk Armen. SSSR*, Matem., 10:3 (1975), 255–263.

ARZUMANIAN, V. A.; VERŠIK, A. M.
1. Factor representations of the crossed product of an abelian C^*-algebra and a semi-group of its endomorphisms (Russian), *Dokl. Akad Nauk*, 238 (1978), 513–516.

ASIMOW, L. A.; ELLIS, A. J.
1. On hermitian functionals on unital Banach algebras, *Bull. London Math. Soc.*, **4** (1972), 333–336.

ATIYAH, M. F.
1. Elliptic operators, discrete groups and von Neumann algebras, *Astérisque*, No. **32–33**, 1976, 43–72.

ATIYAH, M. F.; SCHMID, W.
1. A geometric construction of the discrete series for semisimple Lie groups, *Invent. Math.*, **42** (1977), 1–62.

AUBERT, P.-L.
1. *Automorphismes de W^*-algèbres*, Thèse, Neuchâtel, 1976.
2. Théorie de Galois pour une W^*-algèbre, *Comment. Math. Helvet.*, **51** (1976), 411–433.

AUPÉTIT, B.
1. Le théorème de Russo-Dye pour les algèbres de Banach involutives, *CR Acad. Sc. Paris*, **284** (1977), 151–153.

AUSLANDER, L.; MOORE, C. C.
1. *Unitary representations of solvable Lie groups*, Memoirs AMS, No. 62, 1966.

AZOFF, E. A.
1. Spectrum and direct integral, *Trans. AMS*, **197** (1974), 211–223.
2. Invariant linear manifolds and the selfadjointness of operator algebras, *Amer. J. Math.*, **99** (1977), 121–138.

AZOFF, E. A.; FONG, C. K.; GILFEATHER, F.
1. A reduction theory for non-self-adjoint operator algebras, *Trans. AMS*, **224** (1976), 351–366.

AZOFF, E.; GILFEATHER, F.
1. Measurable choice and the invariant subspace problem, *Bull. AMS*, **80** (1974), 893–895.

BADE, W. G.
1. On boolean algebras of projections and algebras of operators, *Trans. AMS*, **86** (1955), 345–360.
2. *The Banach space $C(S)$*, Aarhus Universitet, Lecture Notes, No. 26, 1971.

BADE, W. G.; CURTIS, P. C. JR.
1. The continuity of derivations of Banach algebras, *J. Fnal Analysis*, **16** (1974), 377–387.
2. Homomorphisms of commutative Banach algebras, *Amer. J. Math.*, **82** (1960), 589–608.

BAKER, B. M.
1. Free states of the gauge invariant canonical anticommutation relations, I, II, *Trans. AMS*.

BALSLEV, E.; MANUCEAU, J.; VERBEURE, A.
1. Representations of anticommutation relations and Bogoliubov transformations, *Comm. Math. Phys.*, **8** (1968), 315–326.

BALSLEV, E.; VERBEURE, A.
1. States on Clifford algebras, *Comm. Math. Phys.*, **7** (1968), 55–76.

BARGMANN, V.
1. Irreducible unitary representations of the Lorentz group, *Ann. Math.*, **48** (1947), 568–640.
2. On the unitary ray representations of continuous groups, *Ann. Math.*, **59** (1954), 1–46.
3. On the representations of the rotation group, *Rev. Mod. Phys.*, **34** (1962), 829.

BARNES, B. A.
1. Algebras with spectral expansion property, *Illinois J. Math.*, **11** (1967), 284–290.
2. On the existence of minimal ideals on a Banach algebra, *Trans. AMS*, **133** (1968), 511–518.
3. Algebras with minimal left ideals which are Hilbert spaces, *Pacific J. Math.*, **35** (1970), 537–548.

4. Closed one-sided ideals in certain B^*-algebras, *Illinois J. Math.*, **15** (1971), 515–524.
5. Irreducible algebras of operators which contain a minimal idempotent, *Proc. AMS*, **30** (1971), 337–342.
6. Examples of modular annihilator algebras, *Rocky Mountain J. Math.*, **1** (1971), 657–665.
7. Pure states with restriction property, *Proc. AMS*, 33 (1972), 491–494.
8. Strictly irreducible $*$-representations of Banach $*$-algebras, *Trans. AMS*, **170** (1972), 459–470.
9. Density theorems for algebras of operators and annihilator Banach algebras, *Michigan Math. J.*, **19** (1972), 149–155.
10. Locally B^*-equivalent representations, I, II, *Trans. AMS*, **167** (1972), 435–442; **176** (1973), 297–304.
11. Operators with a strictly cyclic vector, *Proc. AMS*, **41** (1973), 480–486.
12. Representations of B^*-algebras on Banach spaces, *Pacific J. Math.*, **50** (1974), 7–18.
13. Representations of normed algebras with minimal left ideals, *Proc. Edinburgh Math. Soc.*, **19** (1974), 173–190.
14. Linear functionals continuous on abelian $*$-subalgebras of a B^*-algebra, *J. London Math. Soc.*, **10** (1975), 320–328.
15. The similarity problem for representations of a B^*-algebra, *Michigan Math. J.*, **22** (1975), 25–32.
16. The thin operators relative to an ideal of a von Neumann algebra, *Acta Sci. Math.*, **38** (1976), 265–274.
17. When is a representation of a Banach $*$-algebra Naimark related to a $*$-representation, *Pacific J. Math.*, **72** (1977), 5–26.
18. Ideal and representation theory of the L^1-algebra of a group with polynomial growth.

BARNES, B. A.; DUNCAN, J.
1. The Banach algebra $l^1(S)$, *J. Fnal Analysis*, **18** (1975), 96–113.

DE BARRA, G.
1. An inner measure associated with a von Neumann algebra, *Proc. Cambridge Phil. Soc.*, **64** (1968), 645–650.
2. A class of measurable sets associated with a von Neumann algebra, *J. London Math. Soc.*, **5** (1972), 123–126.
3. Some algebras of operators with closed convex numerical range, *Proc. Roy. Irish Acad.*, **72** (1972), 149–154.

BATTY, C. J. K.
1. On relative commutants in tensor products of C^*-algebras, *Math. Z.*, **151** (1976), 215–218.
2. Unbounded derivations of commutative C^*-algebras.
3. Dissipative mappings and well-behaved derivations.

BEAUME, R.; MANUCEAU, J.; PELLET, A.; SIRUGUE, M.
1. Translation invariant states in quantum mechanics, *Comm. Math. Phys.*, **38** (1974), 29–45.

BEHNCKE, H.
1. Structure of certain non-normal operators, I, II, *J. Math. Mech.*, **18** (1968), 103–107; *Indiana Univ. Math. J.*, **22** (1972), 301–308.
2. A remark on C^*-algebras, *Comm. Math. Phys.*, **12** (1969), 142–144.
3. Automorphisms of crossed products, *Tohoku Math. J.*, **21** (1969), 580–600.
4. A central decomposition of automorphisms, *Math. Ann.*, **189** (1970), 308–310.
5. Generators of W^*-algebras, I, II, III, *Tohoku Math. J.*, **22** (1970), 541–546; **24** (1972), 371–382; 383–388.
6. A note on the Gelfand-Neumark conjecture, *Comm. Pure Appl. Math.*, **23** (1970), 189–200.
7. Projections in Hilbert spaces, I, II, *Tohoku Math. J.*, **22** (1970), 181–183; **23** (1971), 349–352.
8. Nilpotent elements in group algebras, *Bull. Acad. Polon. Sci.*, **19** (1971), 197–198.
9. Generators of finite W^*-algebras, *Tohoku Math. J.*, **24** (1972), 401–408.
10. Topics in C^*- and von Neumann algebras, *Lecture Notes in Math.*, No. 247, 1–54, Springer Verlag, 1972.

11. Nilpotent elements in Banach algebras, *Proc. AMS*, **37** (1973), 137–141.
12. Outer automorphic representations of locally compact groups on certain W^*-algebras.
13. C^*-algebras with completely ord ered system of ideals.
14. Finite dimensional perturbations.

BEHNCKE, H.; BÖS, W.
1. A class of C^*-algebras, *Proc. AMS*, **40** (1973), 128–134.
2. *JB*-algebras with exceptional ideals.

BEHNCKE, H.; CUNTZ, J.
1. Local completeness of operator algebras, *Proc. AMS*, **62** (1977), 95–100.

BEHNCKE, H.; KRAUS, F.; LEPTIN, H.
1. C^*-Algebren mit geordneten ideal Folgen, *J. Fnal Analysis*, **10** (1972), 204–211.

BEHNCKE, H.; LEPTIN, H.
1. C^*-algebras with two point dual, *J. Fnal Analysis*, **10** (1972), 330–335.
2. C^*-algebras with finite duals, *J. Fnal Analysis*, **14** (1973), 253–268.
3. Classification of C^*-algebras with a finite dual, *J. Fnal Analysis*, **16** (1974), 241–257.

BEKES, R. A.
1. Algebraically irreducible representations of $L^1(G)$, *Pacific J. Math.*, **60** (1975), 11–26.

BELLISSARD, J.; JOCHUM, B.
1. Homogeneous selfdual cones versus Jordan algebras, ZIF Bielefeld, 1976.
2. Spectral theory for homogeneous symmetric selfdual cones, Marseille, 1978.

BELLISSARD, J.; JOCHUM, B.; LIMA, R.
1. Cones autopolaires homogènes et facialement homogènes, *CR Acad. Sc. Paris*, **282** (1976), 1363–1366.

BENDAT, J.; SHERMAN, S.
1. Monotone and convex operator functions, *Trans. AMS*, **79** (1955), 58–71.

BENFATTO, G.; NICOLO, F.
1. The local von Neumann algebras for the massless scalar free field and the free electromagnetic field.

BENNETT, G.
1. Some ideals of operators on Hilbert space, *Studia Math.*, **55** (1976), 27–40.

BERBERIAN, S. K.
1. On the projection geometry of a finite AW^*-algebra, *Trans. AMS*, **83** (1956), 493–509.
2. Reduction of the projection geometry of finite AW^*-algebras, *Bull. AMS*, **62** (1956), 348.
3. The regular ring of a finite AW^*-algebra, *Ann. Math.*, **65** (1957), 224–240.
4. $N \times N$ matrices over an AW^*-algebra, *Amer. J. Math.*, **80** (1958), 37–44.
5. Note on a theorem of Fuglede and Putnam, *Proc. AMS*, **10** (1959), 175–182.
6. *Notes on spectral theory*, Van Nostrand, 1966.
7. Trace and convex hull of the spectrum in a von Neumann algebra of finite class, *Proc. AMS*, **23** (1969), 211–212.
8. A note on the algebra of measurable operators of an AW^*-algebra, *Tohoku Math. J.*, **22** (1970), 613–618.
9. The regular ring of a finite Baer $*$-ring, *J. Algebra*, **23** (1972), 35–65.
10. Equivalence of projections, *Proc. AMS*, **33** (1972), 485–490.
11. *Baer $*$-rings*, Springer Verlag, 1972.
12. Borel spaces.
13. Normal derivations in operator algebras.
14. Extensions of a theorem of Fuglede and Putnam.
15. *Introduction to Hilbert space*, Chelsea, N.Y., 1976.

BERBERIAN, S. K.; ORLAND, G. H.
1. On the closure of numerical range of an operator, *Proc. AMS*, **18** (1967), 499–503.

BERCOVICI, H.; VOICULESCU, D.
1. Tensor operations on characteristic fu nctions, *Acta Sci. Math.*, **39** (1977), 205–231.

BEREZIN, F. A.
1. On commutation relations (Russian), *Uspekhi Mat. Nauk*, **24** (1969), 65–88.
2. Quantization (Russian), *Izvestya Akad. Nauk*, **38**: 5 (1974).
3. General concept of quantization, *Comm. Math. Phys.*, **40** (1975), 153–174.

BERGER, C. A.; COBURN, L. A.
1. C^*-algebras of translation and multipliers, *Bull. AMS*, **74** (1968), 1008–1012.
2. One parameter semigroups of isometries, *Bull. AMS*, **76** (1970), 1125–1129.

BERGER, C. A.; COBURN, L. A.; LEBOW, A.
1. C^*-algebras generated by commuting isometries, I, II, *Bull. AMS*, **81** (1975), 747–749.
2. Representation and index theory for C^*-algebras generated by commuting isometries, *J. Fnal Analysis*, **27** (1978), 51–99.

BERGLUND, M. C. F.
1. Ideal C^*-algebras, *Duke Math. J.*, **40** (1973), 241–258.

BERKSON, E.
1. Some characterization of C^*-algebras, *Illinois J. Math.*, **10** (1966), 1–8.
2. Action of W^*-algebras in Banach spaces, *Math. Ann.*, **189** (1970), 261–271.
3. A characterization of complex Hilbert spaces, *Bull. London Math. Soc.*, **2** (1970), 313–315.
4. Prehermitian elements and B^*-algebras, *Math. Ann.*, **195** (1972), 192–198.

BERKSON, E.; DOWSON, H. R.; ELLIOTT, G. A.
1. On Fuglede's theorem and scalar type operators, *Bull. London Math. Soc.*, **4** (1972), 13–16.

BERKSON, E.; FLEMING, R. J.; JAMISON, J.
1. Groups of isometries on certain ideals of Hilbert space operators, *Math. Ann.*, **220** (1976), 151–156.

BERKSON, E.; PORTA, H.
1. Representations of $B(X)$, *J. Fnal Analysis*, **3** (1969), 126–135.

BERNAT, P.; DIXMIER, J.
1. Sur le dual d'un groupe de Lie, *CR Acad. Sc. Paris*, **250** (1960), 1778–1779.

BERNSTEIN, I. N.
1. All reductive algebraic groups are liminal (Russian), *Funkt. Analiz. i pril.*, **8**: 2 (1974), 3–6.

BICHTELER, K.
1. A generalization to the non separable case of Takesaki's duality theorem for C^*-algebras, *Invent. Math.*, **9** (1969), 89–98.

BIRELL, I. D.
1. Maximal simple C^*-algebras, I, II, *Bull. London Math. Soc.*, **6** (1974), 141–144.

BISOGNANO, J. J.; WICHMANN, E. H.
1. On the duality for a hermitian scalar field, *J. Math. Phys.*, **16** (1975), 985–1007.

BLACKADAR, B. E.
1. Infinite tensor products of C^*-algebras, *Pacific J. Math.*, **72** (1977), 313–334.
2. The regular representation of restricted direct product groups. *J. Fnal Analysis*, **25** (1977), 267–274.
3. Weak expectations and injectivity in operator algebras, *Proc. AMS*, **68**(1978), 49–53.
4. Weak expectations and nuclear C^*-algebras.

BLATTNER, R. J.
1. Automorphic group representations, *Pacific J. Math.*, **8** (1958), 665–677.
2. On induced representations, I, II, *Amer. J. Math.*, **83** (1961), 79–98; 499–512.
3. On a theorem of G. W. Mackey, *Bull. AMS*, **68** (1962), 585–587.
4. Positive definite measures, *Proc. AMS*, **14** (1963), 423–428.
5. Group extension representations and the structure space, *Pacific J. Math.*, **15** (1965), 1102–1113.

BOHNENBLUST, H. F.; KARLIN, S.
1. Geometrical properties of the unit sphere of Banach algebras, *Ann. Math.*, **62** (1955), 217–229.

BONNET, P.
1. Une théorie spectrale dans certains espaces du type A(K), *CR Acad. Sc. Paris*, **282** (1976), 207–210.
2. *Une théorie spectrale dans certains espaces de Banach ordonnés*, Thèse III[e] cycle, Univ. St-Etienne.

BONSALL, F. F.
1. A survey of Banach algebra theory., *Bull London Math. Soc.*, **2** (1970), 257–274.
2. Jordan algebras spanned by hermitean elements of a Banach algebra, *Math. Proc. Cambridge Phil. Soc.*, **81** (1977), 3–14.

BONSALL, F. F.; DUNCAN, J.
1. Dual representations of Banach algebras, *Acta Math.*, **117** (1967), 79–102.
2. *Numerical Ranges*, I, II, Lecture Notes, London Math. Soc., 1971,1972.
3. *Complete normed algebras*, Springer Verlag, 1973.

BONSALL, F. F.; GOLDIE, A. W.
1. Annihilator algebras, *Proc. London Math. Soc.*, **3** (1954), 154–167.

BONSALL, F. F.; ROSENTHAL, P.
1. Certain Jordan operator algebras and double commutant theorems, *J. Fnal Analysis*, **21** (1976), 155–186.

BONSALL, F. F.; STIRLING, D. S. G.
1. Square roots in Banach $*$-algebras, *Glasgow Math. J.*, **13** (1972), 74.

BORCHERS, H. J.
1. On the structure of the algebra of field operators, I, II, *Nuovo Cimento*, **24** (1962), 214–236; *Comm. Math. Phys.*, **1** (1965), 49–56.
2. On the vacuum state in quantum field theory. II, *Comm. Math. Phys.*, **1** (1965), 57–79.
3. Local rings and connection of spin with statistics, *Comm. Math. Phys.*, **1** (1965), 281–307.
4. Energy and momentum as observables in quantum field theory, *Comm. Math. Phys.*, **2** (1966), 49–54.
5. A remark on a theorem of B. Misra, *Comm. Math. Phys.*, **4** (1967), 315–323.
6. On the converse of Reeh-Schlieder theorem, *Comm. Math. Phys.*, **10** (1968), 269–273.
7. On the theory of local observables, *Cargese Lectures in Phys.*, **3** (1968), 3–39.
8. On the implementability of automorphism groups, *Comm. Math. Phys.*, **11** (1969), 305–314.
9. On group of automorphisms with semibounded spectrum, *CNRS Conf. on systems with infinitely many degrees of freedom*, May, 1969, 125–142.
10. Strongly continuous automorphism groups on C^*-algebras, *Cargese Lectures in Phys.*, **4** (1970), 31–59.
11. C^*-algebras and locally compact groups of automorphisms.
12. Characterization of inner $*$-automorphisms of W^*-algebras, *Publ. RIMS, Kyoto Univ.*, **10** (1974), 11–50.
13. Über C^*-Algebren mit lokalkompakter Symmetrie Gruppen, *Nachr. Akad. Wiss., Göttingen*, 1973, 1–18.
14. Über Ableitungen von C^*-algebren, *Nachr. Akad. Wiss. Göttingen*, 1973, 19–35.
15. On the algebra of test functions, *Prépublication de la R. C. P.*, no. 25, vol. 15, Strasburg, (1973).
16. On fields of C^*-algebras, in *Symposia Math.*, XX, 117–136, Academic Press, 1976.
17. Algebraic aspects of Wightman field theory, in *Statistical Mechanics and Field Theory Lectures*, 1971; Haifa Summer School, New York-Jerusalem-London, 1972, 31–79.
18. Physical observables and symmetry groups, in *Lectures in theoretical physics*, vol. XII, p. 101–135, Gordon and Breach, 1971.
19. Decomposition of families of unbounded operators, *Acta Phys. Austriaca, Suppl. XVI*, 1976, 15–46.

BORCHERS, H. J.; HAAG, R.; SCHRÖER, B.
1. The vacuum state in quantum field theory, *Nuovo Cimento*, **29** (1963), 148–162.

BORCHERS, H. J.; HEGERFELDT, G. C.
1. The structure of space time transformations, *Comm. Math. Phys.*, 28 (1972), 259–266.

BORCHERS, H. J; SEN, R. N.
1. Relativity group in the presence of matter, *Comm. Math. Phys.*, 42 (1975), 101–126.

BORCHERS, H. J.; YNGVASON, J.
1. On the algebra of field operators. The weak commutant and integral decomposition of states, *Comm. Math. Phys.*, 42 (1975), 231–252.
2. Integral representations for Schwinger functionals and the moment problem over nuclear spaces, *Comm. Math. Phys.*, 43 (1975), 255–272.
3. Necessary and sufficient conditions for integral representations of Wightman functionals at Schwinger points, *Comm. Math. Phys.*, 47 (1976), 197–214.

BORCHERS, H. J; ZIMMERMANN, W.
1. On the selfadjointness of field operators, *Nuovo Cimento*, 31 (1964), 1047.

BÖS, W.
1. Direct integrals of selfdual cones and standard forms of von Neumann algebras, *Invent. Math.*, 37 (1976), 241–251.
2. A remark on a paper of A. Connes, Osnabrück, 1976.
3. A classification for selfdual cones in Hilbert space, Osnabrück, 1976.
4. The structure of finite homogeneous cones and Jordan algebras, Osnabrück, 1976.

BOURBAKI, N.
1. *Espaces vectoriels topologiques*, Hermann, Act. Sc. Ind. 1189, 1229, Paris.
2. *Intégration*, Hermann, Act. Sc. Ind. 1175, 1244, 1281, 1306, Paris.
3. *Théories spectrales*, Hermann, Act. Sc. Ind. 1332, Paris.

BOUVIER, A.-M.
1. La formule de Plancherel pour les groupes localement compacts séparables, *CR Acad. Sc. Paris*, 285 (1977), 357–360.

BOWEN, R.
1. Anosov foliations are hyperfinite, *Ann. Math.*, 106 (1977), 549–566.

BOYER, R.; MARTIN, R.
1. The group C^*-algebra of the De Sitter group.

BRAND, D.
1. When is a matrix positive?, *Canad. Math. Bull.*, 17 (1974), 409–410.

BRATTELI, O.
1. Conservation of estimates in quantum field theory, *Comm. Pure Appl. Math.*, 25 (1972), 759–780.
2. Inductive limits of finite dimensional C^*-algebras, *Trans. AMS*, 171 (1972), 195–234.
3. Structure spaces of approximately finite-dimensional C^*-algebras, I, *J. Fnal Analysis*, 16 (1974), 192–204.
4. The center of approximately finite-dimensional C^*-algebras, *J. Fnal Analysis*, 21 (1976), 195–202.
5. Self-adjointness of unbounded derivations on C_*-algebras, *Symposia Math.*, XX, 137–148, Academic Press, 1976.
6. Local norm convergence of states on the zero time Bose fields, *Trans. AMS*, 188 (1974), 269–280.
7. Unbounded derivations and invariant states, ZIF, Bielefeld, 1976.
8. A non-simple crossed product of a simple C^*-algebra by a properly outer automorphic action, Marseille, 1977.
9. Unbounded derivations of operator algebras, Marseille, 1977.

BRATTELI, O.; ELLIOTT, G. A.
1. Structure spaces of approximately finite dimensional C_*-algebras, II, Marseille, 1977.

BRATTELI, O.; HAAGERUP, U.
1. Unbounded derivations and invariant states, Odense, 1977.

Bratteli, O.; Herman, R. H.; Robinson, D. W.
1. Quasianalytic vectors and derivations of operator algebras, *Math. Scand.*, **39** (1976), 371–381.
2. Perturbations of flow on Banach spaces and operator algebras, ZIF, Bielefeld, 1976.

Bratteli, O.; Kastler, D.
1. Relaxing the clustering condition in the derivation of the KMS property, *Comm. Math. Phys.*, **46** (1976), 37–42.

Bratelli, O.; Kishimoto, A.
1. Generation of semigroups, and two-dimensional quantum lattice systems, Marseille, 1977.

Bratteli, O.; Kishimoto, A.; Robinson, D. W.
1. Stability properties and the KMS condition, Marseille, 1977.

Bratteli, O.; Robinson, D. W.
1. Unbounded derivations of C^*-algebras, I, II, *Comm. Math. Phys.*, **42** (1975), 253–268; **46** (1976), 11–30.
2. Unbounded derivations of von Neumann algebras, *Ann. IHP*, **25** (1976), 139–164.
3. Unbounded derivations and invariant trace states, *Comm. Math. Phys.*, **46** (1976), 31–35.
4. Green's functions, Hamiltonians and modular automorphisms, *Comm. Math. Phys.*, **50** (1976), 133–156.

Breitenecker, M.; Grumm, H. R.
1. Note on trace inequalities, *Comm. Math. Phys.*, **26** (1972), 276–279.

Breuer, M.
1. Banachalgebren mit Anwendungen auf Fredholmoperatoren und singulare Integralgleichungen, *Bonn Math. Schr.*, **24** (1965), 108 p.
2. A generalization of Kuiper's theorem to factors of type II, *J. Math. Mech.*, **16** (1967), 917–925.
3. Homotopy triviality of the group of regular elements of von Neumann algebras of type I, II_∞, *Univ. Kansas, Tech. Rep.*, **15** (1968).
4. Fredholm-theories in von Neumann algebras, I, II, *Math. Ann.*, **178** (1968), 243–254; **180** (1969), 313–325.
5. Theory of Fredholm operators and vector bundles relative to von Neumann algebras, *Rocky Mountain J. Math.*, **3** (1973), 383–429.
6. On the homotopy type of the group of regular elements of semifinite von Neumann algebras, *Math. Ann.*, **185** (1970), 61–74.
7. The periodicity theorem relative to von Neumann algebras, *Univ. Kansas, Tech. Rep.*, **2** (1970).

Breuer, M.; Butcher, R.
1. A generalized Riesz-Schauder decomposition theorem, *Math. Ann.*, **203** (1973), 221–230.
2. Fredholm theories of mixed type with analytic index functions, *Math. Ann.*, **209** (1974), 31–42.

Breuer, M.; Cordes, H. O.
1. On Banach algebras with σ-symbol, I, II, *J. Math. Mech.*, **13** (1964), 313–323; **14** (1965), 299–314.

Brezin, J.
1. *Unitary representation theory for solvable Lie groups*, Memoirs AMS, No. **79**, 1968.

Brinke, G. Ten; Winnink, M.
1. Spectra of Liouville operators, *Comm. Math. Phys.*, **51** (1976), 135–150.

Britton, O. L.
1. Primitive ideals of twisted group algebras, *Trans. AMS*, **202** (1975), 221–241.

Broise, M.
1. Une caractérisation des représentations unitaires, *Bull. Sc. Math.*, **88** (1964), 59–64.
2. Sur les isomorphismes de certaines algèbres de von Neumann, *Ann. Ec. Norm. Sup.*, **83** (1966), 91–111.
3. Sur certaines applications unitaires de l'espace des opérateurs de Hilbert-Schmidt, *CR Acad. Sc. Paris*, **263** (1966), 722–725.

4. Une caractérisation de représentations unitaires de certains sémigroupes, *Bull. Sc. Math.* 90 (1966), 67–69.
5. Sur les vecteurs séparateurs des algèbres de von Neumann, *CR Acad. Sci. Paris*, 264 (1967), 937–940; *J. Fnal Analysis*, 1 (1967), 281–289.
6. Commutateurs dans le groupe unitaire d'un facteur, *J. Math. Pures Appl.*, 46 (1967), 299–312.

Brown, A.
1. The unitary equivalence of binormal operators, *Amer. J. Math.*, 76 (1954), 414–434.
2. On the absolute equivalence of projections, *Bul. Inst. Politehnic Iaşi*, 4 (1958), fasc. 3–4, p. 5–6.
3. A version of multiplicity theory, *Topics in operator theory, Math. Surveys*, No. 13, 129–160.

Brown, A.; Pearcy, C.
1. Structure of commutators of operators, *Ann. Math.*, 82 (1965), 112–127.
2. Multiplicative commutators of operators, *Can. J. Math.*, 18 (1966), 737–749.
3. Spectra of tensor products of operators, *Proc. AMS*, 17 (1966), 162–166.
4. Commutators in factors of type III, *Can. J. Math.*, 18 (1966), 1152–1160.

Brown, A.; Pearcy, C.; Topping, D. M.
1. Commutators and the strong radical, *Duke Math. J.*, 35 (1968), 853–860.

Brown, L. G.
1. Operator algebras and algebraic *K*-theory, *Bull. AMS*, 81 (1975), 1119–1121.
2. Completeness, separability, metrizability and extensions of topological groups.
3. Group cohomology of topological groups.
4. Foundations of ergodic theory on locally compact groups.
5. Extensions of C^*-algebras, *Notices AMS*, 23 : 3 (1976), 735-B 12.
6. Extensions and the structure of C^*-algebras, *Symposia Math.*, XX, 539–565, Academic Press, 1976.
7. Stable isomorphisms of hereditary subalgebras of C^*-algebras, *Pacific J. Math.*, 71 (1977), 335–348.
8. Some techniques in the theory of C^*-algebras, Japan-US Seminary, 1977.
9. Characterising Ext., *Lectures Notes in Math.*, No. 575 (1977), 10–19.

Brown, L. G.; Douglas, R. G.; Fillmore, P. A.
1. Unitary equivalence modulo the compact operators and extensions of C^*-algebras, *Lecture Notes in Math.*, No. 345, 58–128, Springer Verlag, 1973.
2. Extensions of C^*-algebras, operators with compact self-commutators and K-homology, *Bull. AMS*, 79 (1973), 973–978.
3. Extensions of C^*-algebras and K-homology, *Ann. Math.*, 105 (1977), 265–324.

Brown, L. G.; Green, P.; Rieffel, M. A.
1. Stable isomorphisms and strong Morita equivalence of C^*-algebras, *Pacific J. Math.*, 71 (1977), 349–363.

Brown, L. G.; Schochet, C.
1. K_1 of the compact operators is zero, *Proc. AMS*, 59 (1967), 119–122.

Brünning, J.
1. Indextheorie für eine C^*-algebra von Toeplitzoperatoren, *Math. Ann.*, 216 (1975), 113–121.

Brünning, J.; Willgerodt, W.
1. Eine Verallgemeinerung eines Satzes von Kuiper, *Math. Ann.*, 220 (1976), 47–58.

Buchholz, D.
1. Product states for local algebras, *Comm. Math. Phys.*, 36 (1974), 287–304.
2. On the structure of local quantum fields with non-trivial interaction.

Buchholz, D.; Roberts, J. E.
1. Bounded perturbations of dynamics, Preprint, Th. 2043–CERN, 1975.

Buck, R. C.
1. Operator algebras and dual spaces, *Proc. AMS*, 3 (1952), 681–686.

BULINSKIĬ, A. V. (in Russian)
1. On the problem of the classes of C^*-algebras satisfying the Haag-Kastler axioms, *Teor. Mat. Fiz.*, **8** : 3 (1971), 328–334.
2. On a result of Kadison and Ringrose, *Teor. Mat. Fiz.*, **12** ; 2 (1972), 197–203.
3. Sufficient conditions for the covariance of representation of C^*-algebras, *Teor. Mat. Fiz.*, **17** : 1 (1973), 19–23.
4. Continuous groups of automorphisms of operator algebras, *Sci. Conf. Moscow phys.-tech. inst.*, 1975; *Ser. Aerophysics and Appl. Math.*, 1976, 157–159.

BUNCE, J.
1. Characters of singly generated C^*-algebras, *Proc. AMS*, **25** (1970), 297–303.
2. The ordered vector space structure of JC-algebras, *Proc. London Math. Soc.*, **22** (1971), 359–368.
3. A note on two-sided ideals in C^*-algebras, *Proc. AMS*, **28** (1971), 635.
4. The joint spectrum of commuting normal operators, *Proc. AMS*, **29** (1971), 499–504.
5. Representations of strongly amenable C^*-algebras, *Proc. AMS*, **32** (1972), 241–246.
6. Characters of amenable and strongly amenable C^*-algebras, *Pacific J. Math.*, **43** (1973), 563–572.
7. Irreducible representations of the C^*-algebra generated by a quasinormal operator, *Trans. AMS*, **183** (1973), 487–494.
8. Automorphisms and tensor products of algebras, *Proc. AMS*, **44** (1974), 93–95.
9. Finite operators and amenable C^*-algebras, *Proc. AMS*, **56** (1976), 145–151.
10. A universal diagram property of minimal normal extensions.

BUNCE, J.; DEDDENS, J. A.
1. Irreducible representations of the C^*-algebra generated by an n-normal operator, *Trans. AMS*, **171** (1972), 301–308.
2. C^*-algebras generated by weighted shifts, *Indiana Univ. Math. J.*, **23** (1973), 257–272.
3. C^*-algebras with Hausdorff spectrum, *Trans. AMS*, **212** (1975), 199–218.
4. Subspace approximants and GCR operators, *Indiana Univ. Math. J.*, **24** (1974), 341–349.
5. A family of simple C^*-algebras related to weighted shift operators, *J. Fnal Analysis*, **19** (1975), 13–24.

BUNCE, J.; PASCHKE, W. L.
1. Quasi-expectations and injective operator algebras, *Japan-US Seminary*, 1977.
2. Quasi-expectations and amenable von Neumann algebras.

BUNCE, J.; SALINAS, N.
1. Completely positive maps on C^*-algebras and the left matricial spectrum of an operator, *Duke Math. J.*, **43** (1976), 747–774.

BURES, D.
1. Certain factors constructed as infinite tensor products, *Compositio Math.*, **15** (1963), 169–191.
2. Tensor products of W^*-algebras, *Pacific J. Math.*, **27** (1968), 13–37.
3. An extension of Kakutani's theorem on infinite product measures to the tensor product of semifinite W^*-algebras, *Trans. AMS*, **135** (1969), 199–212.
4. Representations of infinite weak product groups, *Compositio Math.*, **22** (1970), 7–18.
5. *Abelian subalgebras of von Neumann algebras*, Memoirs AMS, No. 110, 1971.

BUSBY, R. C.
1. On structure space and extensions of C^*-algebras, *J. Fnal Analysis*, **1** (1967), 370–377.
2. Double centralizers and extensions of C^*-algebras, *Trans. AMS*, **132** (1968), 79–99.
3. Extensions in certain topological algebraic categories, *Trans. AMS*, **159** (1971), 41–56.
4. On a theorem of Fell, *Proc. AMS*, **30** (1971), 133–140.
5. On the equivalence of twisted group algebras and Banach $*$-algebraic bundles, *Proc. AMS*, **37** (1973), 142–148.
6. Centralizers of twisted group algebras, *Pacific J. Math.*, **47** (1973), 357–392.
7. Double centralizers and covariance algebras.

BUSBY, R.C.; SMITH, H. A.
1. Representations of twisted group algebras, *Trans. AMS*, **149** (1970), 503–539.

BYRNES, C. I.
1. *Lectures on the theory of harmonic algebras*, Univ. of Mass., 1974.
2. *On the algebraic foundations of differential geometry*, Thesis, Univ. of Mass., 1975.

CALKIN, J. W.
1. Two-sided ideals and convergence in the ring of bounded operators in Hilbert space, *Ann. Math.*, 42 (1941), 839–873.

CANNIÈRE, J. de
1. Produit croisé d'une algèbre de Kac par un groupe localement compact, Leuven, 1977.
2. On the intrinsic group of a Kac algebra, Leuven, 1978.

CAPPON, F.
1. Non-relativistic limit in the algebraic approach of relativistic quantum mechanics.

CAREY, A. L.; GAFFNEY, J. M.; HURST, C. A.
1. A C^*-algebraic formulation of the quantization of the electromagnetic field, *J. Math. Phys.*, 18 (1977), 629–640.

CAREY, R. W.; PINCUS, J. D.
1. The structure of intertwining partial isometries, I, II, *Indiana Univ. Math. J.*, 22 (1973), 679–703.
2. An invariant for certain operator algebras, *Proc. Nat. Acad. Sci. USA*, 71 (1974), 1952–1956.
3. Unitary equivalence modulo the trace class for selfadjoint operators, *Amer. J. Math.*, 98 (1976), 481–514.
4. Mosaics, principal functions and mean motion in von Neumann algebras, *Acta Math.*, 138 (1977), 153–218.
5. Commutators, symbol and determining functions, *J. Fnal Analysis*, 19 (1975), 50–80.
6. Almost commuting algebras, *Lecture Notes in Math.*, No. 575 (1977), 19–43.

CARLING, L. N.
1. On the restriction map of the Fourier-Stieltjes algebras $B(G)$ and $B_p(G)$, *J. Fnal Analysis*, 25 (1977), 236–243.

CASTEREN, J. A.
1. A characterization of C^*-subalgebras, *Proc. AMS.*

CECCHINI, C.
1. On neighbouring type I von Neumann factors, *Bolletino U.M.I.*, 11 (1975), 103–106.
2. Operators on $VN(G)$ commuting with $A(G)$, *Publ. Ist. Mat. Univ. Genova*, No. 182 (1976), 8 p.

CHACRON, M.
1. A generalization of a theorem of Kaplansky and ring with involution, *Michgan Math. J.*, 20 (1973), 45–54.

CHAIKEN, J. M.
1. Finite particle representations and states of the canonical commutation relations, *Ann. Phys.*, 42 (1967), 23–80.
2. Number operators for representations of the canonical commutation relations, *Comm. Math. Phys.*, 8 (1968), 164–184.

CHANDLER, J. D. Jr.
1. Extensions of monotone operator functions, *Proc. AMS*, 54 (1976), 221–224.

CHANG, S. Y.
1. A characterization of Douglas subalgebras, *Acta Math.*, 137 (1976), 81–90.
2. Structure of subalgebras between L^∞ and H^∞, *Trans. AMS*, 227 (1977), 319–332.
3. On the structure and characterization of some Douglas subalgebras, *Amer. J. Math.*, 99 (1977), 530–578.

CHANG, T.; SKJELBRED, T.
1. The topological Schur lemma and related results, *Ann. Math.*, 100 (1974), 307–321.

CHE, K.
1. A sufficient condition that an operator be normal, *Michigan Math. J.*, **21** (1974), 161 – 162.

CHEN, E.
1. *Über die C*-Algebra lokaler Observablen in der Quantenfeldtheorie*, Thesis, Univ. Marburg, 1966.
2. Continuity of automorphisms of quasilocal algebras, *J. Math. Phys.*, **12** (1971), 591–593.
3. On S-property of C^*-algebras, *Notices AMS*, **18** (1971), 682–46–22.
4. On annihilator of energy momentum spectra in quantum field theory, *Nuovo Cimento*, B(11), **2** (1971), 161–166.
5. *J. Math. Phys.*, **12** (1971), 2364.
6. Structure of simplexes of equilibrium states in quantum statistical mechanics, *J. Math. Phys.*, **13** (1972), 1130–1135.
7. Choquet simplexes of states in physical systems, *J. Math. Phys.*, **15** (1974), 2022–2025.
8. On the duality of local observables, *Ann. IHP*, A. **22**, 123–130.
9. Facial aspect of superposition principle in algebraic quantum theory, *J. Math. Phys.*, **14** (1973), 1462–1465.
10. Entropy and the superposition principle, *Rep. Math. Phys.*, **11** (1977), 189–195.
11. Markovian subdynamics in quantum dynamical systems, *J. Math. Phys.*, **17** (1976), 1785–1789.

CHEN, E.; DEMOEN, B.
1. Poisson brackets and a commutation theorem, Leuven, 1977.

CHERNOFF, P. R.
1. Representations, automorphisms and derivations of some operator algebras, *J. Fnal Analysis*, **12** (1973), 275–289.
2. *Product formulas, non-linear semigroups and addition of unbounded operators*, Memoirs AMS, No. 140, 1974.
3. Notes on product formulas of operator semigroups, *J. Fnal Analysis*, **2** (1968), 238–242.
4. Quasi-analytic vectors and quasi-analytic functions, *Bull. AMS*, **81** (1975), 637–646.

CHEVALLEY, C.
1. *Theory of Lie groups*, I, Princeton Univ. Press, 1946.

CHI, DONG-PYO
1. Derivations in C^*-algebras, *Notices AMS*, **23** : 1 (1976), 731–46–21; Philadelphia, 1976.

CHING, W. M.
1. Non-isomorphic non-hyperfinite factors, *Can. J. Math.*, **21** (1969), 1293–1308.
2. A continuum of non isomorphic non-hyperfinite factors, *Comm. Pure Appl. Math.*, **23** (1970), 921–938.
3. Free products of von Neumann algebras, *Trans. AMS*, **178** (1973), 147–163.
4. Topologies on the quasi-spectrum of a C^*-algebra, *Proc. AMS*, **46** (1974), 273–276.
5. On some non-hyperfinite factors of type III, *Canad. Math. Bull.*, **18** (1975), 643–648.
6. The structure of standard C^*-algebras and their representations, *Pacific J. Math.*, **67** (1976), 131–153.

CHING, W. M.; TSAU, C. S.
1. A family of type III factors without property L, *Math. Ann.*, **222** (1976), 195–204.

CHING, W. M.; WILLIG, P.
1. Non asymptotically abelian factors of type III, *Proc. AMS*, **34** (1972), 102–104.

CHO, S. J.
1. Strong extensions versus weak extensions of C^*-algebras.

CHODA, H.
1. On automorphisms of abelian von Neumann algebras, *Proc. Jap. Acad.*, **41** (1965), 280–283.
2. On the crossed product of abelian von Neumann algebras, I, II, *Proc. Jap. Acad.*, **43** (1967), 111–116; 198–201.
3. On Maharam subfactors of finite factors, I, II, *Proc. Jap. Acad.*, **43** (1967), 451–455; 937–940.

4. An extremal property of the polar decomposition in von Neumann algebras, *Proc. Jap. Acad.*, **46** (1970), 341–344.
5. On egodic and abelian automorphism groups of von Neumann algebras, *Proc. Jap. Acad.*, **47** (1971), suppl. I, 982–985.
6. On a simple representation of groups as outer automorphism groups of the hyperfinite II_1 factor, *Mem. Osaka Kyoiku Univ.*, **20** (1971), 113–115.
7. On a decomposition of automorphisms of von Neumann algebras, *Proc. Jap. Acad.*, **49** (1973), 809–811.
8. A comment on the Galois type theory for finite factors, *Proc. Jap. Acad.*, **50** (1974), 619–622.
9. On freely acting automorphisms of operator algebras, *Kodai Math. Sem. Rep.*, **26** (1974), 1–21.
10. Expectations and extreme points in von Neumann algebras, *Math. Jap.*, **19** (1974), 297–302.
11. A Galois correspondence in a von Neumann algebra.
12. A scholium on the commutativity for operator algebras, *Math. Jap.*, **22** (1977), 57–60.

CHODA, H.; CHODA, M.
1. On theorems of Korovkin, *Proc. Jap. Acad.*, **39** (1963), 107–108.
2. Some remarks on von Neumann algebras with an algebraic property, *Proc. Jap. Acad.*, **41** (1965), 71–72.
3. On extensions of automorphisms of abelian von Neumann algebras, *Proc. Jap. Acad.*, **43** (1967), 295–299.
4. On the minimality of the polar decomposition in finite factors, *Proc. Jap. Acad.*, **44** (1968), 798–800.
5. Some characterizations of certain von Neumann algebras, *Proc. Jap. Acad.*, **46** (1970), 1086–1091.

CHODA, H.; ECHIGO, M.
1. A new algebraical property of certain von Neumann algebras, *Proc. Jap. Acad.*, **39** (1963), 651–656.
2. Some remarks on von Neumann algebras with property Q, *Mem. Osaka Gagukei Univ.*, **13** (1964), 13–21.
3. A remark on the construction of finite factors, I, II, *Proc. Jap. Acad.*, **40** (1964), 474–478, 479–481.

CHODA, H.; KIJIMA, Y.; NAKAGAMI, Y.
1. Some extremal properties in the unit ball of von Neumann algebras, *Kodai Math. Sem. Rep.*, **21** (1969), 175–181.

CHODA, H.; MATOBA, K.
1. On a theorem of Levine, *Proc. Jap. Acad.*, **37** (1961).

CHODA, H.; NAKAMURA, M.
1. Elementary proofs of Gleason-Kahane-Zelazko theorem for B^*-algebras, *Mem. Osaka Kyoiku Univ.*, **20** (1971), 111–112.

CHODA, M.
1. On the conditional expectation of a partial isometry in a certain von Neumann algebra, *Proc. Jap. Acad.*, **41** (1965), 277–279.
2. A remark on the normal expectations, I, II, *Proc. Jap. Acad.*, **44** (1968), 462–466; **49** (1973), 252–256.
3. A von Neumann algebra continuous over a von Neumann subalgebra, *Proc. Jap. Acad.*, **49** (1973), 174–178.
4. Abelian projections over a von Neumann subalgebra, *Proc. Jap. Acad.*, **48** (1972), 384–388.
5. On types over von Neumann subalgebras and the Dye correspondence, *Publ. RIMS, Kyoto Univ.*, **9** (1973), 45–60.
6. Shift automorphism groups of von Neumann algebras, *Proc. Jap. Acad.*, **50** (1974), 470–475.

7. Normal expectations and crossed products of von Neumann algebras, *Proc. Jap. Acad.*, **50** (1974), 738–742.
8. The fixed algebra of a von Neumann algebra under an automorphism group, *Tohoku Math. J.*, **28** (1976), 227–234.
9. Correspondence between subgroups and subalgebras in the compact crossed product of a von Neumann algebra, *Math. Jap.*, **21** (1976), 51–59.
10. A characterization of crossed products of factors by discrete outer automorphism groups, Osaka, 1977.
11. Automorphism of finite factors on free groups, I, *Math. Jap.*, **22** (1977), 219–226.
12. Some relations of II_1-factors on free groups, *Math. Jap.*
13. Extensions of the inner automorphism groups of a factor, Osaka, 1977.

CHODA, M.; KASAHARA, I.; NAKAMOTO, R.
1. Dependent elements of an automorphism of a C^*-algebra, *Proc. Jap. Acad.*, **48** (1972), 561–565.

CHODA, M.; NAKAMURA, M.
1. A remark on the concept of channels, I, II, III, *Proc. Jap. Acad.*, **38** (1962), 307–309; **46** (1970), 932–935; **47** (1971), 464–469.

CHOI, M. D.
1. Positive linear maps on C^*-algebras, *Can. J. Math.*, **24** (1972), 520–529.
2. *Positive linear maps on C^*-algebras*, Thesis, Univ. Toronto, 1971.
3. A Schwartz inequality for positive linear maps on C^*-algebras, *Illinois J. Math.*, **18** (1974), 565–574.
4. Completely positive linear maps on complex matrices, *Linear Algebra and Appl.*, **10** (1975), 285–290.
5. Positive semidefinite biquadratic forms, *Linear Algebra and Appl.*, **12** (1975), 95–100.
6. The extension of nuclear C^*-algebras, *Notices AMS*, **23**: 3 (1976), 735-B 2.
7. A simple C^*-algebra generated by two finite order unitaries.

CHOI, M. D.; EFFROS, E. G.
1. Injectivity and operator spaces, *J. Fnal Analysis*, **24** (1977), 156–209.
2. Nuclear C^*-algebras and the approximation property, *Amer. J. Math.*
3. Separable nuclear C^*-algebras and injectivity, *Duke Math. J.*, **43** (1976), 309–322.
4. The completely positive lifting problem, *Ann. Math.*, **104** (1976), 585–609.
5. Nuclear C^*-algebras and injectivity; the general case, *Indiana Univ. Math. J.*
6. Quotients, extensions and liftings for operator algebras, *Notices AMS*, **23**: 3 (1976), 734-B 5.
7. Lifting problems and the cohomology of C^*-algebras, *Can. J. Math.*, **29** (1977), 1092–1111.

CHOW, T. R.
1. The spectral radius of direct integral of operators, *Proc. AMS*, **26** (1970), 593–597.
2. A spectral theory for direct integrals of operators, *Math. Ann.*, **188** (1970), 285–303.

CHRISTENSEN, E.
1. Non-commutative integration for monotone sequential closed C^*-algebras, *Math. Scand.*, **31** (1972), 171–190.
2. Perturbations of type I von Neumann algebras, *J. London Math. Soc.*, **9** (1975), 395–405.
3. Perturbations of operator algebras, I, II, *Invent. Math.*, **43** (1977), 1–14; *Indiana Univ. Math. J.*, **26** (1977), 891–904.
4. Extensions of derivations, *J. Fnal Analysis*, **27** (1978), 234–247.
5. Derivations of nest algebras, *Math. Ann.*, **229** (1977), 155–162.
6. Subalgebras of a finite algebra, Copenhagen, 1977.
7. Near inclusions of C^*-algebras, Copenhagen, 1978.

CHRISTENSEN, E.; ELLIOTT, G. A.; PEDERSEN, G. K.
1. Separability and hereditary C^*-subalgebras, Copenhagen, 1976.

CHRISTENSEN, J. P. R.
1. On some properties of Effors-Borel structure on spaces of closed subsets, *Math. Ann.*, **195** (1972), 17–23.
2. *Topology and Borel structure*, North-Holland Publ., 1974.

CHU, CHO-HO
1. Prime faces in C^*-algebras, *J. London Math. Soc.*, **7** (1973), 175–180.
2. Idéaux premiers dans $A(X)$, *CR Acad. Sc. Paris*, **275** (1972), 1179–1182.
3. Anti-lattices and prime sets, *Math. Scand.*, **31** (1972), 151–165.
4. On convexity theory and C^*-algebras, *Proc. London Math. Soc.*, **31** (1975), 257–288.
5. On central traces and groups of symmetries of order unit Banach spaces.

CHU, CHO-HO; MAITLAND-WRIGHT, J. D.
1. Une théorie des types pour une classe d'espace de Banach ordonnés, *CR Acad. Sc. Paris*, **281** (1975), 633–636.
2. A theory of types for convex sets and ordered Banach spaces.

CHUAN, J. C.
1. *One-sided ideals in a C^*-algebra*, Thesis, Berkeley, 1977.

CHUI, C. K.; LEGG, D. A.; SMITH, P. W.; WARD, J. D.
1. On a question of Olsen concerning compact perturbations of operators.

CIORĂNESCU, I.; ZSIDÓ, L.
1. Analytic generators for one-parameter groups, *Tohoku Math. J.*, **28** (1976), 327–362.
2. On spectral subspaces of some unbounded groups of operators, *Rev. Roum. Math. Pures Appl.*, **21** (1976), 817–850.

CIVIN, P.; YOOD, B.
1. The second conjugate space of a Banach algebra as an algebra, *Pacific J. Math.*, **11** (1961), 847–870.
2. Involutions on Banach algebras, *Pacific J. Math.*, **9** (1959), 415–436.

CLEVELAND, S. B.
1. Homomorphisms of non-commutative $*$-algebras, *Pacific J. Math.*, **13** (1963), 1097–1109.

COBURN, L. A.
1. Weyl's theorem for non-normal operators, *Michigan Math. J.*, **13** (1966), 285–288.
2. The C^*-algebra generated by an isometry, I, II, *Bull. AMS*, **73** (1967), 722–726; *Trans. AMS*, **137** (1969), 211–219.
3. Singular integral operators and Toeplitz operators on odd spheres, *Indiana Univ. Math. J.*, **23** (1973), 433–440.
4. Representations and index theory for C^*-algebras generated by commuting isometries: some remarks, *Symposia Math.*, XX, 479–486, Academic Press, 1976.

COBURN, L. A.; DOUGLAS, R. G.
1. Translation operators on the half-line, *Proc. Nat. Acad. Sci. USA*, **62** (1969), 1010–1013.
2. C^*-algebras of operators on a half-space, I, *Publ. IHES*, **40** (1972), 59–67.

COBURN, L. A.; DOUGLAS, R. G.; SCHAEFFER, D.; SINGER, I. M.
1. C^*-algebras of operators on a half-space, II, *Publ. IHES*, **40** (1972), 69–79.

COBURN, L. A.; LEBOW, A.
1. Algebraic theory of Fredholm operators, *J. Math. Mech.*, **15** (1966), 577–584.
2. Components of invertible elements in quotient algebras of operators, *Trans. AMS*, **130** (1968), 359–365.

COBURN, L. A.; MOYER, R. D.; SINGER, I. M.
1. C^*-algebras of almost periodic pseudo-differential operators, *Acta Math.*, **130** (1973), 279–308.

COBURN, L. A.; SCHECHTER, M.
1. Joint spectra and interpolation of operators, *J. Fnal Analysis*, **2** (1968), 226–237.

COLELLA, P.; CORDES, H. O.
1. The C^*-algebra of the elliptic boundary problem.

COLOJOARĂ, I.; FOIAŞ, C.
1. *Theory of generalized spectral operators*, Gordon and Breach, 1968.

COMBES, F.

1. Relations entre formes positives sur une C^*-algèbre, *CR Acad. Sc. Paris*, **260** (1965), 5435–5438.
2. Représentations d'une C^*-algèbre et formes linéaires positives, *CR Acad. Sc. Paris*, **260** (1965), 5993–5996.
3. Etude des représentations tracées d'une C^*-algèbre, *CR Acad. Sc. Paris*, **262** (1966), 114–117.
4. Etude des poids définis sur une C^*-algèbre, *CR Acad. Sc. Paris*, **265** (1967), 340–343.
5. Sur les états factoriels d'une C^*-algèbre, *CR Acad. Sc. Paris*, **265** (1967), 736–739.
6. Eléments semi-continus associés à une C^*-algèbre, *CR Acad. Sc. Paris*, **267** (1968), 986–989.
7. Poids sur une C^*-algèbre, *J. Math. Pures Appl.*, **47** (1968), 57–100.
8. Sur les faces d'une C^*-algèbre, *Bull. Sc. Math.*, **93** (1969), 37–62.
9. Quelques propriétés des C^*-algèbres, *Bull. Sc. Math.*, **94** (1970), 165–192.
10. Poids associés à une algèbre hilbertienne à gauche, *CR Acad. Sc. Paris*, **270** (1970), 33–36; *Compositio Math.*, **23** (1971), 49–77.
11. *C^*-algèbres et espaces de Banach ordonnés*, Thèse, Paris, 1970.
12. Poids et espérances conditionnelles dans les algèbres de von Neumann, *Bull. Soc. Math. France*, **99** (1971), 73–112.
13. Les facteurs de von Neumann de type III, Sém. Bourbaki, 1974/1975, *Lecture Notes in Math.*, No. 514, 1976, 124–137.

COMBES, F.; DELAROCHE, C.

1. Groupe modulaire d'une espérance conditionnelle dans une algèbre de von Neumann, *Bull. Soc. Math. France*, **103** (1975), 385–426.
2. *Séminaire sur les algèbres d'opérateurs*, Orléans, 1973/74.

COMBES, F.; PERDRIZET, F.

1. Certains idéaux dans les espaces vectoriels ordonnés, *CR Acad. Sc. Paris*, **268** (1969), 1552–1555; *J. Math. Pures Appl.*, **49** (1970), 29–60.

CONNES, A.

1. Un nouvel invariant pour les algèbres de von Neumann, *CR Acad. Sc. Paris*, **273** (1971), 900–903.
2. Calcul des deux invariants d'Araki et Woods par la théorie de Tomita et Takesaki, *CR Acad. Sc. Paris*, **274** (1972), 175–178.
3. États presque périodiques sur une algèbre de von Neumann, *CR Acad. Sc. Paris*, **274** (1972), 1402–1405.
4. Groupe modulaire d'une algèbre de von Neumann, *CR Acad. Sc. Paris*, **274** (1972), 1923–1926.
5. Une classification des facteurs de type III, *CR Acad. Sc. Paris*, **275** (1972), 523–525.
6. Une classification des facteurs de type III, *Ann. Ec. Norm. Sup.*, **6** (1973), 133–252.
7. Caractérisation des espaces vectoriels ordonnées sousjacents aux algèbres de von Neumann, *Ann. Inst. Fourier*, **24** (1974), 121–155.
8. Almost periodic states and Krieger's factors, Marseille, 1973.
9. Almost periodic states and factors of type III_1, *J. Fnal Analysis*, **16** (1974), 415–445.
10. On hyperfinite factors of type III_0 and Krieger's factors, *J. Fnal Analysis*, **18** (1975), 318–327.
11. On the hierarchy of Krieger, *Illinois J. Math.*, **19** (1975), 428–432.
12. Sur le théorème de Radon-Nikodym pour les poids normaux fidèles semifinis, *Bull. Sc. Math.*, **97** (1973), 253–258.
13. Applications of Tomita-Takesaki theory to classification of factors of type III, in *"C^*-algebras and their applications to statistical mechanics and quantum field theory"*, p. 29–46, North-Holland, 1976.
14. Periodic automorphisms of the hyperfinite factor of type II_1, *Acta Sci. Math.*, **39** (1977), 39–66.
15. A factor not antiisomorphic to itself, *Ann. Math.*, **101** (1975), 536–554.
16. A factor not antiisomorphic to itself, *Bull. London Math. Soc.*, **7** (1975), 171–174.
17. Classification of the automorphisms of the hyperfinite factors of type II_1 and II_∞ and applications to type III factors, *Bull. AMS*, **80** (1975), 1090–1092.

18. Outer conjugacy classes of automorphisms of factors, *Ann. Ec. Norm. Sup.*, 8 (1975), 383–419.
19. Outer conjugacy of automorphisms of factors, *Symposia Math.*, XX, 149–160, Academic Press, 1976.
20. Conditional expectations and Krieger's factor.
21. Sur la classification des facteurs de type II, *CR Acad. Sc. Paris*, 281 (1975), 13–15.
22. Classification of injective factors, *Ann. Math.*, 104 (1976), 73–116.
23. Structure theory for type III factors, *Proc. Int. Congr. Math.*, 1974, Vancouver, vol. II, 87–91.
24. On the classification of von Neumann algebras and their automorphisms, *Symposia Math.*, XX, 435–478, Academic Press, 1976.
25. On the cohomology of operator algebras, *J. Fnal Analysis*.
26. On a spatial theory of von Neumann algebras, Paris, 1977.
27. The von Neumann algebra of a foliation, Paris, 1977.
28. Sur la théorie non-commutative de l'intégration, Paris, 1977.

CONNES, A.; VAN DAELE, A.
1. The group property of the invariant S, *Math. Scand.*, 32 (1973), 187–192.

CONNES, A.; GHEZ, P.; LIMA, R.; TESTARD, D.; WOODS, E. J.
1. Review of Crossed products of von Neumann algebras (Golodeč V. I., [22]), Marseille, 1975.

CONNES, A.; KRIEGER, W.
1. Measure space automorphisms, the normalizers of their full groups and approximate finiteness, *J. Fnal Analysis*, 24 (1977), 336–352.
2. Outer conjugacy of non-commutative Bernoulli shifts.

CONNES, A.; STÖRMER, E.
1. Entropy of automorphisms of II_1 von Neumann algebras, *Acta Math.*, 134 (1975), 289–306.
2. Homogeneity of the state space of factors of type III_1, ZIF, Bielefeld, 1976.

CONNES, A.; TAKESAKI, M.
1. Flot des poids sur les facteurs de type III, *CR Acad. Sc. Paris*, 278 (1974), 945–948.
2. The flow of weights on a factor of type III, *Tohoku Math. J.*, 29 (1977).

CONNES, A.; WOODS, E. J.
1. Existence des facteurs infinis asymptotiquement abéliens, *CR Acad. Sc. Paris*, 279 (1974), 189–191.

CONWAY, J. B.
1. The numerical range and a certain convex set in an infinite factor, *J. Fnal Analysis*, 5 (1970), 428–435.
2. On the Calkin algebra and covering homotopy property I, II, *Trans. AMS*, 211 (1975), 135–142; *Canad. J. Math.*, 29 (1977), 210–215.
3. The direct sum of normal operators, *Indiana Univ. Math. J.*, 26 (1977), 277–290.

CONWAY, J. B.; OLIN, R. F.
1. A functional calculus for subnormal operators, I, II, *Bull. AMS*, 82 (1976), 259–261; *Memoirs AMS*, No. 184, 1977.

CONWAY, J. B.; SZÜCS, J.
1. The weak sequential closure of certain sets of extreme points in a von Neumann algebra, *Indiana Univ. Math. J.*, 22 (1973), 763–768.

CONWAY, J. B.; WU, P.
1. The splitting of $\mathfrak{A}(T_1 \oplus T_2)$ and related questions, *Indiana Univ. Math. J.*, 26 (1977), 41–56.

CONZE, J. P.; DANG-NGOC, N.
1. Ergodic theorems for non-commutative dynamical systems, *Bull. AMS*, 83 (1977), 1297–1299; *Invent. Math.*

CORDES, H. O.
1. On a class of C^*-algebras, *Math. Ann.*, 170 (1967), 283–313.
2. On a generalized Fredholm theory, *J. Reine Angew. Math.*, 227 (1967), 121–149.

3. An algebra of singular integral operators with two symbol homomorphisms, *Bull. AMS*, **75** (1969), 37–42.
4. *Banach algebras and partial differential operators*, Lecture Notes, Lund, 1971.

CORDES, H. O.; MC OWEN, R. C.
1. The C^*-algebra of a singular elliptic problem on a non-compact Riemannian manifold, *Math. Z.*, **153** (1977), 101–106.

CORDESSE, A.; RIDEAU, G.
1. On some representations of anticommutation relations, I, II, III, *Nuovo Cimento*, **45** (1966) 1–14; **46** (1966), 624–636; **50** (1967), 244–255.

CORTET, J.-C.
1. États G-factoriels, *CR Acad. Sc. Paris*, **275** (1972), 563–566; **276** (1973), 1141–1143.
2. Opérateurs fermés affiliés à une algèbre de von Neumann, *CR Acad. Sc. Paris*, **281** (1975), 53–55.
3. Sur l'intégrabilité des algèbres de Lie de dimension finie de dérivations d'une C^*-algèbre, *CR Acad. Sc. Paris*, **285** (1977), 187–190.

COURBAGE, M.; MIRACLE-SOLE, S.; ROBINSON, D. W.
1. Normal states and representations of the canonical commutation relations, *Ann. Inst. H. Poincaré*, **14** (1971), 171–178.

CRABB, M. J.; DUNCAN, J.; MC GREGOR, C. M.
1. Characterizations of commutativity for C^*-algebras, *Glasgow Math. J.*, **15** (1974), 172–175.

CRAW, I. C.
1. Axiomatic cohomology of operator algebras, *Bull. Soc. Math. France*, **100** (1972), 449–460.
2. Axiomatic cohomology for Banach modules, *Proc. AMS*, **38** (1973), 68–74.

CUCULESCU, I.
1. Supermartingales on W^*-algebras, *Rev. Roum. Math. Pures Appl.*, **14** (1969), 759–773.
2. Spectral families and stochastic integrals, *Rev. Roum. Math. Pures Appl.*, **15** (1970), 201–221.
3. An example of a II_∞-factor representation of a connected Lie group, *Bull. Fac. Sci., Teheran Univ.*, **3**: 2 (1971), 47–49.
4. Martingales on von Neumann algebras, *J. Multiv. Analysis*, **1** (1971), 17–27.
5. A proof of $(A \otimes B)' = A' \otimes B'$ for von Neumann algebras, *Rev. Roum. Math. Pures Appl.*, **16** (1971), 665–670.

CUNTZ, J.
1. Locally C^*-equivalent algebras, *J. Fnal Analysis*, **23** (1976), 95–106.
2. On the continuity of seminorms on operator algebras, *Math. Ann.*, **220** (1976), 171–184.
3. Eine Klasse von postliminalen gewichteten Shiftoperatoren, *Arch. Math.*, **27** (1976), 188–198.
4. The structure of multiplication and addition in simple C^* algebras, *Math. Scand.*, **40** (1977), 215–233.
5. Simple C^*-algebras generated by isometries, *Comm. Math. Phys.*, **57** (1977), 173–185.
6. Dimension functions on simple C^*-algebras, *Math. Ann.*, 1978.
7. Non-commutative Haar measure and algebraic condition for simple C^*-algebras, Marseille, 1977.
8. Murray-von Neumann equivalence of projections in infinite simple C^*-algebras, *Rev. Roum. Math. Pures Appl.*, 1978.

CUNTZ, J.; PEDERSEN, G. K.
1. Equivalence and traces on C^*-algebras, Copenhagen, 1977.
2. Equivalence and KMS-states on periodic C^*-dynamical systems, Copenhagen, 1977.

CURTIS, P.
1. Derivations of commutative Banach algebras, *Bull. AMS*, **67** (1961), 271–273.

VAN DAELE, A.
1. Quasi-equivalence of quasi-free states on the Weyl algebra, *Comm. Math. Phys.*, **21** (1971), 171–191.

2. The upper envelope of invariant functionals majorized by an invariant weight, *Pacific J. Math.*, **46** (1973), 283–302.
3. The Tomita-Takesaki theory for von Neumann algebras with a separating and cyclic vector, in "*C*-algebras and their applications to statistical mechanics and quantum field theory*", p. 19–28, North-Holland, 1976.
4. A new approach to the Tomita-Takesaki theory of generalized Hilbert algebras, *J. Fnal Analysis*, **15** (1974), 378–393.
5. A Radon-Nikodym theorem for weights on von Neumann algebras, *Pacific J. Math.*, **61** (1976), 527–542.
6. On the spectrum of the analytic generator, *Math. Scand.*, **37** (1975), 307–318.
7. On a certain class of semisimple subalgebras of a semisimple Lie algebra, *Ann. Inst. H. Poincaré*, A **13** (1970), 195–213.
8. A bounded operator approach to the Tomita-Takesaki theory, *Symposia Math.*, XX, 219–232, Academic Press, 1976.
9. A framework to study commutation problems.
10. Fixed points and commutation theorems, *Japan-US Seminary*, 1977.
11. Crossed products of von Neumann algebras, *Lecture Notes Series, London Math. Soc.*, 1978.
12. The structure of type III von Neumann algebras, *Lecture Notes Series, London Math. Soc.*, 1978.

VAN DAELE, A.; ROUSSEAU, R.
1. Crossed products of commutation systems.

VAN DAELE, A.; VERBEURE, A.
1. Unitary equivalence of Fock representations on the Weyl algebra, *Comm. Math. Phys.*, **20** (1971), 268–278.

DAGUE, P.
1. Détermination de la topologie de Fell sur le dual du groupe de Poincaré, *CR Acad. Sc. Paris*, **283** (1976), 293–296.

DALES, H. G.
1. Automatic continuity; a survey, Leeds, 1977.

DANG-NGOC, N.
1. Sur la classification des systèmes dynamiques non commutatifs, *J. Fnal Analysis*, **15** (1974), 188–201.
2. On the integral representation of states on a C^*-algebra, *Comm. Math. Phys.*, **40** (1975), 223–233.
3. On the classification of dynamical systems, *Ann. Inst. H. Poincaré*, B **9** (1973), 397–425.
4. Convergence forte des espérances conditionnelles et des projecteurs d'un espace de Hilbert, *Ann. Inst. H. Poincaré*, B **6** (1970), 9–13.
5. Σ^*-algèbres, probabilités non-commutatives et applications, *Bull. Soc. Math. France, Mémoire* **35** (1973), 145–189.
6. Classification of dynamical systems and classification of von Neumann algebras, *Bull. Soc. Math. France*.
7. Partie finie d'un système dynamique et deux nouvelles démonstrations du théorème de Hopf, *Z. Wahr. verv. Gebiete*, **27** (1973), 131–140.
8. Décomposition et classification des systèmes dynamiques, *Bull. Soc. Math. France*, **103** (1975), 149–175.
9. Invariant weights on decomposable von Neumann algebras.
10. Plancherel measure for type I traces.
11. Integral of semi-finite traces and σ-finite measures.
12. Produits croisés restreints et extensions des groupes.
13. On the integrability of the maximal ergodic function.
14. Pointwise convergence of martingales on von Neumann algebras.
15. Random ergodic theorem in von Neumann algebras.

DANG-NGOC, N.; LEDRAPPIER, F.
1. Sur les systèmes dynamiques simpliciaux, *CR Acad. Sc. Paris*, **277** (1973), 777–779.

Darsow, W. F.
1. Positive definite functions and states, *Ann. Math.*, **60** (1954), 447–453.

Darst, R. B.
1. On a theorem of Nikodym with applications to weak convergence and von Neumann algebras, *Pacific J. Math.*, **23** (1967), 473–477; *Bull. AMS*, **74** (1968), 283–284.

Dash, A. T.
1. Joint spectrum in the Calkin algebra, *Bull. AMS*, **81** (1975), 1083–1085.

Davenport, J.
1. The strict dual of a B^*-algebra, *Proc. AMS*, **65** (1977), 309–312.

Davidson, K. R.
1. On operators commuting with Toeplitz operators modulo the compact operators, *J. Fnal Analysis*, **24** (1977), 291–302.

Dauns, J.
1. Categorical W^*-tensor product, *Trans. AMS*, **166** (1972), 439–456.
2. The primitive ideal space of a C^*-algebra, *Can. J. Math.*, **26** (1974), 42–49.

Dauns, J.; Hofmann, K. H.
1. *Representations of rings by continuous sections*, Memoirs AMS, No. 83, 1968.
2. Spectral theory of algebras and adjunction of identity, *Math. Ann.*, **179** (1969), 175–202.

Davies, E. B.
1. On the Borel structure of C^*-algebras, *Comm. Math. Phys.*, **8** (1968), 147–163.
2. The structure of Σ^*-algebras, *Quart. J. Math., Oxford*, **20** (1969), 351–366.
3. Decompositions of traces on separable C^*-algebras, *Quart. J. Math., Oxford*, **20** (1969), 97–111.
4. Quantum stochastic processes, I, II, III, *Comm. Math. Phys.*, **15** (1969), 277–304; **19** (1970), 83–105; **20** (1971), 51–70.
5. On the repeated measurement of continuous observables in quantum mechanics, *J. Fnal Analysis*, **6** (1970), 318–346.
6. Involutory automorphisms of operator algebras, *Trans. AMS*, **158** (1971), 115–142.
7. Hilbert space representations of Lie algebras, *Comm. Math. Phys.*, **23** (1971), 159–168.
8. A generalization of Kaplansky's theorem, *J. London Math. Soc.*, **4** (1972), 435–436.
9. *Quantum theory of open systems*, Academic Press, 1976.
10. Generators of dynamical semigroups.
11. Dilations of completely positive maps.
12. Irreversible dynamics of infinite fermion systems, *Comm. Math. Phys.*, **55** (1977), 231–258.

Davies, E. B.; Lewis, J. T.
1. An operational approach to quantum mechanics, *Comm. Math. Phys.*, **17** (1970), 239–260.

Davis, C.
1. Generators of the ring of bounded operators, *Proc. AMS*, **6** (1955), 970–972.
2. Various averaging operations into subalgebras, *Illinois J. Math.*, **3** (1959), 538–553.
3. Operator valued entropy of a quantum mechanical measurement, *Proc. Jap. Acad.*, **37** (1961), 533–538.
4. A Schwarz inequality for convex operator functions, *Proc. AMS*, **8** (1957), 42–44.
5. Notions generalizing convexity for functions defined on spaces of matrices, *Proc. Symp. Pure Math.*, vol. 7, *AMS*, Providence, (1962), 187–201.

Day, M. M.
1. Amenable semigroups, *Illinois J. Math.*, **1** (1957), 509–544.

Deckard, D.; Pearcy, C.
1. On matrices over the ring of continuous complex valued functions on a stonian space, *Proc. AMS*, **14** (1963), 322–328.
2. On continuous matrix-valued functions on a stonian space, *Pacific J. Math.*, **14** (1964), 857–869.

DEDDENS, J. A.
1. Another description of nest algebras.

DEDDENS, J. A.; WOGEN, W. R.
1. On operators with the double commutant property, *Duke Math. J.*, 43 (1976), 359–364.

DEEL, J. C.
1. Derivations of AW^*-algebras, *Proc. AMS*, 42 (1974), 85–95.

DELAROCHE, C.
1. Sur les centres des C^*-algèbres, I, II, *CR Acad. Sc. Paris*, 265 (1967), 465–466; *Bull. Sc. Math.*, 91 (1967), 105–112; 92 (1968), 111–128.
2. Spectres des extensions des C^*-algèbres, *CR Acad. Sc. Paris*, 269 (1969), 1003–1005.
3. Extensions localement quasicompactes d'un espace localement quasicompact par un autre, *CR Acad. Sc. Paris*, 269 (1969), 953–955.
4. Limites et valeurs d'adhérence d'un filtre sur le spectre d'une C^*-algèbre, *CR Acad. Sc. Paris*, 271 (1970), 434–437.
5. Sur les extensions des C^*-algèbres, *CR Acad. Sc. Paris*, 272 (1971), 727–730.
6. Extensions de C^*-algèbres, *Bull. Soc. Math. France*, Suppl. 29 (1972).
7. Quelques problèmes concernant la structure des C^*-algèbres, *Séminaire Choquet*, 1966/67, fasc. 2, exp. 17, 7 p.

DELAROCHE, C.; KIRILLOV, A. A.
1. Relations entre l'espace dual d'un groupe et la structure de ses sous-groupes fermés, *Sém. Bourbaki*, 1967/68, no. 343.

DELL'ANTONIO, G. F.
1. On some groups of automorphisms of physical observables, *Comm. Math. Phys.*, 2 (1966), 384–397.
2. On the limits of sequences of normal states, *Comm. Pure Appl. Math.*, 20 (1967), 413–429.
3. Structure of the algebras of some free systems, *Comm. Math. Phys.*, 9 (1968), 81–117.
4. Can local gauge transformations be implemented?, *J. Math. Phys.*, 12 (1971), 148–156.

DELL'ANTONIO, G. F.; DOPLICHER, S.; RUELLE, D.
1. A theorem on canonical commutation and anticommutation relations, *Comm. Math. Phys.*, 2 (1966), 223–230.

DELODE, C.
1. Champs measurables d'espaces sousliniens, *CR Acad. Sc. Paris*, 283 (1976), 831–833; *Ann. IHP*, 13 (1977), 181–191.

DELODE, C.; ARINO, O.; PENOT, J.-P.
1. Champs mesurables d'espaces polonais, *CR Acad. Sc. Paris*, 281 (1975), 617–620.
2. Champs mesurables et multisections, *Ann. IHP*, B 12 (1976), 11–42.

DELORME, P.
1. Sur la 1-cohomologie des groupes localement compacts et produits tensoriels continus de représentations, *CR. Acad. Sc. Paris*, 280 (1975), 1101–1104.
2. *1-cohomologie et produits tensoriels continus de représentations*, Thèse, Paris, 1975.
3. Sur la 1-cohomologie des représentations unitaires des groupes de Lie semi-simples et résolubles, *CR Acad. Sc. Paris*, 282 (1976), 499–501.

DEMOEN, B.; VANHEERVERZWIJN, P.; VERBEURE, A.
1. Completely positive maps on the CCR-algebras, *Lett. Math. Phys.*, 2 (1977), 161–166.
2. Completely quasi-free maps on the CCR-algebra, Leuven, 1977.

DESKINS, W. E.
1. When is an automorphism inner?, *J. Indian Math. Soc.*, 38 (1974), 37–41.

DEVINATZ, A.; NUSSBAUM, A. E.
1. On the permutability of normal operators, *Ann. Math.*, 65 (1957), 144–152.

DEVINATZ, A.; NUSSBAUM, A. E.; VON NEUMANN, J.
1. On the permutability of selfadjoint operators, *Ann. Math.*, 62 (1955), 199–203.

DIGERNES, T.

1. A new characterization of separable GCR algebras, *Proc. AMS*, **36** (1972), 448–450.
2. On classes of projections in a von Neumann algebra, *Math. Scand.*, **31** (1972), 191–200.
3. Poids duals sur un produit croisé, *CR Acad. Sc. Paris*, **278** (1974), 937–940.
4. Dual weight and the commutation theorem for crossed products of W^*-algebras.
5. *Duality for weights on covariant systems and its applications*, Dissertation, Los Angeles, 1975.

DIGERNES, T.; HALPERN, H.

1. On open projections for GCR algebras, *Can. J. Math.*, **24** (1972), 978–982.

DIXMIER, J.

1. Sur une classe nouvelle de variétés et d'opérateurs linéaires de l'espace de Hilbert, *CR Acad. Sc. Paris*, **223** (1946), 971–972.
2. Propriétés géométriques des domaines d'existence des opérateurs linéaires fermés de l'espace de Hilbert, *CR Acad. Sc. Paris*, **224** (1947), 180–181.
3. Position relative de deux variétés linéaires fermées dans un espace de Hilbert, *Rev. Sc.*, **86** (1948), 387–399.
4. Sur un théorème de Banach, *Duke Math. J.*, **15** (1948), 1057–1061.
5. Les fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert, *CR Acad. Sc. Paris*, **227** (1948), 948–950.
6. Sur les variétés de Julia d'un espace hilbertien, *J. Math. Pures Appl.*, **28** (1949), 321–358.
7. Etude sur les variétés et les opérateurs de Julia avec quelques applications, *Bull. Soc. Math. France*, **77** (1949), 11–101.
8. L'adjoint du produit de deux opérateurs fermés, *Ann. Fac. Univ. Toulouse*, (4), **11** (1949), 101–106.
9. Les idéaux dans l'ensemble des variétés J d'un espace hilbertien, *Ann. Fac. Univ. Toulouse*, (4), **10** (1949), 91–114.
10. Mesure de Haar et trace d'un opérateur, *CR Acad. Sc. Paris*, **228** (1949), 152–154.
11. Les opérateurs permutables à l'opérateur intégral, *Portug. Math.*, **8** (1949), 73–84.
12. Les anneaux d'opérateurs de classe finie, *Ann. Ec. Norm. Sup.*, **66** (1949), 209–261.
13. Sur les opérateurs self-adjoints d'un espace de Hilbert, *CR Acad. Sc. Paris*, **230** (1950), 267–269.
14. Les fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert, *Ann. Math.*, **51** (1950), 387–408.
15. Applications $\natural$ dans les anneaux d'opérateurs, *CR Acad. Sc. Paris*, **230** (1950), 607–608; *Compositio Math.*, **10** (1952), 1–55.
16. Les moyennes invariantes dans les semi-groupes et leurs applications, *Acta Sci. Math.*, **12** (1959), 213–227.
17. Sur certains espaces considérés par M. H. Stone, *Summa Bras. Math.*, 2 (1951), fasc. 11, 151–181.
18. Sur la réduction des anneaux d'opérateurs, *Ann. Ec. Norm. Sup.*, **68** (1951), 185–202.
19. Algèbres quasi-unitaires, *CR Acad. Sc. Paris*, **233** (1951), 837–839; *Comment. Math. Helv.*, **26** (1952), 275–322.
20. Remarques sur les applications $\natural$, *Arch. Math.*, **3** (1952), 290–297.
21. Sur les bases orthonormales dans les espaces préhilbertiens, *Acta Sci. Math.*, **15** (1953), 29–30.
22. Sur une inégalité de E. Heinz, *Math. Ann.*, **126** (1953), 75–78.
23. Formes linéaires sur un anneau d'opérateurs, *Bull. Soc. Math. France*, **81** (1953), 9–39.
24. Sur les anneaux d'opérateurs dans les espaces hilbertiens, *CR Acad. Sc. Paris*, **238** (1954), 439–441.
25. Sous-anneaux abéliens maximaux dans les facteurs de type fini, *Ann. Math.*, **59** (1954), 279–286.
26. *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)*, Gauthier-Villars, Paris, 1957, 1969.
27. Sur les représentations unitaires des groupes de Lie algébriques, *Ann. Inst. Fourier*, **7** (1957), 315–328.
28. Sur les représentations unitaires des groupes de Lie nilpotentes, I, II, III, IV, V, VI, *Amer. J. Math.*, **81** (1959), 160–170; *Bull. Soc. Math. France*, **85** (1957), 325–388; *Can. J. Math.*, **10** (1958), 321–348; **11** (1959), 321–344; *Bull. Soc. Math. France*, **87** (1959), 65–79; *Can. J. Math.*, **12** (1960), 324–352.

29. Sur les structures boréliennes du spectre d'une C^*-algèbre, *Publ. IHES*, 6 (1960), 297–303.
30. Opérateurs de rang fini dans les représentations unitaires, *Publ. IHES*, 6 (1960), 13–25.
31. Sur les C^*-algèbres, *Bull. Soc. Math. France*, 88 (1960), 95–112.
32. Points isolés dans le dual d'un groupe localement compact, *Bull. Sc. Math.*, 85 (1961), 91–96.
33. Représentations intégrables du groupe de De Sitter, *Bull. Soc. Math. France*, 89 (1961), 9–41.
34. Points séparés dans le spectre d'une C^*-algèbre, *Acta Sci. Math.*, 22 (1961), 115–128.
35. Sur le revêtement universel d'un groupe de Lie de type I, *CR Acad. Sc. Paris*, 252 (1961), 2805–2806.
36. Dual et quasi-dual d'une algèbre de Banach involutive, *Trans. AMS*, 104 (1962), 278–283.
37. Sur les représentations unitaires des groupes de Lie résolubles, *Math. J. Okayama Univ.*, 11 (1962), 1–18.
38. Quasi-dual d'un idéal dans une C^*-algèbre, *Bull. Sc. Math.*, 87 (1963), 7–11.
39. Champs continus d'espaces hilbertiens et de C^*-algèbres, II, *J. Math. Pures Appl.*, 42 (1963), 1–20.
40. Traces sur les C^*-algèbres, I, II, *Ann. Inst. Fourier*, 13 (1963), 219–262; *Bull. Sc. Math.*, 88 (1964), 39–57.
41. Utilisation des facteurs hyperfinis dans la théorie des C^*-algèbres, *CR Acad. Sc. Paris*, 258 (1964), 4184–4187.
42. *Les C^*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964, 1968.
43. Existence des traces non-normales, *CR Acad. Sc. Paris*, 262 (1966), 1107–1108.
44. Sur les automorphismes des algèbres de Banach, *CR Acad. Sc. Paris*, 264 (1967), 729–731.
45. On some C^*-algebras considered by J. Glimm, *J. Fnal Analysis*, 1 (1967), 182–203.
46. Localization of a theorem of Glimm, *Proc. AMS*, 19 (1968), 364–366.
47. Ideal center of a C^*-algebra, *Duke Math. J.*, 35 (1968), 375–382.
48. Sur les espaces localement quasi-compacts, *Can. J. Math.*, 20 (1968), 1093–1100.
49. Algèbres d'opérateurs, *Proc. Int. School Phys. "E. Fermi"*, 15 (1968), 1–31.
50. Sur la représentation régulière d'un groupe localement compact connexe, *Ann. Ec. Norm. Sup.*, 2 (1969), 423–436.
51. Sur les groupes d'automorphismes normiquement continus, *CR Acad. Sc. Paris*, 269 (1969), 643–644.
52. Quelques propriétés des suites centrales dans les facteurs de type II_1, *Invent. Math.*, 7 (1969), 215–225.
53. Facteurs: classification, dimension, trace, *Séminaire Bourbaki*, 1949/50, exp. 30.
54. Les algèbres hilbertiennes modulaires de Tomita (d'après Takesaki), *Séminaire Bourbaki*, 1969/70, *Lecture Notes in Math.*, no. 180, 1971.
55. Bicontinuité dans la méthode du petit groupe de Mackey, *Bull. Sc. Math.*, 97 (1973), 233–240.
56. Sur les algèbres de Weyl, I, II, *Bull. Soc. Math. France*, 96 (1968), 209–242; *Bull. Sc. Math.*, 94 (1970), 289–301.
57. *Algèbres enveloppantes*, Gauthier-Villars, Paris, 1974.
58. Simple C^*-algebras, Lecture Notes.
59. Groupes liminaires.

DIXMIER, J.; DOUADY, A.
1. Champs continus d'espaces hilbertiens et de C^*-algèbres, *Bull. Soc. Math. France*, 91 (1963), 227–284.

DIXMIER, J.; LANCE, E. C.
1. Deux nouveaux facteurs de type II_1, *Invent. Math.*, 7 (1969), 226–234.

DIXMIER, J.; MARÉCHAL, O.
1. Vecteurs totalisateurs d'une algèbre de von Neumann, *Comm. Math. Phys.*, 22 (1971), 44–50.

DIXON, P. G.
1. Generalized B^*-algebras, I, II, *Proc. London Math. Soc.*, 21 (1970), 693–715; *J. London Math. Soc.*, 5 (1972), 159–165.
2. Unbounded operator algebras, *Proc. London Math. Soc.*, 23 (1971), 53–69.

3. Approximate identities in normed algebras, I, II, *Proc. London Math. Soc.*, **26** (1973), 485–496.
4. Semiprime Banach algebras, *J. London Math. Soc.*, **6** (1973), 676–678.
5. Locally finite Banach algebras, *J. London Math. Soc.*, **8** (1974), 325–328.
6. A symmetric normed $*$-algebra whose completion is not symmetric.
7. Non-separable Banach algebras whose squares are pathological, *J. Fnal Analysis*, **26** (1977), 190–200.
8. A Jacobson semi-simple Banach algebra with dense nil subalgebra, *Colloq. Math.*, **37** (1977).
9. Varieties of Banach algebras, *Quart. J. Math.*, **27** (1976), 481–487.
10. A characterization of closed subalgebras of $B(H)$, *Proc. Edinburgh Math. Soc.*, **20** (1976–77), 215–217.

DOODS, P. G.
1. The order dual of an abelian von Neumann algebra, *J. Australian Math. Soc.*, **18** (1974), 153–160.

DOOLEY, A. H.
1. The spectral theory of posets and its applications to C^*-algebras, *Trans. AMS*, **224** (1976), 143–155.

DOPLICHER, S.
1. An algebraic spectrum condition, *Comm. Math. Phys.*, **1** (1965), 1–5.
2. Superselection rules and fields for local observables, *Cargese Lectures in Phys.*, **4** (1970), 79–93.
3. Fock representations and massless particles, *Comm. Math. Phys.*, **3** (1966), no. 3.
4. Quantum field theory and operator algebras of local observables, in *Symposia Math.*, XX, 109–116, Academic Press, 1976.
5. Fields, observables and gauge transformations, in *Systèmes à un nombre infini de degrés de liberté*, Gif-sur-Yvette, 1969, Ed. CNRS-1970, 113–123.
6. A remark on a theorem of Powers and Sakai, *Comm. Math. Phys.*, **45** (1975), 59–62.
7. Locality and particle statistics, *Trudy Mat. Inst. Akad. Nauk SSSR*, **135** (1975), 107–114.
8. The superselection structure in local quantum theories, *Proc. Int. School Math. Phys.*, Univ. Camerino, 1974, 1–36.

DOPLICHER, S.; GALLAVOTTI, G.; RUELLE, D.
1. Almost periodic states on C^*-algebras (Unpublished report, 1966).

DOPLICHER, S.; GUICHARDET, A.; KASTLER, D.
1. Désintégration des états quasi-invariants des C^*-algèbres (Unpublished report, 1969).

DOPLICHER, S.; HAAG, R.; ROBERTS, J. E.
1. Fields, observables and gauge transformations, I, II, *Comm. Math. Phys.*, **13** (1969), 1–24; **15** (1969), 173–201.
2. Local observables and particle statistics, I, II, *Comm. Math. Phys.*, **23** (1971), 199–230; **35** (1974), 49–85.

DOPLICHER, S.; KADISON, R. V.; KASTLER, D.; ROBINSON, D. W.
1. Asymptotically abelian systems, *Comm. Math. Phys.*, **6** (1967), 101–120.

DOPLICHER, S.; KASTLER, D.
1. Ergodic states in a non-commutative ergodic theory, *Comm. Math. Phys.*, **7** (1968), 1–20.

DOPLICHER, S.; KASTLER, D.; ROBINSON, D. W.
1. Covariance algebras in field theory and statistical mechanics, *Comm. Math. Phys.*, **3** (1966), 1–28.

DOPLICHER, S.; KASTLER, D.; STÖRMER, E.
1. Invariant states and asymptotic abelianess, *J. Fnal Analysis*, **3** (1969), 419–434.

DOPLICHER, S.; POWERS, R. T.
1. On the simplicity of the even CAR algebra and free field models, *Comm. Math. Phys.* **7** (1968), 77–92.

DOPLICHER, S.; REGGE, T.; SINGER, I. M.
1. A geometrical model showing the independence of locality and positivity of the energy, *Comm. Math. Phys.*, **7** (1968), 51–54.

DOPLICHER, S.; ROBERTS, J. E.
1. Fields, statistics and non-abelian gauge groups, *Comm. Math. Phys.*, **28** (1972), 331–348.

DORAN, R. S.; WICHMANN, J.
1. The Gelfand-Naimark theorem for C^*-algebras, *L'Enseignement Math.*, II, **23** (1977), 153–180.

DOUGLAS, R. G.
1. On hyperinvariant subspaces for isometries, *Math. Z.*, **107** (1968), 297–300.
2. On the C^*-algebra of a one-parameter semigroup of isometries, *Acta Math.*, **128** (1972), 143–152.
3. *Banach algebra techniques in operator theory*, Academic Press, 1972.
4. *Banach algebra techniques in the theory of Toeplitz operators, Lectures given at a CBMS Reg. Conf. Math. Univ. Georgia, Athens*, 1972.
5. Canonical models, *Topics in operator theory, Math. Surveys*, No. 13 (1974), 161–218.
6. The relation of Ext to K-theory, in *Symposia Math.*, XX, 513–530, Academic Press, 1976.
7. Extensions of C^*-algebras and K-homology, *Lecture Notes in Math.*, No. 575 (1977), 44–53.

DOUGLAS, R. G.; HOWE, R.
1. On the C^*-algebra of Toeplitz operators on the quarter plane, *Trans. AMS*, **158** (1971), 203–218.

DOUGLAS, R. G.; PEARCY, C.
1. A characterization of thin operators, *Acta Sci. Math.*, **29** (1968), 295–297.
2. On a topology for invariant subspaces, *J. Fnal Analysis*, **2** (1968), 323–341.
3. Von Neumann algebras with a single generator, *Michigan Math. J.*, **16** (1969), 21–26.

DOWSON, H. R.; GILLESPIE, T. A.; SPAIN, P. G.
1. A commutativity theorem for hermitean operators, *Math. Ann.*, **220** (1976), 215–217.

DRIESSLER, W.
1. On the type of local algebras in quantum field theory, *Comm. Math. Phys.*, **53** (1977), 295–298.

DRIESSLER, W.; FRÖHLICH, J.
1. The reconstruction of local observable algebras from the Euclidean Green's functions of a relativistic quantum field theory, *Ann. IHP*, (A) **27** (1977), 221–236.

DRIESSLER, W.; WILDE, I. F.
1. Stochastic independence in non-commutative probability theory and factoring states on von Neumann algebras, London, 1977.

DUBIN, D. A.
1. Bosons in thermal contact: a C^*-algebraic approach, *Comm. Math. Phys.*, **32** (1973), 1–19.

DUBOIS-VIOLETTE, M.
1. A generalization of the classical moment problem on $*$-algebra with applications to relativistic quantum theory, I, II, *Comm. Math. Phys.*, **43** (1975), 225–254; **54** (1977), 151–172.
2. A theory of regularity for linear forms on C^*-algebras, preprint, LPTHE Orsay, 75/17.

DUBUC, S.
1. Fonctionnelles linéaires positives extrémales, *CR Acad. Sc. Paris*, **270** (1970), 1502–1505.

DUFLO, M.; MOORE, C. C.
1. On the regular representation of a non-unimodular locally compact group, *J. Fnal Analysis*, **21** (1976), 209–243.

DUFLO, M.; RAÏS, M.
1. Sur l'analyse harmonique sur les groupes de Lie résoluble, *Ann. Sc. Ec. Norm. Sup.*, **9** (1976), 107–144.

DUNCAN, J.
1. The continuity of the involution in Banach $*$-algebras, *J. London Math. Soc.*, **41** (1966), 701–706.

2. B^*-modular annihilator algebras, *Proc. Edinburgh Math. Soc.*, **15** (1966/67), 89–112.
3. The evaluation functionals associated with an algebra of operators, *Glasgow Math. J.*, **10** (1969), 73–76.

DUNCAN, J.; TAYLOR, P. J.
1. Norm inequalities for C^*-algebras, Proc. Roy. Soc. Edinburgh, A **75**, (1976), 119–129.

DUNFORD, N.
1. Resolution of the identity for commutative B^*-algebras of operators, *Acta Sci. Math.*, **12** B (1950), 51–56.

DUNFORD, N.; SCHWARTZ, J.
1. *Linear Operators*, I, II, III, Intersci. Publ., 1958, 1963, 1970.

DUNKL, C. F.; RAMIREZ, D. E.
1. *Topics in harmonic analysis*, Appleton-Century-Crafts, 1971.
2. C^*-algebras generated by measures, *Bull. AMS*, **77** (1971), 411–412.
3. C^*-algebras generated by Fourier-Stieltjes transforms, *Trans. AMS*, **164** (1972), 435–442.

DUPRÉ, M. J.
1. Classifying Hilbert bundles, I, II, *J. Fnal Analysis*, **15** (1974), 244–278; **22** (1976), 295–322.
2. Hilbert bundles with infinite dimensional fibres, *Memoires AMS*, **148** (1974), 165–176.
3. Classifying C_1^*-bundles, *Notices AMS*, **22**: 5 (1975), 75 T-B 155.
4. Cohomology for classifying categories of Banach bundles (summary), Tulane Univ.
5. The classification and structure of C^*-algebra bundles.
6. Duality for C^*-algebras, Tulane, 1977.

DYE, H. A.
1. The Radon-Nikodym theorem for finite rings of operators, *Trans. AMS*, **72** (1952), 243–280.
2. The unitary structure in finite rings of operators, *Duke Math. J.*, **20** (1953), 55–69.
3. On the geometry of projections in certain operator algebras, *Ann. Math.*, **61** (1955), 73–89.
4. On groups of measure preserving transformations, I, II, *Amer. J. Math.*, **81** (1959), 119–159; **85** (1963), 551–576.

DYER, J. A.
1. Concerning AW^*-algebras, *Notices AMS*, **17** (1970), 677–47–5; *J. Fnal Analysis*.

DYER, J.; PEDERSEN, E.; PORCELLI, P.
1. An equivalent formulation of the invariant subspace conjecture, *Bull. AMS*, **78** (1972), 1020–1023.

DYER, J. A.; PORCELLI, P.; ROSENFELD, M.
1. Spectral characterization of two-sided ideals in $B(H)$, *Israel J. Math.*, **10** (1971), 26–31.

ECKMANN, J. P.; FRÖHLICH, J.
1. Unitary equivalence of local algebras in the quasi-free representation, *Ann. Inst. H. Poincaré*, A **20** (1974), 201–209.

ECKMANN, J. P.; GUENIN, M.
1. *Méthodes algébriques en mécanique statistique*, Lecture Notes in Math., No. 81, Springer Verlag, 1969.

ECKMANN, J. P.; OSTERWALDER, K.
1. An application of Tomita's theory of modular Hilbert algebras: Duality for free Bose fields, *J. Fnal Analysis*, **13** (1973), 1–12.

EDWARDS, C. M.
1. C^*-algebras of central group extensions, I, *Ann. Inst. H. Poincaré*, **10** (1969), 229–246.
2. The operational approach to algebraic quantum mechanics, I, *Comm. Math. Phys.*, **16** (1970), 207–230.
3. Classes of operations in quantum theory, *Comm. Math. Phys.*, **20** (1971), 26–57.
4. The theory of pure operations, *Comm. Math. Phys.*, **24** (1972), 260–288.
5. Spectral theory for $A(X)$, *Math. Ann.*, **267** (1974), 67–85.

6. The spectrum of a real Banach space, *Proc. London Math. Soc.*, **28** (1974), 654–670.
7. On global central decomposition for a universal cap, *Math. Ann.*, **213** (1975), 223–230.
8. Sets of simple observables in the operational approach to quantum theory, *Ann. IHP*, A **15** (1971), 1–14.
9. Alternative axioms for statistical physical theories, *Ann. IHP*, **22** (1975), 81–95.
10. On the monotone σ-envelope of $A(K)$, *Math. Proc. Cambr. Phil.* Soc., **78** (1975), 315–316.
11. The centre of a physical system, *Lecture Notes in Physics*, **29** (1974), 199–205.
12. Ideal theory in *JB*-algebras, *J. London Math. Soc.*, **16** (1977), 507–513.

EDWARDS, C. M.; GERZON, M. A.
1. Monotone convergence in partially ordered vector spaces, *Ann. Inst. H. Poincaré*, **12** (1970), 323–328.

EDWARDS, C. M; LEWIS, J. T.
1. Twisted group algebras, I, II, *Comm. Math. Phys.*, **13** (1969), 119–130; 131–140.

EFFROS, E. G.
1. A decomposition theory for representations of C^*-algebras, *Trans. AMS*, **107** (1963), 83–106.
2. Order ideals in a C^*-algebra and its dual, *Duke Math. J.*, **30** (1963), 391–412.
3. Convergence of closed subsets in a topological space, *Proc. AMS*, **16** (1965), 929–931.
4. Transformation groups and C^*-algebras, *Ann. Math.*, **81** (1965), 38–55.
5. The Borel space of von Neumann algebras on separable Hilbert space, *Pacific J. Math.*, **15** (1965), 1153–1164.
6. Global structure in von Neumann algebras, *Trans. AMS*, **121** (1966), 434–454.
7. Structure in simplexes, I, II, *Acta Math.*, **117** (1967), 103–121; *J. Fnal Analysis*, **1** (1967), 379–391.
8. The canonical measures for a separable C^*-algebra, *Amer. J. Math.*, **92** (1970), 56–60.
9. *Injectives and tensor products for convex sets and C^*-algebras, Lecture given at the NATO Conf., Univ. Coll. Swansea, Wales,* 1972.
10. On a class of real Banach spaces, *Israel J. Math.*, **9** (1971), 430–458.
11. On a class of complex Banach spaces, *Illinois J. Math.*, **18** (1974), 48–59.
12. Property Γ and inner amenability, *Proc. AMS*, **47** (1975), 483–486.
13. Approximation problems for C^*-algebras.
14. Aspects of non-commutative geometry, Marseille, 1977.
15. Aspects of non-commutative order, *Japan-US Seminary*, 1977.
16. Nuclearity and the C^*-algebraic flip.

EFFROS, E. G.; GLEIT, A. M.
1. Structures in simplexes, III, *Trans. AMS*, **142** (1969), 335–380.

EFFROS, E. G.; HAHN, F.
1. Locally compact transformation groups and C^*-algebras, *Bull. AMS*, **73** (1967), 222–226; *Memoirs AMS*, No. 75 (1967).

EFFROS, E. G.; LANCE, E. C.
1. Tensor products of operator algebras, *Adv. Math.*, **25** (1977), 1–34.

EFFROS, E. G.; ROSENBERG, J.
1. C^*-algebras with approximately inner flip.

EFFROS, E. G.; STÖRMER, E.
1. Jordan algebras of selfadjoint operators, *Trans, AMS*, **127** (1967), 313–316.

EHRENPREIS, L.; MAUTNER, F. I.
1. Uniformly bounded representations of groups, *Proc. Nat. Acad. Sci. USA*, **41** (1955), 231–233.
2. Some properties of the Fourier transform on semisimple Lie groups, I, II, III, *Ann. Math.*, **61** (1955), 406–439; *Trans. AMS*, **84** (1957), 1–55; **90** (1959), 431–484.

EKMAN, K. E.
1. Indices on C^*-algebras through representation in the Calkin algebra, *Duke Math. J.*, **41** (1974), 413–432.

2. Unitaries and partial isometries in a real W^*-algebra, *Proc. AMS*, **54** (1976), 138–140.

ELLIOTT, G. A.

1. An application of holomorphic functional calculus to sharpen a classical matrix commutator theorem, *Indiana Math. J.*, **20** (1970), 407–409 (Erratum, **23** (1973), 367–368).
2. Derivations of matroid C^*-algebras, I, II, *Invent. Math.*, **9** (1970), 253–269; *Ann. Math.*, **100** (1974), 407–422.
3. A weakening of the axioms for a C^*-algebra, *Math. Ann.*, **189** (1970), 257–260.
4. Ideal preserving automorphisms of postliminary C^*-algebras, *Proc. AMS*, **27** (1971), 107–109.
5. An extension of some results of Takesaki in the reduction theory of von Neumann algebras, *Pacific J. Math.*, **39** (1971), 145–148.
6. Another weak Stone-Weierstrass theorem for C^*-algebras, *Can. Math. Bull.*, **15** (1972), 355–358.
7. Convergence of automorphisms in certain C^*-algebras, *J. Fnal Analysis*, **11** (1972), 204–206.
8. Perspectivity in the projection lattice of a W^*-algebra, *Proc. AMS*, **38** (1973), 367–368.
9. Some C^*-algebras with outer derivations, I, II, III, *Rocky Mountain J. Math.*, **3** (1973), 501–506; *Can. J. Math.*, **26** (1974), 185–189; *Ann. Math.*, **106** (1977), 121–143.
10. A characterization of compact groups, *Proc. AMS*, **29** (1971), 621.
11. On lifting and extending derivations of approximately finite-dimensional C^*-algebras, *J. Fnal Analysis*, **17** (1974), 395–408.
12. Finite projections in tensor product von Neumann algebras, *Trans. AMS*, **212** (1975), 47–60.
13. An abstract Dauns-Hofmann-Kaplansky multiplier theorem, *Can. J. Math.*, **27** (1975), 827–836.
14. On the classification of inductive limits of sequence of finite dimensional algebras, *J. Algebra*, **38** (1976), 29–44.
15. On the Radon-Nikodym derivative with a chain rule in von Neumann algebras, *Canad. Math. Bull.*, **18** (1975), 661–669.
16. Derivations determined by multipliers on ideals of a C^*-algebra, *Publ. RIMS, Kyoto Univ.*, **10** (1975), 721–728.
17. Automorphisms determined by multipliers on ideals of a C^*-algebra, *J. Fnal Analysis*, **23** (1976), 1–10.
18. The Mackey Borel structure on the spectrum of an approximately finite-dimensional separable C^*-algebra, *Trans. AMS*, **233** (1977), 59–68.
19. On approximately finite-dimensional von Neumann algebras, I, II, *Math. Scand.*, **39** (1976), 91–101.
20. Appendix to: Compact abelian groups of automorphisms of simple C^*-algebras, *Invent. Math.*, **39** (1977), 65–66.
21. On derivations of AW^*-algebras, *Tohoku Math. J.*
22. On totally ordered groups, Copenhagen, 1978.

ELLIOTT, G. A.; OLESEN, D.

1. A simple proof of the Dauns-Hofmann theorem, *Math. Scand.*, **34** (1974), 231–234.

ELLIOTT, G. A.; WOODS, E. J.

1. The equivalence of various definitions of hyperfiniteness of a properly infinite von Neumann algebra, *Proc. AMS*, **60** (1976), 175–178.

EMBRY, M. R.

1. Selfadjoint strictly cyclic operator algebras, *Pacific J. Math.*, **52** (1974), 53–57.

EMCH, G. G.

1. The definition of states in quantum statistical mechanics, *J. Math. Phys.*, **7** (1966), 1413–1420.
2. *Algebraic methods in statistical mechanics and quantum field theory*, John Wiley and Sons, New York, 1972.
3. Positivity of the K-entropy of non-abelian K-flows, *Z. Wahr. verw. Geb.*, **29** (1974), 241–252.

4. Non-abelian special *K*-flows, *J. Fnal Analysis*, **19** (1975), 1–12.
5. *Rigorous results in non-equilibrium statistical mechanics, Lectures in Physics*, Boulder, Colorado, vol. VIII A.
6. The minimal *K*-flows, in *Phys. Reality and Math. Description*, (Ch. Enz and J. Mehra, Eds.) Reidel Publ., Dordrecht 1974, 477–493.
7. Algebraic *K*-flows, *Proc. Int. Conf. Dynamical Systems in Math. Phys.*, Rennes, 1975, *Astérisque*, **40** (1976), 63–65.
8. An algebraic approach to the theory of *K*-flows and *K*-entropy, *Lecture Notes in Phys.*, **39** (1975), 315–318.
9. Non-equilibrium quantum statistical mechanics, *Lecture Notes, XV International Universitätswochen Schladming, Austria, Feb. 28, 1976.*
10. Generalized *K*-flows, *Comm. Math. Phys.*, **49** (1976), 191–216.
11. Quantum generalization of the Kolmogorov entropy, ZIF, Bielefeld, 1976.
12. An algebraic approach for spontaneous symmetry breaking in quantum statistical mechanics.
13. Phase transitions, approach to equilibrium, and structural stability.
14. Stochasticity and irreversibility in infinite mechanical systems.

EMCH, G. G; ALBEVERIO, S.; ECKMANN, J. P.
1. Quasi-free generalized *K*-flows, *Rep. Math. Phys.*

EMCH, G. G; KNOPS, H. J. F.
1. Pure thermodynamical phases as extremal KMS-states, *J. Math. Phys.*, **11** (1970), 3008–3018.

ENOCK, M.
1. Produit croisé d'une algèbre de von Neumann par une algèbre de Kac, *J. Fnal Analysis*, **26** (1977) 16–47.
2. Kac algebras and crossed products, Marseille, 1977.

ENOCK, M.; SCHWARTZ, J. M.
1. Une dualité dans les algèbres de von Neumann, *CR Acad. Sc. Paris*, **277** (1973), 683–685; **279** (1974), 643–645; *Bull. Soc. Math. France*, Mémoire No. 44, 1975, 144 p.
2. Une nouvelle construction du poids dual sur le produit croisé d'une algèbre de von Neumann par un groupe localement compact, *CR Acad. Sc. Paris*, **282** (1976), 415–418.
3. Algèbres de Kac et produits croisés, *CR Acad. Sc. Paris*, **283** (1976), 321–323.
4. Produit croisé d'une algèbre de von Neumann par une algèbre de Kac, II.

ENOMOTO, M.
1. An automorphism on tensor products of Banach algebras, *Math. Jap.*, **20** (1975), 87–88.
2. Automorphisms and tensor products of C^*-algebras.

ENOMOTO, M.; FUJII, M.; TAKEHANA, H.
1. On a conjecture of Breuer, *Math. Jap.*, **21** (1976), 387–389.

ENOMOTO, M.; TAMAKI, K.
1. On a theorem of Pearcy and Ringrose, *Math. Jap.*, **18** (1973), 253–256.
2. Freely acting automorphisms of abelian C^*-algebras, *Nagoya Math. J.*, **56** (1974), 7–12.

ERDÖS, J. A.
1. On certain elements of C^*-algebras, *Illinois J. Math.*, **15** (1971), 682–693.
2. An abstract characterization of nest algebras, *Quart. J. Math.*, **22** (1971), 47–63.

ERNEST, J.
1. A decomposition theory for unitary representations of locally compact groups, *Bull. AMS*, **67** (1961), 385–388; *Trans. AMS*, **104** (1962), 252–277.
2. Notes on duality theorem of non-commutative non-compact topological groups, *Tohoku Math. J.*, **16** (1964), 291–296.
3. A new group algebra for locally compact groups, I, II, *Amer. J. Math.*, **86** (1964), 467–492; *Can. J. Math.*, **17** (1965), 604–615.
4. The representation lattice of locally compact group, *Illinois J. Math.*, **10** (1966), 127–135.
5. Hopf-von Neumann algebras, *Lecture Fnal Anal. Conf. Irwine*, 1966; Academic Press, 1967, 195–215.

6. The enveloping algebra of a covariant system, *Comm. Math. Phys.*, **17** (1970), 61–74.
7. A duality theorem for the automorphism group of a covariant system, *Comm. Math. Phys.*, **17** (1970), 75–91.
8. A strong duality theorem for separable locally compact groups, *Trans. AMS*, **156** (1971), 287–308.
9. On the topology of the spectrum of a C^*-algebra, *Math. Ann.*, **216** (1975), 149–153.
10. *Charting the operator terrain*, Memoirs AMS, No. 177, 1976.

EVANS, D. E.
1. On the spectral type of one-parameter groups on operator algebras, *Proc. AMS*, **61** (1976), 351–352.
2. On the spectrum of a one parameter strongly continuous representation, *Math. Scand.*, **39** (1976), 80–82.
3. Positive linear maps on operator algebras, *Comm. Math. Phys.*, **48** (1976), 15–22.
4. Completely positive bounded linear maps, Dublin, 1975.
5. Unbounded completely positive linear maps on C^*-algebras, *Pacific J. Math.*, **66** (1976), 325–346.
6. Complete positivity and asymptotic abelianess, *Ann, IHP*, (A) **26** (1977), 213–218.
7. Scattering in the CAR algebra, *Comm. Math. Phys.*, **48** (1976), 23–30.
8. Time-dependent perturbations and scattering of strongly continuous groups on Banach spaces, *Math. Ann.*
9. Smooth perturbations in non-reflexive Banach spaces, *Math. Ann.*, **221** (1976), 183–194.
10. Conditionally completely positive maps on operator algebras, *Quart. J. Math.*, **28** (1977), 271–284.
11. Smooth perturbations of one-parameter groups on W^*-algebras, Dublin, 1976.
12. Irredecible quantum dynamical semigroups, *Comm. Math. Phys.*, **54** (1977), 293–297.
13. Completely positive quasi-free maps on the CAR algebra, Copenhagen, 1978.

EVANS, D. E.; HANCHE-OLSEN, H.
1. The generators of positive semigroups, *J. Fnal Analysis.*

EVANS. D. E.; HOEGH-KROHN, R.
1. Spectral properties of positive maps on C^*-algebras, *J. London Math. Soc.*

EVANS, D. E; LEWIS, J. T.
1. Dilations of dynamical semigroups, *Comm. Math. Phys.*, **50** (1976), 219–228.
2. Completely positive maps on the CCR algebra, Dublin, 1976.
3. Some semigroups of completely positive maps on the CCR algebra, *J. Fnal Analysis*, **26** (1977), 369–377.
4. Completely positive maps on some C^*-algebras, Dublin, 1976.
5. *Lecture notes on dynamical semigroups.*
6. Dilations of irreversible evolutions in algebraic quantum theory, *Comm. Dublin Inst. Adv. Studies*, A, **24**, 1977.

EVANS, D. E.; SUND, T.
1. Spectral subspaces for compact actions.

EYMARD, P.
1. L'algèbre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France*, **92** (1964), 181–236.
2. *Moyennes invariantes et représentations unitaires*, Lecture Notes in Math., No. 300, Springer Verlag, 1972.
3. Initiation à la théorie des groupes moyennables, in *Lecture Notes in Math.*, No. 497, 1975, 89–107.

FACK, T.
1. Construction de nouveaux facteurs de type II_1, *CR Acad. Sc. Paris*, **280** (1975), 1369–1372; Thèse de 3^e cycle, Paris, 1975.

FACK, T.; MARÉCHAL, O.
1. Sur la classification des symmétries des C^*-algèbres UHF, *CR Acad. Sc. Paris*, **285** (1977), 625–627.
2. Symmetries of UHF-algebras, Paris, 1967.

FANNES, M.
1. The entropy density of quasifree states, *Comm. Math. Phys.*, **31** (1973), 279–290.
2. Quasifree states and automorphisms of the CCR algebra, *Comm. Math. Phys.*, **51** (1976), 55–66.
3. The entropy density of quasi-free states for a continuous boson system.

FANNES, M.; VERBEURE, A.
1. On the time evolution automorphisms of the CCR-algebra for quantum mechanics, *Comm. Math. Phys.*, **35** (1974), 257–264.
2. Gauge transformations and normal states of the CCR algebra, *J. Math. Phys.*, **16** (1975), 2086–2088 (Erratum, ibidem, **17** (1976), 284).
3. Correlation inequalities and equilibrium states, I, II, *Comm. Math. Phys.*, **55** (1977), 125–132; **57** (1977), 165–171.
4. Global thermodynamical stability and correlation inequalities.

FANNES, M.; VERBEURE, A.; WEDER, R.
1. On momentum states in quantum mechanics, *Ann. IHP*, A **20** (1974), 291–296.

FARIS, W. G.
1. *Self-adjoint operators*, Lecture Notes in Math., No. 433, Springer Verlag, 1975.

FARIS, W.G.; SIMON, B.
1. Degenerate and non-degenerate ground states for Schrödinger operators, *Duke Math. J.*, **42** (1975), 559–567.

FELDMAN, J.
1. Isomorphisms of finite type II rings of operators, *Ann. Math.*, **63** (1956), 565–571.
2. Some connections between topological and algebraic properties in rings of operators, *Duke Math. J.*, **23** (1956), 365–370.
3. Embedding of AW^*-algebras, *Duke Math. J.*, **23** (1956), 303–308.
4. Nonseparability of certain finite factors, *Proc. AMS*, **7** (1956), 23–26.
5. The uniformly closed ideals in an AW^*-algebra, *Bull. AMS*, **62** (1958), 245–246.
6. Borel sets of states and of representations, *Michigan Math. J.*, **12** (1965), 363–366.
7. Borel structure and invariants for measurable transformations, *Proc. AMS*, **46** (1974), 383–394.
8. Ergodic equivalence relations and factors, in *Symposia Math.*, XX, 317–326, Academic Press, 1976.

FELDMAN, J.; FELL, J.M.G.
1. Separable representations of rings of operators, *Ann. Math.*, **65** (1957), 241–249.

FELDMAN, J.; HAHN, P. F.; MOORE, C. C.
1. Measure groupoids coming from group actions, *Notices AMS*, **23**: 3 (1976), 735-B 17.
2. Sections for group action and some applications, Marseille, 1977.
3. Orbit structure and countable sections for actions of continuous groups.

FELDMAN, J.; KADISON, R. V.
1. The closure of the regular operators in a ring of operators, *Proc. AMS*, **5** (1954), 909–916.

FELDMAN, J.; LIND, D. A.
1. Hyperfiniteness and the Halmos-Rohlin theorem for non-singular abelian actions, *Proc. AMS*, **55** (1976), 339–344.

FELDMAN, J.; MOORE, C. C.
1. Ergodic equivalence relations, cohomology and von Neumann algebras, I, II, *Bull. AMS*, **81** (1975), 921–924: *Trans. AMS*, **234** (1977), 289 – 324, 325 – 361.

FELL, J. M. G.
1. Representations of weakly closed algebras, *Math. Ann.*, **133** (1957), 118–126.
2. The dual space of C^*-algebras, *Trans. AMS*, **94** (1960), 365–403.
3. C^*-algebras with smooth dual, *Illinois J. Math.*, **4** (1960), 221–230.
4. The structure of algebras of operator fields, *Acta Math.*, **106** (1961), 233–280.
5. A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space, *Proc. AMS*, **13** (1962), 472–476.

6. A new proof that nilpotent groups are CCR, *Proc. AMS*, **13** (1962), 93–99.
7. Weak containment and induced representations, I, II, *Can. J. Math.*, **14** (1962), 237–268; *Trans. AMS*, **110** (1964), 424–447.
8. Weak containment and Kronecker product of group representations, *Pacific J. Math.*, **13** (1963), 503–510.
9. The dual spaces of Banach algebras, *Trans. AMS*, **114** (1965), 227–250.
10. Non unitary dual spaces of groups, *Acta Math.*, **114** (1965), 267–310.
11. Algebras and fiber bundles, *Pacific J. Math.*, **16** (1966), 497–503.
12. Conjugated representations and related results on semisimple Lie groups, *Trans. AMS*, **127** (1967), 405–426.
13. An extension of Mackey's method to algebraic bundles over finite groups, *Amer. J. Math.*, **91** (1969), 203–238.
14. *An extension of Mackey's method to Banach *-algebraic bundles*, Memoirs *AMS*, No. 90, 1969.
15. *Group representation theory.*
16. A new look at Mackey's imprimitivity theorem, *Conference on Harmonic Analysis, Lecture Notes in Math.* No. 266, 1972, 43–58.

FELL, J. M. G.; KELLEY, J. L.

1. An algebra of unbounded operators, *Proc. Nat. Acad. Sci. USA*, **38** (1952), 592–598.

FELL, J.M.G.; THOMA, E.

1. Einige Bemerkungen über vollsymmetrisch Banachsche Algebren, *Arch. Math.*, **12** (1961), 69–70.

FILLMORE, P. A.

1. The dimension theory of certain cardinal algebras, *Trans. AMS*, **117** (1965), 21–36.
2. Perspectivity in projection lattices, *Proc. AMS*, **16** (1965), 383–387.
3. *Notes on operator theory*, Van Nostrand, 1970.
4. Extensions relative to semifinite factors, in *Symposia Math.*, XX, 487–496, Academic Press, 1976.

FILLMORE, P. A.; STAMPFLI, J. G.; WILLIAMS, J. P.

1. On the essential numerical range, the essential spectrum and a problem of Halmos, *Acta Sci. Math.*, **33** (1972), 179–192.

FILLMORE, P. A.; TOPPING, D. M.

1. Sums of irreducible operators, *Proc. AMS*, **20** (1969), 131–133.
2. Operator algebras generated by projections, *Duke Math. J.*, **34** (1967), 333–336.
3. A direct integral decomposition for certain operator algebras, *Amer. J. Math.*, **91** (1969), 11–18.

FILLMORE, P. A.; WILLIAMS, J. P.

1. On operator ranges, *Advances in Math.*, **7** (1971), 254–281.

FLENSTED-JENSEN, M.

1. A note on disintegration, type and global type of von Neumann algebras, *Math. Scand.*, **24** (1969), 232–238.

FLORY, V.

1. Estimating norms in C^*-algebras of discrete groups, *Math. Ann.*, **224** (1976), 41–52.

FOIAŞ, C.

1. Spectral and semispectral measures (Romanian), *Stud. Cerc. Mat.*, **18** (1966), 7–56.
2. Invariant paraclosed subspaces, *Indiana Univ. Math. J.*, **21** (1972), 887–906.

FOIAŞ, C.; KOVÁCS, I.

1. Une caractérisation nouvelle des algèbres de von Neumann finies, *Acta Sci. Math.*, **23** (1962), 274–278.

FOIAŞ, C.; PASNICU, C.; VOICULESCU, D.

1. Weak limits of approximately invariant projections, Bucharest, 1978.

FORD, J. W. M.

1. A square root lemma for Banach star algebras, *J. London Math. Soc.*, **42** (1967), 521–522.

FORELLI, F.
1. Analytic and quasi-invariant measures, *Acta Math.*, **118** (1967), 33–59.
2. Conjugate functions and flows, *Quart. J. Math.*, 20 (1969), 215–233.
3. The F. and M. Riesz theorem, *Proc. AMS*, 47 (1975), 431.

FORMANEK, E.
1. The type I part of the regular representation, *Canad. J. Math.*, 26 (1974), 1086–1089.

FREDENHAGEN, K.
1. Implementation of automorphisms and derivation of the CAR algebra, *Comm. Math. Phys.*, 52 (1977), 255–266.

FRIGERIO, A.
1. Quantum dynamical semigroups and approach to equilibrium, Milano, 1977.

FUGLEDE, B.
1. A commutativity theorem for nomal operator, *Proc. Nat. Acad. Sci. USA*, 36 (1950), 35–40.

FUGLEDE, B.; KADISON, R. V.
1. On a conjecture of Murray and von Neumann, *Proc. Nat. Acad. Sci. USA*, 37 (1951), 420–425.
2. On determinants and a property of the trace in finite factors, *Proc. Nat. Acad. Sci. USA*, 37 (1951), 425–431.
3. Determinant theory in finite factors, *Ann. Math.*, 55 (1952), 520–530.

FUJII, M.
1. Extensions of homomorphisms and the spectral theorem, *Mem. Osaka Univ.*, 19 (1970), 113–115.
2. On hypercentral states of C^*-algebras, *Math. Jap.*, 18 (1973), 215–220.

FUJITA, M.
1. Banach algebra structure in Fourier spaces and generalization of harmonic analysis on locally compact groups.

FUKAMIYA, M.
1. On B^*-algebras, *Proc. Jap. Acad.*, 27 (1951), 321–327.
2. On a theorem of Gelfand and Naimark and the B^*-algebras, *Kumamoto J. of Science*, 1 (1952), 17–22.

FUKAMIYA, M.; MISONOU, Y.; TAKEDA, Z.
1. On order and commutativity of B^*-algebras, *Tohoku Math.* J., 6 (1954), 89–93.

FUNAKOSI, S.
1. Borel structures in topological $*$-algebras and their duals, *Proc. Jap. Acad.*, 50 (1974), 309–312.

GAJENDRAGADKAR, P.
1. Norm of a derivation on a von Neumann algebra, *Trans. AMS*, 170 (1972), 165–170.

GALLAVOTTI, G.; PULVIRENTI, M.
1. Classical *KMS* condition and Tomita-Takesaki theory, *Comm. Math. Phys.*, 46 (1976), 1–9.

GALLAVOTTI, G.; VERBOVEN, E.
1. On the classical *KMS* boundary condition, *Nuovo Cimento*, B 28 (1975), 274–286.

GANGOLLI, R.
1. On the symmetry of L^1-algebras of locally compact motion groups and the Wiener tauberian theorem, *J. Fnal Analysis*, 25 (1977), 244–252.

GANIHODJAEV, N.
1. The Krieger algebra of a class of dynamical systems, *Leipzig Conference*, September 1977,

GARBACZEWSKI, P.
1. Representations of the CAR generated by representations of the CCR, I, II, *Bull. Acad. Sc. Pol.*, 24 (1976), 201–206.

GARBACZEWSKI, P.; POPOVICZ, Z.
1. Representations of the CAR generated by representations of the CCR, III, *Rep. Math. Phys.*, **11** (1977), 73–80.

GARDING, L.; WIGHTMAN, A.
1. Representations of the anticommutation relations, *Proc. Nat. Acad. Sci. USA*, **40** (1954), 617–621.
2. Representations of commutation relations, *Proc. Nat. Acad. Sci. USA*, **40** (1954), 622–626.

GARDNER, L. T.
1. A note on isomorphisms of C^*-algebras, *Bull. AMS*, **70** (1964), 788–791.
2. On isomorphisms of C^*-algebras, *Amer. J. Math.*, **87** (1965), 384–396.
3. An invariance theorem for representations of Banach algebras, *Proc. AMS*, **16** (1965), 983–986.
4. On the "third" definition of topology on the spectrum of a C^*-algebra, *Can. J. Math.*, **23** (1971), 445–450.
5. On the Mackey-Borel structure, *Can. J. Math.*, **23** (1971), 674–678.
6. Notes on regularity conditions for operator algebras and the proof of Mackey's conjecture (with an appendix on Godement topology for $\hat{G}$).
7. Uniformly closed Fourier algebras, *Acta Sci. Math.*, **33** (1972), 211–216.

GARLING, D. H. J.
1. On ideals of operators in Hilbert spaces, *Proc. London Math. Soc.*, **17** (1967), 115–138.

GARTENBERG, M.
1. Extensions of the index in factors of type II_∞, *Proc. AMS*, **43** (1974), 163–166.
2. Approximation by invertibles, *Proc AMS*, **61** (1976), 341–346.

GELFAND, I. M.*)
1. Normierte Ringe, *Matem. Sb.*, **9** (1941), 3–24.

GELFAND, I. M.; GRAEV, M. I.
1. Unitary representations of the real unimodular group (Russian), *Izvestya Akad. Nauk*, **17** (1953), 189–248.
2. An analogue of Plancherel formula for classical groups (Russian), *Trudy Mosk. Mat. Obšč.*, **4** (1955), 375–404.

GELFAND, I. M;. GRAEV, M. I.; PIATEČKII-ŠAPIRO, I. I.
1. *The theory of representations and automorphic forms* (Russian), Moscow, 1966.

GELFAND, I. M.; GRAEV, M. I.; VILENKIN, N. I.
1. *Integral geometry and problems on the theory of representations* (Russian), Moscow, 1962.

GELFAND, I. M.; MINLOS, P. A.; ŠAPIRO, Z. I.
1. *The representations of rotations group and Lorentz group* (Russian), Moscow, 1958.

GELFAND, I. M.; NAIMARK, M. A.
1. On the imbedding of normed rings into the ring of operators in Hilbert space, *Matem. Sb.*, **12** (1943), 197–213.
2. Normed rings with involutions and their representations (Russian), *Izvestya Akad. Nauk*, **12** (1948), 445–480.
3. Unitary representations of classical groups (Russian), *Trudy Mat. Inst. Steklova*, **36** (1948), 445–480.

GELFAND, I. M.; RAIKOV, D. A.
1. Irreducible unitary representations of local bicompact groups (Russian), *Matem. Sb.*, **13** (1943), 301–316.

GEORGESCU, V.
1. Sur l'existence des opérateurs d'onde dans la théorie algébrique de la diffusion, *Ann. IHP*, A **27** (1977), 9–30.

*) For a complete bibliography of I. M. Gelfand and co-workers, see *Uspekhi Mat. Nauk*, **19** : 3 (1964) and **29** : 1 (1974).

GHEZ, P.

1. Spectre du hamiltonien et équivalence algébrique des états quasi-libres du champ de fermions, *Ann. IHP*, **16** (1972), 235–246.

GHEZ, P.; LIMA, R; TESTARD, D.

1. Une extension d'un théorème de A. Connes sur les facteurs constructibles, *Comm. Math. Phys.*, **32** (1973), 305–311.

GHEZ, P.; TESTARD, D.

1. Central sequences and property $L'_{1/2}$ in II_∞ factors.

GILES, R.

1. Foundations of quantum mechanics, *J. Math. Phys.*, **11** (1970), 2139–2160.

GILES, R.; KUMMER, H.

1. A non-commutative generalization of topology, *Indiana Univ. Math. J.*, **21** (1971), 91–102.
2. A matrix representation of a pair of projections in a Hilbert space, *Can. Math. Bull.*, **14** (1971), 35–44.

GILLE, J. F.

1. An exponentiation theorem for unbounded derivations, *Ann. Inst. H. Poincaré*, **13** (1970), 215–220.
2. On a question of A. Verbeure, *Comm. Math. Phys.*, **34** (1973), 131–134.
3. Non quasi-free classes of product states of the CCR-algebra, *Comm. Math. Phys.*, **34** (1973), 223–228.

GILLE, J. F.; MANUCEAU, J.

1. Gauge transformations of second type and their representations, I, II, *J. Math. Phys.*, **13** (1972), 2002 – 2007 (corrections, ibidem, **14** (1973), 1152); *Ann. IHP*, A **20** (1974), 297 – 312.

GILLETTE, R. M.; TAYLOR, D. C.

1. A characterization of the Pedersen ideal of $C_0(T, B_0(H))$ and a counterexample, *Proc. AMS*, **68** (1978), 59 – 63.

GLASER, M. S.

1. Asymptotic abelianess of infinite factors, *Trans. AMS*, **178** (1973), 41–56.

GLEASON, A. M.

1. Measures on the closed subspaces of a Hilbert space, *J. Rat. Mech. Anal.*, **6** (1957), 885 – 894.
2. Projective topological spaces, *Illinois J. Math.*, **2** (1958), 482–489.
3. A characterization of maximal ideals, *J. Analyse Math.*, **19** (1967), 171–172.

GLICKFELD, W. B.

1. A metric characterization of $C(X)$ and its generalizations to C^*-algebras, *Illinois J. Math.*, **10** (1956), 547–556.

GLIMM, J.

1. On a certain class of operator algebras, *Trans. AMS*, **95** (1960), 318–340.
2. A Stone-Weierstrass theorem for C^*-algebras, *Ann. Math.*, **72** (1960), 216–244.
3. Type I C^*-algebras, *Ann. Math.*, **73** (1961), 572–612.
4. Locally compact transformation groups, *Trans. AMS*, **101** (1961), 124–138.
5. What are C^*-algebras? *Conf. Columbia Univ.*, 1961.
6. Families of induced representations, *Pacific J. Math.*, **12** (1962), 885–911.
7. Models for quantum field theory, in *Local Quantum Theory*, Academic Press, 1969, 97–119.
8. Integration in function spaces, in *Local Quantum Theory*, Academic Press, 1969, 227–233.
9. The foundations of quantum mechanics, *Adv. Math.*, **3** (1969), 101–125.
10. Mathematical problems in the foundations of quantum field theory, *Lecture Notes in Math.*, **140**, 1970, 58–67.
11. *Lectures on harmonic analysis (non abelian)*, Lecture Notes Courant Institute Math. Sci., New York Univ., 1965.
12. The mathematics of quantum fields, *Adv. Math.*, **16** (1975), 221–232.

GLIMM, J.; JAFFE, A. M.
1. The $\lambda(\varphi^4)_2$ quantum field without cutoffs, I, II, III, IV, *Phys. Rev.*, **176** (1968), 1945–1951; *Ann. Math.*, **91** (1970), 362–401; *Acta Math.*, **125** (1970), 203–267; *J. Math. Phys.*, **13** (1972), 1568–1583.
2. Rigorous quantum field theory models, *Bull. AMS*, **76** (1970), 407–410.
3. The energy momentum spectrum and vacuum expectations values in quantum field theory, I, II, *J. Math. Phys.*, **11** (1970), 3335–3338; *Comm. Math. Phys.*, **22** (1971), 1–22.
4. The Yukawa$_2$ quantum field theory without cutoffs, *J. Fnal Analysis*, **7** (1971), 323–357.
5. Boson quantum field models, in *Mathematics of contemporary physics* (Ed. R. F. Streater), Academic Press, 1972, 77–144.

GLIMM, J.; KADISON, R. V.
1. Unitary operators in C^*-algebras, *Pacific J. Math.*, **10** (1960), 547–556.

GODEMENT, R.
1. (A series of notes on functions of positive type), *CR Acad. Sc. Paris*, **221** (1945), 69–71, 134–136, 686–687, **222** (1946), 36–37, 213–215, 529–531.
2. Analyse harmonique dans les groupes centraux, I, II, *CR Acad. Sc. Paris*, **225** (1947), 19–21, 221–223.
3. Sur les relations d'orthogonalité de V. Bargmann, I, II, *CR Acad. Sc. Paris*, **225** (1947), 521–523, 657–659.
4. Les fonctions de type positif et la théorie des groupes, *Trans. AMS*, **63** (1948), 1–84.
5. Théorie générale des sommes continues d'espaces de Banach, *CR Acad. Sc. Paris*, **228** (1949), 1321–1323.
6. Sur la théorie des caractères, I, *CR Acad. Sc. Paris*, **229** (1949), 967–969.
7. Some unsolved problems in the theory of group representations, *Proc. Int. Congr. Math., Cambridge Mass.*, AMS Providence, 1952, vol. II, 106–111.
8. Mémoire sur la théorie des caractères dans les groupes localement compacts unimodulaires, *J. Math. Pures Appl.*, **30** (1951), 1–110.
9. Sur la théorie des représentations unitaires, *Ann. Math.*, **53** (1951), 68–124.
10. A theory of spherical functions, I, *Trans. AMS*, **73** (1952), 496–556.
11. Théorie des caractères, I, II, *Ann. Math.*, **59** (1954), 47–62; 63–85.
12. Sur la transformation de Fourier dans les groupes discrets, *CR Acad. Sc. Paris*, **228** (1949), 627–628.

GOLDMAN, M.
1. Structure of AW^*-algebras, I, *Duke Math. J.*, **23** (1956), 23–34.
2. On subfactors of factors of type II_1, *Michigan Math. J.*, **6** (1959), 167–172.

GOLODEČ, V. YA. (in Russian)
1. On the irreducible representations of commutation and anticommutation relations, *Uspekhi Mat. Nauk*, **20** : 2 (1965), 175–182.
2. On factor type II representations for commutation and anticommutation relations, *Uspekhi Mat. Nauk*, **20** : 6 (1965), 68–72.
3. On factor representations of anticommutation relations, *Dokl. Akad. Nauk*, **167** : 1 (1966), 19–22.
4. On factor type II_1 representations of Clifford algebra, *Dokl. Akad. Nauk*, **173**: 4 (1967), 745–747.
5. On the equivalence of quasiinvariant measures, *Matem. Sb.*, **72** : 4 (1967), 68–82.
6. Some properties of the representations of infinite dimmensional Clifford algebra, *Dokl. Akad. Nauk*, **178** : 5 (1968), 771–773.
7. On quasiinvariant ergodic measures, *Matem. Sb.*, **72** (1967), 558–572.
8. The study of approximate finite von Neumann algebras with finite trace, *Dokl. Akad. Nauk*, **181** : 6 (1968), 1307–1310.
9. Finite groups of automorphisms of hyperfinite factors, *Dokl. Akad. Nauk*, **188** : 6 (1969), 1214–1216.
10. On approximate finite groups of transformations on a measure space, *Uspekhi Mat. Nauk*, **24** (1969), 195–196.
11. The description of the representations of anticommutation relations, *Uspekhi Mat. Nauk*, **24** : 4 (1969), 3–64.

12. On hyperfinite type III factors, *Mat. Fiz. Funkt. Analiz.*, **1** (1969), 220–228.
13. Approximate finite groups of transformation and approximate finite factors, *Mat. Fiz. Funkt. Analiz.*, **1** (1969), 250–253.
14. Type II and III factor representations of commutation relations, *Mat. Sb.*, **78** (1969), 501–511.
15. On transformations group of measure spaces for which the measure is quasi-invariant, preprint, Kharkov, 1969.
16. Equivalent modular operators, preprint, Kharkov, 1970.
17. On the transformations for which the measure is quasiinvariant, *Mat. Zametki*, **7** (1970), 223–227.
18. On a class of approximate finite factors, *Funkt. Analiz i pril.*, **4** : 4 (1970), 14–20.
19. Factor representations of anticommutation relations, *Trudy Mosk. Mat. Obšč.*, **22** (1970), 3–62.
20. On type II_∞ and type III hyperfinite factors, *Funkt. Analiz i pril.*, **5** : 2 (1971), 37–44.
21. Tensor and crossed products of type III factors, *Funkt. Analiz i pril.*, **5** : 3 (1971), 96–97.
22. Crossed products of von Neumann algebras, *Uspekhi Mat. Nauk*, **26** : 5 (1971), 3–50.
23. Spectral properties of modular operators, *Funkt. Analiz i pril.*, **6** : 1 (1972), 70–72.
24. Conditional expectations and the modular automorphisms of von Neumann algebras, *Funkt. Analiz i pril.*, **6**: 3 (1972), 68–69.
25. Conditional expectations, modular automorphisms and crossed products of type III algebras, preprint, Kharkov, 1971.
26. Modular operators and asymptotic ratio set, *Funkt. Analiz. i pril.*, **7** : 3 (1973), 77–78.
27. Asymptotic algebra and its applications to modular operators and their spectral properties, *Dokl. Akad. Nauk, SSSR*, **220** : 1 (1975), 15–18.
28. Spectral properties of modular operators and asymptotic ratio set, *Izvestya Akad. Nauk SSSR*, **39** : 3 (1975), 635–656.
29. On the structure of von Neumann algebras which are dual of the algebras associated to dynamical systems, *Funkt. Analiz. i pril.*, **9** : 3 (1975), 87–88.
30. On the reduction of type III algebras, *Funkt. Analiz i pril.*, **9** : 4 (1975), 1–17.
31. On local perturbations of infinite dynamical systems, *Teor. i Mat. Fiz.*, **23** : 3 (1975), 300–309.
32. Factor representations of type II and III for commutation relations, *Proc. Summer School in Spectral Theory of Operators*, Baku, 1968, 78–89.
33. Modular operators and the asmyptotic ratio, *Mat. Fiz. i Funkt. Analiz.*, **5** (1974), 106–122; **158** *Akad. Nauk Ukrain. SSR, Fiz. Tehn. Inst. Nizkih Temperatur, Kharkov.*
34. Local perturbations of the dynamics of a system, *Mat. Fiz. i Funkt. Analiz*, **5** (1974), 151–155; **159** *Akad. Nauk Ukrain. SSR, Fiz. Tehn. Inst. Nizkih Temperatur, Kharkov.*
35. The asymptotic algebra, its properties and some applications, Kiev, 1976.
36. Asymptotic abelianess in von Neumann algebras, *Dokl. Akad. Nauk SSSR*, **233** (1977), 269–272.
37. Modular operators and asymptotic abelianess in von Neumann algebras, *Uspekhi Mat. Nauk*, **33**: 1 (1978), 43–94.

GOLODEČ, V. YA.; STIOPIN, G.
1. On type II_1 factors with property Γ (Russian), *Mat. Fiz. Funkt. Analiz*, **1** (1969), 229–249.

GONSHOR, H.
1. Spectral theory for a class of non-normal operators, I, II, *Can. J. Math.*, **8** (1956), 449–461; **10** (1958), 97–102.
2. Injective hulls of C^*-algebras, I, II, *Trans. AMS*, **131** (1968), 315–322; *Proc. AMS*, **24** (1970), 486–492.
3. Remarks on the algebra of bounded functions, *Math. Z.*, **108** (1969), 325–328.

GOODEARL, K. R.; HANDELMAN, D.
1. Rank functions and K_0 of regular rings, *J. Pure Appl. Algebra*, **7** (1976), 195–216.

GOOTMAN, E. C.
1. Primitive ideals of C^*-algebras associated with transformation groups, *Trans. AMS*, **170** (1972), 97–108.

2. The type of some C^* and W^*-algebras associated with transformation groups, *Pacific J. Math.*, **48** (1973), 93–106.
3. Weak containment and weak Frobenius reciprocity, *Proc. AMS*, **54** (1976), 417–422.
4. Local eigenvectors for group representations, *Studia Math.*, **53** (1975), 135–138.
5. Induced representations and finite volume homogeneous spaces, I, II, *J. Fnal Analysis*.

GOOTMAN, E. C.; KALLMAN, P. R.
1. The left regular representation of a p-adic algebraic group is type I, *Adv. Math.*

GOOTMAN, E. C.; KANNAN, D.
1. Zero-one law in finite W^*-algebras, *J. Math. Anal. Appl.*, **55** (1976), 743–756.

GORINI, V.; SUDARSHAN, E. C. G.
1. Inversibility and dynamical maps of statistical operators, *Lecture Notes in Physics*, No. 29, 1974, 260–268.

GOTO, M.
1. On a class of solvable Lie groups of type I.

GOULET DE RUGY, A.
1. *Géométrie des simplexes*, Centre Doc. Univ. et SEDEX Réunis, Paris, 1968.
2. Faces parallélisables et topologies faciales sur l'espace des états d'une algèbre stellaire, *CR Acad. Sc. Paris*, **270** (1970), 376–379.
3. Simplex spaces, *Lectures Notes in Physics*, No. 29, 1974, 33–40.

GRANIRER, E. E.
1. Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group, *Trans. AMS*, **189** (1974), 371–382.
2. Properties of the set of topologically invariant means of P. Eymard's W^*-algebra $VN(G)$, *Indag. Math.*, **36** (1974), 116–124.
3. A characterization of discreteness for locally compact groups in terms of the Banach algebras $A_p(G)$, *Proc. AMS*, **54** (1976), 189–192.

GREEN, M. D.
1. The lattice structure of C^*-algebras and their duals, *Math. Proc. Cambridge Phil. Soc.*, **81** (1977), 245–248.

GREEN, P.
1. C^*-algebras of transformation groups with smooth orbit space, *Pacific J. Math.*, **72** (1977), 71–98.
2. Morita equivalence of C^*-algebras, *Notices AMS*, **23** (1976), 735-B 20.
3. The local structure of twisted covariance algebras, *Acta Math.*, **140** (1978), 191–250.

GREEN, P.; ROSENBERG, J.
1. C^*-algebras of transformation groups with smooth orbit spaces and applications to the computation of some group C^*-algebras, *Notices AMS*, **23** : 3 (1976), 735-B 14.

GREEN, W. L.
1. Topological dynamics and C^*-algebras, *Trans. AMS*, **210** (1975), 107–121.
2. Compact groups of automorphisms of von Neumann algebras, *Math. Scand.*, **37** (1975), 284–296.

GREEN, W. L.; LAU, A.T.-M.
1. Strongly finite von Neumann algebras, *Math. Scand.*, **40** (1977), 105–112.

GREENE, W. A.
1. Ambrose modules, *Memoirs AMS*, **148** (1974), 109–133.

GREENLEAF, F. P.
1. *Invariant means on topological groups and their applications*, Van Nostrand, 1969.
2. Amenable actions of locally compact groups, *J. Fnal Analysis*, **4** (1969).

GREENLEAF, F. P.; MOSKOWITZ, M.
1. Cyclic vectors for representations of locally compact groups, *Math. Ann.*. **190** (1971), 265–288.

GRIFFIN, E. L.
1. Isomorphisms of rings of type III, *Bull. AMS*, 58 (1952), 480.
2. Some contributions to the theory of rings of operators, I, II, *Trans. AMS*, 75 (1953), 471–504; 79 (1955), 389–400.
3. Everywhere defined linear transformations affiliated with rings of operators, *Pacific J. Math.*, 18 (1968), 489–493.

GROSS, L.
1. *Harmonic Analysis on Hilbert space*, Memoirs AMS, No. 46, 1963.
2. A non-commutative extension of Perron-Frobenius theorem, *Bull. AMS*, 77 (1971), 343–347.
3. Existence and uniqueness of physical ground states, *J. Fnal Analysis*, 10 (1972), 52–109.
4. Potential theory on Hilbert space, *J. Fnal Analysis*, 1 (1967), 123–181.
5. Analytic vectors for representations of the canonical commutation relations and non-degeneracy of ground states, *J. Fnal Analysis*, 17 (1974), 104–111.

GROTHENDIECK, A.
1. Réarrangements de fonctions et inégalités de convexité dans les algèbres de von Neumann munies d'une trace, *Sém. Bourbaki*, Mars 1955.
2. Un résultat sur le dual d'une C^*-algèbre, *J. Math. Pures Appl.*, 36 (1957), 97–108.

GUDDER, S. P.
1. A Radon-Nikodym theorem for $*$-algebras.

GUDDER, S. P.; HUDSON, R. L.
1. A non-commutative probability theory, Nottingham, 1976.

GUDDER, S. P.; MARCHAND, J. P.
1. Non-commutative probability on von Neumann algebras, *J. Math. Phys.*, 13 (1972), 799–806.
2. Conditional expectations on von Neumann algebras: a new approach.

GUDDER, S. P.; MARCHAND, J. P.; W. WYSS
1. Bures distance and relative entropy.

GUDDER, S. P.; SCRUGGS, W.
1. Unbounded representations of $*$-algebras.

GUENIN, M.
1. On the interaction picture, *Comm. Math. Phys.*, 3 (1966), 120–132.
2. Practical use of algebraic concepts in quantum field theory, *V-th Annual Winter School Theor. Phys., Karpacz, Univ. Wroclaw*, 1968.
3. Axiomatic foundations of quantum theories, *J. Math. Phys.*, 7 (1968), 271–282.
4. *Algebraic methods in quantum field theory*, Boulder, Colorado, 1966, 187–253.
5. *Introduction to algebraic techniques in quantum field theory and statistical mechanics*, Boulder, Colorado, 1969, 23–99.

GUENIN, M.; MISRA, B.
1. On the von Neumann algebras generated by the field operators, *Nuovo Cimento*, 30 (1963), 1272–1290.

GUENIN, M.; WEHRL, A.; THIRRING, W.
1. *Introduction to algebraic technique, Lectures given in the Theoretical Seminar series at CERN, CERN* 69–14, 1969.

GUERRA, F.
1. Local algebras in Euclidian quantum field, in *Symposia Math.*, vol. XX, 13–26, Academic Press, 1976.

GUERRA, F.; ROSEN, L.; SIMON, B.
1. The $P(\Phi)_2$ Euclidean quantum field theory as classical statistical mechanics, I, II, *Ann. Math.*, 101 (1975), 111–190; 191–259.

GUICHARDET, A.

1. Sur un problème posé par G. W. Mackey, *CR Acad. Sc. Paris*, **250** (1960), 962–963.
2. Une caractérisation des algèbres de von Neumann discrètes, *CR Acad. Sc. Paris*, **248** (1959), 3398–3400. *Bull. Soc. Math. France*, **89** (1961), 77–101.
3. Caractères des algèbres de Banach involutives, *Ann. Inst. Fourier*, **13** (1962), 1–81.
4. Sur la décomposition des mesures quasi-invariantes en mesures ergodiques, *CR Acad. Sc. Paris*, **257** (1963), 1747–1748.
5. Caractères et représentations des produits tensoriels de C^*-algèbres, *Ann. Éc. Norm. Sup.*, **81** (1964), 189–206.
6. Sur la décomposition des représentations des C^*-algèbres, *CR Acad. Sc. Paris*, **258** (1964), 768–770.
7. On the tensor product of C^*-algebras (Russian), *Dokl. Akad. Nauk*, **160** (1965), 986–989 (Erratum, **168** (1965), 1231).
8. Utilisation des sous-groupes distingués ouverts dans l'étude des représentations unitaires des groupes localement compacts, *Compositio Math.*, **17** (1965), 1–35.
9. Sur l'homologie et la cohomologie des algèbres de Banach, *CR Acad. Sc. Paris*, **262** (1966), 38–41.
10. Produits tensoriels infinis et représentations des relations d'anticommutation, *Ann. Éc. Norm .Sup.*, **83** (1966), 1–52.
11. Sur la catégorie des algèbres de von Neumann, *Bull. Sc. Math.*, **90** (1966), 41–64.
12. Produits tensoriels continus d'espaces de Banach et d'algèbre de Banach, *Comm. Math. Phys.*, **5** (1967), 262–287.
13. Sur un théorème de Sakai, *CR Acad. Sc. Paris*, **266** (1968), 974–975.
14. *Algèbres d'observables associés aux relations de commutation*, Armand Colin, Paris, 1968.
15. *Tensor products of C^*-algebras*, Aarhus Univ., Lecture Notes Series, No. **12**, **13**, 1969.
16. Sur la cohomologie des groupes topologiques, I, II, III, *Bull. Sc. Math.*, **95** (1971), 161–176; **96** (1972), 305–332; **98** (1974), 201–208.
17. *Symmetric Hilbert spaces and related topics*, Lecture Notes in Math., No. 261, Springer Verlag, 1972.
18. *Systèmes dynamiques non-commutatifs*, *Astérisque*, No. 13–14, 1974.
19. Sur les systèmes dynamiques non-commutatifs, *CR Acad. Sc. Paris*, **277** (1973), 289–290.
20. *Intégration non-commutative et théorie de Tomita-Takesaki (Introduction élémentaire)*, CNRS, Paris, 1973.
21. Désintégration centrale des états (d'après S. Sakai).
22. Cohomology of topological groups and positive definite functions, *J. Multivariate Analysis*, **3** (1973), 249–261.
23. 1-cohomologie des groupes résolubles de type (R) et propriété (P), *CR Acad. Sc. Paris*, **280** (1975), 101–103.
24. Cohomologie des groupes localement compacts et produits tensoriels continus de représentations.
25. Automorphismes et type de certaines algèbres de von Neumann, *Proc. London Math. Soc.*, **35** (1977), 541 – 554.
26. Group cohomology and dynamical systems, in *Symposia Math.*, XX, 187–196, Academic Press, 1976.
27. Classification of automorphisms of measure spaces and classification of von Neumann factors, *Operator Algebra Newsletter*, May 26, 1975.
28. Disintegration of quasi-invariant states on C^*-algebras (report of a paper by S. Doplicher, D. Kastler and A. Guichardet), *Proc. Fnal. Analysis Week*, Aarhus Univ., 1969, 20–25.
29. Sur l'homologie et la cohomologie des groupes localement compacts, *CR Acad. Sc. Paris*, **262** (1966), 118–120.
30. *Entropie des systèmes dynamiques commutatifs et non-commutatifs*, Centre de Math. Ec. Polytech., Paris, 1974.
31. Sur la 1-cohomologie de certains groupes localement compacts, *CR Acad. Sc. Paris*, **282** (1976), 571–573.

GUICHARDET, A.; KASTLER, D.

1. Désintégration des états quasi-invariants des C^*-algèbres, *J. Math. Pures Appl.*, **49** (1970), 349–380.

GUICHARDET, A.; WULFSON, A.
1. Sur les produits tensoriels continus d'espaces hilbertiens, *J. Fnal. Analysis*, 2 (1968), 371–377.
2. Continuous tensor product states which are translation invariant but not quasi-free, *Comm. Math. Phys.*, 17 (1970), 133–142.

GUREVIČ, A.
1. Unitary representations in Hilbert space of a compact topological group, *Mat. Sb.*, 13 (1943), 79–86.

HAAG, R.
1. On quantum field theories, *Dan. Math. Fys. Medd.*, 29 (1955), No. 12, 37 p.
2. Discussion des axiomes et des propriétés assymptotiques d'une théorie des champs locale avec particules composées, *Coll. Int. Lille*, 1957, 151–162.
3. Remarks on the mathematical structure of quantum field theories, *Proc. Conf. Th. Appl. Anal.* (Ed. W. T. Martin and I. E. Segal), M.I.T. Press, 1964.
4. Quantum field theory, in *Mathematics of contemporary physics* (Ed. R. F. Streater), Academic Press, 1972, 1–16.

HAAG, R.; HUGENHOLTZ, N. M.; WINNINK, M.
1. On the equilibrium states in quantum statistical mechanics, *Comm. Math. Phys.*, 5 (1967), 215–236.

HAAG, R.; KADISON, R. V.; KASTLER, D.
1. Nets of C^*-algebras and classification of states, *Comm. Math. Phys.*, 16 (1970), 81–104.
2. Asymptotic orbits in a free Fermi gas, *Comm. Math. Phys.*, 33 (1973), 1–22 (Corrections, 43 (1975), 198).

HAAG, R.; KASTLER, D.
1. An algebraic approach to quantum field theory, *J. Math. Phys.*, 5 (1964), 848–861.

HAAG, R.; KASTLER, D.; MICHEL, L.
1. Central decomposition of ergodic states (Unpublished report, 1968).

HAAG, R.; KASTLER, D.; TRYCH-POHLMEYER, E. B.
1. Stability and equilibrium states, *Comm. Math. Phys.*, 38 (1974), 173–194.

HAAG, R.; SCHRÖER, B.
1. Postulates of quantum field theory, *J. Math. Phys.*, 3 (1962), 248–256.

HAAG, R.; SWIECA, J. A.
1. When does a quantum field theory describe particles? *Comm. Math. Phys.*, 1 (1965), 308–320.

HAAG, R.; TRYCH-POHLMEYER, E. B.
1. Stability properties of equilibrium states, *Comm. Math. Phys.*, 56 (1977), 213–225.

HAAGERUP, U.
1. Normal weights on W^*-algebras, *J. Fnal Analysis*, 19 (1975), 302–318.
2. The standard form of von Neumann algebras, *Math. Scand.*, 37 (1975), 271–283.
3. Operator valued weights and crossed products, *Symposia Math.*, XX, 241–252, Academic Press, 1976.
4. L_p-spaces associated with an arbitrary von Neumann algebra, Marseille, 1977.
5. An example of a weight with type III centralizer, *Proc AMS*, 62 (1977), 278–280.
6. On the dual weight for crossed products of von Neumann algebras, I, II, Odense Univ., Math. Inst., preprints no. 10, 11/October 1975, *Math. Scand.*
7. Operator valued weights in von Neumann algebras, Odense Univ., Math. Inst., preprint no. 12/November 1975.
8. Tomita's theory for von Neumann algebras with a cyclic separating vector, preprint, Copenhagen, 1972.
9. An example of a non-nuclear C^*-algebra which has the metric approximation property, Odense, 1977.

HADWIN, D. W.
1. An asymptotic double commutant theorem for C^*-algebras.
2. Continuous functions of operators; a functional calculus.

HAGA, Y.
1. A note on traces of von Neumann algebras, *Proc. Jap. Acad.*, **44** (1968), 933–938.
2. On subalgebras of crossed product von Neumann algebra, *Tohoku Math. J.*, **25** (1973), 291–305.
3. On approximately finite algebras, *Tohoku Math. J.*, **26** (1974), 325–332.
4. Crossed products of von Neumann algebras by compact groups, *Tohoku Math. J.*, **28** (1976), 511–522.
5. Monoidal spectrum and crossed products (Japanese), *J. Fac. Eng. Ibaraki Univ.*, **23** (1975), 167–170.

HAGA, Y.; TAKEDA, Z.
1. Correspondence between subgroups and subalgebras in a cross product von Neumann algebra, *Tohoku Math. J.*, **24** (1972), 167–190.

HAHN, P. F.
1. *Convolution algebras and Haar measure on ergodic grupoids*, Thesis, Harvard, 1975.
2. Haar measure for measure groupoids, Berkeley, 1975.
3. The regular representations of measure groupoids, Berkeley, 1976.
4. The σ-representations of amenable groupoids.

HAJIAN, A.; ITO, Y.; KAKUTANI, S.
1. Full groups and a theorem of Dye, *Adv. Math.*, **17** (1975), 48–59.

HAKEDA, J.
1. On property P of von Neumann algebras, I, II, *Tohoku Math. J.*, **19** (1967), 238–242; *Bull. Yamagata Univ.*, **7** (1968), 5–6.

HAKEDA, J.; TOMIYAMA, J.
1. On some extension properties of von Neumann algebras, *Tohoku Math. J.*, **19** (1967), 315–323.

HALL, A.
1. Derivations of certain C^*-algebras, *J. London Math. Soc.*, **5** (1972), 321–329.

HALMOS, P. R.
1. *Introduction to Hilbert space and the theory of spectral multiplicity*, Chelsea Publ. Comp., New York, 1951.
2. *Finite-dimensional vector spaces*, Van Nostrand, 1958.
3. *Lectures on boolean algebras*, Van Nostrand, 1963.
4. What does the spectral theorem say? *Amer. Math. Monthly*, **70** (1963), 241–247.
5. *A Hilbert space problem book*, Van Nostrand, 1967.
6. Continuous functions of Hermitian operators, *Proc. AMS*, **31** (1972), 130–132.
7. Ten problems in Hilbert space, *Bull. AMS*, **76** (1970), 887–933.

HALPERIN, I.
1. A remark on a preceding paper by J. von Neumann, *Ann. Math.*, **41** (1940), 554–555.
2. Introduction to von Neumann algebras and continuous geometries, *Can. Math. Bull.*, **3** (1960), 273–288 (Corrections, ibidem, **5** (1962), 59).

HALPERN, H.
1. The maximal GCR ideal in an AW^*-algebra, *Proc. AMS*, **17** (1966), 906–914.
2. An integral representation of a normal functional on a von Neumann algebra, *Trans. AMS*, **125** (1966), 32–46.
3. Finite sums of irreducible functionals on C^*-algebras, *Proc. AMS*, **18** (1957), 352–358.
4. A spectral decomposition for self adjoint elements in the maximum GCR ideal of a von Neumann algebra with applications to non-commutative integration theory, *Trans. AMS*, **133** (1968), 281–306.
5. Module homomorphisms of a von Neumann algebra into its center, *Trans. AMS*, **140** (1969), 183–194.
6. Irreducible module homomorphisms of a von Neumann algebra into its center, *Trans. AMS*, **140** (1969), 195–221.
7. Commutators in properly infinite von Neumann algebras, *Trans. AMS*, **139** (1969), 55–74.

8. Commutators modulo the center in a properly infinite von Neumann algebra, *Trans. AMS*, **150** (1970), 55–68.
9. Embedding as a double commutator in a type I AW^*-algebra, *Trans. AMS*, **148** (1970), 85–98.
10. A generalized dual for C^*-algebras, *Trans. AMS*, **153** (1971), 139–157.
11. Unitary implementation of automorphism groups on von Neumann algebras, *Comm. Math. Phys.*, **25** (1972), 253–276.
12. Integral decompositions of functionals on C^*-algebras, *Trans. AMS*, **168** (1972), 371–385.
13. Essential central spectrum and range for elements of a von Neumann algebra, *Pacific J. Math.*, **43** (1972), 349–380.
14. Primitive ideals in von Neumann algebras, *Proc. AMS*, **39** (1973), 563–566.
15. Open projections and Borel structures for C^*-algebras, *Pacific J. Math.*, **50** (1974), 81–98.
16. Mackey-Borel structure for the quasi-dual of a separable C^*-algebra, *Can. J. Math.*, **26** (1974), 621–628.
17. Proper values in the maximal GCR ideal of a von Neumann algebra.
18. Quasi-equivalence classes of normal representations for a separable C^*-algebra, *Trans. AMS*, **203** (1975), 129–140.
19. One-parameter automorphism groups of generalized KH-algebras.
20. The norm of an inner derivation on an AW^*-algebra.
21. Essential central range and self-adjoint commutators in properly infinite von Neumann algebras, *Trans. AMS*, **228** (1977), 117–146.
22. Spectra of operator equations on von Neumann algebras, *Notices AMS*, **22**: 6 (1975), 75 T B 227.
23. Normal expectations and integral decomposition of type III von Neumann algebras.

HAMACHI, T.
1. Equivalent measures on product spaces, *Mem. Fac. Sci. Kyushu Univ.*, **27** (1973), 335–341.
2. Equivalence of measure and classification of groups of non-singular transformations, *Math. Rep. College Gen. Ed. Kyushu Univ.*, **10** (1975), 21–44.

HAMACHI, T.; OKA, Y.; OSIKAWA, M.
1. A classification of ergodic non-singular transformation groups, *Memoirs Fac. Sc. Kyushu Univ.*, **28** (1974), 113–133.
2. Flows associated with ergodic non-singular transformation groups, *Publ. RIMS, Kyoto Univ.*, **11** (1975), 31–50.

HAMANA, M.
1. Injective envelopes of C^*-algebras.
2. Injective envelopes of operator systems.
3. On linear topological properties of some C^*-algebras, *Tohoku Math. J.*, **29** (1977), 157–163.

HANDELMAN, D.
1. K_1 of non commutative von Neumann regular rings, *J. Pure Appl. Algebra*, **8** (1976), 105–118.
2. K_0 of von Neumann and AF-C^*-algebras.
3. Stable ranges in AW^*-algebras.
4. Rickart C^*-algebras and their properties, I, II, *Adv. Math.*

HANDELMAN, D.; LAWRENCE, J.
1. Finite Rickart C^*-algebras, *Bull. AMS*, **84** (1978), 157–158.

HANSEN, F.
1. Inner one-parameter groups acting on a factor, *Math. Scand.*, **41** (1977), 113–116.

HANSEN, F.; OLESEN, D.
1. Perturbations of centre-fixing dynamical systems, Copenhagen, 1976.

HARATIAN, S. G.
1. von Neumann algebras of observable with non abelian commutant and superselection rules (Russian), *Teor. Mat. Fiz.*, **14**: 3 (1973), 306–313.
2. Generalization of Wigner's theorem on symmetries in the C^*-algebraic context (Russian), *Teor. Mat. Fiz.*, **20**: 2 (1974), 177–180.

HARISH-CHANDRA
1. Representations of semisimple Lie groups, I, II, III, IV, V, VI, *Trans. AMS*, **75** (1953), 185–243; **76** (1954), 26–65; 234–253; *Amer. J. Math.*, **77** (1955), 743–777; **78** (1956), 1–41, 546–628.
2. Discrete series for semisimple Lie groups, I, II, *Acta Math.*, **113** (1965), 241–318; **116** (1966), 1–111.
3. Harmonic analysis on semisimple Lie groups, *Bull. AMS*, **78** (1970), 529–551.
4. Harmonic analysis on real reductive groups, I, II, III, *J. Fnal Analysis*, **19** (1975), 104–204; *Invent. Math.*, **36** (1976), 1–55; *Ann. Math.*, **104** (1976), 117–201.

DE LA HARPE, P.
1. *Classical Banach-Lie algebras and Banach-Lie groups of operators in Hilbert spaces*, Lecture Notes in Math., No. 285, Springer Verlag, 1972.

DE LA HARPE, P.; KAROUBI, M.
1. Perturbations compactes des représentations d'un groupe dans un espace de Hilbert, *Bull. Soc. Math. France, Mémoire* No. **46** (1976), 41–65.

HARRIS, L. A.
1. Schwartz's lemma in normed linear spaces, *Proc. Nat. Acad. Sci. USA*, **62** (1969), 1014–1017.
2. Banach algebras with involution and Möbius transformations, *J. Fnal Analysis*, **11** (1972), 1–16.
3. Operator extreme points and the numerical range, *Indiana Math. J.*, **23** (1974), 937–947.

HARRIS, R. T.
1. A direct integral construction, *Duke Math. J.*, **33** (1966), 535–537.

HASUMI, M.
1. The extension property of complex Banach spaces, *Tohoku Mat. J.*, **10** (1958), 135–142.
2. A continuous selection theorem for extremally disconnected spaces, *Math. Ann.*, **179** (1969), 83–89.

HAVET, J. F.
1. Espérances conditionnelles permutables à un groupe d'automorphismes sur une algèbre de von Neumann, *CR Acad. Sc. Paris*, **282** (1976), 1095–1098.
2. *Quelques propriétés des systèmes dynamiques non-commutatifs*, Thèse 3-ème cycle, Orléans, 1976.

HAYDON, R.
1. A new proof that every polish space is the extreme boundary of a simplex, *Bull. London Math. Soc.*, **7** (1975), 97–100.

HAYDON, R.; WASSERMANN, S.
1. A commutation result for tensor products of C^*-algebras, *Bull. London Math. Soc.*, **5** (1973), 283–287.

VAN HEESWIJCK, L.
1. Duality in the theory of crossed products.

HEINZ, E.
1. Beiträge zur Störungstheorie der Spektralzerlegung, *Math. Ann.*, **123** (1951), 415–438.

HELEMSKIĬ, A. I. (in Russian)
1. Strong decomposability of finite dimensional extensions of C^*-algebras, *Vestnik Mosk. Univ.*, **22** (1974), 50–52.
2. The global dimension of a Banach function algebra is different from one, *Trudy Kaf. Teor. Funkt. Analiz, MGU*, **1** (1974), 39–58; *Funkt. Analiz i pril.*, **6**: 2 (1972), 95–96.
3. On the homological dimension of normed modules over Banach algebras, *Mat. Sbornik*, **81** (1970), 430–444.

HELEMSKIĬ, A.; SINAĬ, YA. G.
1. A description of differentiations in algebras of the type of algebras of local observables of spin systems (Russian), *Funkt. Analiz i pril.*, **6**: 3 (1973), 99–100.

HELSON, H.; PARRY, W.

1. Cocycles and spectra, Warwick, 1976.

HELTON, J. W.

1. Operator algebras and the propagation of singularities in solutions to a differential equation, *Indiana Univ. Math. J.*, **26** (1977), 997–1018.

HELTON, J. W.; HOWE, R. E.

1. Integral operators: commutators, traces, index and homology, *Proc. Conference on Operator Theory, Lecture Notes in Math.*, No. 345, 1973.
2. Traces of commutators of integral operators, *Acta Math.*, **135** (1975), 271–305.

HENLE, M.

1. Spatial representations of groups of automorphisms of von Neumann algebras with properly infinite commutant, *Comm. Math. Phys.*, **19** (1970), 273–275.
2. A Lebesgue decomposition for C^*-algebras, *Can. Math. Bull.*, **15** (1972), 87–92.
3. Galois theory of W^*-algebras.

HENRY, J. N.; TAYLOR, D. C.

1. The β topology for W^*-algebras, *Pacific J. Math.*, **60** (1975), 123–140.

HERMAN, R. H.

1. Some remarks on expectations, *Acta Sci. Math.*, **31** (1970), 125–128.
2. Invariant states, *Trans. AMS*, **158** (1971), 503–512.
3. Perturbations of the modular automorphisms group, *Comm. Math. Phys.*, **28** (1972), 237–243.
4. Centralizers and an ordering for faithful normal states, *J. Fnal Analysis*, **13** (1973), 317–323.
5. Automorphism groups of operator algebras, *Pacific J. Math.*, **52** (1974), 91–100.
6. Spectra of automorphism groups of operator algebras, *Duke Math. J.*, **41** (1974), 667–674.
7. Unbounded derivations, I, II, *J. Fnal Analysis*, **20** (1975), 234–239; *Amer. J. Math.*, **99** (1977), 1003–1013.
8. The centralizer under tensor product, *Math. Scand.*, **36** (1975), 286–290.
9. Implementation and the core problem, preprint, Bielefeld.
10. Faithful normal states on von Neumann algebras, in "*C^*-algebras and their applications to statistical mechanics and quantum field theory*", p. 121–127, North-Holland, 1976.

HERMAN, R. H.; REED, M. C.

1. Covariant representations of infinite tensor product algebras, *Pacific J. Math.*, **40** (1972), 311–326.

HERMAN, R. H.; TAKESAKI, M.

1. States and automorphism groups of operator algebras, *Comm. Math. Phys.*, **19** (1970), 142–160.
2. The comparability theorem for cyclic projections, *Bull. London Math. Soc.*, **9** (1977), 186–187.

HERSTEIN, I. N.

1. *Rings with involution*, The University of Chicago Press, 1976.

HEWITT, E.; MC DUFF, D.

1. Some pathological maximal ideals in operator algebras and measure algebras on groups, (Russian), *Matem. Sb.*, **83** (1970), 527–546.

HEWITT, E.; ROSS, K. A.

1. *Abstract harmonic analysis*, I, II, Springer Verlag, 1963, 1970.

HEYER, H.

1. L'analyse de Fourier non commutative et applications à la théorie des probabilités, *Ann. Inst. H. Poincaré*, **4** (1968), 143–164.
2. *Dualität lokalkompakter Gruppen*, Lecture Notes in Math., No. 150, Springer Verlag, 1970.

HILLE, E.; PHILLIPS, R.

1. *Functional analysis and semigroups*, AMS Providence, 1957.

HILSUM, M.
1. Sur les sous-espaces spectraux d'un opérateur compact relativement à une algèbre de von Neumann, *Ann. Inst. Fourier*, 1978.

HIMMELBERG, C.
1. Measurable selections, *Fund. Math.*, **87** (1975), 53–72.

HIRSCHFELD, R.
1. Sur l'analyse harmonique dans les groupes localement compacts, *CR Acad. Sc. Paris*, **246** (1958), 1138–1140.

HOFMANN, K. H.
1. *Introduction to the theory of compact groups*, Lecture Notes, Tulane Univ., 1967.
2. *The duality of compact semigroups and C^*-bigebras*, Lecture Notes in Math., No. 129, Springer Verlag, 1970.
3. Representations of algebras by continuous sections, *Bull. AMS*, **78** (1972), 291–373.
4. Some bibliographical remarks on "Representations of algebras by continuous sections", *Memoirs AMS*, **148** (1974), 177–182.
5. Sheaves and bundles of Banach spaces.

HOFMANN, K. H.; JAVIER-THAYER, F.
1. Approximately finite dimensional C^*-algebras, Tulane Univ., 1977.

HOLEVO, A. S.
1. Generalized free states of the C^*-algebra of the commutation relations (Russian), I, II, *Teor. Mat. Fiz.*, **6**: 1 (1971), 3–19; **6**: 2 (1971), 145–149.
2. On quasi-equivalence of locally normal states (Russian), *Teor. Mat. Fiz.*, **13** (1972), 184–199.
3. On the quasi equivalence of quasifree states on the C^*-algebra of CAR (Russian), *Teor. Mat. Fiz.*, **14** (1973), 145–151.
4. On the theory of statistical decisions in operator algebras (Russian), *Dokl. Akad. Nauk*, **218**: 1 (1974), 54–57.
5. On a vector-valued integral in the "non commutative" statistical decision theory, *J. Multivariate Analysis*, **5** (1975), 462–465.
6. The general theory of statistical decision (Russian), *Trudy Mat. Inst. Steklov*, No. 74, 1976.
7. Commutation superoperator of a state and its applications to the non-commutative statistics, *Rep. Math. Phys.*, **12** (1977), 251–271.

HOLLAND, S. S.
1. Partial solution to Mackey's problem about modular pairs and completeness, *Can. J. Math.*, **21** (1969), 1518–1525.
2. Remarks on type I Baer and Baer $*$-rings, *J. Algebra*, **27** (1973), 516–522.

HONGO, E.
1. A note on commutator of certain operator algebras, *Bull. Kyushu Inst. Techn.*, **1** (1955), 19–22.
2. On left rings of certain $*$-algebras, *Bull. Kyushu Inst. Tech.*, **2** (1956), 1–15.
3. On quasi-unitary algebras with semifinite left rings, *Bull. Kyushu Inst. Tech.*, **3** (1957), 1–10.
4. On some properties of quasi-unitary algebras, *Bull. Kyushu Inst. Tech.*, **4** (1958), 1–6.
5. On left multiplicative operators in a quasi-unitary algebra, *Bull. Kyushu Inst. Tech.*, **5** (1959), 19–22.
6. A structure theory for semifinite quasi-unitary algebras, *Bull. Kyushu Inst. Tech.*, **7** (1961), 1–17.

HONGO, E.; ORIHARA, M.
1. A remark on a quasi-unitary algebra, *Yokohama Math. J.*, **2** (1954), 69–72.

HOPENWASSER, A.
1. Boundary representations of C^*-algebras with matrix units, *Trans. AMS*, **177** (1973), 483–490.
2. Ergodic automorphisms and linear spaces of operators, *Duke Math. J.*, **41** (1974), 747–757.
3. Boundary representations and tensor products of C^*-algebras.

HOPENWASSER, A.; PLASTIRAS, J.
1. Isometries of quasitriangular operator algebras, *Proc. AMS*, 65 (1977), 242–244.

HORUZIĬ, S. S. (in Russian)
1. Extended locality and other properties of local W^*-algebras, *Teor. Mat. Fiz.*, 1 (1969), 95–100.
2. The structure of local W^*-algebras in quantum field theory, *Teor. Mat. Fiz.*, 2 (1970), 350–360.

HOWE, R.
1. The Brauer group of compact Hausdorff spaces and n-homogeneous C^*-algebras, *Proc. AMS*, 34 (1972), 209–214.

HSIEH, T.
1. *Existence of normal linear positive functionals on a von Neumann algebra invariant with respect to a semigroup of contractions*, Thesis, Univ. British Columbia, 1971.
2. Invariant normal positive functionals, I, II, *J. Math. Analysis Appl.*, 45 (1974), 556–562; 47 (1974), 554–562.

HUDSON, R. L.; MOODY, G. R.
1. Locally normal symmetric states and an analogue of de Finetti's theorem, *Z. Wahr. Verw. Gebiete*, 33 (1975/76), 343–351.

HUGENHOLTZ, N. M.
1. On the factor type of equilibrium states in quantum statistical mechanics, *Comm. Math. Phys.*, 6 (1967), 189–193.
2. States and representations in statistical mechanics, in *Mathematics of contemporary physics* (Ed. R. F. Streater), Academic Press, 1972, 1–16.
3. Certain automorphisms of the CAR-algebra, in *Symposia Math.*, XX, 327–334, Academic Press, 1976.

HUGENHOLTZ, N. M.; KADISON, R. V.
1. Automorphisms and quasi-free states of the CAR algebra, *Comm. Math. Phys.*, 43 (1975), 181–197.

HUGENHOLTZ, N. M.; WIERINGA, J. D.
1. On locally normal states in quantum statistical mechanics, *Comm. Math. Phys.*, 11 (1969), 183–197.

HULANICKI, A.
1. Groups whose regular representation weakly contains all unitary representations, *Studia Math.*, 24 (1964), 37–59.

HURUYA, T.
1. On the structure of certain C^*-algebras, *Proc. Jap. Acad.*, 45 (1969), 354–358.
2. C^*-algebras having property (T), *Sci. Rep. Niigata Univ.*, 8 (1971), 7–12.
3. The second dual of a tensor product of C^*-algebras, I, II, III, *Sci. Rep. Niigata Univ.*, 9 (1972), 35–38; 11 (1974), 21–23; 13 (1976), 25–29.
4. Boundary representations of a tensor product of C^*-algebras, *Sci. Rep. Niigata Univ.*, 11 (1974), 55–59.
5. The normed space numerical index of a C^*-algebra, *Proc. AMS*, 63 (1977), 289–290.

HUSAIN, T.; RIGELHOF, R.
1. Representations of MQ^*-algebras, *Math. Ann.*, 180 (1969), 297–306.

IASNIKOV, V. P. (in Russian)
1. Harmonic functions connected with finite von Neumann algebras, *Trudy Mosk. Mat. Obšč.*, 32 (1975), 211–266.
2. Harmonic functions connected with finite von Neumann algebras, *Uspekhi Mat. Nauk*, 26:5 (1971), 229–230.
3. Some properties of harmonic functions, connected with finite von Neumann algebras, *Uspekhi Mat. Nauk*, 28:1 (1973), 257.

IGARASHI, A.
1. Left Hilbert algebras, I, II, *Bull. Electrotechn. Lab.*, **38** (1974), 405–410; **39** (1975), 478–489.
2. Ideals in a C^*-algebra, *Bull. Electrotechn. Lab.*, **39** (1975), 399–414.

IKUNISHI, A.; NAKAGAMI, Y.
1. Automorphisms groups of von Neumann algebras and semifiniteness of an infinite tensor product of von Neumann algebras.
2. On invariants $G(\sigma)$ and $\Gamma(\sigma)$ for an automorphism group of a von Neumann algebra, *Publ. RIMS, Kyoto Univ.*, **12** (1976), 1–30.
3. On an invariant Γ for an automorphism group of a von Neumann algebra, *Japan-US Seminary on C^*-algebras and Appl. to Phys.*, 1974, 185–189.

IKUNISHI, A.; TAKAI, H.
1. On crossed products with abelian group actions.

IMAI, S.; TAKAI, H.
1. On a duality for C^*-crossed products by a locally compact group.

INGARDEN, S. R.; WORONOWICZ, S. L.
1. On mixed and pure observables, Toruń, 1975.

INOUE, A.
1. Locally C^*-algebras, *Mem. Fac. Sci. Kyushu Univ.*, **25** (1971), 197–235.
2. L^1-space associated with a weight, *Mem. Fac. Sci. Kyushu Univ.*, **27** (1973), 291–308.
3. Conditional expectations in von Neumann algebras with respect to weights, *Mem. Fac. Sci. Kyushu Univ.*, **27** (1973), 309–321.
4. A class of unbounded operator algebras, I, II, III, *Pacific J. Math.*, **65** (1976), 77–96; **66** (1976), 411–431; **69** (1977), 105–115.
5. Unbounded representations of symmetric $*$-algebras, *J. Math. Soc. Japan*, **29** (1977), 219–232 (Corrections, *Fukuoka Univ. Sci. Rep.*, **8** (1978), 1–4).
6. Unbounded generalizations of standard von Neumann algebras.
7. L_p-spaces and maximal unbounded Hilbert algebras.
8. A commutant of an unbounded operator algebra.

INOUE, A.; KURIYAMA, K.
1. Weakly unbounded operator algebras, *Mem. Fac. Sci. Kyushu Univ.*, **31** (1977), 267–284.

IONESCU-TULCEA, A.; IONESCU-TULCEA, C. T.
1. Topics in the theory of lifting, Springer Verlag, 1971.

IONESCU-TULCEA, C. T.
1. Deux théorèmes concernant certains espaces de champs de vecteurs, *Bull. Sc. Math.*, **79** (1955), 1–9.
2. *Hilbert Spaces* (Romanian), Editura Academiei, Bucharest, 1956.

ISTRATESCU, I.; ISTRATESCU, V.
1. On characters of singly generated C^*-algebras, *Proc. Jap. Acad.*, **47** (1971), 42–43.

IWAMURA, T.
1. On continuous geometries, I, II, *Jap. J. Math.*, **19** (1944), 57–71; **2** (1950), 148–164.

JACOBSON, N.
1. *Structure of rings*, AMS Providence, 1946.
2. A topology for the set of primitive ideals in arbitrary ring, *Proc. Nat. Acad. Sci. USA*, **31** (1945), 333–338.
3. *Structure and representations of Jordan algebras*, Amer. Math. Soc. Colloq. Publ., Providence, No. 39, 1968.

JACOBSON, N.; RICKART, C. E.
1. Homomorphisms of Jordan rings, *Trans. AMS*, **69** (1950), 479–502.

JADCZYK, A. Z.
1. On spectrum of internal symmetries of the algebraic quantum field theory, *Comm. Math. Phys.*, **12** (1969), 58–63.

2. On some groups of automorphisms of von Neumann algebras with cyclic separating vector, *Comm. Math. Phys.*, **13** (1969), 142–154.
3. Note on canonical commutation relations, *Bull. Acad. Pol. Sc.*, **22** (1974), 963–965.
4. On a theorem of Mackey, Stone and von Neumann for projective imprimitivity systems.

JAFFE, A. M.
1. Constructing the $\lambda(\varphi^4)_2$ theory, in *Local Quantum Theory*, Academic Press, 1969, 120–151.
2. Progress in constructive field theory, *Contemp. Phys.*, **2** (1969), 463–469.
3. Mathematical problems motivated by quantum field theory, in *Symposia Math.*, XX, 207–218, Academic Press, 1976.

JAJTE, R.
1. Quelques résultats dans la théorie non commutative des probabilités, *Ann. Sc. Univ. Clermont-Ferrand*, No. 58, Math., No. 12 (1976), 11–17.

JANSSEN, G.
1. Bemerkungen zum Dichtigkeitssatz von Kaplansky, *Math. Z.*, **139** (1974), 159–164.
2. Lattices of order ideals.

JAVIER-THAYER, F.
1. The Weyl-von Neumann theorem for approximately finite C^*-algebras, *Indiana Univ. Math. J.*, **24** (1975), 875–878.
2. Quasi-diagonal C^*-algebras, *J. Fnal Analysis*, **25** (1977), 50–57.
3. Obstructions to lifting $*$-morphisms into the Calkin algebra, *Illinois J. Math.*, **12** (1975/76), 322–328.
4. Almost periodic operator of inner automorphism groups, *Proc. AMS*, **63** (1977), 119–124.

JELINEK, F.
1. BCS-spin-model, its thermodynamic representations and automorphisms, *Comm. Math. Phys.*, **9** (1968), 169–175.

JELOBENKO, D. P.
1. *Compact Lie groups and their representations* (Russian), Moscow, 1970.

JENKINS, J. W.
1. On the characterization of abelian W^*-algebras, *Proc. AMS*, **35** (1972), 436–438.

JOCHUM, B.
1. *Cones autopolaires dans les espaces de Hilbert*, Thesis, Marseille, 1975.

JOHNSON, B. E.
1. An introduction to the theory of centralizers, *Proc. London Math. Soc.*, **14** (1964), 299–330.
2. Continuity of centralizers on Banach algebras, *J. London Math. Soc.*, **41** (1966), 639–640.
3. The uniqueness of the (complete) norm topology, *Bull. AMS*, **73** (1967), 537–539.
4. Continuity of linear operators, *Trans. AMS*, **128** (1967), 88–102.
5. AW^*-algebras are QW^*-algebras, *Pacific J. Math.*, **23** (1967), 97–99.
6. A commutative semisimple annihilator Banach algebra which is not dual, *Bull. AMS*, **73** (1967), 407–409.
7. Continuity of homomorphisms of algebras of operators, I, II, *J. London Math. Soc.*, **42** (1967), 537–541; **1** (1969), 81–84.
8. The Wedderburn decomposition of Banach algebras with finite dimensional radical, *Amer. J. Math.*, **90** (1968), 866–876.
9. Continuity of derivations on commutative algebras, *Amer. J. Math.*, **91** (1969), 1–10.
10. Approximate diagonals and cohomology of certain annihilator Banach algebras, *Amer. J. Math.*, **94** (1972), 685–698.
11. *Cohomology of Banach algebras*, Memoirs AMS, No. 127 (1972).
12. Automorphisms of commutative Banach algebras, *Proc. AMS*, **40** (1974), 497–499.
13. A class of II_1 factors without property P but with zero second cohomology, *Ark. Math.*, **12** (1974), 153–159.
14. Banach algebras: introductory course, in *Algebras in Analysis* (Ed. J. H. Williamson), Academic Press, 1975, 63–83.

15. Introduction to cohomology in Banach algebras, in *Algebras in Analysis* (Ed. J. H. Williamson), Academic Press, 1975, 84–100.
16. Norming $C(\Omega)$ and related algebras, *Trans. AMS,* **220** (1976), 37–50.
17. Perturbations of Banach algebras, *Proc. London Math. Soc.*, **34** (1977), 439–458.
18. Derivations of C^*-algebras; some problems and results.

JOHNSON, B. E.; KADISON, R. V.; RINGROSE, J. R.
1. Cohomology of operator algebras, III, *Bull. Soc. Math. France*, **100** (1972), 73–96.

JOHNSON, B. E.; PARROTT, S. K.
1. Operators commuting with a von Neumann algebra modulo the set of compact operators, *J. Fnal Analysis,* **11** (1972), 39–61.

JOHNSON, B. E.; RINGROSE, J. R.
1. Derivations of operator algebras and discrete group algebras, *Bull. London Math. Soc.*, **1** (1969), 70–74.

JOHNSON, B. E.; SINCLAIR, A. M.
1. Continuity of derivations and a problem of Kaplansky, *Amer. J. Math.*, **90** (1968), 1067–1073.
2. Continuity of linear operators commuting with continuous linear operators, II, *Trans. AMS,* **146** (1969), 533–540.

JOHNSON, B. E.; WILLIAMS, J. P.
1. The range of a normal derivation, *Pacific J. Math.*, **58** (1975), 105–122.

JONES, V.
1. Sur la conjugaison de sous-facteurs de facteurs de type II_1, *C.R. Acad. Sc. Paris,* **284** (1977), 597–598.

JORDAN, P.; VON NEUMANN, J.; WIGNER, E.
1. On an algebraic generalization of the quantum mechanical formalism, *Ann. Math.*, **35** (1934), 29–64.

JORGENSEN, P. E. T.
1. On one-parameter groups of automorphisms and extensions of symmetric operators associated with unbounded derivations in operator algebras.
2. Distribution representations of Lie groups.
3. Approximately reducing subspaces for unbounded linear operators, *J. Fnal Analysis,* **23** (1976), 392–414.
4. Trace states and *KMS* states for approximately inner dynamical one-parameter groups of $*$-automorphisms, *Comm. Math. Phys.*, **53** (1977), 135–142.
5. Approximately invariant subspaces for unbounded linear operators, *Math. Ann.*, **227** (1977), 177–182.
6. Unbounded derivations in operator algebras and extensions of states, *Tohoku Math. J.*
7. Commutation properties for automorphism groups of von Neumann algebras and exponentiation of spatial derivations.

JORGENSEN, P. E. T.; RADIN, C.
1. Approximately inner dynamics.

JULIA, G.
1. *Introduction mathématique aux théories quantiques*, Gauthier-Villars, 1953.

JURZAK, J.-P.
1. Poids strictement sémi-finis et groupe d'automorphismes modulaire sur une algèbre de von Neumann décomposable, *CR Acad. Sc. Paris,* **279** (1974), 221–224.
2. Decomposable operators. Applications to *KMS* weight in a decomposable von Neumann algebra, *Rep. Math. Phys.*, **8**: 2 (1975), 203–228.
3. Topologies pour les $*$-algèbres d'opérateurs non bornés, *CR Acad. Sc. Paris,* **280** (1975), 1283–1286.
4. Simple facts about algebras of unbounded operators, *J. Fnal Analysis,* **21** (1976), 469–482.

Kadison, R. V.
1. *A representation theory for commutative topological algebras*, Memoirs AMS, No. 7 (1951).
2. Isometries of operator algebras, *Ann. Math.*, 54 (1951), 325–338.
3. Order properties of bounded selfadjoint operators, *Proc. AMS*, 2 (1951), 505–510.
4. Infinite unitary groups, *Trans. AMS*, 72 (1952), 386–399.
5. A generalized Schwartz inequality and algebraic invariants for operator algebras, *Ann. Math.*, 56 (1952), 494–503.
6. Infinite general linear groups, *Trans. AMS*, 76 (1954), 66–91.
7. On the additivity of trace in finite factors, *Proc. Nat. Acad. Sci. USA*, 41 (1955), 385–387.
8. The general linear group of infinite factors, *Duke Math. J.*, 22 (1955), 119–122.
9. On the orthogonalization of operator representations, *Amer. J. Math.*, 77 (1955), 600–621.
10. Isomorphisms of factors of infinite type, *Canadian J. Math.*, 7 (1955), 322–327.
11. Report on operator algebras, *Intern. Conf. on operator theory and group representations*, Arden House, Harriman, New York (1955), 4–10, Publ. 387.
12. Multiplicity theory for operator algebras, *Proc. Nat. Acad. Sci. USA*, 41 (1955), 169–173.
13. Operator algebras with a faithful weakly closed representation, *Ann. Math.*, 64 (1956), 175–181.
14. Unitary invariants for representations of operator algebras, *Ann. Math.*, 66 (1957), 304–379.
15. Irreducible operator algebras, *Proc. Nat. Acad. Sci. USA*, 43 (1957), 273–276.
16. Theory of operators II: Operator algebras, *Bull. AMS*, 64 (1958), 61–85.
17. The trace in finite operator algebras, *Proc. AMS.*, 12 (1961), 973–977.
18. States and representations, *Trans. AMS*, 103 (1962), 304–319.
19. Normalcy in operator algebras, *Duke Math. J.*, 29 (1962), 459–464.
20. *The trace in finite operator algebras*, Mimeographed, Columbia Univ., 1963.
21. Remarks on type of von Neumann algebras of local observables in quantum field theory, *J. Math. Phys.*, 4 (1963), 1511–1516.
22. Transformations of states in operator theory and dynamics, *Topology*, 3 (1965), 177–198.
23. Derivations of operator algebras, *Ann. Math.*, 83 (1966), 280–293.
24. The energy momentum spectrum of quantum fields, *Comm. Math. Phys.*, 4 (1967), 258–260.
25. Lectures on operator algebras, *Cargese Lecture on Phys.*, 3 (1967), 41–82.
26. Strong continuity of operator functions, *Pacific J. Math.*, 26 (1968), 121–129.
27. Some analytic methods in the theory of operator algebras, *Lecture Notes in Math.*, No. 140, 1969, 8–29.
28. Problems on von Neumann algebras.
29. On the problem of isomorphism of hyperfinite II_∞-factors. A critique of the article by Golodets (*Soviet J. Fnal. Anal. and its Appl.*, 5 (1971), 37–44).
30. Appendix to Davies, E. B. [1].
31. A note on derivations of operator algebras, *Bull. London Math. Soc.*, 7 (1975), 41–44.
32. Notes on the Fermi gas, in *Symposia Math.*, XX, 425–431, Academic Press, 1976.
33. Operator algebras, in *Algebras in Analysis* (Ed. J. H. Williamson), Academic Press, 1975, 101–117.
34. Review of Anastasio, S.; Willig, P. [1], *Bull. AMS*, 81 (1975), 522–527.
35. Normal states and unitary equivalence of von Neumann algebras, in "*C*-algebras and their applications to statistical mechanics and quantum field theory*", p. 1–18, North-Holland, 1976.
36. Unbounded similarity.

Kadison, R. V.; Kastler, D.
1. Perturbations of von Neumann algebras, I: Stability of type, *Amer. J. Math.*, 94 (1972), 38–54.

Kadison, R. V.; Lance, E. C.; Ringrose, J. R.
1. Derivations and automorphisms of operator algebras, II, *J. Fnal Analysis*, 1 (1967), 204–221.

Kadison, R. V.; Pedersen, G. K.
1. Equivalence in operator algebras, *Math. Scand.*, 27 (1970), 205–222.

KADISON, R. V.; RINGROSE, J. R.

1. Automorphisms of operator algebras, *Bull. AMS*, **72** (1966), 1059–1063.
2. Derivations of operator group algebras, *Amer. J. Math.*, **88** (1966), 562–576.
3. Derivations and automorphisms of operator algebras, *Comm. Math. Phys.*, **4** (1967), 32–63.
4. Cohomology of operator algebras, I, II, *Acta Math.*, **126** (1971), 227–243; *Ark. Math.*, **9** (1971), 55–63.
5. Algebraic automorphisms of operator algebras, *J. London Math. Soc.*, **8** (1974), 329–334.

KADISON, R. V.; SINGER, I. M.

1. Some remarks on representations of connected groups, *Proc. Nat. Acad. Sci. USA*, **38** (1952), 419–423.
2. Three test problems in operator theory, *Pacific J. Math.*, **7** (1957), 1101–1106.
3. Extensions of pure states, *Amer. J. Math.*, **81** (1959), 383–400.
4. Triangular operator algebras, *Amer. J. Math.*, **82** (1960), 227–259.

KAFTAL, V.

1. On the theory of compact operators in von Neumann algebras, I, II, *Indiana Univ. Math. J.*

KAHANE, J. P.; ZELAZKO, W.

1. A characterization of maximal ideals in commutative Banach algebras, *Studia Math.*, **29** (1968), 339–343.

KAHN, D. S.; KAMINKER, J.; SCHOCHET, C.

1. Generalized homology theories on compact metric spaces.

KALIMAN, S. I.; SELIVANOV, I. V.

1. Cohomologies of operator algebras (Russian), *Vestnik Moskov. Univ.*, **29**: 5 (1974), 24–27.

KALLMAN, R. R.

1. Uniform continuity, unitary groups and compact operators, *J. Fnal Analysis*, **1** (1967), 245–253.
2. A characterization of uniformly continuous representations of connected locally compact groups, *Michigan Math. J.*, **16** (1969), 257–263.
3. A remark on a paper by J. F. Aarnes, *Comm. Math. Phys.*, **14** (1969), 13–15.
4. A generalization of free action, *Duke Math. J.*, **36** (1969), 781–789.
5. Unitary groups and automorphisms of operator algebras, *Amer. J. Math.*, **91** (1969), 785–806.
6. The strong bounded topology on groups of automorphisms of a von Neumann algebra, *Proc. AMS*, **23** (1969), 367–372.
7. One parameter groups of $*$-automorphisms of II_1 von Neumann algebras, *Proc. AMS*, **24** (1970), 336–340.
8. Certain topological groups are type I, parts I, II, III, *Bull. AMS*, **76** (1970), 404–406; *Adv. Math.*, **10** (1973), 221–255.
9. A theorem on discrete groups and some consequences of Kazdan's thesis, *J. Fnal Analysis*, **6** (1970), 203–207.
10. A generalization of a theorem of Berger and Coburn, *J. Math. Mech.*, **19** (1970), 1005–1010.
11. A problem of Gelfand on rings of operators and dynamical systems, *Can. J. Math.*, **22** (1970), 514–517.
12. A decomposition theorem for automorphisms of von Neumann algebras, in *Inequalities*, Academic Press, 1970, vol. II, 33–35.
13. Spatially induced groups of automorphisms of certain von Neumann algebras, *Trans. AMS*, **156** (1971), 505–515.
14. Groups of inner automorphisms of von Neumann algebras, *J. Fnal Analysis*, **7** (1971), 43–60.
15. A theorem on the restriction of type I representations of a group to certain of its subgroups, *Proc. AMS*, **40** (1973), 291–296.
16. On the type question for some topological groups.

17. A theorem in the representation theory of totally disconnected groups, *Proc. AMS*, **45** (1974), 457–460.
18. The existence of invariant measures on certain quotient spaces, *Adv. Math.*, **11** (1973), 387–391.
19. Certain quotient spaces are countable separated, I, II, III, *Illinois J. Math.*, **19** (1975), 378–388; *J. Fnal Analysis*, **21** (1976), 52–62; *J. Fnal Analysis*, **22** (1976), 225–241.
20. The reduced dual determines countable separability (unpublished).
21. There is no nontrivial Borel measure on S_∞ which is quasiinvariant under inner automorphisms.

KALLMANN, R. R.; MAULDIN, R. D.
1. A cross section theorem and an application to C^*-algebras.

KAMEI, E.
1. Topological properties of normal elements of the Calkin algebra, *Math. Jap.*, **22** (1977), 239–242.

KAMEI, E.; KATO, Y.
1. Partial isometries of the Calkin algebra, *Math. Jap.*, **22** (1977), 73–78.
2. Homotopical properties of partial isometries of the Calkin algebra, *Math. Jap.*, **22** (1977), 83–88.

KAMINKER, J.; SCHOCHET, C.
1. Ext (X) from a homological point of view, *Lecture Notes in Math.*, No. 345, 1973, 129–142.
2. Steenrod homology and operator algebras, *Bull. AMS*, **81** (1975), 431–434.
3. K-theory and Steenrod homology: Applications to the Brown-Douglas-Fillmore theory of operator algebras, *Trans. AMS*, **227** (1977), 63–108.
4. Analytic equivariant K-homology.

KAMOWITZ, H.; SCEINBERG, S.
1. The spectrum of automorphisms of Banach algebras, *J. Fnal Analysis*, **4** (1969), 268–276.

KANDELAKI, T. K. (in Russian)
1. Vector bundles and K-theory over commutative unitary C^*-algebras, *Bull. Akad. Sc. Georgian SSR*, **82**: 1 (1976), 27–30.
2. On the equivalence of the categories of vector bundles and projection modules of finite type over a Banach algebra, *Bull. Acad. Sc. Georgian SSR*, **83** (1976), 35–37.

KANIUTH, E.
1. Der Type der regulären Darstellungen diskreter Gruppen, *Math. Ann.*, **182** (1969), 334–339.
2. Die Struktur der regulären Darstellung lokalkompakter Gruppen mit invarianter Umgebungsbasis der Eins.

KANNAN, D.
1. Expectations in normed $*$-algebras, *J. Math. Anal. Appl.*, **41** (1973), 14–19.

KAPLAN, S.
1. The second dual of the space of continuous functions, I, II, III, IV, *Trans. AMS*, **86** (1957), 70–90; **93** (1959), 329–350; **101** (1961), 34–51; **113** (1964), 512–546.

KAPLANSKY, I.
1. Topological rings, *Amer. J. Math.*, **69** (1947), 153–193.
2. Dual rings, *Ann. Math.*, **49** (1948), 689–701.
3. Rings with a polynomial identity, *Bull. AMS*, **54** (1948), 575–580.
4. Regular Banach algebras, *J. Indian Math. Soc.*, **12** (1948), 57–62.
5. Locally compact rings, I, II, *Amer. J. Math.*, **70** (1948), 447–459; **73** (1951), 20–24.
6. Normed algebras, *Duke Math. J.*, **16** (1949), 399–418.
7. Groups with representations of bounded degree, *Can. J. Math.*, **1** (1949), 105–112.
8. Primary ideals in group algebras, *Proc. Nat. Acad. Sci. USA*, **35** (1949), 133–136.
9. Forms in infinite dimensional spaces, *An. Acad. Bras. Ci.*, **22** (1950), 1–17.
10. Quelques résultats sur les anneaux d'opérateurs, *CR Acad. Sc. Paris*, **231** (1950), 485–486.
11. A theorem on rings of operators, *Pacific J. Math.*, **1** (1951), 227–232.

12. The structure of certain operator algebras, *Trans. AMS*, **70** (1951), 219–255.
13. Projections in Banach algebras, *Ann. Math.*, **53** (1951), 235–249.
14. Group algebras in the large, *Tohoku Math. J.*, **3** (1951), 249–256.
15. Representations of separable algebras, *Duke Math. J.*, **19** (1952), 219–222.
16. Symmetry in Banach algebras, *Proc. AMS*, **3** (1952), 336–339.
17. Algebras of type I, *Ann. Math.*, **56** (1952), 460–472.
18. Modules over operator algebras, *Amer. J. Math.*, **75** (1953), 839–853.
19. Products of normal operators, *Duke Math. J.*, **20** (1953), 257–260.
20. Ring isomorphisms of Banach algebras, *Can. J. Math.*, **6** (1954), 374–381.
21. Any orthocomplemented complete modular lattice is a continuous geometry, *Ann. Math.*, **61** (1955), 524–541.
22. *Rings of operators*, Lecture Notes, Chicago Univ., 1955; Benjamin, 1968.
23. *Functional Analysis. Some aspects of analysis and probability*, J. Wiley & Sons, 1958, 1–34.
24. Von Neumann characterization of factors of type II_1, published in J. von Neumann [18], 1961, 562–563.
25. A survey of operator algebras, Queen's Univ., 1969.
26. *Algebraic and analytic aspects of operator algebras*, Publ. AMS, 1970.
27. *Lie algebras and locally compact groups*, The University of Chicago Press, 1971.
28. Derivations of Banach algebras, *Seminar on analytic functions*, Sept. 2–14, 1957, Inst. for Adv. Study, Princeton, vol. II, 254–258.
29. *Bialgebras*, Lecture Notes in Math., Dept. Math. Univ. Chicago, 1975.

KAPLANSKY, I. et al.

1. Operator theory and group representations, *Publ. AMS*, **387** (1955).

KAROUBI, M.

1. Matrices de Jacobi, périodicité de Bott et C^*-algèbres, *CR Acad. Sc. Paris*, **268** (1969), 1091–1094.

KASAHARA, Y., TAKAI, H.

1. Approximate proper values and characters of C^*-algebras, *Proc. Jap. Acad.*, **48** (1972), 91–93.

KASTLER, D.

1. A C^*-algebra approach to field theory, *Proc. Conf. Th. Appl. Analysis*, MIT Press, 1964.
2. The C^*-algebra of a free Boson field, I, *Comm. Math. Phys.*, **1** (1965), 14–48.
3. Topics in the algebraic approach to field theory, *Cargese Lecture in Phys.*, **3** (1967), 289–302.
4. Broken symmetries and the Goldstone theorem in axiomatic field theory, *Conf. on particles and fields, Inst. Theor. Phys.*, Rochester, 1967.
5. B_1-equivalence and B_1-decompositions of states or representations of a C^*-algebra. Applications of groups of automorphisms and local or asymptotic properties with respect to local subalgebras funnels and locally normal states, *Systèmes à un nombre infini de degrés de liberté*, Gif-sur-Yvette, Coll. Inst. CNRS, **181** (1970), 21–31.
6. Stability and equilibrium in quantum statistical mechanics, preprint, Marseille, 1974.
7. Equilibrium states of the matter and operator algebras, in *Symposia Math.*, XX, 49–108, Academic Press, 1976.
8. Fondements de la mécanique statistique de l'équilibre, Strasbourg, 1976.

KASTLER, D.; MEBKHOUTH, M.; LOUPIAS, G.; MICHEL, L.

1. Central decomposition of invariant states. Applications to the groups of time translations and of Euclidian transformations in algebraic field theory, *Comm. Math. Phys.*, **27** (1972), 195–222.

KASTLER, D.; POOL, J. C. T.; THUE-POULSEN, E.

1. Quasi-unitary algebras attached to temperature states in statistical mechanics; a comment on the work of Haag, Hugenholtz and Winnink, *Comm. Math. Phys.*, **12** (1969), 175–192.

KASTLER, D.; ROBINSON, D. W.

1. Invariant states in statistical mechanics, *Comm. Math. Phys.*, **3** (1966), 151–180.

KASTLER, D.; ROBINSON, D. W.; SWIECA, J. A.
1. Conserved currents and associated symmetries; Goldstone theorem, *Comm. Math. Phys.*, 2 (1966), 108–120.

KATAVOLOS, A.
1. Isometries of non-commutative L_p spaces, *Canad. J. Math.*, 28 (1976), 1180–1186.
2. Are non-commutative L_p-spaces really non-commutative?

KATAVOLOS, A.; KOCH, I.
1. Gibbs states and the Tomita-Takesaki theory on the algebra of the canonical commutation relations.

KATAYAMA, Y.
1. The tensor product of weights, *Proc. Jap. Acad.*, 50 (1974), 430–432.
2. Non-existence of a normal conditional expectation in a continuous crossed product, *Japan-US Seminary*, 1977.
3. Isomorphisms of the Fourier algebras in crossed products.

KATO, Y.
1. A proof of Olsen's theorem, *Math. Jap.*, 21 (1976), 127–128.

KATO, Y.; MAEDA, S.
1. A remark on a theorem of Szymański, *Math. Jap.*, 20 (1975), 89–94.

KATO, Y.; NAKAMURA, M.
1. A characterization of finiteness of von Neumann algebras, *Math. Jap.*, 22 (1977), 69–71.

KATZ, G. I. (in Russian)
1. Group rings and the duality principle, I, II, *Trudy Mosk. Mat. Obšč.*, 12 (1963), 259–301; 13 (1964), 84–113.
2. Group extensions which are group rings, *Matem. Sb.*, 76 (1968), 473–496.
3. Generalization of group duality principle, *Dokl. Akad. Nauk, SSSR*, 138 : 2 (1961), 275–278.
4. Finite ring groups, *Dokl. Akad. Nauk, SSSR*, 147 : 1 (1962), 21–24.
5. Compact and discrete ring groups, *Ukrainsk. Mat. J.*, 14 : 3 (1962), 260–269.
6. Representations of compact ring groups, *Dokl. Akad. Nauk, SSSR*, 145 : 5 (1962), 989–992.
7. Some arithmetical properties of group rings, *Funkt. Analiz. i pril.*, 6 : 2 (1972), 88–90.

KATZ, G. I.; PALIUTKIN, V. G.
1. Finite group rings (Russian), *Trudy Mosk. Mat. Obšč.*, 15 (1966), 224–262.
2. Example of ring groups generated by Lie groups (Russian), *Ukrainsk. Mat. J.*, 16: 1 (1964), 99–105.

KAUP, W.; UPMEIER, H.
1. Jordan algebras and symmetric Siegel domains in Banach spaces, *Math. Z.*, 157 (1977), 179–200.

KAWAMURA, S.
1. A note on the spectra of weakly continuous representations of a locally compact abelian group on von Neumann algebras, *Bull. Yamagata Univ.*, 9 (1976/77), 205–210.

KAWAMURA, S.; TOMIYAMA, J.
1. On subdiagonal algebras associated with flows in operator algebras, *J. Math. Soc. Japan*, 29 (1977), 73–90.

KAZDAN, D. A.
1. On the relations between the dual space of a group and the structure of its closed subgroups (Russian), *Funkt. Analiz i pril.*, 1 (1967), 71–74.

KEHLET, E. T.
1. On the monotone sequential closure of a C^*-algebra, *Math. Scand.*, 25 (1969), 59–70.
2. A proof of the Mackey-Blattner-Nielsen theorem.
3. Disintegration theory on a constant field of non-separable Hilbert spaces

4. A non separable measurable choice principle related to induced representations, *Math. Scand.*, **42** (1978).

KELLEY, J. L.
1. Commutative operator algebras, *Proc. Nat. Acad. Sci. USA*, **38** (1952), 598–605.
2. Duality for compact groups, *Proc. Nat. Acad. Sci. USA*, **49** (1963), 457–458.
3. Measures in Boolean algebras, *Pacific J. Math.*, **9** (1959), 1165–1177.

KELLEY, J. L.; VAUGHT, R. L.
1. The positive cone in Banach algebras, *Trans. AMS*, **74** (1953), 44–55.

KEOWN, R.
1. Some new Hilbert algebras, *Trans. AMS*, **128** (1967), 71–87.

KERVIN, M. R.
1. The trace class of a full Hilbert algebra, *Trans. AMS*, **178** (1973), 259–270.

KIRCHBERG, E.
1. C^*-nuclearity implies CPAP, *Math. Nachr.*, **76** (1977), 203–212.
2. On the KMS condition for weights on C^*-algebras.
3. Representations of coinvolutive Hopf-W^*-algebras and non abelian duality, *Bull. Acad. Pol. Sc.*, **25** (1977), 117–122.
4. Über Darstellungen von Hopf-W^*-Algebren.
5. Ein Radon-Nikodym Satz für Gewichte auf W^*-Algebren.
6. Hopf-W^*-Algebren mit linksinvarianten Gewicht.
7. Die nicht-kommutative Dualitätstheorie als Funktor.

KIRILLOV, A. A. (in Russian)
1. Unitary representations of nilpotent Lie groups, *Uspekhi Mat. Nauk*, **17** (1962), 57–110.
2. Dynamical systems, factors and group representations, *Uspekhi Mat. Nauk*, **22** (1967), 67–80.
3. The characters of unitary representations of Lie groups, *Funkt. Analiz i pril.*, **2** : 2 (1968), 40–55.
4. *Elements of the theory of representations*, Moscow, 1972.
5. The representations of infinite dimmensional unitary group, *Dokl. Akad. Nauk, SSSR*, **212** (1973), 288–290.

KISHIMOTO, A.
1. Dissipations and derivations, *Comm. Math. Phys.*, **47** (1976), 25–32.
2. Some remarks on Fröhlich's condition in $P(\Phi)_2$ euclidian field theory, *Comm. Math. Phys.*, **47** (1976), 117–129.
3. On uniqueness of KMS states of one-dimensional quantum lattice systems, *Comm. Math. Phys.*, **47** (1976), 167–170.
4. On invariant states and the commutant of a group of quasi-free automorphisms of the CAR algebra, I, II, III.
5. On the fixed point algebra of a UHF algebra under a periodic automorphism of product type, *Publ. RIMS, Kyoto Univ.*, **13** (1977), 777–791.
6. Remarks on compact automorphism groups of a certain von Neumann algebra.

KISHIMOTO, A.; TAKAI, H.
1. On the invariant $\Gamma(\alpha)$ in C^*-dynamical systems, *Tohoku Math. J.*, **30** (1978), 83–94.
2. Some topics in C^*-dynamical systems based on a compact abelian group.
3. Some properties of C^*-crossed products by a compact abelian group and their applications.

KISSIN, E. V.
1. C^*-algebras generated by dynamical systems and by weighted shifts (Russian), *Dokl. Akad. Nauk, SSSR*, **216** (1974), 1215–1218; **219** (1974), 1061–1064.

KITCHEN, J. W.; ROBBINS, D. A.
1. Gelfand representation of Banach modules.

KLEIN, A.; RUSSO, B.
1. Sharp inequalities for Weyl operators and Heisenberg groups, Irwine, 1977.

KLEPPNER, A.
1. Continuity and measurability of multipliers and projective representations, *J. Fnal Analysis*, **17** (1974), 214–226.

KLEPPNER, A.; LIPSMAN, R. L.
1. The Plancherel formula for group extensions, I, II, *Ann. Ec. Norm. Sup.*, **5** (1972), 459–516; **6** (1973), 103–132.

KNOPS, H. J. F.
1. *Ergodic states and symmetry breaking in phase transition. A C^*-algebraic approach*. Thesis, Nijmegen, 1969.

KONDO, M.
1. Sur la notion de dimension, *Proc. Imp. Acad. Tokyo*, **19** (1943), 215–223.
2. Les anneaux d'opérateurs et les dimensions, I, II, *Proc. Imp. Acad. Tokyo*, **20** (1944), 389–398; 689–693.
3. Sur les sommes directes des espaces linéaires, *Proc. Imp. Acad. Tokyo*, **20** (1944), 425–431.
4. Sur la réductibilité des anneaux d'opérateurs, *Proc. Imp. Acad. Tokyo*, **20** (1944), 432–438.
5. Le produit kroneckerien infini des espaces linéaires, *Proc. Imp. Acad. Tokyo*, **20** (1944), 569–579.
6. Les anneaux d'opérateurs sur un espace de S. Banach et quelques problèmes qui s'y rattachent, I, *J. Math. Tokyo*, **1** (1951), 35–54.

DE KORVIN, A.
1. Expectations in von Neumann algebras, *Bull. AMS*, **74** (1968), 912–914.
2. Normal expectations in von Neumann algebras, *Pacific J. Math.*, **27** (1968), 333–338.
3. Stable maps and Schwartz maps, *Trans. AMS*, **148** (1970), 283–291.
4. Complete sets of expectations on von Neumann algebras, *Quart. J. Math.*, **22** (1971), 135–142.

DE KORVIN, A.; EASTON, R. J.
1. Expectations on B^*-algebras, *J. Reine Angew. Math.*, **251** (1971), 1–6.

KOVÁCS, I.
1. Un complément à la théorie de l'intégration non-commutative, *Acta Sci. Math.*, **21** (1960), 7–11.
2. Théorèmes ergodiques non-commutatifs, *CR Acad. Sc. Paris*, **253** (1961), 770–771.
3. Sur certains automorphismes des algèbres hilbertiennes, *Acta Sci. Math.*, **22** (1961), 234–242.
4. Ergodic theorems for gages, *Acta Sci. Math.*, **24** (1963), 103–118.

KOVÁCS, I.; MOCANU, GH.
1. Unitary dilations and C^*-algebras, *Acta Sci. Math.*, **38** (1976), 79–82.

KOVÁCS, I.; SZÜCS, J.
1. Théorèmes de type ergodique dans les algèbres de von Neumann, *CR Acad. Sc. Paris*, **262** (1966), 341–344.
2. Ergodic type theorems in von Neumann algebras, *Acta Sci. Math.*, **27** (1966), 233–246.
3. A note on invariant linear forms on von Neumann algebras, *Acta Sci. Math.*, **30** (1969), 35–37.

KRALJEVIČ, H.
1. Induced representations of locally compact groups on Banach spaces, *Glasnik Math.*, **4** (1969), 183–196.

KRALJEVIČ, H.; MILICIČ, D.
1. The C^*-algebra of the universal covering group of SL (2, R), *Glasnik Math.*, **7** (1972), 35–48.

KRAUS, J. E.
1. Compact abelian groups of automorphisms of von Neumann algebras, SUNY, Buffalo, 1978.

KRAUS, K.

1. An algebraic spectrum condition, *Comm. Math. Phys.*, **16** (1970), 138–141.
2. Algebras of observables associated with continuous representations of symmetry groups, *Comm. Math. Phys.*, **7** (1968), 99–111.
3. Operations and effects in the Hilbert space formulation of quantum theory, *Lecture Notes in Phys.*, No. 29, 1974, 206–229.

KRAUSS, F.

1. *Structure theory of C*-algebras*, Thesis, Tulane Univ., 1973.

KRAUSS, F.; LAWSON, T. C.

1. Examples of homogeneous C^*-algebras, *Memoirs AMS*, **148** (1974), 153–164.

KREĬN, M. G. (in Russian)

1. A duality principle for bicompact groups, *Dokl. Akad. Nauk, SSSR*, **69** (1949), 725–728.
2. The theory of selfadjoint extensions of semibounded hermitian operators, I, II, *Matem. Sb.*, **20** (1947), 431–498; **21** (1947), 366–404.

KRIEGER, W.

1. On the isomorphy problem for ergodic equivalence relations, *Math. Z.*, **103** (1968), 78–84.
2. On non-singular transformations of a measure space, I, II, *Z. Wahr. verw. Geb.*, **11** (1969), 83–97; 98–119.
3. On constructing non-isomorphic hyperfinite factors of type III, *J. Fnal Analysis*, **6** (1970), 97–109.
4. On the Araki-Woods asymptotic ratio set and non-singular transformations of a measure space, *Lecture Notes in Math.*, No. 160, 1970, 158–177.
5. On entropy and generators of measure-preserving transformations, *Trans. AMS*, **149** (1970), 453–464.
6. On a class of hyperfinite factors which arise from null-recurrent Markov chains, *J. Fnal Analysis*, **7** (1971), 27–42.
7. On the hyperfinite factors and non-singular transformations of a measure space.
8. On non singular transformations that arise from Kolmogoroff systems.
9. On the infinite product construction of non-singular transformations of a measure space, *Invent. Math.*, **15** (1972), 144–163 (Erratum, ibidem, **26** (1974), 323–328).
10. On ergodic flows and the isomorphism of factors, *Math. Ann.*, **223** (1976), 19–70.
11. On the Dye weak equivalence theorem, *Proc. Int. School Math. Phys., Univ. Camerino*, 1974, 355–365.
12. On quasi-invariant measures with uniquely ergodic systems, *Invent. Math.*, **14** (1971), 184–196.
13. On the entropy of groups of measure preserving transformations.
14. Automorphisms of measure spaces and automorphisms of von Neumann algebras, *Convegno sulle algebre C* e loro applicazioni in fisica teoretica*, Roma, March, 1975.
15. On the uniqueness of the equilibrium state, *Math. System Theory*, **8** (1975), 97–104.
16. On Borel automorphisms and their quasi-invariant measures, *Math. Z.*, **15** (1976), 19–24.
17. On the construction of factors from ergodic non-singular transformations, in *C*-algebras and their applications to statistical mechanics and quantum field theory*, p. 114–120, North-Holland, 1976.
18. On a dimension for a class of homeomorphism groups.

KRÖGER, P.

1. Unbounded operator algebras with C*-subalgebras, Leipzig, 1977.

KRUSZYNSKI, P.

1. Probability measures on operator algebras, *Rep. Math. Phys.*, **7** (1975), 395–401.
2. Automorphisms of quantum logics, *Rep. Math. Phys.*, **10** (1976), 213–217.
3. On the existence of KMS states for invariantly approximately inner dynamics, *Bull. Acad. Polon. Sc.*, **24** (1976), 299–301.
4. On positive maps on C^*-algebras.

KRUSZYNSKI, P.; NAPIÓRKOWSKI, K.

1. On the independence of local algebras, II, *Rep. Math. Phys.*, **4** (1973), 303–306.

KUBO, F.

1. On decompositions of linear mappings among operator algebras, *Proc. Jap. Acad.*, **51** (1975), 113–116.
2. On theorems of Phelps, Russo and Dye, *Math. Jap.*, **20** (1975), 69–72.

KUBO, F.; KUBO, K.

1. Extreme points of the unit ball in an operator algebra.
2. Some extremal properties in the convex sets in operator algebras.

KUBO, F.; TAMAKI, K.

1. Contractive mappings and projections among operator algebras, *Math. Jap.*, **19** (1974), 247–250.

KUIPER, N.

1. The homotopy type of the unitary group of Hilbert space, *Topology*, **3** (1965), 19–30.

KUNZE, R. A.

1. L^p-Fourier transforms on locally compact unimodular groups, *Trans. AMS*, **89** (1958), 519–540.
2. A note on square integrable representations, *J. Fnal Analysis*, **6** (1970), 454–459.
3. Positive definite operator valued kernels and unitary representations, *Proc. Conf. Funct. Analysis*, Irwine, 1966, 235–247.
4. On the Frobenius reciprocity theorem for square integrable representations, *Pacific J. Math.*, **53** (1974), 465–471.

KUNZE, R. A.; STEIN, E. M.

1. Uniformly bounded representations, I, II, III, IV, *Amer. J. Math.*, **82** (1960), 1–62; **83** (1961), 723–786; **89** (1967), 385–442; *Adv. Math.*, **11** (1973), 1–71.

KURANISHI, M.

1. On the non connected maximally almost periodic groups, *Tohoku Math. J.*, **2** (1950), 40–46.

KYLE, J.

1. *Norms, spectra and numerical ranges of derivations*, Thesis, Newcastle upon Tyne, 1976.
2. Norms of derivations, *J. London Math. Soc.*, **16** (1977), 297–312.
3. Spectra of derivations, *Math. Proc. Cambridge Phil. Soc.*, **82** (1977), 49–57.
4. $W_\delta(T)$ is convex, *Pacific J. Math.*, **72** (1977), 483–486.
5. Ranges of Lyapunov transformations in Hilbert spaces, *Glasgow Math. J.*, **19** (1978), 99–101.

KURIYAMA, K.

1. On left Hilbert algebras with respect to Minkowski forms, *Mem. Fac. Sci. Kyushu Univ.*, **30**:1 (1976).
2. Remarks on left Hilbert algebras with respect to Minkowski forms, *Mem. Fac. Sci. Kyushu Univ.*, **31** (1976), 173–176.
3. Crossed products of left Hilbert algebras with respect to Minkowski forms, *Mem. Fac. Sci. Kyushu Univ.*, **31** (1976), 267–284.

KWAK, H. C.

1. A note on C^*-algebras, *J. Korean Math. Soc.*, **12** (1975), 5–11.

LAISON, D.

1. On W^*-embedding of AW^*-algebras, *Proc. AMS*, **35** (1972), 499–502.

LAISON, D.; LAISON, G.

1. Topological dynamics on C^*-algebras, *Trans. AMS*, **204** (1975), 197–205.
2. Weak almost periodicity on C^*-algebras, *Math. Ann.*, **227** (1977), 135–144.

LAISON, G.

1. A semigroup associated with an invariant measure on a transformation group, *Math. System Theory*, **8** (1974), 276–288.

LANCE, E. C.

1. Automorphisms of postliminal C^*-algebras, *Pacific J. Math.*, **23** (1967), 547–555.

2. Inner automorphisms of UHF algebras, *J. London Math. Soc.*, **43** (1968), 681—688.
3. Some properties of nest algebras, *Proc. London Math. Soc.*, **19** (1969), 45—68.
4. Automorphisms of certain operator algebras, *Amer. J. Math.*, **91** (1969), 160—174.
5. On nuclear C^*-algebras, *J. Fnal Analysis*, **12** (1973), 157—176.
6. Quadratic forms on Banach space, *Proc. London Math. Soc.*, **25** (1972), 341—357.
7. Direct integrals of left Hilbert algebras, *Math. Ann.*, **216** (1975), 11—28.
8. Non commutative ergodic theorems, in *Symposia Math.*, XX, 39—48, Academic Press, 1975.
9. *Notes on the Glimm-Sakai theorem, Lecture Notes*, Newcastle upon Tyne, July 1971.
10. Refinement of direct integral decompositions, *Bull. London Math. Soc.*, **8** (1976), 49—56.
11. Tensor products of C^*-algebras, in *"C^*-algebras and their applications to statistical mechanics and quantum field theory"*, p. 154—166, North-Holland, 1976.
12. Tensor products of non-unital C^*-algebras, *J. London Math. Soc.*, **12** (1976), 160—168.
13. Ergodic theorems for convex sets and operators algebras, *Invent. Math.*, **37** (1976), 201—214.
14. A strong non commutative ergodic theorem, *Bull. AMS.*, **82** (1976), 925—926.
15. Almost uniform convergence in operator algebras.
16. Martingale convergence in operator algebras.

LANCE, E. C.; NIKNAM, A.
1. Unbounded derivations of group C^*-algebras, *Proc. AMS*, **61** (1976), 310—314.

LANDSTAD, M.
1. Duality theory for covariant systems, Trondheim, 1974; *Trans. AMS.*
2. Covariant systems over a compact group.
3. Duality for dual covariant algebras, *Comm. Math. Phys.*, **52** (1977), 191—202.

LANDSTAD, M.; OLESEN, D.; PEDERSEN, G. K.
1. Towards a Galois theory for crossed products of C^*-algebras, Copenhagen, 1978.

LANFORD III, O.
1. Quantum spin systems, *Cargese Lectures in Phys.*, **4** (1970), 454—459.

LANFORD III, O.; ROBINSON, D. W.
1. Statistical mechanics of quantum spin systems, III, *Comm. Math. Phys.*, **9** (1968), 327—338.
2. Mean entropy of states in quantum statistical mechanics, *J. Math. Phys.*, **9** (1968), 1120—1125.

LANFORD III, O.; RUELLE, D.
1. Integral representations of invariant states on B^*-algebras, *J. Math. Phys.*, **8** (1967), 1460—1463.
2. Observables at infinity and states with short range correlations in statistical mechanics, *Comm. Math. Phys.*, **13** (1969), 194—215.

LANGERHOLC, J.; SCHRÖER, B.
1. On the structure of the von Neumann algebras generated by local functions of the free Bose field, *Comm. Math. Phys.*, **1** (1965), 215—239.

LASRY, J.-M.
1. Un théorème de convexité dans les algèbres de von Neumann, *CR Acad. Sc. Paris*, **281** (1975), 153—154.

LASSNER, G.
1. Topological algebras of operators, *Rep. Math. Phys.*, **3** (1972), 279—293.
2. O^* topologies on the test function algebras, *Publ. Dept. Math., Lyon*, **12**: 1 (1975), 25—38.
3. On the structure of the test function algebra, *Rep. Math. Phys.*
4. The β-topology on operator algebras, Leipzig, 1977.
5. (Herausgegeben von G. Lassner) *Beiträge zur Theorie nicht-normierbarer topologischer Algebren*, Leipzig, 1975.
6. Topologische Operatorenalgebren und Darstellungen topologischer Algebren, *Beiträge Anal.*, **8** (1976), 85—94.

ASSNER, C.; LASSNER, G. A.
1. Completely positive mappings on operators, Leipzig, 1975.

2. Completely positive mappings and unbounded observables, *Rep. Math. Phys.*, **11** (1977), 133–140.
3. On the continuity of the entropy.

LASSNER, G.; TIMMERMANN, W.
1. Normal states on algebras of unbounded operators, *Rep. Math. Phys.*, **3** (1972), 295–305.
2. Classification of domains of closed operators, *Rep. Math. Phys.*, **9** (1976), 157–170.
3. Classification of domains of operator algebras, *Rep. Math. Phys.*, **9** (1976), 205–217.
4. The strong topology on the algebra of polynomials, *Rep. Math. Phys.*, **11** (1977), 81–87.

LASSNER, G.; UHLMANN, A.
1. On positive functionals on algebras of test functions for quantum fields, *Comm. Math. Phys.*, **7** (1968), no. 2.

LAU, A.
1. W^*-algebras and invariant functionals, *Studia Math.*, **56** (1976), 55–63.
2. Some fixed point theorems and their applications to W^*-algebras, *Proc. Sem. Dalhousie Univ., Halifax, N. S.*, 1975, Academic Press, 1976, p. 121–129.
3. Characterization of amenable Banach algebras.

LAURSEN, K. B.
1. Tensor products of Banach algebras with involution, *Trans. AMS*, **136** (1969), 467–489.
2. A note on lifting of matrix units in C^*-algebras, *Math. Scand.*, **33** (1973), 338–342.
3. Continuity of linear maps on C^*-algebras, *Pacific J. Math.*, **61** (1976), 483–491.
4. Some remarks on automatic continuity, *Lecture Notes in Math.*, **512** (1976), 96–108.
5. Automatic continuity of generalized intertwining operators, *Diss. Math.*

LAURSEN, K. B.; SINCLAIR, A. M.
1. Lifting matrix units in C^*-algebras, II, *Math. Scand.*, **37** (1975), 167–172.

LAURSEN, K. B.; STEIN, J. D.
1. Automatic continuity in Banach spaces and algebras, *Amer. J. Math.*, **95** (1973), 495–506.

LAZAR, A. J.; TAYLOR, D. C.
1. Double centralizers of Pedersen's ideal of a C^*-algebra, I, II, *Bull. AMS*, **78** (1972), 992–997; **79** (1973), 361–366.
2. Multipliers of Pedersen's ideal, *Memoirs AMS*, **169** (1976).
3. A Dauns-Hofmann theorem for $\Gamma(K)$, *Memoirs AMS*, **148** (1974), 135–144.

LEBOW, A.
1. A Schröder-Bernstein theorem for projections, *Proc. AMS*, **19** (1968), 144–145.
2. Spatial homomorphisms of operator algebras, *Indiana Univ. Math. J.*, **24** (1975), 865–874.

LEE, RU-YING
1. On the C^*-algebras of operator fields, *Indiana Univ. Math. J.*, **25** (1976), 303–314.
2. Full algebras of operators fields trivial except at one point, *Indiana Univ. Math. J.*, **26** (1977), 351–372.

LEGISA, P.
1. Well embedded Hilbert subspaces in C^*-algebras, II, *Glasnik Mat.*, **11** (1976), 53–57.
2. Hilbert subspaces in the C^*-algebra $B(H)$, *Glasnik Math.*, **11** (1976), 59–62.

LENNON, M. J. J.
1. Direct integrals of locally measurable operator, *Math. Scand.*, **32** (1973), 123–132.
2. On sums and products of unbounded operators in Hilbert space, *Trans. AMS*, **198** (1974), 273–285.

LEPTIN, H.
1. Reduktion linearer Funktionale auf Operatorringen, *Abstr. Math. Sem.*, Univ. Hamburg, **22** (1958), 98–113.
2. Zur Reduktionstheorie Hilbertscher Raüme, *Math. Z.*, **69** (1958), 40–58.
3. Ideale endlicher Codimension in L^1-Algebren, *Abhand. Math. Sem. Univ. Hamburg*, **28** (1965), 215–218.
4. On a certain invariant of a locally compact group, *Bull. AMS*, **72** (1966), 870–874.

5. Verallgemeinerte L^1-Algebren un projektive Darstellungen lokalkompakter Gruppen, I, II, *Invent. Math.*, **3** (1967), 257–281; **4** (1967), 68–86.
6. Sur l'algèbre de Fourier d'un groupe localement compact, *CR Acad. Sc. Paris*, **266** (1968), 1180–1182.
7. Darstellungen verallgemeinerter L^1-Algebren, I, II, *Invent. Math.*, **5** (1968), 192–215, *Lecture Notes in Math.*, No. 247 (1972), 251–307.
8. Zur harmonischen Analyse klassenkompakter Gruppen, *Invent. Math.*, **5** (1968), 249–254.
9. A separable postliminal C^*-algebra without maximal closed ideals, *Trans. AMS*, **159** (1971), 489–496.
10. Harmonische Analyse auf gewissen nilpotenten Lieschen Gruppen, *Studia Math.*, **48** (1973), 201–205.
11. On group algebras of nilpotent Lie groups, *Studia Math.*, **47** (1973), 37–49.
12. On symmetry of some Banach algebras, *Pacific J. Math.*, **53** (1974), 203–206.
13. Symmetrie in Banachschen Algebren, *Arch. Math.*, **27** (1976), 394–400.
14. Ideal theory in group algebras of locally compact groups, *Invent. Math.*, **31** (1976), 259–278.

LERESCHE, G.
1. Algèbres d'opérateurs non bornés sur un espace de Hilbert, *Comment. Math. Helv.*, **40** (1966), 281–324.

LE RICHE, L. R.
1. Pure states and non-homogeneity, *J. London Math. Soc.*, **13** (1976), 138–144.

LI, BING-REN
1. Real C^*-algebras (Chinese), *Acta Math. Sinica*, **13**: 3 (1975), 216–218.

LIEB, E. H.
1. Convex trace functions and the Wigner-Yanase-Dyson conjecture, *Adv. Math.*, **11** (1973), 267–288.
2. Some convexity and subadditivity properties of entropy, *Bull. AMS*, **81** (1975), 1–13.
3. Inequalities for some operator and matrix functions, *Adv. Math.*

LIEB, E. H.; RUSKAI, M. B.
1. Proof of the strong subadditivity of quantum-mechanical entropy, *J. Math. Phys.*, **14** (1973), 1938–1941.
2. Some operator inequalities of the Schwartz type, *Adv. Math.*, **12** (1974), 269–273.
3. A fundamental property of quantum mechanical entropy, *Phys. Rev. Letter*, **30** (1973), 434–436.

LIEBERMANN, A.
1. Measurable vectors for von Neumann algebras, *J. Fnal Analysis*, **18** (1975), 191–212.
2. The structure of certain unitary representations of infinite symmetric groups, *Trans. AMS*, **164** (1972), 189–198.
3. Some representations of the automorphism group of an infinite continuous homogeneous measure algebra, *Pacific J. Math.*, **44** (1973), 607–612.
4. Continuous representations of infinite symmetric groups on reflexive Banach spaces, *Illinois J. Math.*, **17** (1973), 450–457.
5. *The structure of certain unitary representations of infinite symmetric groups*, Thesis, MIT, 1970.
6. Special distribution of the sum of self-adjoint operators, *Pacific J. Math.*, **53** (1974), 211–216.
7. Entropy of states of a gage space, *Acta Sci. Math.*, **40** (1978), 99–106.
8. Adjoint representations of factor groups, *Michigan Math. J.*, **24** (1977), 109–113.

LIND, D. A.
1. Locally compact measure preserving flows, *Adv. Math.*, **15** (1975), 175–193.

LINDBLAD, G.
1. Entropy information and quantum measurement, *Comm. Math. Phys.*, **33** (1973), 305–322.
2. Expectations and entropy inequalities for finite quantum systems, *Comm. Math. Phys.*, **39** (1974), 111–119.

3. Completely positive maps and entropy inequalities, *Comm. Math. Phys.*, 40 (1975), 147–152.
4. On the generators of quantum dynamical semigroups, *Comm. Math. Phys.*, 48 (1976), 119–130.
5. Dissipative operators and cohomology of operator algebras, *Lett. Math. Phys.*, 1 (1975/76), 219–224.
6. Brownian motion of a quantum harmonic oscillator, *Rep. Math. Phys.*, 10 (1976), 393–406.
7. Non-markovian quantum stochastic processes and their entropy, Stockholm, 1977.

LIPSMAN, R.
1. Non-abelian Fourier analysis, *Bull. Sc. Math.*, 98 (1974), 209–233.
2. *Group representations. A survey of some current topics*, Lecture Notes in Math., No. 388, Springer Verlag, 1974.
3. Representation theory for almost connected groups, *Pacific J. Math.*, 42 (1972), 453–467.
4. The CCR property for algebraic groups, *Amer. J. Math.*, 97 (1975), 741–752.
5. Harmonic analysis on SL (n, R), *J. Fnal Analysis*, 3 (1969), 126–155.

LIU, T. S.; VAN ROOIJ, A.
1. Invariant means on a locally compact group, *Monatsch. Math.*, 78 (1974), 356–359.

LIUBIČ, IU. I.
1. On the spectrum of a representation of an abelian topological group (Russian), *Dokl. Akad. Nauk*, SSSR, 200: 4 (1971), 777–780.

LODKIN, A. A.
1. Any measure on the projections of a W^*-algebra extends to a state (Russian), *Funkt. Analiz i pril.*, 8: 4 (1974), 54–58.
2. A lemma about the approximation of finite dimensional $*$-algebras (Russian), *Zap. Nauk Sem. Leningradsk. otdel. Mat. Inst. Steklov., AN SSSR*, 47 (1974), 175–178, 194.

LOEBL, R. I.
1. Injective von Neumann algebras, *Proc. AMS*, 44 (1974), 46–48.
2. Contractive linear maps on C^*-algebras, *Michigan Math. J.*, 22 (1975), 361–366.
3. *Flows on C^*-algebras*, Thesis, Berkeley, 1973.
4. A Hahn decomposition for linear maps, *Pacific J. Math.*, 65 (1976), 119–134.
5. On the range of completely bounded maps.

LOEBL, R. I.; MUHLY, P. S.
1. Reductive algebras and automorphisms groups of von Neumann algebras, *Bull. AMS*, 81 (1975), 759–760.
2. Analyticity and flows on von Neumann algebras, *J. Fnal Analysis*.

LOEBL, R. I.; SCHOCHET, C.
1. Covariant representations of the Calkin algebra, I.

LOGINOV, A. I.; SULMAN, V. S.
1. The hereditary and intermediate reflexivity for W^*-algebras (Russian), *Dokl. Akad. Nauk, SSSR*, 212 (1973), 810–812; *Izv. Akad. Nauk, SSSR*, 39: 6 (1975), 1260–1273.
2. On Sarason's theorem and the Radjavi-Rosenthal hypothesis (Russian), *Dokl. Akad. Nauk, SSSR*, 205 (1972), 284–285.

LOMONOSOV, V. I.
1. Invariant subspaces for operators commuting with compact operators (Russian), *Funkt. Analiz i pril.*, 7: 3 (1973), 55–56.

LONGO, R.
1. On perturbed derivations of C^*-algebras, *Rep. Math. Phys.*, 12 (1977), 119–124.
2. A simple proof of the existence of modular automorphisms in approximately finite dimensional von Neumann algebras, *Pacific J. Math.*
3. Automatic relative boundedness of derivations in C^*-algebras.
4. Some aspects of C^*-dynamics.

LOOMIS, L. H.
1. *An introduction to abstract harmonic analysis*, Van Nostrand, 1953.

2. *The lattice theoretic background of the dimension theory of operator algebras*, Memoirs AMS, No. 18 (1955).
3. Note on a theorem of Mackey, *Duke Math. J.*, **19** (1952), 641–645.
4. Positive definite functions and induced representations, *Duke Math. J.*, **27** (1960), 569–579.
5. The spectral characterization of a class of almost periodic functions, *Ann. Math.*, **72** (1960), 362–368.
6. Unique direct integral decomposition of convex sets, *Amer. J. Math.*, **84** (1962), 509–526.

LORENZ, F.
1. Die Epimorphismen des Rings von Operatoren, *Arch. Math.*, **20** (1969), 48–53.

LOUPIAS, G.; MIRACLE-SOLE, S.
1. C^*-algèbres de systèmes canoniques, I, II, *Comm. Math. Phys.*, **2** (1966), 31–48; *Ann. Inst. H. Poincaré*, **6** (1966), 39–54.

LÖWNER, K.
1. Über monokone Matrixfunktionen, *Math. Z.*, **38** (1934), 177–216.

LUBIN, A.
1. Extensions of measures and the von Neumann selection theorem, *Proc. AMS*, **43** (1974), 118–122.

LUFT, E.
1. The two-sided closed ideals of the algebras of bounded linear operators of a Hilbert space, *Czech. Math. J.*, **18** (1968), 595–605.

LUMER, C.
1. Semi-inner product spaces, *Trans. AMS*, **100** (1961), 29–43.
2. Etats, algèbres quotients et sous-espaces invariants, *CR Acad. Sc. Paris*, **274** (1972), 1308–1311.

LUNDBERG, L. E.
1. Quasifree "second quantization", *Comm. Math. Phys.*, **50** (1976), 103–112.
2. Observable algebra approach to the Thirring-Schwinger model, Copenhagen, 1976.

LUXEMBURG, W. A. J.
1. On R. Pallu de la Barrière's characterization of normal states, *Lecture Notes in Math.*, **541** (1976), 303–307.

MACKEY, G. W.
1. Note on a theorem of Murray, *Bull. AMS*, **52** (1946), 322–325.
2. The Laplace transform for locally compact groups, *Proc. Nat. Acad. Sci. USA*, **34** (1948), 156–162.
3. A theorem of Stone and von Neumann, *Duke Math. J.*, **16** (1949), 313–326.
4. Imprimitivity for representations of locally compact groups, I, II, III, *Proc. Nat. Acad. Sci. USA*, **35** (1949), 537–545; *CR Acad. Sc. Paris*, **230** (1950), 808–809; 908–909.
5. Functions on locally compact groups, *Bull. AMS*, **56** (1950), 385–412.
6. On induced representations of groups, *Amer. J. Math.*, **73** (1951), 576–592.
7. Induced representations of locally compact groups I, II, *Ann. Math.*, **55** (1952), 101–139; **58** (1953), 193–221.
8. *The theory of group representations*, Lecture Notes, Chicago Univ., 1955.
9. Les ensembles boréliens et les extensions des groupes, *J. Math. Pures Appl.*, **36** (1957), 171–178.
10. Borel structures in groups and their duals, *Trans. AMS*, **85** (1957), 134–165.
11. Unitary representations of group extensions, I, *Acta Math.*, **99** (1958), 265–311.
12. Point realizations of transformation groups, *Illinois J. Math.*, **6** (1962), 327–335.
13. Infinite dimensional group representations, *Bull. AMS*, **69** (1963), 628–686.
14. Ergodic theory, group theory and differential geometry, *Proc. Nat. Acad. Sci. USA*, **50** (1963), 1184–1191.
15. Ergodic transformation groups with a pure point spectrum, *Illinois J. Math.*, **8** (1964), 593–600.
16. Ergodic theory and virtual groups, *Math. Ann.*, **166** (1966), 187–207.

17. *Group representations and non-commutative harmonic analysis*, Lecture Notes, Berkeley, 1965.
18. *Group representations and applications*, Lecture Notes, Oxford, 1966/67, 539 p.
19. *Induced representations of groups and quantum mechanics*, Benjamin, 1968.
20. *Infinite dimensional group representations and their applications*, CIME, 1970; Ed. Cremonese, Roma, 1971, 221–330.
21. Ergodicity in the theory of group representations, *Proc. Int. Congr. Math.*, Nice, 1970, vol. II, 401–405.
22. Ergodic theory in statistical mechanics and probability theory, *Adv. Math.*, 12 (1974), 178–268.

MAEDA, F.
1. Relative dimensionality in operator rings, *J. Sci. Hiroshima Univ.*, 11 (1941), 1–6.

MAEDA, F.; MAEDA, S.
1. *Theory of symmetric lattices*, Springer Verlag, 1971.

MAEDA, S.
1. Dimension functions of certain general lattices, *J. Sci. Hiroshima Univ.*, 19 (1955), 211–237.
2. Lengths of projections in rings of operators, *J. Sci. Hiroshima Univ.*, 20 (1956), 5–11.
3. *Kontinuerliche Geometrien*, Springer Verlag, 1958.
4. On the lattice of projections of a Baer $*$-ring, *J. Sci. Hiroshima Univ.*, 22 (1958), 76–88.
5. Decomposition of general lattices into direct summands of types I, II, III, *J. Sci. Hiroshima Univ.*, 23 (1959), 161–170.
6. On $*$-rings satisfying the square root axiom, *Proc. AMS*, 52 (1975), 188–190.
7. Equivalence of projections in Baer $*$-rings, *J. Algebra*, 39 (1976), 150–159.
8. On arcs in the space of projections of a C^*-algebra, *Math. Jap.*, 21 (1976), 371–374.
9. On the distance between two projections in a C^*-algebra, *Math. Jap.*, 22 (1977), 61–66.

MAHARAM, D.
1. On homogeneous measure algebras, *Proc. Nat. Acad. Sci. USA*, 28 (1942), 108–111.
2. The representation of abstract measure functions, *Trans. AMS*, 65 (1949), 279–330.

MAITLAND-WRIGHT, J. D.
1. An extension theorem and a dual proof of a theorem of Gleason, *J. London Math. Soc.*, 43 (1968), 699–702.
2. A spectral theorem for normal operators on a Kaplansky-Hilbert module, *Proc. London Math. Soc.*, 19 (1969), 258–268.
3. Every monotone σ-complete C^*-algebra is the quotient of its Baire σ-envelope by a two-sided σ-ideal, *J. London Math. Soc.*, 6 (1973), 210–214.
4. On minimal σ-completions of C^*-algebras, *Bull. London Math. Soc.*, 6 (1974), 168–174.
5. Regular σ-completions of C^*-algebras, *J. London Math. Soc.*, 12 (1976), 299–309.
6. On von Neumann algebras whose pure states are separable, *J. London Math. Soc.*, 12 (1976), 385–388.
7. On AW^*-algebras of finite type, *J. London Math. Soc.*, 12 (1976), 431–439.
8. Wild AW^*-factors and Kaplansky-Rickard algebras, *J. London Math. Soc.*, 13 (1976), 83–89.
9. On semifinite AW^*-algebras, *Math. Proc. Cambridge Phil. Soc.*, 79 (1976), 443–446.
10. On approximating concave functions by convex functions, *Bull. London Math. Soc.*, 5 (1976), 221–222.
11. Jordan C^*-algebras, *Michigan Math. J.*
12. Une caractérisation algébrique des algèbres de von Neumann dans un espace hilbertien séparable, *CR Acad. Sc. Paris*, 286 (1978), 263–264.
13. On C^*-algebras which are almost separably representable.

MAITLAND-WRIGHT, J. D.; YOUNGSON, M. A.
1. On isometries of Jordan algebras, *J. London Math. Soc.*
2. A Russo-Dye theorem for Jordan C^*-algebras.

MALVIYA, B. D.
1. Finite dimensionality and duality of B^*-algebras, *Collectanea Math.*, 25 (1974), 3–5.

2. On locally C^*-algebras, *Publ. Math. Debrecen*, **21** (1974), 47–52.
3. On complemented Banach $*$-algebras, *Rev. Roum. Math. Pures Appl.*, **21** (1976), 337–341.

MALVIYA, B. D.; TOMIUK, B. J.
1. Multiplier operators on B^*-algebras, *Proc. AMS*, **31** (1972), 505–510.

MANUCEAU, J.
1. Étude de quelques automorphismes de la C^*-algèbre du champ des bosons libres, *Ann. IHP*, **8** (1968), 117–138.
2. C^*-algèbres des relations de commutation, *Ann. IHP*, **8** (1968), 139–161.

MANUCEAU, J.; NAUDTS, J.; VERBEURE, A.
1. Entropy and normal states, *Comm. Math. Phys.*, **27** (1972), 327–338.

MANUCEAU, J.; ROCCA, F.; SIRUGUE, M.; VERBEURE, A.
1. États quasi-libres, *Cargese Lectures in Phys.*, **4** (1970), 303–362.

MANUCEAU, J.; ROCCA, F.; TESTARD, D.
1. On the product form of quasi-free states, *Comm. Math. Phys.*, **12** (1969), 43–57.

MANUCEAU, J.; SIRUGUE, M.; TESTARD, D.; VERBEURE, A.
1. The smallest C^*-algebra for canonical commutation relations, *Comm. Math. Phys.*, **32** (1973), 231–244.

MANUCEAU, J.; TROTIN, J. C.
1. On lattice spin systems, *Ann. IHP*, **10** (1969), 359–380.

MANUCEAU, J.; VERBEURE, A.
1. Quasi-free states of the CCR algebra and Bogoliubov transformations, *Comm. Math. Phys.*, **9** (1968), 293–302.
2. Non-factor quasi-free states of the CAR algebra, *Comm. Math. Phys.*, **18** (1970), 319–329.

MARÉCHAL, O.
1. Opérateurs décomposables dans les champs mesurables d'espaces de Hilbert, *CR Acad. Sc. Paris*, **266** (1968), 710–713.
2. Opérateurs décomposables dans les espaces L^p_F, *CR Acad. Sc. Paris*, **266** (1968), 1043–1046.
3. Décomposition des opérateurs dans les champs mesurables d'espaces de Hilbert, *CR Acad. Sc. Paris*, **267** (1968), 636–639.
4. Champs mesurables d'espaces hilbertiens, *Bull. Sc. Math.*, **93** (1969), 113–143.
5. Champs mesurables d'espaces hilbertiens, *CR Acad. Sc. Paris*, **270** (1970), 1316–1319.
6. Topologie et structure borélienne sur l'ensemble des algèbres de von Neumann, *CR Acad. Sc. Paris*, **276** (1973), 847–850.
7. Une remarque sur un théorème de Glimm, *Bull. Sc. Math.*, **99** (1975), 41–44.

MASANI, P.; ROSENBERG, M.
1. When is an operator the integral of a given spectral measure?, *J. Fnal Analysis*, **21** (1976), 88–121.

MATOBA, K.
1. A remark on a theorem of H. Araki, *Proc. Jap. Acad.*, **42** (1966), 344–346.

MATSUMOTO, K.; MORIYA, N.
1. On regular completely positive maps among C^*-algebras, *Math. Jap.*, **21** (1976), 265–268.

MATSUSHITA, S.
1. Positive linear functionals in self-adjoint B^*-algebras, *Proc. Jap. Acad.*, **29** (1953), 427–430.
2. Positive functionals and representation theory on Banach algebras, I, *J. Inst. Polytech., Osaka Univ.*, **6** (1955), 1–18.

MATVEIČUK, M. S. (in Russian)
1. Random norms and the properties of probability measures on the projections of a factor, *Veroyat. Met. Kibern., Kazan*, **9** (1971), 73–77.
2. Measures in an approximately finite factor, *Veroyat. Met. Kibern., Kazan*, **10/11** (1974), 100–111.

3. On the convergence of operators in a factor, *Veroyat. Met. Kibern., Kazan*, **10/11** (1974), 112–115.
4. Finite measures in an approximately finite factor, *Izv. Vysš. Učebn. Zaved. Mat.*, no. 5 (168), 79–85, 1976.

MAUCERI, G.
1. Square integrable representations and the Fourier algebra of a locally compact unimodular group, *Pacific J. Math.*, **73** (1977), 143–154.

MAURES, J.
1. Positive lineare Abbildungen auf der n-dimensionalen Matrizenalgebra, *Arch. Math.*, **28** (1977), 193–199.

MAUREY, B.
1. Tout opérateur d'une C^*-algèbre dans un espace de cotype 2 se factorise par un Hilbert, *Sém. Maurey-Schwartz*, 1975/76, exp. 21.

MAURIN, K.
1. *Methods on Hilbert spaces* (Polish, Russian), Warsaw, 1959.
2. On some theorems of H. J. Borchers, *Bull. Acad. Polon. Sci.*, **11** (1963), 121–123.
3. *General eigenfunctions expansions and unitary representations of topological groups*, Warsaw, 1968.

MAUTNER, F. I.
1. Unitary representations of locally compact groups, I, II, *Ann. Math.*, **51** (1950), 1–25; **52** (1950), 528–556.
2. The structure of the regular representation of certain discrete groups, *Duke Math. J.*, **17** (1950), 437–441.
3. Infinite dimensional irreducible representations of certain groups, *Proc. AMS*, **1** (1950), 582–584.
4. Induced representations, *Amer. J. Math.*, **74** (1952), 737–758.
5. Note on Fourier inversion formula on groups, *Trans. AMS*, **78** (1955), 371–384.

MAYER, M. E.
1. Differentiable cross-sections in Banach $*$-algebraic bundles, *Cargese Lectures in Phys.*, **4** (1970), 368–387.
2. Automorphism groups of C^*-algebras, Fell bundles, W^*-bigebras and the description of internal symmetries in algebraic quantum theory, *Acta Phys. Austriaca*, suppl. **8**, 1971, 117–226.

MC CHAREN, E. A.
1. A new topology on B^*-algebras arising from the Arens product, *Proc. AMS*, **37** (1973), 77–83.
2. A characterization of dual B^*-algebras, *Proc. AMS*, **37** (1973), 84.

MC CHARTHY, C. A.
1. Optimal conditioning of operators on Hilbert spaces, *Functional Analysis*, Academic Press, 1970, 107–125.
2. The norm of certain derivations, *Pacific J. Math.*, **53** (1974), 515–518.

MC DUFF, D.
1. A countable infinity of II_1 factors, *Ann. Math.*, **90** (1969), 361–371.
2. Uncountable many II_1 factors, *Ann. Math.*, **90** (1969), 372–377.
3. The structure of type II_1-factors (Russian), *Uspekhi Mat. Nauk*, **25** (1970), 29–51.
4. Central sequences and the hyperfinite factor, *Proc. London Math. Soc.*, **21** (1970), 443–461.
5. On residual sequences in II_1-factors, *J. London Math. Soc.*, **3** (1971), 273–280.

MC GOVERN, R. J.
1. Quasi-free derivations of the canonical anticommutation algebra, *J. Fnal Analysis*, **26** (1977), 89–101.

MC INTOSH, A.
1. Functions and derivations of C^*-algebras.

MC KENNON, K.
1. *Multipliers, positive functionals, positive definite functions and Fourier transform*, *Memoirs AMS*, No. **111** (1971).
2. The strict topology and the Cauchy structure of the spectrum of a C^*-algebra, *Gen. Top. and Appl.*, **5**: 3 (1975), 249–262.

MC SHANE, E. J.
1. Families of measures and representations of algebras of operators, *Trans. AMS*, **102** (1962), 328–345.

MICHELE, L.; SOARDI, P. M.
1. A non-commutative extension of Helson's translation lemma, *Boll. UMI*, **9** (1974), 800–806.

MIERS, C. R.
1. Lie isomorphisms in factors, *Trans. AMS*, **147** (1970), 55–63.
2. Lie homomorphisms of operator algebras, *Pacific J. Math.*, **38** (1971), 717–735.
3. Lie derivations of von Neumann algebras, *Duke Math. J.*, **40** (1973), 403–410.
4. Derived ring isomorphisms of von Neumann algebras, *Canad. J. Math.*, **25** (1973), 1254–1268.
5. Polynomially ideal C^*-algebras, *Amer. J. Math.*, **98** (1976), 165–170 (Erratum, ibidem, **99** (1977), 225–226).
6. Lie $*$-triple homomorphisms into von Neumann algebras, *Proc. AMS*, **58** (1976), 169–172.

MILES, P. E.
1. Order isomorphisms of B^*-algebras, *Trans. AMS*, **107** (1963), 217–236.
2. B^*-algebra unit ball extreme points, *Pacific J. Math.*, **14** (1964), 627–637.
3. Derivations of B^*-algebras, *Pacific J. Math.*, **14** (1964), 1359–1366.

MILICIČ, D.
1. A note on invariant order ideals in the predual of a von Neumann algebra, *Glasnik Mat.*, **6** (1971), 71–72.
2. Topological representations of the group C^*-algebra of SL(2, R), *Glasnik Mat.*, **6** (1971), 231–246.
3. On C^*-algebras with bounded trace, *Glasnik Mat.*, **8** (1973), 7–22.
4. The dual spaces of almost connected reductive groups, *Glasnik Mat.*, **9** (1974), 273–288.
5. Representations of almost connected groups, *Proc. AMS*, **47** (1975), 517–518.

MILMAN, D.
1. On the theory of rings with involution (Russian), *Dokl. Akad. Nauk SSSR*, **76** (1951), 349–352.

MIRACLE-SOLE, S.; ROBINSON, D. W.
1. The physical states of Fermi systems, *Comm. Math. Phys.*, **14** (1969), 235–270.
2. Statistical mechanics of quantum mechanical particles with hard cores, II: The equilibrium states, *Comm. Math. Phys.*, **19** (1970), 204–218.

MISONOU, Y.
1. On a weakly central operator algebra, *Tohoku Math. J.*, **4** (1952), 194–202.
2. Unitary equivalence of factors of type III, *Proc. Jap. Acad.*, **29** (1953), 482–485.
3. Operator algebras of type I, *Kodai Math. Sem. Rep.*, **5** (1953), 87–90.
4. On the direct product of W^*-algebras, *Tohoku Math. J.*, **6** (1954), 189–204.
5. Generalized approximately finite W^*-algebras, *Tohoku Math. J.*, **7** (1955), 192–205.
6. On divisors of factors, *Tohoku Math. J.*, **8** (1956), 63–69.

MISONOU, Y.; NAKAMURA, M.
1. Centering of an operator algebra, *Tohoku Math. J.*, **3** (1951), 243–248.

MIURA, Y.
1. A note on the von Neumann algebras with cyclic and separating vector, *Tohoku Math. J.*, **29** (1977), 77–80.

MIYATA, H.
1. On clustering states, *Comm. Math. Phys.*, **34** (1973), 1–6.

MOFFAT, J.

1. Continuity of automorphic representations, *Proc. Cambridge Phil. Soc.*, **74** (1973), 461–465.
2. On groups of automorphisms of the tensor product of von Neumann algebras, *Math. Scand.*, **34** (1974), 226–230.
3. Connected topological groups acting on von Neumann algebras, *J. London Math. Soc.*, **9** (1975), 411–417.
4. On groups of automorphisms of operator algebras, *Math. Proc. Cambridge Phil. Soc.*, **81** (1977), 237–244.

MOORE, C. C.

1. Invariant measures on product spaces, *Proc. Vth Berkeley Symp. Math. Stat. Prob.*, 1967, vol. II, part II, 447–459.
2. Restrictions of unitary representations to subgroups and ergodic theory, *Lecture Notes in Phys.*, No. 6, 1–35, Springer Verlag, 1970.
3. On the Frobenius reciprocity theorem for locally compact groups, *Pacific J. Math.*, **12** (1962), 359–365.
4. Group extensions and cohomology for locally compact groups, I, II, III, IV, *Trans. AMS*, **113** (1964), 40–63; 64–88; **221** (1976), 1–33; 34–58.
5. Decomposition of unitary representations defined by discrete subgroups of nilpotent groups, *Ann. Math.*, **82** (1965), 146–182.
6. The Plancherel formula for non-unimodular groups, *Int. Congr. Fnal Analysis., Univ. Maryland*, College Park, 1971.
7. Square integrable primary representations, Berkeley, 1976.

MOORE, C. C.; ROSENBERG, J.

1. Group with T_1 primitive ideal space, *J. Fnal Analysis*.

MOORE, R. L.

1. Reductivity in C^*-algebras and essentially reductive operators, *Pacific J. Math*.

MOORE, R. T.

1. Hermitian functional on B^*-algebras and duality characterization of C^*-algebras, *Trans. AMS*, **162** (1971), 253–265.
2. An asymptotic Fuglede theorem, *Proc. AMS*, **50** (1975), 138–142.
3. Analytic structure in the state space of Banach algebras.

MOORE, S. M.

1. An algebraic formulation of ergodic problems, *Rev. Columbiana Mat.*, **9** (1975), 189–204.
2. An algebraic study of the Kolmogorov entropy, *Rev. Columbiana Mat.*, **10** (1976), 57–68.

MOSAK, R.

1. The L^1- and C^*-algebras of $[FIA]_B^-$ groups and their representations, *Trans. AMS*, **163** (1972), 277–310.
2. *Banach algebras*, The University of Chicago Press, 1975.
3. *Lectures on abstract harmonic analysis*.

MOSCOVICI, H.

1. Generalized induced representations, *Rev. Roum. Math. Pures Appl.*, **14** (1969), 1539–1551.
2. On a conjecture of Kirillov, *Rev. Roum. Math. Pures Appl.*, **19** (1974), No. 3.
3. Coherent state representations of nilpotent Lie groups, *Comm. Math. Phys.*, **54** (1977), 63–68.
4. Topological Frobenius properties for nilpotent Lie groups, *Rev. Roum. Math. Pures Appl.*, **19** (1974), 421–425.

MOSCOVICI, H.; VERONA, A.

1. Holomorphically induced representations of solvable Lie groups, *Bull. Sc. Math.*, 1978.
2. Cocycle representations of solvable Lie groups, *Math. Z.*, **160** (1978), 183–194.
3. Coherent states and square integrable representations, *Trans. AMS*, 1978.

MOULIS, N.

1. *Structures de Fredholm sur les variétés hilbertiennes*, Lecture Notes in Math., No. 259, Springer Verlag, 1972.

MOYER, R. D.

1. Computations of symbols on C^*-algebras of singular integral operators, *Bull. AMS*, **77** (1971), 615–620.
2. Fourier integral operators and C^*-algebras.

MUHLY, P. S.

1. A structure theory for isometric representations of a class of semigroups, *J. Reine Angew. Math.*, **255** (1972), 135–154.
2. Function algebras and flows, I, II, III, IV, *Acta Sci. Math.*, **35** (1973), 111–121; *Arkiv Math.*, **11** (1973), 203–213; *Math. Z.*, **136** (1974), 253–260; *Trans. AMS*, **203** (1975), 55–66.

MULLER, M. A.

1. A singly generated operator algebra, *J. London Math. Soc.*, **14** (1976), 437–440.

MURAKAMI, K.

1. On the W^*-tensor product of $L^\infty(G)$ and M, Tokyo, 1975.

MURAVEVA, G. P. (in Russian)

1. On maximal abelian subrings in approximately finite factors, *Sibirsk. Mat. J.*, **9**: 3 (1968), 614–622.
2. Finite transforms of X-generators into a maximal abelian subalgebra of an approximately finite factor, *Izvestya vysš. uchebn. zaved., Mat.*, **10** (1970), 61–68.
3. On regular abelian subalgebras of an approximately finite factor, *Sibirsk. Mat. J.*, **13**: 4 (1972), 805–816.

MURPHY, I. S.

1. Continuity of positive linear functionals on Banach algebras, *Bull. London Math. Soc.*, **1** (1969), 171–173.
2. A note on B^*-algebras, *Glasgow Math. J.*, **14** (1973), 185–186.

MURRAY, F. J.

1. Bilinear transformations in Hilbert space, *Trans. AMS*, **45** (1939), 474–507.
2. Theory of operators, I: Single operators, *Bull. AMS*, **64** (1958), 57–60.

MURRAY, F. J.; VON NEUMANN, J.

1. On rings of operators, *Ann. Math.*, **37** (1936), 116–229.
2. On rings of operators, II, *Trans. AMS*, **41** (1937), 208–248.
3. On rings of operators, IV, *Ann. Math.*, **44** (1943), 716–808.

NACHBIN, L.

1. On the finite dimensionality of every irreducible representation of a compact group, *Proc. AMS*, **12** (1961), 11–12.
2. *The Haar integral*, Van Nostrand, 1965.

NAGEL, B.

1. Some results on non-commutative ergodic theory, *Comm. Math. Phys.*, **26** (1972), 247–258.
2. Mean ergodic semigroups on W^*-algebras.

NAǏMARK, M. A. (in Russian)

1. Rings with involution, *Uspekhi Mat. Nauk*, **3** (1948), 52–145.
2. Rings of operators in Hilbert spaces, *Uspekhi Mat. Nauk*, **4** (1949), 83–147.
3. On some problems in the theory of rings with involution, *Uspekhi Mat. Nauk*, **6** (1951), 160–164.
4. On the description of all unitary representations of complex classical groups, I, II, *Matem. Sb.*, **35** (1954), 317–356; **37** (1955), 121–140.
5. A continuous analogue of Schur's lemma and its applications to Plancherel formula for complex classical groups, *Dokl. Akad. Nauk, SSSR*, **98** (1954), 185–188; *Izvestya Akad. Nauk SSSR*, **20** (1956), 3–16.
6. *Normed rings*, Moscow, 1956, 1968.
7. *Linear representations of the Lorentz group*, Moscow, 1968.
8. Factor representations of a locally compact group, *Dokl. Akad. Nauk, SSSR*, **134** (1960), 275–277.

9. The decomposition in factor representations of the unitary representations of locally compact groups, *Sibirsk. Mat. J.*, **2** (1961), 89–99.
10. The structure of factor representations of a locally compact group, *Dokl. Akad. Nauk SSSR*, **148** (1963), 775–778.
11. *The theory of group representations*, Moscow, 1976.

NAĬMARK, M. A.; FOMIN, S. V.
1. Continuous direct sums of Hilbert spaces and some applications (Russian), *Uspekhi Mat. Nauk*, **10** (1955), 111–142.

NAĬMARK, M. A.; KATZ, G. I.
1. Unitary representations of locally compact groups (Russian), *Proc. IVth Unional Math. Congr., Leningrad*, 1961, vol. II, 308–317.

NAKAGAMI, Y.
1. Infinite tensor products of von Neumann algebras I, II, *Kodai Math. Sem. Rep.*, **22** (1970), 341–354; *Publ. RIMS, Kyoto Univ.*, **6** (1970), 257–292.
2. A remark on the probabilistical definiteness for self-adjoint operators, *Kodai Math. Sem. Rep.*, **18** (1967), 229–235.
3. Covariance operators of skew distributions, *Publ. RIMS, Kyoto Univ.*, **7** (1971), 69–83.
4. Infinite tensor products of operators, *Publ. RIMS, Kyoto Univ.*, **10** (1974), 111–146.
5. Duality for crossed product of von Neumann algebras by locally compact groups, *Bull. AMS*, **81** (1975), 1106–1109.
6. Dual action on a von Neumann algebra and Takesaki's duality for a locally compact group, *Publ. RIMS, Kyoto Univ.*, **12** (1976).
7. Essential spectrum $\Gamma(\beta)$ of a dual action on a von Neumann algebra, *Pacific J. Math.*

NAKAGAMI, Y.; SUTHERLAND, C. E.
1. Takesaki's duality for regular extensions of von Neumann algebras, *Japan-US Seminary*, 1977.

NAKAI, M.
1. Some expectations in AW^*-algebras, *Proc. Jap. Acad.*, **34** (1958), 411–416.

NAKAMOTO, R.
1. On the operator equation THT = K, *Math. Jap.*, **18** (1973), 251–252.

NAKAMURA, M.
1. The two-sided representation of an operator algebra, *Proc. Jap. Acad.*, **27** (1951), 172–176.
2. Complete continuities of linear operators, *Proc. Jap. Acad.*, **27** (1951), 544–547.
3. Center of closure operators and decomposition of a lattice, *Math. Jap.*, **3** (1954), 49–52.
4. On the direct product of finite factors, *Tohoku Math. J.*, **6** (1954), 205–207.
5. A remark on the integral decomposition of a measure, *Mem. Osaka Univ.*, **3** (1954), 25–28.
6. On operators of Schaeffer class in the theory of singular integral equations, *Proc. Jap. Acad.*, **33** (1957), 455–456.
7. A proof of a theorem of Takesaki, *Kodai Math. Sem. Rep.*, **10** (1958), 189–190.
8. A remark on a paper by Greub and Rheinboldt, *Proc. Jap. Acad.*, **36** (1960), 198–199.

NAKAMURA, M.; OTOMI, Y.; URATA, Y.
1. A remark on the carrier projection in abelian operator algebras, *Mem. Osaka Gagukei Univ.*, **13** (1964), 25–26.

NAKAMURA, M.; TAKEDA, Z.
1. Group representations and Banach limit, *Tohoku Math. J.*, **3** (1951), 132–135.
2. The Radon-Nikodym theorem of traces for certain operator algebras, *Tohoku Math. J.*, **4** (1952), 275–283.
3. Normal states of commutative operator algebras, *Tohoku Math. J.*, **5** (1953), 109–121.
4. On some elementary properties of the crossed products of von Neumann algebras, *Proc. Jap. Acad.*, **34** (1958), 489–494.
5. On certain examples of the crossed product of finite factors, I, II, *Proc. Jap. Acad.*, **34** (1958), 495–499; 500–502.
6. On the extension of finite factors, I, *Proc. Jap. Acad.*, **35** (1959), 149–154.
7. A Galois theory for finite factors, *Proc. Jap. Acad.*, **36** (1960), 258–260.

8. On the fundamental theorem of Galois theory for finite factors, *Proc. Jap. Acad.*, **36** (1960), 313–318.
9. On inner automorphisms of finite factors, *Proc. Jap. Acad.*, **37** (1961), 31–32.
10. On outer automorphisms of certain finite factors, *Proc. Jap. Acad.*, **37** (1961), 215–216.

NAKAMURA, M.; TAKEDA, Z.; TURUMARU, T.
1. On some extended principal axis theorems for completely continuous operators, *Tohoku Math. J.*, **5** (1953), 190–193.

NAKAMURA, M.; TAKESAKI, M.; UMEGAKI, H.
1. A remark on the expectation of operator algebras, *Kodai Math. Sem. Rep.*, **12** (1960), 82–90.

NAKAMURA, M.; TURUMARU, T.
1. On the representation of positive definite functions and stationary functions on topological groups, *Tohoku Math. J.*, **4** (1952), 1–9.
2. Simple algebras of completely continuous operators, *Tohoku Math. J.*, **4** (1952), 303–308.
3. On a proof of a theorem of Rosenberg, *Proc. Jap. Acad.*, **29** (1953), 501–502.
4. Completely continuous operators with property F, *Tohoku Math. J.*, **6** (1954), 174–176.
5. Expectations in an operator algebra, *Tohoku Math. J.*, **6** (1954), 182–188.
6. On extensions of pure states of an abelian operator algebra, *Tokohu Math. J.*, **6** (1954), 253–257.

NAKAMURA, M.; UMEGAKI, H.
1. A remark on theorems of Stone and Bochner, *Proc. Jap. Acad.*, **27** (1951), 506–507.
2. On a proposition of von Neumann, *Kodai Math. Sem. Rep.*, **8** (1956), 142–144.
3. On the Blackwell theorem in operator algebras, *Proc. Jap. Acad.*, **37** (1961), 312–315.
4. A note on the entropy for operator algebras, *Proc. Jap. Acad.*, **37** (1961), 149–154.
5. On von Neumann's theory of measurements in quantum statistics, *Math. Jap.*, **7** (1962), 151–157.
6. Heisenberg commutation relations and the Plancherel theorem, *Proc. Jap. Acad.*, **37** (1961), 239–242.

NAKANO, H.
1. Reduction of Bochner's theorem to Stone's theorem, *Ann. Math.*, **49** (1948), 278–280.
2. Hilbert algebras, *Tohoku Math. J.*, **2** (1950), 4–23.

NAMIOKA, I.; ASPLUND, E.
1. A geometric proof of Ryll-Nardzewski's fixed point theorem, *Bull. AMS*, **73** (1967), 443–445.

NAPIORKOWSKI, K.
1. On the independence of local algebras, *Rep. Math. Phys.*, **3** (1972), 33–35.
2. Continuous tensor products of Hilbert spaces and representations of CCR, *Bull. Acad. Polon. Sci.*, **18** (1970), 267–271.
3. Continuous tensor products of Hilbert spaces and product operators, *Studia Math.*, **39** (1971), 307–327.
4. On a class of representations of CCR, *Rep. Math. Phys.*, **1** (1971), 235–243.

NAPIORKOWSKI, K.; PUSZ, W.
1. Particle representations of CCR, *Rep. Math. Phys.*, **3** (1972), 221–225.

NARNHOFER, H.
1. Self-adjoint operators, derivations and automorphisms of C^*-algebras, *J. Math. Phys.*, **16** (1975), 2192–2194.
2. Kommutative Automorphismen und Gleichgewichtszustände, *Acta Phys. Austriaca*, **47** (1977), 1–29.

NARNHOFER, H.; ROBINSON, D. W.
1. Dynamical stability and pure thermodynamical phases, *Comm. Math. Phys.*, **41** (1975), 89–97.

NAUDTS, J.
1. A generalized entropy function, *Comm. Math. Phys.*, **37** (1974), 175–182.

NAUDTS, J.; VERBEURE, A.
1. Number operator states on CAR algebra, *Ann. Soc. Sc. Bruxelles*, **86** (1972), 101–108.
2. Bounds on the admittance for KMS states, *J. Math. Phys.*, **17** (1976), 419–423.

NAUDTS, J.; VERBEURE, A.; WEDER, R.
1. Linear response theory and the KMS condition, *Comm. Math. Phys.*, **44** (1975), 87–100.

NELSON, E.
1. Analytic vectors, *Ann. Math.*, **70** (1959), 572–615.
2. Notes on non-commutative integration, *J. Fnal Analysis*, **15** (1974), 103–116.

NELSON, E.; STINESPRING, W. F.
1. Representations of elliptic operators in an enveloping algebra, *Amer. J. Math.*, **81** (1959), 547–560.

NEST, R.
1. Invariant weights of operator algebras satisfying the KMS condition.
2. A non-commutative version of the maximal ergodic theorem for invariant traces, Copenhagen, 1976.
3. Measurability of "subgroup of inner action" map in disintegration of dynamical systems, Copenhagen, 1974.
4. On the martingale theorem in a non-commutative setting.

VON NEUMANN, J.
1. Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, *Math. Ann.*, **102** (1929), 49–131.
2. Zur Algebra der Funktionaloperationen und Theorie der Normalen Operatoren, *Math. Ann.*, 102 (1929/30), 370–427.
3. Über Funktionen von Funktionaloperatoren, *Ann. Math.*, **32** (1931), 191–226.
4. *Mathematische Grundlagen der Quantenmechanik*, Springer-Verlag, 1932.
5. Almost periodic functions on a group, I, *Trans. AMS*, 36 (1934), 445–492.
6. *Characterisierung des Spektrums eines Integraloperatoren*, Act. Sci. Ind., 229, Hermann, 1935,
7. On regular rings, *Proc. Nat. Acad. Sci. USA*, 22 (1936), 707–713.
8. *Continuous geometry*, *Proc. Nat. Acad. Sci. USA*, 22 (1936); Lecture Notes, I, II, III, Michigan 1936, 1937; Princeton Univ. Press, 1960.
9. On a certain topology for rings of operators, *Ann. Math.*, 37 (1936), 111–115.
10. On an algebraic generalization of the quantum mechanical formalism, I, *Mat. Sb.*, 1 (1936), 415–484.
11. Algebraic theory of continuous geometries, *Proc. Nat. Acad. Sc. USA*, 23 (1937), 16–22.
12. On infinite direct products, *Compositio Math.*, 6 (1938), 1–77.
13. On rings of operators, III, *Ann. Math.*, 41 (1940), 94–161.
14. On some algebraical properties of operator rings, *Ann. Math.*, 44 (1943), 709–715.
15. On rings of operators. Reduction theory, *Ann. Math.*, 50 (1949), 401–485.
16. *Functional operators*, I, II, *Ann. Math. Studies*, No. 21, 22, Princeton Univ. Press, 1950.
17. Eine Spektraltheorie für allgemeine Operatoren eines unitären Räumes, *Math. Nachr.*, 4 (1951), 258–281.
18. *Collected works*, vol. III *(On rings of operators)*, Pergamon Press, 1961.

VON NEUMANN, J.; SCHATTEN, R.
1. The cross space of linear transformations, I, II, III, *Ann. Math.*, 47 (1946), 37–84; 608–630; 49 (1948), 557–582.

NG, K. F.
1. On a theorem of Dixmier, *Math. Scand.*, 29 (1971), 279–280.

NIELSEN, O. A.
1. Maximal abelian subalgebras in hyperfinite factors, I, II, *Bull. AMS*, 75 (1969), 579–581; *Trans. AMS*, 146 (1969), 259–272; *J. Fnal Analysis*, 6 (1970), 192–202.
2. A note on the product measures and representations of the canonical commutation relations, *Comm. Math. Phys.*, 22 (1971), 23–36.
3. The asymptotic ratio set and direct integral decomposition of a von Neumann algebra, *Can. J. Math.*, 23 (1971), 598–607.

4. An example of a von Neumann algebra of global type, II, *J. Fnal Analysis*, **11** (1972), 207–210.
5. Borel sets of von Neumann algebra, *Amer. J. Math.*, **95** (1973), 145–164.
6. The Mackey-Blattner theorem and Takesaki's generalized commutation relation for locally compact groups, *Duke Math. J.*, **40** (1973), 105–114.
7. New proofs of two theorems of Sakai.
8. New proof of Sakai's theorem on global von Neumann algebras.
9. Reduced crossed products of von Neumann algebras, I, preprint, Aarhus Univ., 1974/75, no. 24.

NIIRO, F.
1. Sur l'unicité de la décomposition d'une trace, *Sci. Papers College Gen. Ed. Univ. Tokyo*, **13** (1963), 159–162.

NOSOV, V. N. (in Russian)
1. Strongly closed ideals in $*$-algebras of operators, *Vestn. Mosk. Univ.*, **25** : 4 (1970), 33–34.
2. Extensions of C^*-algebras with continuous trace, *Funkt. Analiz i pril.*, **6** : 2 (1972), 91–92.
3. The S invariant, weights and states on W^*-algebras, *Funkt. Analiz i pril.*, **9** : 2 (1975), 85–86.
4. The S invariant and the reduction of von Neumann algebras, *Uspekhi Mat. Nauk*, **13** (1976), 255–256.

NOUYRIGAT, P.
1. Sur le prédual de l'algèbre de von Neumann associé à une représentation unitaire d'un groupe localement compact, *Publ. Dépt. Math.*, **9** (1972), 30–59.

NUSSBAUM, A. E.
1. On the reduction of C^*-algebras, *Proc. AMS*, **15** (1964), 567–573.
2. Reduction theory for unbounded closed operators in Hilbert space, *Duke Math. J.*, **31** (1964), 33–44.
3. On the integral representation of positive linear functionals, *Trans. AMS*, **128** (1967), 460–473.
4. A commutativity theorem for unbounded operators, *Trans. AMS*, **140** (1969), 485–493.

O'DONOVAN, D. P.
1. Weighted shifts and covariance algebras, *Trans. AMS*, **208** (1975), 1–26.
2. Quasi-diagonality in the Brown-Douglas-Fillmore theory, *Duke Math. J.*, **44** (1977), 767–776.

OGASAWARA, T.
1. Some general theorems and convergence theorems in vector lattices, *J. Sc. Hiroshima Univ.*, **14** (1949), 13–25.
2. Finite dimensionality of certain Banach algebras, *J. Sc. Hiroshima Univ.*, **17** (1954), 359–364.
3. Topologies on rings of operators, *J. Sc. Hiroshima Univ.*, **19** (1955), 255–272.
4. A structure theorem for complete quasi-unitary algebras, *J. Sc. Hiroshima Univ.*, **19** (1955), 79–85.
5. A theorem on operator algebras, *J. Sc. Hiroshima Univ.*, **18** (1955), 307–309.

OGASAWARA, T.; MAEDA, S.
1. A generalization of a theorem of Dye, *J. Sc. Hiroshima Univ.*, **20** (1956), 1–4.

OGASAWARA, T.; YOSHINAGA, K.
1. Weakly completely continuous Banach algebras, *J. Sc. Hiroshima Univ.*, **18** (1954), 15–36.
2. A characterization of dual B^*-algebras, *J. Sc. Hiroshima Univ.*, **18** (1954), 179–182.
3. Extensions of $\natural$-applications to unbounded operators, *J. Sc. Hiroshima Univ.*, **19** (1955), 273–299.
4. A non-commutative theory of integration for operators, *J. Sc. Hiroshima Univ.*, **18** (1955), 311–347.

OKA, Y.
1. Uniformly hyperfinite algebras and locally compact transformation groups, *J. Math. Soc. Japan*, **25** (1973), 357–362.

OKAYASU, T.
1. On the tensor product of C^*-algebras, *Tohoku Math. J.*, **18** (1966), 325–331.
2. A structure theorem of automorphisms of von Neumann algebras, *Tohoku Math. J.*, **20** (1968), 199–206.
3. On GCR operators, *Tohoku Math. J.*, **21** (1969), 573–579.
4. On representation of tensor products of involutive Banach algebras, *Proc. Jap. Acad.*, **46** (1970), 404–408.
5. Some cross-norms which are not uniformly cross, *Proc. Jap. Acad.*, **46** (1970), 54–57.
6. Polar decomposition for isomorphisms of C^*-algebras, *Tohoku Math. J.*, **26** (1974), 541–554.
7. Roots of operators, *Proc. Jap. Acad.*, **51** (1975), 554–557.
8. Spectral measures for $*$-automorphisms, *Japan-US Seminary*, 1977.

OKAYASU, T.; TAKESAKI, M.
1. Dual spaces of tensor products of C^*-algebras, *Tohoku Math. J.*, **18** (1966), 332–337.

OLESEN, D.
1. Derivations of AW^*-algebras are inner, *Pacific J. Math.*, **53** (1974), 555–562.
2. Inner $*$-automorphisms of simple C^*-algebras, *Comm. Math. Phys.*, **44** (1975), 175–190.
3. On norm continuity and compactness of spectrum, *Math. Scand.*, **35** (1974), 223–236.
4. On spectral subspaces and their applications automorphism groups, *Symposia Math.*, XX, 253–296, Academic Press, 1976.
5. A generalization of a theorem of Bochner, *Proc. AMS*, **57** (1976), 115–118.
6. Appendix to: Applications of the Connes spectrum to C^*-dynamical systems.

OLESEN, D.; PEDERSEN, G. K.
1. Derivations of C^*-algebras have semi-continuous generator, *Pacific J. Math.*, **53** (1974), 563–572.
2. Groups of automorphisms with spectrum condition and the lifting problem, *Comm. Math. Phys.*, **51** (1976), 85–95.
3. Applications of the Connes spectrum to C^*-dynamical systems, *J. Fnal Analysis.*
4. Some C^*-dynamical systems with a single *KMS* state, *Math. Scand.*

OLESEN, D.; PEDERSEN, G. K.; STÖRMER, E.
1. Periodic automorphisms of simple C^*-algebras, Copenhagen, 1976.
2. Compact abelian groups of automorphisms of simple C^*-algebras, *Invent. Math.*, **39** (1977), 55–64.

OLSEN, C. L.
1. Thin operators in a von Neumann algebra, *Acta Sci. Math.*, **35** (1974), 211–216 (Corrections, ibid., **36** (1974), 377–378).
2. A characterization of thin operators in a von Neumann algebra, *Proc. AMS*, **39** (1973), 571–578.
3. Approximation by unitary operators, *Notices AMS*, **23**: 4 (1976), 76 T – B 88.

OLSEN, C. L.; PLASTIRAS, J. K.
1. Quasialgebraic operators, compact perturbations and the essential norm, *Michigan Math. J.*, **21** (1974), 385–397.

OLSEN, C. L.; ZAME, W. R.
1. Some C^*-algebras with a single generator, *Trans. AMS*, **215** (1976), 205–217.

OLSON, P. M.
1. The self-adjoint operators of a von Neumann algebra from a conditionally complete lattice, *Proc. AMS*, **28** (1971), 537–544.

ONO, T.
1. Local theory of rings of operators, I, II, *J. Math. Soc. Jap.*, **10** (1958), 184–216; 438–458.
2. Note on a B^*-algebra, I, II, *J. Math. Soc. Jap.*, **11** (1959), 146–158; *Bull. Nagoya Inst. Tech.*, **21** (1969), 93–95.
3. Local theory in function analysis, *J. Math. Soc. Jap.*, **15** (1963), 9–30.
4. A real analogue of the Gelfand-Neumark theorem, *Proc. AMS*, **25** (1970), 159–160.

5. In a finite AW^*-algebra, $x^*x \leqslant xx^*$ implies $x^*x = xx^*$, I, II, III (Japanese), *Bull. Nagoya Inst. Tech.*, **24** (1972), 61–66; **25** (1973), 85–90; **26** (1974), 119–122.

ORIHARA, M.
1. Sur les anneaux d'opérateurs, I, II, *Proc. Imp. Acad. Tokyo*, **20** (1944), 399–405; 545–553.
2. Rings of operators and their traces, *Mem. Fac. Sci. Kyushu Univ.*, **5** (1950), 107–138 (Corrections, ibidem, **8** (1953), 89–91).

ORNSTEIN, D. S.
1. *Ergodic theory, randomness and dynamical systems*, Yale Math. Monographs, Yale Univ. Press, 1974.

ORSTED, B.
1. Induced representations and Poulsen's new proof of the imprimitivity theorem.

OSIKAWA, M.
1. Point spectra of non-singular flows, *Publ. RIMS, Kyoto Univ.*, **13** (1977), 167–172.

OSTERWALDER, K.
1. Duality for free Bose fields, *Comm. Math. Phys.*, **29** (1973), 1–14.

OTA, S.
1. On a representation of a C^*-algebra in a Lorentz algebra, *Acta Sci. Math.*, **39** (1977), 129–133.
2. Certain operator algebras induced by $*$-derivations in C^*-algebras on an indefinite inner product space.
3. Locally closedness of unbounded derivations in C^*-algebras.

OVTCHINNIKOV, V. I. (in Russian)
1. Symmetric spaces of measurable operators, *Dokl. Akad. Nauk SSSR*, **191** (1970), 769–771.
2. On completely continuous operators related to a von Neumann algebra, *Funkt. Analiz i pril.*, **6** : 1 (1972), 37–40.
3. *s*-numbers of measurable operators, *Funkt. Analiz i pril.*, **4** : 3 (1970), 78–85.

OVTCHINNIKOV, V. I.; PILJUGIN, E. N.
1. Von Neumann algebras and non linear operators (Russian), *Voronež. Gos Univ. Trudy Mat. Fak., vyp.* **16** (1975), 28–34.

PADMANABHAN, A.
1. Some dominated convergence theorems in a von Neumann algebra, *Proc. Jap. Acad.*, **42** (1966), 347–350.
2. Stability and mixing in von Neumann algebras, *Kodai Math. Sem. Rep.*, **18** (1966), 335–342.
3. Convergence in measure and related results in finite rings of operators, *Trans. AMS*, **128** (1967), 359–378 (Corrections, ibidem, **132** (1967), 563).

PAGE, W.
1. Characterizations of B^*- and A^*-algebras, *Rev. Roum. Math. Pures Appl.*, **18** (1973), 1241–1244.

PALIUTKIN, V. G.
1. On the equivalence of two definitions of a finite ring group (Russian), *Ukrainsk. Mat. J.*, **16**: 3 (1964), 402–406.

PALLU DE LA BARRIÈRE, R.
1. Algèbres autoadjointes faiblement fermées et algèbres hilbertiennes de classe finie, *CR Acad. Sc. Paris*, **232** (1951), 1994–1995.
2. Décomposition des opérateurs non bornés dans les sommes continues d'espaces de Hilbert, *CR Acad. Sc. Paris*, **232** (1951), 2071–2073.
3. Algèbres unitaires et espaces d'Ambrose, *CR Acad. Sc. Paris*, **233** (1951), 997–999; *Ann. Éc.Norm. Sup.*, **70** (1953), 381–401.
4. Isomorphismes des $*$-algèbres faiblement fermées d'opérateurs, *CR Acad. Sc. Paris*, **234** (1952), 795–797.
5. Sur les algèbres d'opérateurs dans les espaces hilbertiens, *Bull. Soc. Math. France*, **82** (1954), 1–52.

PALMER, W. T.

1. Unbounded normal operators on Banach spaces, *Trans. AMS*, **133** (1968), 385–414.
2. Characterizations of C^*-algebras, I, II, *Bull. AMS*, 74 (1968), 538–540; *Trans. AMS*, **148** (1970), 577–588.
3. Real C^*-algebras, *Pacific J. Math.*, **35** (1970), 195–204.
4. $*$-representations of U^*-algebras, *Indiana Univ. Math. J.*, **20** (1971), 929–933.
5. The Gelfand-Naimark pseudonorm on Banach algebras, *J. London Math. Soc.*, **3** (1971), 59–67.
6. Hermitian Banach $*$-algebras, *Bull. AMS*, **78** (1972), 522–524.
7. Arens multiplication and a characterization of W^*-algebras, *Proc. AMS*, **44** (1974), 81–87.
8. *The general theory of $*$-algebras.*
9. Characterizations of $*$-homomorphisms and expectations, *Proc. AMS*, **46** (1974), 265–272.
10. Jordan $*$-homomorphisms between reduced Banach $*$-algebras, *Pacific J. Math.*, **58** (1975), 169–178.

PARRY, W.; SCHMIDT, K.

1. A note on cocycles of unitary representations, *Proc. AMS*, **55** (1976), 185–190.

PARTHASARATHY, K. R.; SCHMIDT, K.

1. Infinite divisible projective representations, cocycles and Levy-Khinchin-Araki formula on locally compact groups, *Lecture Notes*, Manchester, 1970.
2. *Positive definite kernels, continuous tensor products and central limit theorems of probability theory*, Lecture Notes in Math., No. 272, Springer Verlag, 1973.
3. On the cohomology of a hyperfinite action, *Monatsch. Math.*

PASCHKE, W. L.

1. Completely positive maps on U^*-algebras, *Proc. AMS*, **34** (1972), 412–416.
2. Inner product modules over B^*-algebras, *Trans. AMS*, **182** (1973), 443–468.
3. Orthogonality of states on C^*-algebras, *Notices AMS*, **21** (1974), 711–46–10.
4. Left ideal annihilating sets of states on C^*-algebras (not published).
5. A factorable Banach algebra without bounded approximate unit, *Pacific J. Math.*, **46** (1973), 249–251.
6. The double B-dual of an inner product module over a C^*-algebra B, *Canad. J. Math.*, **26** (1974), 1272–1280.
7. *Hilbert B-modules and completely positive maps*, Thesis, Univ. Oregon, 1972.
8. Inner product modules arising from compact automorphism groups of von Neumann algebras, *Trans. AMS*, **224** (1976), 87–102.
9. Integrable group actions on von Neumann algebras, *Math. Scand.*, **40** (1977), 234–248.
10. Relative commutant of a von Neumann algebra in its crossed product by a group action.
11. Inner amenability and conjugation operators.

PASCHKE, W. L.; SALINAS, N.

1. Matrix algebras over O_n.

PATERSON, A. L. T.

1. Isometries between B^*-algebras, *Proc. AMS*, **22** (1969), 570–572.
2. Amenability and locally compact semigroups, Aberdeen, 1977.

PATERSON, A. L. T.; SINCLAIR, A. M.

1. Characterization of isometries between C^*-algebras, *J. London Math. Soc.*, **5** (1972), 755–761.

PAWLIK, B.; SADOWSKI, P.

1. On the construction of the maximal additive field of von Neumann algebras in quantum field theory, *Rep. Math. Phys.*, **4** (1973), 223–225.

PEARCY, C.

1. A complete set of unitary invariants for operators generating finite W^*-algebras of type I, *Pacific J. Math.*, **12** (1962), 1405–1416.
2. W^*-algebras with a single generator, *Proc AMS*, **13** (1962), 831–832.

3. On unitary equivalence of matrices over the ring of continuous complex valued functions on a Stonean space, *Canad. J. Math.*, **15** (1963), 323–331.
4. On certain von Neumann algebras which are generated by partial isometries, *Proc. AMS*, **15** (1964), 393–395.
5. Entire functions on infinite von Neumann algebras of type I, *Michigan Math. J.*, **11** (1964), 1–7.
6. On commutators of operators on Hilbert space, *Proc. AMS*, **16** (1965), 53–59.
7. Some unsolved problems in operator theory.
8. Some recent developments in operator theory, Ann Arbor, 1976.

PEARCY, C.; RINGROSE, J. R.
1. Trace preserving isomorphisms on finite operator algebras, *Amer. J. Math.*, **90** (1968), 444–455.

PEARCY, C.; SALINAS, N.
1. Finite dimensional representations of C^*-algebras and the reducing matricial spectra of an operator, *Bull. AMS*, **80** (1974), 970–972; *Rev. Roum. Math. Pures Appl.*, **20** (1975), 567–598.
2. The reducing essential matricial spectra of an operator, *Duke Math. J.*, **42** (1975), 423–434.
3. Extensions of C^*-algebras and the reducing essential matricial spectra of an operator, *Lecture Notes in Math.*, No. 575 (1977), 96–112.

PEARCY, C.; SHIELDS, A.
1. A survey of the Lomonosov technique in the theory of invariant subspaces, Topics in Operator Theory, *Math. Surveys*, No. 18, 1974.

PEARCY, C.; TOPPING, D. M.
1. Sums of small numbers of idempotents, *Michigan Math. J.*, **14** (1967), 453–465.
2. Commutators and certain II_1 factors, *J. Fnal Analysis*, **3** (1969), 69–78.

PEDERSEN, E. A.
1. *A decomposition theory for rings of operators*, Thesis, Louisiana State Univ., 1966.

PEDERSEN, G. K.
1. Measure theory for C^*-algebras, I, II, III, IV, *Math. Scand.*, **19** (1966), 131–145; **22** (1968), 63–74; **25** (1969), 71–93, 121–127.
2. A decomposition theorem for C^*-algebras, *Math. Scand.*, **22** (1968), 266–268.
3. On weak and monotone σ-closure of C^*-algebras, *Comm. Math. Phys.*, **11** (1969), 221–226.
4. Atomic and diffuse functionals on a C^*-algebra, *Pacific J. Math.*, **37** (1971), 795–800.
5. The "Up-Down" problem for operator algebras, *Proc. Nat. Acad. Sc. USA*, **68** (1971), 1896–1897.
6. *C^*-integrals; an approach to non-commutative measure theory*, Thesis, Copenhagen, 1971.
7. Monotone closures in operator algebras, *Amer. J. Math.*, **94** (1972), 955–961.
8. Operator algebras with weakly closed abelian subalgebras, *Bull. London Math. Soc.*, **4** (1972), 171–175.
9. Some operator monotone functions, *Proc. AMS*, **36** (1972), 309–310.
10. Applications of weak semicontinuity to C^*-algebra theory, *Duke Math. J.*, **39** (1972), 431–450.
11. Inner derivations of certain tensor products.
12. Weights on operator algebras, *Int. School Phys. "E. Fermi"*, Varenna,,1973.
13. Borel structure in operator algebras, *Danske Vid. Math.-Fys. Medd.*, **39**, 5 (1974).
14. Some Borel sets in the factor spectrum of a C^*-algebra,
15. A non-commutative version of Suslin's theorem, *Bull. London Math. Soc.*, **8** (1976), 87–90.
16. The trace in semi-finite von Neumann algebras, *Math. Scand.*, **37** (1975), 142–144.
17. On the operator equation $HT + TH = 2K$.
18. Lifting derivations from quotients of separable C^*-algebras, *Proc. Nat. Acad. Sc. USA*, **73** (1976), 1414–1415.
19. Lifting groups of automorphisms, *Symposia Math.*, vol. XX, 161–168, Academic Press, 1976.

20. Similarities among UHF-algebras, *J. Fnal Analysis.*
21. *An introduction to C*-algebra theory* (Chapters I–V, VII), Copenhagen Univ. Publikationserie, No. 1, 2, 3 (1973), 1, 3 (1974), 1, 2 (1977).
22. Derivations of operator algebras, in *C*-algebras and their applications to statistical mechanics and quantum field theory*, p. 139–146, North-Holland, 1976.
23. Spectral formulas in quotient C^*-algebras, *Math. Z.*, 148 (1976), 299–300.
24. Groupes localement compacts d'automorphismes d'une C^*-algèbre et conditions spectrales, *CR Acad. Sc. Paris.*
25. Maximal temperature?
26. *C*-algebras and their automorphism groups*, Academic Press.
27. Approximating derivations on ideals of C^*-algebras.

PEDERSEN, G. K.; PETERSEN, N. H.
1. Ideals in a C^*-algebra, *Math. Scand.*, 27 (1970), 193–204.

PEDERSEN, G. K.; STÖRMER, E.
1. Automorphisms and equivalence in von Neumann algebras, II, *Indiana Univ. Math. J.*, 23 (1973), 121–130.

PEDERSEN, G. K.; TAKESAKI, M.
1. The operator equation THT = K, *Proc. AMS*, 36 (1972), 311–312.
2. The Radon-Nikodym theorem for von Neumann algebras, *Acta Math.*, 130 (1973), 53–88

PEDERSEN, N. V.
1. Duality for induced representations and induced weights, Copenhagen, 1978.

PEETRE, J.; SPARR, G.
1. Interpolation and non-commutative integration, *Ann. Mat. Pura Appl.*, 104 (1975), 187–207.

PELIGRAD, C.
1. Some transitive algebras, *Rev. Roum. Math. Pures Appl.*, 19 (1974), 245–250.
2. Invariant subspaces of von Neumann algebras, I, II, *Acta Sci. Math.*, 37 (1975), 273–277; Bucharest, 1977.
3. Reductive algebras which contain n-strictly cyclic algebras, *Rev. Roum. Math. Pures Appl.*, 19 (1974), 1041–1042.
4. Remarks on a theorem of Douglas and Pearcy, *Rev. Roum. Math. Pures Appl.*, 20 (1975), 359–362.
5. A sufficient condition that a transitive algebra be equal to $B(H)$, *Bucharest*, 1977.

PELIGRAD, C.; ZSIDÓ, L.
1. A Riesz decomposition theorem in W^*-algebras, *Acta Sci. Math.* 34 (1973), 317–322.

PELLETIER, J. W.
1. Categorical characterisation of commutative von Neumann algebras, *Bull. Acad. Polon. Sc.*, 25 (1977), 111–116.

PENNEY, R.
1. Abstract Plancherel theorems and Frobenius reciprocity theorem, *J. Fnal Analysis*, 18 (1975), 177–190.
2. Self-dual cones in Hilbert space, *J. Fnal Analysis*, 21 (1976), 305–315.

PERDRIZET, F.
1. Sur certains espaces de Banach ordonnés, *Bull. Sc. Math.*, 92 (1968), 129–141.
2. Sur les sous-algèbres involutives abéliennes d'un facteur fini, *CR Acad. Sc. Paris*, 268 (1969), 872–875.
3. Espaces de Banach ordonnés et idéaux, *CR Acad. Sc. Paris*, 269 (1969), 393–396; *J. Math. Pures Appl.*, 49 (1970), 61–98.
4. Éléments positifs relatifs à une algèbre hilbertienne à gauche, *CR Acad. Sc. Paris*, 270 (1970), 322–325; *Compositio Math.*, 23 (1971), 25–47.
5. Trace et topologie dans les C^*-algèbres, *CR Acad. Sc. Paris*, 271 (1970), 427–430; *Bull. Soc. Math. France*, 99 (1971), 193–239.

PEREZ, J. F.; WILDE, I. F.
1. On positivity preservation for free quantum fields, *J. Math. Analysis and Appl.*, **51** (1975), 94–99.

PETERS, J.
1. Groups with completely regular primitive dual space, *J. Fnal Analysis*, **20** (1975), 136–148.

PETERSEN, N. H.
1. Invariant weights on semifinite von Neumann algebras, *Math. Scand.*, **32** (1973), 133–144.

PHELPS, R. R.
1. *Lectures on Choquet's theorems*, Van Nostrand, 1966.

PHILLIPS, J.
1. *Tomita's generalized Hilbert algebras and locally compact groups*, Thesis, Oregon, 1972.
2. Positive integrable elements relative to a left Hilbert algebra, *J. Fnal Analysis*, **13** (1973), 390–409.
3. Perturbations of C^*-algebras, *Indiana Univ. Math. J.*, **23** (1974), 1167–1176.
4. Perturbation of type I von Neumann algebras, *Pacific J. Math.*, **52** (1974), 505–512.
5. Complementation for right ideals in generalized Hilbert algebras, *Trans. AMS*, **197** (1974), 409–417.
6. Neighbouring representations of certain C^*-algebras.
7. Automorphisms of full II_1-factors with applications to factors of type III, *Duke Math. J.*, **43** (1976), 375–385.
8. A note on square integrable representations, *J. Fnal Analysis*, **20** (1975), 83–92.
9. Nearest normal approximation for certain operators, *Proc. AMS*, **67** (1977), 236–240.

PHILLIPS, J.; RAEBURN, I.
1. On extensions of *AF*-algebras.

PIMSNER, M.
1. On the Ext-group of an *AF*-algebra, II, *Rev. Roum. Math. Pures Appl.*, 1978.

PIMSNER, M.; POPA, S.
1. On the Ext-group of an *AF*-algebra, *Rev. Roum. Math. Pures Appl.*, **23** (1978), 251–267.
2. The Ext-group of some C^*-algebras considered by J. Cuntz, *Rev. Roum. Math. Pures Appl.*, 1978.

PIMSNER, M.; POPA, S.; VOICULESCU, D.
1. Homogeneous C^*-extensions of $C(X) \otimes K(H)$, Bucharest, 1977.

PLASTIRAS, J.
1. Compact perturbations of C^*-algebras, *Symposia Math.*, vol. XX, 507–512, Academic Press, 1976.
2. Compact perturbations of certain von Neumann algebras, *Trans. AMS*, **234** (1977), 561–577.
3. Quasitriangular operator algebras, *Pacific J. Math.*, **64** (1976), 543–549.
4. C^*-algebras isomorphic after tensoring, *Proc. AMS*, **66** (1977), 276–278.

PLESNER, A. I.
1. *Spectral theory of linear operators* (Russian), Moscow, 1940, 1966.

PLYMEN, R. J.
1. C^*-algebras and Mackey's axioms, *Comm. Math. Phys.*, **8** (1968), 132–146.
2. Dispersion-free normal states, *Nuovo Cimento*, **54** (1968), 862–870.
3. A modification of Piron's axioms, *Helv. Acta Phys.*, **41** (1968), 69–74.
4. Spinors in Hilbert space, *Math. Proc. Cambridge Phil. Soc.*, **80** (1976), 337–348.
5. Recent results on infinite dimensional spin-groups, *Adv. Math.*
6. Projective representations of the infinite orthogonal group, *Mathematika*, **24** (1977), 115–121.

PORCELLI, P.; PEDERSEN, E. A.
1. On rings of operators, *Bull. AMS*, **73** (1967), 142–144.

PORTA, H.
1. Two-sided ideals of operators, *Bull. AMS*, **75** (1969), 599–602.

2. A note on homomorphisms of operator algebras, *Colloq. Math.*, **20** (1969), 117–119.
3. Ideals and universal representations of certain C^*-algebras, *Rev. Union Mat. Argentina*, **25** (1970), 27–36.

PORTA, H.; SCHWARTZ, J. T.
1. Representations of the algebra of all operators in Hilbert space and related analytic function algebras, *Comm. Pure Appl. Math.*, **20** (1967), 457–492.

POWER, S. C.
1. C^*-modules and odd-even decomposition for C^*-algebras, *Bull. London Math. Soc.*, **8** (1976), 268–272.
2. C^*-algebras generated by Hankel operators and Toeplitz operators.

POWERS, R. T.
1. Representations of uniformly hyperfinite algebras and their associated von Neumann algebras, *Bull. AMS*, **73** (1967), 572–575; *Ann. Math.*, **86** (1967), 138–171.
2. *Representations of canonical anticommutation relations*, Thesis, Princeton, 1968.
3. UHF algebras and their applications to representations of the anticommutation relations, *Cargese Lectures in Physics*, **4** (1970), 137–168.
4. Fermi field algebra, *Cargese Lectures in Physics*, **4** (1970), 363–367.
5. Self-adjoint algebras of unbounded operators, I, II, *Comm. Math. Phys.*, **21** (1971), 85–124, *Trans. AMS*, **187** (1974), 261–294.
6. Simplicity of the C^*-algebra associated with the free group on two generators, *Duke Math. J.*, **42** (1975), 151–156.
7. A remark on the domain of an unbounded derivation of a C^*-algebra, *J. Fnal Analysis*, **18** (1975), 85–95.
8. Resistance inequalities of KMS states of the isotropic Heisenberg model, *Comm. Math. Phys.*, **51** (1976), 151–156.
9. Resistance inequalities for the isotropic Heisenberg model, *J. Math. Phys.*

POWERS, R. T.; SAKAI, S.
1. Existence of ground states and KMS states for approximately inner dynamics, *Comm. Math. Phys.*, **39** (1975), 273–288.
2. Unbounded derivations in operator algebras, *J. Fnal Analysis*, **19** (1975), 81–95.

POWERS, R. T.; STÖRMER, E.
1. Free states of the canonical anticommutation relations, *Comm. Math. Phys.*, **16** (1970), 1–33.

PRESUTTI, E.; SCACCIATELLI, E.
1. Locally bounded perturbations of KMS states, *J. Math. Phys.*, **15** (1974), 1620–1626.

PRESUTTI, E.; SCACCIATELLI, E.; SEWELL, B. L.; WANDERLINGH, F.
1. Studies in the C^*-algebraic theory of non-equilibrium statistical mechanics: dynamics of open mechanically driven systems, *J. Math. Phys.*, **13** (1972), 1085–1098.

PRICE, J. F.
1. On positive definite functions over locally compact groups, *Canad. J. Math.*, **22** (1970), 892–896.

PRIESTLEY, W. M.
1. A non-commutative Korovkin theorem, *J. Approx. Theory*, **16** (1976), 251–260.

PROGUVECKI, E.
1. *Quantum mechanics in Hilbert space*, Academic Press, 1971.

PROMISLOW, D.
1. The Kakutani theorem for tensor product of W^*-algebras, *Pacific J. Math.*, **36** (1971), 507–514.
2. Semi-metrics on the normal states of a W^*-algebra, *Canad. J. Math.*, **25** (1973), 1238–1253.

PROSSER, R. T.
1. *On the ideal structure of operator algebras*, Memoirs AMS, 45 (1963).

2. A comparison theorem for operators with compact resolvent, *Proc. AMS*, **27** (1971), 519–521.

PTÁK, V.

1. On the spectral radius in Banach algebras with involution, *Bull. London Math. Soc.*, **2** (1970), 327–334.
2. Banach algebras with involution, *Manuscripta Math.*, **6** (1972), 245–290.
3. A theorem of the closed graph type, *Manuscripta Math.*, **13** (1974), 109–130.

PUKÁNSZKY, L.

1. On a theorem of Mautner, *Acta Sci. Math.*, **15** (1954), 145–148.
2. The theorem of Radon-Nikodym in operator rings, *Acta Sci. Math.*, **15** (1954), 149–156.
3. On the theory of quasi-unitary algebras, *Acta Sci. Math.*, **16** (1955), 103–121.
4. Some examples of factors, *Publ. Math. Debrecen*, **4** (1956), 135–156.
5. On maximal abelian subrings of factors of type II_1, *Canad. J. Math.*, **12** (1960), 289–296.
6. *Leçons sur les représentations des groupes*, Dunod, Paris, 1967.
7. On the theory of exponential groups, *Trans. AMS*, **126** (1967), 487–507.
8. On the characters and the Plancherel formula of nilpotent groups, *J. Fnal Analysis*, **1** (1967), 255–280.
9. On the unitary representations of exponential groups, *J. Fnal. Analysis*, **2** (1968), 37–113.
10. Characters of algebraic solvable groups, *J. Fnal Analysis*, **3** (1969), 435–494.
11. Sur la représentation régulière des groupes de Lie résolubles connexes, I, II, *CR Acad. Sc. Paris*, **268** (1969), 1077–1079, 1172–1173.
12. Unitary representations of solvable Lie groups, *Ann. Sci. Ec. Norm. Sup.*, **4** (1971), 457–608.
13. Action of algebraic groups of automorphisms on the dual of a class of type I groups, *Ann. Sci. Ec. Norm. Sup.*, **5** (1972), 379–396.
14. Characters of connected Lie groups, *Bull. AMS*, **80** (1974), 709–712; *Acta Math.*, **133** (1974), 81–137.
15. The primitive ideal space of a solvable Lie group, *Invent. Math.*, **22** (1973), 75–118.
16. Lie groups with completely continuous representations, *Bull. AMS*, **81** (1975), 1061–1063.

PULVIRENTI, M.

1. Stability, KMS, and selfadjointness of the Liouville operator in classical systems, Camerino, 1977.

PUSZ, W.; WORONOWICZ, S. L.

1. Functional calculus for sesquilinear forms and the purification map, *Rep. Math. Phys.*, **8** (1975), 159–170.
2. Positive states and KMS states for general quantum systems, Marseille, 1977.

PUTNAM, C. R.

1. *Commutation properties of Hilbert space operators*, Springer Verlag, 1967.

RADIN, CH.

1. Approach to equilibrium in a simple model, *J. Math. Phys.*, **11** (1970), 2945–2955.
2. Non-commutative mean ergodic theory, *Comm. Math. Phys.*, **21** (1971), 291–302.
3. Automorphisms of von Neumann algebras as point transformations, *Proc. AMS*, **39** (1973), 343–346.
4. Ergodicity in von Neumann algebras, *Pacific J. Math.*, **48** (1973), 235–239.
5. Dynamics of limit models, *Comm. Math. Phys.*, **33** (1973), 283–292.
6. Gentle perturbations, *Comm. Math. Phys.*, **23** (1971), 189–198.
7. A non-commutative L^1-mean ergodic theorem, *Adv. Math.*, **21** (1976), 110–111.
8. Pointwise ergodic theory and operator algebras.

RADJAVI, H.

1. Non-selfadjoint representations of C^*-algebras, *Proc. AMS*, **47** (1975), 133–136.

RADJAVI, H.; ROSENTHAL, P.

1. *Invariant subspaces*, Springer Verlag, 1973.

RAEBURN. I.; TAYLOR, J. L.
1. Hochschild cohomology and perturbations of Banach algebras, *J. Fnal Analysis*, **25** (1977), 258–266.

RAÏKOV, D. A. (in Russian)
1. On the theory of normed rings with an involution, *Dokl. Akad. Nauk SSSR*, **54** (1946), 387–390.
2. On some types of convergence for positive definite functions, *Dokl. Akad. Nauk SSSR*, **58** (1947), 1279–1282.

RALL, C.
1. Über-Boolsche Algebren von Projektionen, *Math. Z.*, **153** (1977), 199–217.

RAMSAY, A.
1. Virtual groups and group actions, *Adv. Math.*, **6** (1971), 253–322.
2. Boolean duals of virtual groups, *J. Fnal Analysis*, **15** (1974), 56–101.
3. Subobjects of virtual groups.
4. Non-transitive quasi-orbits in Mackey's analysis of group extensions.

REED, M. C.
1. On self-adjointness in infinite tensor product spaces, *J. Fnal Analysis*, **5** (1970), 94–124.
2. Torus invariance for the Clifford algebra, I, II, *Trans. AMS*, **154** (1971), 177–183; *J. Fnal Analysis*, **8** (1971), 450–468.

REED, M. C.; SIMON, B.
1. A spectral mapping theorem for tensor products of unbounded operators, *Bull. AMS*, **78** (1972), 730–732.
2. Tensor products of closed operators on Banach spaces, *J. Fnal Analysis*, **13** (1973), 107-124.
3. *Methods of modern mathematical physics*, vols. I, II, III, Academic Press.

REEH, H.; SCHLIEDER, S.
1. Bemerkungen zur Unitarequivalenz von Lorentzinvarianten Feldern, *Nuovo Cimento*, **22** (1961), 1051–1068.

REID, G. A.
1. A generalization of W^*-algebras, *Pacific J. Math.*, **15** (1965), 1019–1026.
2. Epimorphisms and surjectivity, *Invent. Math.*, **9** (1970), 295–307.
3. On a question of Dixmier, *J. London Math. Soc.*, **3** (1971), 544–548.
4. On the Calkin representation, *Proc. London Math. Soc.*, **23** (1971), 547–564.

REITER, H.
1. *Classical harmonic analysis and locally compact groups*, Clarendon Press, Oxford, 1968.
2. *L^1-algebras and Segal algebras*, Lecture Notes in Math., No. 231, Springer Verlag, 1971.

RENAUD, P. F.
1. General ergodic theorems for locally compact groups, *Amer. J. Math.*, **93** (1971), 52–63.
2. Invariant means on a class of von Neumann algebras, *Trans. AMS*, **170** (1972), 285–291.

RENOUARD, P.; FUCHS, G.
1. Séminaire sur la théorie constructive des champs d'après Glimm et Jaffe. La théorie $\lambda(\varphi^{2n})_2$ sans cutoffs, Éc. Polytechnique, Paris, 1971.

RENSHAW, B. B.
1. On the reduction theory for W^*-algebras.
2. Normed modules over W^*-algebras.
3. On the tensor product of W^*-algebras, *Trans. AMS*, **194** (1974), 337–347.

REYNOLDS, M.
1. Perturbation of groups of automorphisms of von Neumann algebras, *Proc. AMS*, **55** (1976), 326–328.

RICHTER, P.
1. Unitary representations of countable infinite dimensional Lie groups, Leipzig, 1977.

RICKART, C. E.

1. Banach algebras with an adjoint operation, *Ann. Math.*, **47** (1946), 528–550.
2. The singular elements of a Banach algebra, *Duke Math. J.*, **14** (1947), 1066–1077.
3. The uniqueness of norm problem in Banach algebras, *Ann. Math.*, **51** (1950), 615–628.
4. Representations of certain Banach algebras on Hilbert space, *Duke Math. J.*, **18** (1951), 27–39.
5. Spectral permanence for certain Banach algebras, *Proc. AMS*, **4** (1953), 191–196.
6. *General theory of Banach algebras*, Van Nostrand, 1960.

RIDEAU, G.

1. On some representations of anticommutation relations, *Comm. Math. Phys.*, **9** (1968), 229–241.

RIEFFEL, M. A.

1. Induced Banach representations of Banach algebras and locally compact groups, *J. Fnal Analysis*, **1** (1967), 443–491.
2. Square integrable representations of Hilbert algebras, *J. Fnal Analysis*, **3** (1969), 265–300.
3. On the continuity of certain intertwining operators, centralizers and positive linear functionals, *Proc. AMS*, **20** (1969), 455–457.
4. Unitary representations induced from compact subgroups, *Studia Math.*, **42** (1972), 145–175.
5. On the uniqueness of the Heisenberg commutation relations, *Duke Math. J.*, **39** (1972). 745–753.
6. Induced representations of C^*-algebras, *Bull. AMS*, **78** (1972), 605–609; *Adv. Math.*, **13** (1974), 176–257.
7. Morita equivalence for C^*-algebras and W^*-algebras, *J. Pure Appl. Alg.*, **5** (1974), 51–96.
8. A commutation theorem and duality for free Bose fields, *Comm. Math. Phys.*, **39** (1974), 153–164.
9. Commutation theorems and generalized commutation relations, *Bull. Soc. Math. France*, **104** (1976), 205–224.
10. Strong Morita equivalence of certain transformation group C^*-algebras, *Math. Ann.*, **222** (1976), 7–22.
11. Induced representations of rings, *Canad. J. Math.*, **27** (1975), 261–270.
12. Unitary representations of group extensions: an algebraic approach to the theory of Mackey and Blattner, *Adv. Math.*
13. The type of group measure space von Neumann algebras.
14. Von Neumann algebras associated with some discrete cocompact subgroups.

RIEFFEL, M. A.; VAN DAELE, A.

1. The commutation theorem for tensor products of von Neumann algebras, *Bull. London Math. Soc.*, **7** (1975), 257–260.
2. A bounded operator approach to Tomita-Takesaki theory, *Pacific J. Math.*, **69** (1977), 187–221.

RIESZ, F.; SZ.-NAGY, B.

1. *Leçons d'analyse fonctionnelle*, Akadémiai Kiadó, Budapest, 1952, 1953, 1954, 1965.

RIGOTTI, C.

1. Remarks on the modular operator and local observables, Roma, 1978.

RINGROSE, J. R.

1. On subalgebras of a C^*-algebra, *Pacific J. Math.*, **15** (1965), 1377–1382.
2. On some algebras of operators, I, II, *Proc. London Math. Soc.*, **15** (1965), 61–83; **16** (1966), 385–402.
3. *Lecture Notes on von Neumann algebras*, Univ. Newcastle upon Tyne, 1966/67.
4. *Lectures on Banach algebras and spectral theory*. Mimeographed Notes.
5. Lectures on the trace in finite von Neumann algebras, *Lecture Notes in Math.*, No. 247, 309–354, Springer Verlag, 1972.
6. Cohomology of operator algebras, *Lecture Notes in Math.*, No. 247, 355–435, Springer Verlag, 1972.
7. Automatic continuity of derivations of operator algebras, *J. London Math. Soc.*, **5** (1972) 432–438.

8. Linear functionals on operator algebras and their abelian subalgebras, *J. London Math. Soc.*, 7 (1974), 553–560.
9. Derivations of quotients of von Neumann algebras, *Proc. LMS*, 36 (1978) 1–26.
10. Linear mappings between operator algebras, *Symposia Math.*, vol. XX, 297–316, Academic Press, 1976.
11. *Compact non-selfadjoint operators*, Van Nostrand, 1971.
12. Operator algebras and their abelian subalgebras, *Proc. Int. Congr. Math.*, Vancouver, 1974, 105–110.
13. The global theory of von Neumann algebras, *Rend. Sem. Mat. Fis., Milano*, **45** (1975), 49–63.

ROBERTS, J. E.
1. Cross products of von Neumann algebras by group duals, *Symposia Math.*, vol. XX, 335–364, Academic Press, 1976.
2. Must there be a gauge group? *Trudy Mat. Inst. Akad. Nauk SSSR*, **135** (1975), 206–209.
3. Local cohomology and superselection structure, *Comm. Math. Phys.*, **51** (1976), 107–120.
4. A survey of local cohomology, Marseille, 1977.

ROBERTS, J. E.; ROEPSTROFF, G.
1. Some basic concepts of algebraic quantum theory, DESY, Hamburg, 1968.

ROBERTSON, A. G.
1. A note on the unit ball in C^*-algebras, *Bull. London Math. Soc.*, 6 (1974), 333–335.
2. Standard states of C^*-algebras, *J. London Math. Soc.*, **11** (1975), 423–433.
3. States which have a trace-like property relative to a C^*-subalgebra of $B(H)$, *Glasgow Math. J.*, **17** (1976), 158–160.
4. On the density of the invertible group in C^*-algebras, *Proc. Edinburgh Math. Soc.*, **20** (1976), 153–156.
5. Best approximation in von Neumann algebras, *Math. Proc. Cambridge Phil. Soc.*, **81** (1977), 233–236.
6. A Korovkin theorem for Schwarz maps on C^*-algebras, *Math. Z.*, **156** (1977), 205–208.

ROBINSON, D. W.
1. Algebraic aspects of relativistic quantum field theory, *Brandeis Univ. Summer School*, 1965, vol. I, Gordon & Breach, 1966.
2. Statistical mechanics of quantum systems, I, II, *Comm. Math. Phys.*, 6 (1967), 151–160; 7 (1968), 337–348.
3. Normal and locally normal states, *Comm. Math. Phys.*, **19** (1970), 219–234.
4. Statistical mechanics of quantum mechanical particles with hard cores, I: The thermodynamical pressure, *Comm. Math. Phys.*, 16 (1970), 290–309.
5. Existence theorems in quantum statistical mechanics, *Cargese Lectures in Phys.*, 4 (1970), 277–302.
6. Return to equilibrium, *Comm. Math. Phys.*, **31** (1973), 171–190.
7. The ground state of the Bose gas, *Comm. Math. Phys.*, 1 (1965), no. 2.
8. A characterization of clustering states, *Comm. Math. Phys.*, **41** (1975), 79–88.
9. Time dependent scattering theory, Marseille, 1974.
10. The approximation of flows, *J. Fnal Analysis*.
11. Unbounded derivations of C^*-algebras, *Lecture Notes in Phys.*, No. 39, 303–310, Springer Verlag, 1975.
12. Dynamics in quantum statistical mechanics, ZIF, Bielefeld, 1976.
13. The decomposition of states. A review, Marseille, 1977.
14. C^*-algebras and quantum statistical mechanics, in *C^*-algebras and their applications to statistical mechanics and quantum field theory*, p. 235–252, North-Holland, 1976.

ROBINSON, D. W.; RUELLE, D.
1. Mean entropy of states in classical statistical mechanics, *Comm. Math. Phys.*, **5** (1967), 288–300.
2. Extremal invariant states, *Ann. Inst. H. Poincaré*, 6 (1967), 299–310.

ROBINSON, G. B.
1. *Representation theory of the symmetric group*, Math. Expositions, No. 12, Univ. Toronto Press, 1961.

ROCCA, F.; SIRUGUE, M.; TESTARD, D.
1. Les états quasi-libres comme états d'équilibre sous les conditions KMS, *CR Acad. Sc. Paris*, **267** (1968), 672–673.
2. États quasi-libres invariants de translation et transformations de Bogoliubov, *CR Acad. Sc. Paris*, **267** (1968), 722–723.
3. Translation invariant quasi-free states and Bogoliubov transformations, *Ann. Inst. H. Poincaré*, **10** (1969), 247–258.
4. On a class of equilibrium states under the KMS boundary conditions, I, II, *Comm. Math. Phys.*, **13** (1969), 317–334; **19** (1970), 119–141.

ROCCA, F.; SISSON, P. N. M.; VERBEURE, A.
1. A class of states on the boson-fermion algebra, *J. Math. Phys.*, **17** (1976), 665–667.

ROHLIN, V. A. (in Russian)
1. Unitary rings, *Dokl. Akad. Nauk, SSSR*, **59** (1948), 643–646.
2. On the fundamental ideas of measure theory, *Matem. Sb.*, **25** (1949), 107–150.

VAN ROOIJ, A. C. M.; SCHIKHOF, W. F.
1. Non-archimedian commutative C^*-algebras, *Indag. Math.*, **35** (1973), No. 4.

ROOS, H.
1. Independence of local algebras in quantum field theory, *Comm. Math. Phys.*, **16** (1970), 238–246.

ROOS, J.-E.
1. Sur l'anneau maximal de fractions des AW^*-algèbres et des anneaux de Baer, *CR Acad. Sc. Paris*, **266** (1968), 120–123.

ROSENBERG, A.
1. The number of irreducible representations of simple rings with no minimal ideals, *Amer. J. Math.*, **75** (1953), 523–530.

ROSENBERG, J.
1. The C^*-algebras of some real and p-adic solvable groups, *Pacific J. Math.*, **65** (1976), 175–192.
2. Square-integrable factor representations of locally compact groups, *Trans. AMS*.
3. Amenability of crossed products of C^*-algebras, *Comm. Math. Phys.*, **57** (1977), 187–191.
4. Frobenius reciprocity for square integrable factor representations, *Illinois J. Math.*
5. A quick proof of Harish-Chandra's Plancherel theorem for spherical functions on a semisimple Lie groups, *Proc. AMS*, **63** (1977), 143–149.

ROSENBLUM, M.
1. On a theorem of Fuglede and Putnam, *J. London Math. Soc.*, **33** (1958), 376–377.

ROUSSEAU, R.
1. The left Hilbert algebra associated to a semi-direct product, *Math. Proc. Cambridge Phil. Soc.*
2. Construction of a left Hilbert algebra with respect to a Minkowski form.
3. The covariance algebra of an extended covariant system.
4. An example in the theory of crossed products.

ROUSSEAU, R.; VAN DAELE, A.; VAN HEESWIJCK, L.
1. A note on the commutation theorem for tensor products of von Neumann algebras, *Proc. AMS*, **61** (1976), 179–180.
2. A necessary and sufficient condition for a von Neumann algebra to be in standard form, *J. London Math. Soc.*, **15** (1977), 147–154.

ROWLANDS, K.
1. A note on multiplier operators and dual B^*-algebras, *Canad. Math. Bull.*, **17** (1974), 563–565.

RUDIN, W.
1. *Fourier Analysis on Groups*, Interscience Publ., 1962.
2. An extension theorem for positive definite functions, *Duke Math. J.*, **37** (1970), 49–54.

RUELLE, D.

1. States of physical systems, *Comm. Math. Phys.*, 3 (1966), 133–150.
2. States of classical statistical mechanics, *J. Math. Phys.*, 8 (1967), 1657–1668.
3. Quantum statistical mechanics and canonical commutation relations, *Cargese Lectures in Phys.*, 3 (1967), 85–106.
4. *Statistical Mechanics. Rigorous results*, Benjamin, 1969.
5. New methods and problems in statistical mechanics, *Lecture Notes in Math.*, No. 140, 68–83, Springer Verlag, 1970.
6. Symmetry breakdown in statistical mechanics, *Cargese Lectures in Phys.*, 4 (1970), 169–194.
7. Integral representations of states on a B^*-algebra, *J. Fnal Analysis*, 6 (1970), 116–151.

RUESS, W.

1. Sur quelques problèmes de topologies strictes, *CR Acad. Sc. Paris*, 283 (1976), 603–606.

RUSKAI, M. B.

1. Time development of quantum lattice systems, *Comm. Math. Phys.*, 20 (1971), 193–204.
2. Inequalities for traces on von Neumann algebras, *Comm. Math. Phys.*, 26 (1972), 280–289.
3. A generalization of entropy using traces on von Neumann algebras, *Ann. Inst. H. Poincaré*, A, 19 (1974), 357–374.

RUSSO, B.

1. Linear mappings of operator algebras, *Proc. AMS*, 17 (1966), 1019–1022.
2. Isometries of L^p-spaces associated with finite von Neumann algebras, *Bull. AMS*, 74 (1968), 228–232.
3. Unimodular contractions in Hilbert spaces, *Pacific J. Math.*, 26 (1968), 163–169.
4. Isometries of the trace class, *Proc. AMS*, 23 (1969), 213.
5. Trace preserving mappings of matrix algebras, *Duke Math. J.*, 36 (1969), 297–300.
6. *Algebre di operatori e gruppi localmente compatti*, Mimeographed Notes, Univ. Genoa, 1972.
7. The norm of L^p-Fourier transform on unimodular groups, I, II, III, *Trans. AMS*, 192 (1974), 293–305; *Canad. J. Math.*, 28 (1976), 1121–1131.
8. On the Hausdorff-Young theorem for integral operators, Irwine, 1976.
9. Operator theory in harmonic analysis, Irwine, 1977.

RUSSO, B.; DYE, H. A.

1. A note on unitary operators in C^*-algebras, *Duke Math. J.*, 33 (1966), 413–416.

RYLL-NARDZEWSKI, C.

1. On fixed points of semigroups of endomorphisms of linear spaces, *Proc. Vth Berkeley Symp. Prob.*, 1967, vol. II, part I, 55–61.

SAÏDEEVA, D. M.

1. Factors of sequentially complete C^*-algebras (Russian), 1977.

SAINTE-BEUVE, M. F.

1. On the extension of von Neumann-Aumann's theorem, *J. Fnal Analysis*, 17 (1974), 112–129.

SAITÔ, K.

1. On the preduals of W^*-algebras, *Tohoku Math. J.*, 19 (1967), 324–331.
2. Non-commutative extension of Lusin's theorem, *Tohoku Math. J.*, 19 (1967), 332–340.
3. On duality for locally compact groups, *Tohoku Math. J.*, 20 (1968), 355–367.
4. On the algebra of measurable operators for a general AW^*-algebra, I, II, III, *Tohoku Math. J.*, 21 (1969), 249–270; 23 (1971), 525–534.
5. A non-commutative integration theory for a semifinite AW^*-algebra and a problem of Feldman, *Proc. Jap. Acad.*, 46 (1970), 463–467; *Tohoku Math. J.*, 22 (1970), 420–461.
6. On the embedding as a double commutator in a type I AW^*-algebra, I, II, *Tohoku Math. J.*, 23 (1971), 541–558; 26 (1974), 333–340.
7. Automorphisms groups of von Neumann algebras and ergodic type theorems, *Acta Sci. Math.*, 36 (1974), 119–130.
8. Groups of $*$-automorphisms and invariant maps of von Neumann algebras, *Pacific J. Math.*, 57 (1975), 553–558.

9. A note on the continuity of automorphic representations of groups, *Tohoku Math. J.*, **28** (1967), 305—310.

SAITO, K.-S.
1. On non-commutative Hardy spaces associated with flows on finite von Neumann algebras, *Japan-US Seminary*, 1977.
2. The Hardy spaces associated with a periodic flow on a von Neumann algebra, *Tohoku Math. J.*, **29** (1977), 69—75.
3. On the maximality of $H^\infty(\alpha)$ in finite von Neumann algebras, *Sci. Rep. Niigata Univ.*, A, **14** (1977), 1—3.

SAITO, T.
1. On incomplete infinite direct products of W^*-algebras, *Tohoku Math. J.*, **10** (1958), 165—171.
2. The direct product and crossed product of rings of operators, *Tohoku Math. J.*, **11** (1959), 299—304.
3. Some remarks on a representation of a group, I, II, *Tohoku Math. J.*, **12** (1960), 383—388; **17** (1965), 206—209.
4. On a representation of a countable infinite group, *Tohoku Math. J.*, **13** (1961), 268—273.
5. On groups of automorphisms of finite factors, *Tohoku Math. J*, **13** (1961), 427—433.
6. Some theorems on the crossed products of finite factors, *Tohoku Math. J.*, **14** (1962), 312—317.
7. Generators of certain von Neumann algebras, *Tokoku Math. J.*, **20** (1968), 101—105.
8. On generators of von Neumann algebras, *Michigan Math. J.*, **15** (1968), 373—376.
9. A remark on generators of von Neumann algebras, *Michigan Math. J.*, **16** (1969), 1—4.
10. Generation of von Neumann algebras, *Lecture Notes in Math.*, No. 247, 436—533, Springer Verlag, 1972.
11. Hyponormal operators and related topics, *Lecture Notes in Math.*, No. 247, 534—665, Springer Verlag, 1972.

SAITO, T.; TOMIYAMA, J.
1. Some results on the product of W^*-algebras, *Tohoku Math. J.*, **12** (1960), 455—458.

SAKAI, S.
1. A remark on Mautner's decomposition, *Kodai Math. Sem. Rep.*, **4** (1952), 107—108.
2. On the group isomorphism of unitaries in AW^*-algebras, *Tohoku Math. J.*, **7** (1955), 87—95.
3. A characterization of W^*-algebras, *Pacific J. Math.*, **6** (1956), 763—773.
4. The absolute value of W^*-algebras of finite type, *Tohoku Math. J.*, **8** (1956), 70—85.
5. On the σ-weak topology of W^*-algebras, *Proc. Jap. Acad.*, **32** (1956), 329—332.
6. On topological properties of W^*-algebras, *Proc. Jap. Acad.*, **33** (1957), 439—444.
7. On linear functionals of W^*-algebras, *Proc. Jap. Acad.*, **34** (1958), 571—574.
8. On some problems of C^*-algebras, *Tohoku Math. J.*, **11** (1959), 453—455.
9. On a conjecture of Kaplansky, *Tohoku Math. J.*, **12** (1960), 31—33.
10. *The theory of W^*-algebras*, Lecture Notes, Yale Univ., 1962.
11. On the reduction theory of von Neumann, *Bull. AMS*, **70** (1964), 393—398.
12. Weakly compact operators on operator algebras, *Pacific J. Math.*, **14** (1964), 659—664.
13. On the central decomposition for positive functionals on C^*-algebras, *Trans. AMS*, **118** (1965), 406—419.
14. On topologies on finite W^*-algebras, *Illinois J. Math.*, **9** (1965), 236—241.
15. A Radon-Nikodym theorem in W^*-algebras, *Bull. AMS*, **71** (1965), 149—151.
16. On pure states of C^*-algebras, *Proc. AMS*, **17** (1966), 86—87.
17. Derivations of W^*-algebras, *Ann. Math.*, **83** (1966,) 273—279.
18. On a problem of Calkin, *Amer. J. Math.*, **88** (1966), 935—941.
19. On a characterization of type I C^*-algebras, *Bull. AMS*, **72** (1966), 508—512.
20. On type I C^*-algebras, *Proc. AMS*, **18** (1967), 861—863.
21. Derivations of uniformly hyperfinite C^*-algebras, *Publ. RIMS, Kyoto Univ.*, **3** (1967), 167—175.
22. Derivations of simple C^*-algebras, I, II, III, IV, *J. Fnal Analysis*, **2** (1968), 202—206; *Bull. Soc. Math. France*, **99** (1971), 259—263; *Tohoku Math. J.*, **23** (1971), 559—564.
23. On the tensor product of W^*-algebras, *Amer. J. Math.*, **90** (1968), 335—341.

24. On the hyperfinite II_1 factor, *Proc. AMS*, **19** (1968), 589–591.
25. Asymptotically abelian II_1 factor, *Publ. RIMS, Kyoto Univ.*, **4** (1968), 299–307.
26. On global W^*-algebras, *J. Fnal Analysis*, **3** (1969), 79–84.
27. An uncountable family of non-hyperfinite type III factors, *Functional Analysis* (Ed. C. O. Wilde), Academic Press, 1970, 65–70.
28. An uncountable number of II_1 and II_∞ factors, *J. Fnal Analysis*, **5** (1970), 236–246.
29. On a Stone-Weierstrass theorem for C^*-algebras, *Tohoku Math. J.*, **22** (1970), 191–199; *Indiana Univ. Math. J.*, **20** (1971), 941.
30. On global type II_1 W^*-algebras, *J. Fnal Analysis*, **8** (1971), 95–100.
31. Derived C^*-algebras of primitive C^*-algebras, *Tohoku Math. J.*, **25** (1973), 307–316.
32. *C^*-algebras and W^*-algebras*, Springer Verlag, 1971.
33. Automorphisms and tensor product of operator algebras, *Amer. J. Math.*, **97** (1975), 889–896.
34. One parameter subgroups of $*$-automorphisms on operator algebras and the corresponding unbounded derivations, *Amer. J. Math.*, **98** (1976), 427–440.
35. On automorphism groups of II_1 factors, *Tohoku Math. J.*, **26** (1974), 423–430.
36. On commutative normal $*$-derivations, I, II, III, *Comm. Math. Phys.*, **43** (1975), 39–40; *J. Fnal. Analysis*, **21** (1976), 203–208.
37. Unbounded derivations in C^*-algebras, *Symposia Math.*, vol. XX, 233–240, Academic Press, 1976.
38. Recent developments in the theory of unbounded derivations in C^*-algebras, *Japan-US Seminary*, 1977.

SALINAS, N.
1. Extensions of C^*-algebras and essentially n-normal operators, *Bull. AMS*, **82** (1976), 143–146.
2. Homotopy invariance of Ext(A), *Duke Math. J.*, **44** (1977), 777–794.

SAMUELIDES, M.
1. Mesure de Haar et W^*-couple d'un groupoïde mesuré, *Bull. Soc. Math. France*.

SAMUELIDES, M.; SAUVAGEOT, J.-L.
1. Représentations des algèbres de Lie localement convexes. Applications aux relations de commutations, *Bull. Sc. Math.*, **98** (1974), 3–16.
2. Algèbre de Krieger d'un système dynamique, *CR Acad. Sc. Paris*, **280** (1975), 709–712.

SANKARAN, S.
1. The $*$-algebra of unbounded operators, *J. London Math. Soc.*, **34** (1959), 337–344.
2. Ordered decompositions on Hilbert spaces, *J. London Math. Soc.*, **36** (1961), 97–107.
3. Decompositions of von Neumann algebras of type I, *Math. Ann.*, **142** (1961), 399–406.
4. Stochastic convergence for operators, *Quart. J. Math.*, **15** (1964), 97–102.

SANKARAN, S.; SELESNIK, S. A.
1. Some remarks on C^*-bigebras and duality, *Semigroup Forum*, **3** (1971), 108–129.

SASAKI, U.
1. Lattices of projections in AW^*-algebras, *J. Sci. Hiroshima Univ.*, **19** (1955), 1–30.

SAUVAGEOT, J. L.
1. Sur le type du produit croisé d'une algèbre de von Neumann par un groupe localement compact d'automorphismes, *CR Acad. Sc. Paris*, **278** (1974), 941–944; *Bull. Soc. Math. France*.
2. Image d'un homomorphisme et flot des poids d'une relation d'équivalence mesurée, *CR Acad. Sc. Paris*, **282** (1976), 619–622; preprint, Paris, 1976.
3. Idéaux primitifs de certains produits croisés, *Math. Ann.*, **231** (1977), 61–76.
4. Idéaux primitifs induits dans les produits croisés, Paris, 1977.

SCHAEFFER, D.
1. An application of von Neumann algebras to finite difference equations, *Ann. Math.*, **95** (1972), 117–129.

SCHATTEN, R.
1. *A theory of cross spaces*, Ann. Math. Studies, No. 26 (1950), Princeton.
2. *Norm ideals of completely continuous operators*, Springer Verlag, 1960.

SCHATZ, J. A.
1. Review of: Fukamiya, M. [2], *Math. Reviews*, **14** (1953), 884.
2. Representations of Banach algebras with involution, *Canad. J. Math.*, **9** (1957), 435–442.

SCHECHTER, M.
1. On the spectra of operators on tensor products, *J. Fnal Analysis*, **4** (1969), 95–99.
2. *Spectra of partial differential operators*, North-Holland Publ., 1971.

SCHELLER, H.
1. *Reversibilitäts-Begriffe für Automorphismen-Gruppen auf von Neumann-Algebren*, Dissertation, Erlangen-Nürnberg, 1969.

SCHLIEDER, S.
1. Some remarks about the localization of states in a quantum field theory, *Comm. Math. Phys.*, **1** (1965), 265–280.
2. Einige Bemerkungen über Projektionsoperatoren, *Comm. Math. Phys.*, **13** (1969), 216–225.

SCHMIDT, K.
1. Cohomology and skew products of ergodic transformations, Warwick, 1974.
2. *Lectures on cocycles of ergodic transformation groups*, Warwick, 1976.

SCHMÜDGEN, K.
1. The order structure of topological $*$-algebras of unbounded operators, I, *Rep. Math. Phys.*, **7** (1975), 215–227.
2. Über LMC*-Algebren, *Math. Nachr.*, **68** (1975), 167–182.
3. Uniform topologies and strong operator topologies on polynomial algebras and on the algebra of CCR, *Rep. Math. Phys.*, **10** (1976), 369–384.
4. Uniform topologies on enveloping algebras, Leipzig, 1977.
5. Positive cones on enveloping algebras, Leipzig, 1977.
6. Trace functionals on unbounded operator algebras, Leipzig, 1977.

SCHOCHETMAN, I.
1. Topology and the duals of certain locally compact groups, *Trans. AMS*, **150** (1970), 477–490.
2. Dimensionality and the duals of certain locally compact groups, *Proc. AMS*, (1970), 514–520.
3. Compact and Hilbert-Schmidt induced representations, *Duke Math. J.*, **41** (1974), 89–102.

SCHOEN, J. A.
1. Factor representations and factor states on a C^*-algebra, *Canad. J. Math.*, **28** (1976), 130–134.

SCHRADER, R.; UHLENBROCK, D. A.
1. Markov structures in Clifford algebras, *J. Fnal Analysis*, **18** (1975), 369–413.

SCHWARTZ, J.-M.
1. Sur la structure des algèbres de Kac, *Ann. Inst. Fourier.*
2. Relations entre "ring groups" et algèbres de Kac, *Bull. Sc. Math.*, **100** (1977), 289–300.
3. Kac algebras, Marseille, 1977.

SCHWARTZ, J. T.
1. Two finite, non-hyperfinite, non-isomorphic factors, *Comm. Pure Appl. Math.*, **16** (1963), 19–26.
2. Non-isomorphism of a pair of factors of type III, *Comm. Pure Appl. Math.*, **16** (1963), 111–120.
3. Type II factors in a central decomposition, *Comm. Pure Appl. Math.*, **16** (1963), 247–252.
4. *W*-algebras*, Gordon & Breach, 1967.
5. Recent progress of the structure theory of factors, *Functional Analysis* (Ed. C. O. Wilde), Academic Press, 1970, 37–53.

SCUTARU, H.
1. Some remarks on covariant completely positive linear maps on C^*-algebras, *Rep. Math. Phys.*
2. Coherent states and covariant semi-spectral measures, *Lettres Math. Phys.*, **2** (1977), 101–107.
3. Estimations of the entropy of a quantum state with the aid of covariant and contravariant symbols, *Rep. Math. Phys.*

SEBESTYÉN, Z.
1. On the definition of C^*-algebras, *Publ. Math. Debrecen*, **21** (1974), 207–217.
2. A weakening of the definition of C^*-algebras, *Acta Sci. Math.*, **35** (1973), 17–20.
3. On a problem of Araki and Elliott, *Ann. Sc. Math.* (Budapest), **18** (1975), 209–221.
4. Remarks on the paper "On the definition of C^*-algebras" of Araki and Elliott, *Ann. Sc. Math.* (Budapest), **17** (1974/75), 35–39.
5. Some local characterizations of boundedness and of C^*-equivalent algebras, *Ann. Sc. Math.* (Budapest), **18** (1975), 197–207.

SEDA, A. K.
1. Un concept de mesures invariantes pour les groupoïdes topologiques, *CR. Acad. Sc. Paris*, **280** (1975), 1603–1605.
2. Transformation groupoids and bundles of Banach spaces.

SEGAL, I. E.
1. Irreducible representations of operator algebras, *Bull. AMS*, **53** (1947), 73–88.
2. The group algebra of a locally compact group, *Trans. AMS*, **61** (1947), 69–105.
3. Postulates for general quantum mechanics, *Ann. Math.*, **48** (1947), 930–948.
4. Two-sided ideals in operator algebras, *Ann. Math.*, **50** (1949), 856–865.
5. An extension of Plancherel's formula to separable unimodular groups, *Ann. Math.*, **52** (1950), 272–292.
6. The two-sided regular representation of a unimodular locally compact group, *Ann. Math.*, **51** (1950), 293–298.
7. A class of operator algebras which are determined by groups, *Duke Math. J.*, **18** (1951), 221–265.
8. Equivalence of measure spaces, *Amer. J. Math.*, **73** (1951), 275–313.
9. *Decomposition of operator algebras, I, II*, Memoirs AMS, No. 9 (1951).
10. Hypermaximality of certain operators on Lie groups, *Proc. AMS*, **3** (1952), 13–15.
11. A non-commutative extension of abstract integration, *Ann. Math.*, **57** (1953), 401–457 (Corrections, ibidem, 595–596).
12. Abstract probability spaces and a theorem of Kolmogoroff, *Amer. J. Math.*, **76** (1954), 721–732.
13. Tensor algebras over Hilbert spaces, I, II, *Trans. AMS*, **81** (1956), 106–134; *Ann. Math.*, **63** (1956), 160–175.
14. The structure of a class of representations of the unitary group on a Hilbert space, *Proc. AMS*, **8** (1957), 197–203.
15. Distributions in Hilbert spaces and canonical systems of operators, *Trans. AMS*, **88** (1958), 12–41.
16. Caractérisation mathématique des observables en théorie quantique des champs et ses conséquences pour la structure des particles libres, *Coll. Int. Probl. Math. Th. Quant. Champs*, Paris, 1959, 57–103.
17. A theorem on the measurability of group invariant operators, *Duke Math. J.*, **26** (1959), 549–552 (Corrections, ibidem, **33** (1966), 821).
18. Foundations of the theory of dynamical systems of infinitely many degrees of freedom, I, II, *Mat.-Fys. Medd. Danske Vid. Selsk.*, **31** (1959), 1–38; *Can. J. Math.*, **13** (1961), 1–18.
19. A note on the concept of entropy, *J. Math. Mech.*, **9** (1960), 623–630.
20. Mathematical characterization of the physical vacuum for a linear Bose-Einstein field, *Illinois J. Math.*, **6** (1962), 500–523.
21. *Mathematical problems of relativistic physics*, AMS Providence, 1963.
22. Algebraic integration theory, *Bull. AMS*, **71** (1965), 419–489.
23. Representations of the canonical commutation relations, *Cargese Lectures in Phys.*, **3** (1967), 107–170.

24. Notes towards the construction of nonlinear relativistic quantum fields, I, II, III, *Proc. Nat. Acad. Sci. USA*, **57** (1967), 1178–1183; *Bull. AMS*, **75** (1969), 1383–1389; 1390–1395.
25. Non-linear functions of weak processes, I, II, *J. Fnal Analysis*, **4** (1969), 404–456, **6** (1970), 29–75.
26. The mathematical theory of quantum fields, *Lecture Notes in Math.*, No. 140 (1970), 19–57.
27. Transformations in Wiener space and squares of quantum fields, *Adv. Math.*, **4** (1970), 91–108.
28. Construction of non-linear local quantum processes, I, II, *Ann. Math.*, **92** (1970), 464–481 (Corrections, ibidem, **93** (1971), 597); *Invent. Math.*, **14** (1971), 211–241.
29. Generalized vectors in Hilbert spaces, I, II (unpublished, 1962).
30. *Operator algebras and non-commutative integration in Hilbert space*, Lecture Notes, Lünd, 1972.

SEGAL, I. E.; VON NEUMANN, J.
1. A theorem on unitary representations of semi-simple Lie groups, *Ann. Math.*, **52** (1950), 509–517.

SELESNICK, S. A.
1. Watts cohomology for a class of Banach algebras and the duality of compact abelian groups, *Math. Z.*, **130** (1973), 313–324.
2. A Hopf algebra approach to the Krein duality theorem, *Duke Math. J.*, **40** (1973), 831–838.
3. Quantization via Swan's theorem.

SERIES, C.
1. The Rokhlin tower theorem and hyperfiniteness for actions of continuous groups.
2. An application of groupoid cohomology.

SHALE, D.
1. Linear symmetries of free boson fields, *Trans. AMS*, **103** (1962).

SHALE, D.; STINESPRING, W. F.
1. States of the Clifford algebra, *Ann. Math.*, **80** (1964), 365–381.
2. Spinor representations of infinite orthogonal groups, *J. Math. Mech.*, **14** (1965), 315–322.

SHERMAN, S.
1. The second adjoint of a C^*-algebra, *Proc. Int. Congr. Math., Cambridge*, 1950, vol. I, 470.
2. Non negative observables squares, *Proc. AMS*, **2** (1951), 31–33.
3. Order in operator algebras, *Amer. J. Math.*, **73** (1951), 227–232.
4. On Segal's postulates for quantum mechanics, *Ann. Math.*, **64** (1956), 593–601.

SHERMAN, T.
1. Positive linear functionals on $*$-algebras of unbounded operators, *J. Math. Anal. Appl.*, **22** (1968), 285–318.

SHERSTNEV, A. N. (in Russian)
1. On the general theory of states on von Neumann algebras, *Funkt. Analiz i pril.*, **8**:3 (1974), 89–90.
2. The general method of integration with respect to a measure on the projections of a Hilbert space, *Veroyat. Met. Kibern., Kazan*, **9** (1971), 108–119.

SHIELDS, P. C.
1. A new topology for von Neumann algebras, *Bull. AMS*, **65** (1959), 267–269.
2. *The theory of Bernoulli shifts*, The University of Chicago Press, 1974.

SHIRALI, S.
1. Symmetry in complex involutory Banach algebras, *Duke Math. J.*, **34** (1967), 741–745.
2. On the Jordan structure of complex Banach $*$-algebras, *Pacific J. Math.*, **27** (1968), 397–404 (Corrections, ibidem, **31** (1969), 834).
3. Symmetry of Banach algebras without identity, *J. London Math. Soc.*, **3** (1971), 143–144.
4. Representability of positive functionals, *J. London Math. Soc.*, **3** (1971), 145–150.
5. On Jordan $*$-representations of Banach algebras, *Notices AMS*, **18** (1971), 682–46–43.

SHIRALI, S.; FORD, J. M. W.
1. Symmetry in complex involutory Banach algebras, II, *Duke Math. J.*, **37** (1970), 275–280.

SHULTZ, F. W.
1. On normed Jordan algebras which are Banach dual spaces, 1976.

SIMON, B.
1. *Euclidean quantum field theory*, Princeton Univ. Press, 1974.

SINAĬ, YA. G.; ANSHELEVIČ, V. V.
1. Some problems in non-commutative ergodic theory (Russian), *Uspekhi Mat. Nauk SSSR*, **31** : 4 (1976), 151–167.

SINCLAIR, A. M.
1. Continuous derivations on Banach algebras, *Proc. AMS*, **20** (1969), 166–170.
2. Jordan homomorphisms and derivations on semi-simple Banach algebras, *Proc. AMS*, **24** (1970), 209–214.
3. Jordan automorphisms on a semi-simple Banach algebra, *Proc. AMS*, **25** (1970), 526–528.
4. The norm of a hermitian element in a Banach algebra, *Proc. AMS*, **28** (1971), 446–450.
5. Annihilator ideals in the cohomology of Banach algebras, *Proc. AMS*, **33** (1972), 361–366.
6. Homomorphisms from C^*-algebras, *Proc. London Math. Soc.*, **29** (1974), 435–452 (Corrigendum, ibidem, **32** (1976), 322).
7. *Automatic continuity of linear operators*, London Math. Soc., Lecture Notes Series, No. 21 (1975).
8. Homomorphisms from $C_0(R)$, *Proc. London Math. Soc.*, **11** (1975), 165–174.
9. The states of a Banach algebra generate the dual, *Proc. Edinburgh Math. Soc.*, **17** (1971), 341–344.
10. The Banach algebra generated by a hermitian operator, *Proc. London Math. Soc.*, **24** (1972), 681–691.
11. Bounded approximate identities, factorization and a convolution algebra, *J. Fnal Analysis.*

SINGER, I. M.
1. Uniformly continuous representations of Lie groups, *Ann. Math.* **56** (1952), 242–247.
2. Automorphism of finite factors, *Amer. J. Math.*, **77** (1955), 117–133.
3. Report on group representations, *Int. Congr. on operator theory and group representations*, Arden House, Harriman, New York, 1955, 11–26.
4. Operator theory and periodicity, *Indiana Univ. Math. J.*, **20** (1971), 947–950.
5. On the classification of UHF-algebras (unpublished).

SINGER, I. M.; WERMER, J.
1. Derivations of commutative Banach algebras, *Math. Ann.*, **129** (1955), 260–264.

SIRUGUE-COLIN, M.; SIRUGUE, M.
1. On the probabilistic structure of quasi-free states of a Clifford algebra, Marseille, 1975.

SIRUGUE, M.; TESTARD, D.
1. Some connections between ground states and temperature states of thermodynamical systems, *Comm. Math. Phys.*, **22** (1971), 223–237.

SIRUGUE, M.; WINNINK, M.
1. Constraints imposed upon a state of a system that satisfies the KMS boundary condition, *Comm. Math. Phys.*, **19** (1970), 161–168.
2. Une généralisation du théorème d'unicité de l'automorphisme modulaire d'une algèbre de von Neumann de genre dénombrable, *CR Acad. Sci. Paris*, **272** (1971), 1185–1187.
3. Translation dans le temps comme groupe d'$*$-automorphismes.

SKAU, C. F.
1. Orthogonal measures on the state space of a C^*-algebra, in *Algebras in Analysis* (Ed. J. H. Williamson), Academic Press, 1975, 272–304.
2. Finite subalgebras of a von Neumann algebra, *J. Fnal Analysis*, **25** (1977), 211–235.
3. Positive selfadjoint extensions of operators affiliated with a von Neumann algebra, Trondheim, 1977.
4. *Commutative projections and abelian subalgebras*, Dissertation, Univ. Pennsylvania, 1973.

SKORNYAKOV, L. A.
1. *Complemented Dedekind lattices and regular rings* (Russian), Moscow, 1961.

SLAWNY, J.
1. Representations of canonical anticommutation relations and implementability of canonical transformations, *Comm. Math. Phys.*, **22** (1971), 104–114.
2. On factor representations and the C^*-algebra of canonical commutation relations, *Comm. Math. Phys.*, **24** (1972), 151–170.
3. On the regular representation, von Neumann uniqueness theorem and the C^*-algebra of canonical commutation and anticommutation relations, Tel-Aviv, TAUP 227–71.

SMITH, H. A.
1. Central twisted group algebras.

SMITH, M.
1. Regular representations of discrete groups, *J. Fnal Analysis*, **11** (1972), 401–406.

SMITH, M. K.
1. Spectra of operator equations, *Michigan Math. J.*, **23** (1976), 151–153.

SMITH, M.-S.B.
1. On automorphism groups of C^*-algebras, *Trans. AMS*, **152** (1970), 623–648.

SMITH, M.-S.B.; SMITH, L.
1. On the classification of UHF algebras (unpublished).

SMYTH, M. R. F.; WEST, T. T.
1. The spectral radius formula in quotient algebras, *Math. Z.*, **145** (1975), 157–161.
2. Invariant subspaces of compact elements in C^*-algebras, *Math. Z.*, **153** (1977), 193–197.

SONIS, M. G.
1. On a class of operators in von Neumann algebras with a Segal measure on projections (Russian), *Matem. Sb.*, **84** (1971), 353–368.

SØRENG, J.
1. Pure states of simple C^*-algebras, *J. Fnal Analysis*, **22** (1976), 390–404.

SPAIN, P. G.
1. On commutative V^*-algebras, I, II, *Proc. Edinburgh Math. Soc.*, **17** (1970), 173–180; *Glasgow Math. J.*, **13** (1972), 129–134.
2. V^*-algebras with weakly compact unit spheres, *J. London Math. Soc.*, **4** (1971), 62–64.
3. The W^*-closure of a V^*-algebra, *J. London Math. Soc.*, **7** (1974), 385–386.
4. The decomposition of the dual of a W^*-algebra, *J. London Math. Soc.*, **7** (1974), 568–570.

SPROSTON, J. P.
1. Derivations and automorphisms of homogeneous C^*-algebras, *Proc. London Math. Soc.*, **32** (1976), 521–536.
2. Derivations and pure states, *Glasgow Math. J.*, **18** (1977), 73–77.

STACEY, P. J.
1. Type I points in a compact convex set, *J. London Math. Soc.*, **10** (1975), 306–308.
2. Admissible split faces in the state space of a separable C^*-algebra, *Quart. J. Math.*, **26** (1975), 485–490.

STAMPFLI, J. G.
1. The norm of a derivation, *Pacific J. Math.*, **33** (1970), 737–748.
2. Derivations on $B(H)$: the range, *Illinois J. Math.*, **17** (1973), 518–524.

STEIN, J. D.
1. Homomorphisms of semi-simple algebras, *Pacific J. Math.*, **26** (1968), 589–594.
2. Homomorphisms of B^*-algebras, *Pacific J. Math.*, **28** (1969), 431–439.
3. Continuity of homomorphisms of von Neumann algebras, *Amer. J. Math.*, **91** (1969), 153–159.
4. Some aspects of automatic continuity, *Pacific J. Math.*, **50** (1974), 187–204.

STEPHANSON, J. R.
1. On a problem of J. Dixmier concerning ideals in a von Neumann algebra, *Math. Scand.*, **24** (1969), 111–112.

ŠTERN, A. I.
1. The relation between the topologies on a locally compact group and on its dual space (Russian), *Funkt. Analiz i pril.*, **5**:4 (1971), 56–63.

STINESPRING, W. F.
1. Positive functions on C^*-algebras, *Proc. AMS*, **6** (1955), 211–216.
2. A semi-simple matrix group is of type I, *Proc. AMS*, **9** (1958), 965–967.
3. Integrability of Fourier transform for unimodular Lie groups, *Duke Math. J.*, **26** (1959), 123–131.
4. Integration theorems for gages and duality for unimodular groups, *Trans. AMS*, **90** (1959), 15–56.

STONE, M. H.
1. *Linear transformations in Hilbert space*, Publ. AMS, No. 15, New York, 1932, 1950.
2. On one-parameter unitary groups in Hilbert space, *Ann. Math.*, **33** (1932), 643–648.
3. Boundedness properties in function lattices, *Canad. J. Math.*, **1** (1949), 176–186.
4. On unbounded operators in Hilbert space, *J. Indian Math. Soc.*, **15** (1951/52), 155–192.

STÖRMER, E.
1. Positive linear maps of operator algebras, *Acta Math.*, **110** (1963), 233–278.
2. On the Jordan structure of C^*-algebras, *Trans. AMS*, **120** (1965), 438–447.
3. Jordan algebras of type I, *Acta Math.*, **115** (1966), 165–184.
4. Two-sided ideals in C^*-algebras, *Bull. AMS*, **73** (1967), 254–257.
5. On antiautomorphisms of von Neumann algebras, *Pacific J. Math.*, **21** (1967), 349–370.
6. Types of von Neumann algebras associated with extremal invariant states, *Comm. Math. Phys.*, **6** (1967), 194–204.
7. Large groups of automorphisms of C^*-algebras, *Comm. Math. Phys.*, **5** (1967), 1–22.
8. Irreducible Jordan algebras of selfadjoint operators, *Trans. AMS*, **130** (1968), 153–166.
9. On partially ordered vector spaces and their duals with applications to simplexes and C^*-algebras, *Proc. London Math. Soc.*, **18** (1968), 245–265.
10. A characterization of pure states of C^*-algebras, *Proc. AMS*, **19** (1968), 1100–1102.
11. Symmetric states of infinite tensor products of C^*-algebras, *J. Fnal Analysis*, **3** (1969), 48–68.
12. The even CAR algebra, *Comm. Math. Phys.*, **16** (1970), 136–137.
13. States and invariant maps of operator algebras, *J. Fnal Analysis*, **5** (1970), 44–65.
14. Automorphisms and invariant states of operator algebras, *Acta Math.*, **127** (1971), 1–9.
15. On infinite tensor products of von Neumann algebras, *Amer. J. Math.*, **93** (1971), 810–818.
16. On projection maps of von Neumann algebras, *Math. Scand.*, **30** (1972), 46–50.
17. Automorphisms and equivalence in von Neumann algebras, *Pacific J. Math.*, **44** (1973), 371–384.
18. Invariant states of von Neumann algebras, *Math. Scand.*, **30** (1972), 253–256.
19. Asymptotically abelian systems, *Cargese Lectures in Phys.*, **4** (1970), 195–213.
20. Hyperfinite product factors, I, II, III, *Arkiv Math.*, **9** (1971), 165–170; *J. Fnal Analysis*, **10** (1972), 471–481; *Amer. J. Math.*, **97** (1975), 589–595.
21. Spectra of states and asymptotically abelian C^*-algebras, *Comm. Math. Phys.*, **28** (1972), 279–294 (Corrections, ibidem, **33** (1974), 341–343).
22. Spectra of ergodic transformations, *J. Fnal Analysis*, **15** (1974), 202–215.
23. C^*-homomorphisms of C^*-algebras, I, II, (unpublished).
24. On extremal maps of operator algebras, (unpublished).
25. Positive linear maps and Jordan homomorphisms of C^*-algebras, (unpublished).
26. Invariant measures and von Neumann algebras, (unpublished).
27. Positive linear maps of C^*-algebras, *Lecture Notes in Phys.*, No. 29, 85–106, Springer Verlag, 1974.
28. Inner automorphisms of von Neumann algebras, *Comm. Math. Phys.*, **36** (1974), 115–122.
29. Entropy in finite von Neumann algebras, *Symposia Math.*, vol. XX, 197–206, Academic Press, 1976.

30. Some aspects of ergodic theory in operator algebras, *Proc. Int. Congr. Math., Vancouver*, 1974, 111–114.
31. On the structure theory of factors, in *C*-algebras and their applications to statistical mechanics and quantum field theory*, p. 101–113, North-Holland, 1976.
32. Spectral subspaces of automorphisms, in *C*-algebras and their applications to statistical mechanics and quantum field theory*, p. 127–138, North-Holland, 1976.
33. Involutory automorphisms of simple C^*-algebras, Oslo, 1976.
34. Stone's theorem and spectral subspaces of automorphisms, *Proc. AMS*.
35. Regular abelian Banach algebras of linear maps of operator algebras, Oslo, 1977.
36. Jordan algebras versus C^*-algebras, *Acta Phys. Austriaca, Suppl.* XVI, 1976, 1–13.

STRĂTILĂ, Ş.
1. Ideals in operator algebras and measure theory on C^*-algebras, in *Theory of operators and operator algebras* (Romanian), 263–488, Editura Academiei, 1973.
2. Central spectral theory in W^*-algebras and applications (Romanian), *Stud. Cerc. Mat.*, **25** (1973), 1167–1259.
3. Twisted Plancherel theorem for weights; a non-commutative case, Marseille, 1977.
4. On the tensor product of weights on W^*-algebras, Bucharest, 1978.
5. *Modular theory in operator algebras.*

STRĂTILĂ, Ş.; VOICULESCU, D.
1. Sur les représentations factorielles infinies de $U(\infty)$, *CR Acad. Sc. Paris*, **280** (1975), 555–558.
2. *Representations of AF-algebras and of the group* $U(\infty)$, Lecture Notes in Math., No. 486, Springer Verlag, 1975.
3. On a class of KMS states for the unitary group $U(\infty)$, *Math. Ann.*, **235** (1978), 87–110.
4. A survey on representations of the unitary group $U(\infty)$, X-th Banach Semester, Warsaw, 1977.

STRĂTILĂ, Ş.; VOICULESCU, D.; ZSIDÓ, L.
1. Sur les produits croisés, *CR Acad. Sc. Paris*, **282** (1976), 779–782.
2. On crossed products, I, II, *Rev. Roum. Math. Pures Appl.*, **21** (1976), 1411–1449; **22** (1977), 83–117.

STRĂTILĂ, Ş.; ZSIDÓ, L.
1. Une théorie algébrique de la réduction pour les W^*-algèbres, *CR Acad. Sc. Paris*, **272** (1971), 1453–1456.
2. An algebraic reduction theory for W^*-algebras, I, II, *J. Fnal Analysis*, **11** (1972), 295–313; *Rev. Roum. Math. Pures Appl.*, **18** (1973), 407–460.
3. Sur la théorie algébrique de la réduction pour les W^*-algèbres, *CR Acad. Sc. Paris*, **275** (1972), 451–454.
4. On the range of a derivation in a W^*-algebra, *Rev. Roum. Math. Pures Appl.*, **18** (1973), 101–103.
5. A spectral characterization of the maximal ideals in factors, *Acta Sc. Math.*, **36** (1974), 155–160.
6. *Operator Algebras*, 343 p., INCREST Bucharest 1977, (unpublished).

STREATER, R. F.
1. Intensive observables in quantum theory, *J. Math. Phys.*, **5** (1964), 581–590.
2. Local fields with the wrong connection between spin and statistics, *Comm. Math. Phys.*, **5** (1966), no. 2.
3. Current commutation relations and continuous tensor products, *Nuovo Cimento*, **53** (1968), 487–495.
4. A continuum analogue of lattice gas, *Comm. Math. Phys.*, **12** (1969), 226–232.
5. Current commutation relations, continuous tensor products and infinitely divisible group representations.
6. Outline of axiomatic relativistic quantum field theory, *Rep. Progr. Phys.*, **38** (1975), 771–846.

STREATER, R. F.; WIGHTMAN, A. S.
1. *P.C.T., spin and statistics and all that*, Benjamin, 1964.

STREATER, R. F.; WULFSOHN, A.
1. Continuous tensor products of Hilbert spaces, *Nuovo Cimento*, **57** (1968), 330–339.

ŠUBIN, M. A. (in Russian)
1. Elliptic almost periodic operators and von Neumann algebras, *Funkt. Analiz i pril.*, **94**: 1 (1975), 89–90.
2. Almost periodic pseudo-differential operators and von Neumann algebras, *Trudy Mosk. Mat. Obšč.*, **35** (1976), 103–161.

SUCIU, I.
1. Dilation and extension theorems for operator valued mappings, in *Theory of operators and operator algebras* (Romanian), 93–130, Editura Academiei, 1973.

SUCIU, I.; VALUSESCU, I.
1. On the hyperbolic metric on Harnack parts, *Studia Math.*, **55** (1976), 97–109 (Corrections, ibidem, **58** (1976), 99–100).

SUGIURA, M.
1. *Unitary representations and harmonic analysis*, Halstead Press, 1975.

ŠULMAN, V. S. (in Russian)
1. On operator algebras in a space with an indefinite metric of type II_1, *Dokl. Akad. Nauk SSSR*, **201** (1971), 44–47.
2. On reflexive operator algebras, *Matem. Sb.*, **87** (1972), 179–187.
3. Multiplication operators in C^*-algebras and the problem of reflexivity of algebras containing a maximal abelian subalgebra, *Funkt. Analiz i pril.*, **8** (1974), 92–93.

SUND, T.
1. Square integrable representations and the Mackey theory, *Trans. AMS*, **194** (1974), 131–139.
2. On the dual topology of [*FC*] groups, Oslo, 1975.
3. Duality theory for locally compact groups with precompact conjugacy classes, I, II, *Trans. AMS*, **211** (1975), 185–202; **224** (1976), 313–321.
4. Isolated points in duals of certain locally compact groups, *Math. Ann.*, **224** (1976), 33–40.
5. Multiplier representations of exponential Lie groups.

SUNOUCHI, C.
1. A generalization of Schatten-von Neumann-Dixmier theorem for type AW^*-algebras, *Tohoku Math. J.*, **23** (1971), 727–734.

SUNOUCHI, H.
1. On rings of operators of infinite classes, I, II, *Proc. Jap. Acad.*, **28** (1952), 9–13; 330–335.
2. The irreducible decomposition of the maximal Hilbert algebras of finite class, *Tohoku Math. J.*, **4** (1952), 207–215.
3. An extension of the Plancherel formula to unimodular groups, *Tohoku Math. J.*, **4** (1952), 216–230.
4. A characterization of the maximal ideal in a factor of the case II_∞, parts I, II, *Kodai Math. Sem. Rep.*, **6** (1954), 7; **7** (1954), 65–66.
5. Infinite Lie groups, *Tohoku Math. J.*, **8** (1956), 291–307.

SUŠKO, V. N.; HORUZIĬ, S. S.
1. Properties of the preimage of vector states of operator algebras (Russian), *Teor. Mat. Fiz.*, **8** (1971), 324–327.

SUTHERLAND, C. E.
1. Direct integral theory for weights and the Plancherel formula, *Bull. AMS*, **80** (1974), 456–461; *J. Fnal Analysis*.
2. Crossed products, direct integrals and Connes classification of type III factors, *Math. Scand.*, **40** (1977), 209–214.
3. Type analysis of the regular representation of a non-modular group.
4. Cohomology and crossed products of von Neumann algebras, I, II.
5. Cohomology and extensions of von Neumann algebras, I, II, Kingstone, 1975, Oslo, 1976.

SUZUKI, K.
1. On the character ring of representations of a compact group, *Tohoku Math. J.*, **20** (1968), 38–45.

SUZUKI, N.
1. On the invariants of W^*-algebras, *Tohoku Math. J.*, **7** (1955), 177–185.
2. On automorphisms of W^*-algebras leaving the center elementwise invariant, *Tohoku Math. J.*, **7** (1955), 186–191.
3. A linear representation of a countably infinite group, *Proc. Jap. Acad.*, **34** (1958), 575–579.
4. Crossed product of ring of operators, *Tohoku Math. J.*, **11** (1959), 113–124.
5. Certain types of groups of automorphisms of a factor, *Tohoku Math. J.*, **11** (1959), 314–320 (Corrections, ibid., **12** (1960), 477–478).
6. Extensions of ring of operators in Hilbert spaces, *Tohoku Math. J.*, **14** (1962), 217–232.
7. Isometries on Hilbert spaces, *Proc. Jap. Acad.*, **39** (1963), 435–438.
8. On the type of completely continuous operators, *Proc. Jap. Acad.*, **40** (1964), 683–685.
9. Algebraic aspects of non-self-adjoint operators, *Proc. Jap. Acad.*, **41** (1965), 706–710.
10. The algebraic structure of non-self-adjoint operators, *Acta Sci. Math.*, **28** (1966), 173–184.
11. On the spectral decomposition of dissipative operators, *Proc. Jap. Acad.*, **42** (1966), 577–582.
12. On a weakly convergent sequence of normal functionals on a von Neumann algebra, *Proc. AMS*, **22** (1969), 697–701.
13. Representations of certain Banach $*$-algebras, *Proc. Jap. Acad.*, **45** (1969), 696–699.
14. Every C-symmetric Banach $*$-algebra is symmetric, *Proc. Jap. Acad.*, **46** (1970), 98–102.

SUZUKI, N.; SAITÒ, T.
1. On the operators which generate continuous von Neumann algebras, *Tohoku Math. J.*, **15** (1963), 277–280.

SZ.-NAGY, B.
1. *Spektraldarstellung linear Transformationen des Hilbertschen Räumes*, Springer-Verlag, 1942, 1967.

SZ.-NAGY, B.; FOIAŞ, C.
1. *Analyse harmonique des operateurs dans l'espace hilbertien*, Akadémiai Kiadó, 1968.

SZÜCS, J.
1. On non-localizable measure spaces, *Acta Sc. Math.*, **37** (1975), 293–295.

SZYMAŃSKY, W.
1. Characters of finitely generated C^*-algebras, *Ann. Pol. Math.*, **27** (1973), 317–322.
2. Decompositions of operators-valued functions in Hilbert spaces, *Studia Math.*, **50** (1974), 265–284.
3. Szegö-type properties in a non-commutative case, *Ann. Pol. Math.*, **34** (1977), 97–105.
4. Antisymmetry of subalgebras of C^*-algebras, *Studia Math.*, **60** (1977), 97–107.

TABOV, I. B.
1. On the theory of categories of Hilbert spaces (Russian), *Vestnik Mosk. Univ.*, **26** : 1 (1971), 65–68.

TAKAHASHI, R.
1. Sur les représentations unitaires des groupes de Lorentz généralisées, *Bull. Soc. Math. France*, **91** (1963), 289–433.

TAKAHASHI, S.
1. A characterization of group rings as a special class of Hopf algebras, *Can. Math. Bull.*, **8** (1956), 465–475.

TAKAHASHI, SIN-EI
1. A note on the theorem of Stone-Weierstrass, *Bull. Fac. Sci. Ibaraki Univ.*, **2** (1969), 5–7.
2. On Urysohn's lemma for spectrum of C^*-algebras, *Bull. Fac. Sci. Ibaraki Univ.*, **3** (1971), 5–11.
3. A remark on Urysohn's lemma for spectrum of C^*-algebras, *Bull. Fac. Sci. Ibaraki Univ.*, **4** (1972), 33–35.

4. A note on the homomorphic images of the center of C^*-algebras, *Bull. Fac. Sci. Ibaraki Univ.*, **6** (1974), 29–32.
5. On locally inner automorphisms of certain C^*-algebras, *Bull. Fac. Sci. Ibaraki Univ.*, **5** (1973), 1–16.
6. On Urysohn lemma and division of the set of pure states, *Bull. Fac. Sci. Ibaraki Univ.*, **7** (1975), 61–66.
7. A simple proof of the Stone-Weierstrass theorem for CCR algebras with Hausdorff spectrum, *Math. Scand.*, **38** (1976), 304–306.
8. A remark on Archbold's result, *Bull. Fac. Sci. Ibaraki Univ.*, **8** (1976), 23–24.
9. A characterization for compact central double centralizers on C^*-algebras, *Pacific J. Math.*, **72** (1977), 255–259.
10. Remarks on the Stone-Weierstrass problem for C^*-algebras, *Bull. Fac. Sci. Ibaraki Univ.*, **9** (1977), 61–63.

TAKAI, H.
1. An algebraic model for von Neumann algebras, I, II, III, *Proc. Jap. Acad.*, **46** (1970), 941–944, 1091–1096; *Memoirs Osaka Kyoiku Univ.*, **20** (1971), 121–123.
2. Dualité dans les produits croisés de C^*-algèbres, *CR Acad. Sc. Paris*, **278** (1974), 1041–1043
3. On a duality for crossed products of C^*-algebras, *J. Fnal Analysis*, **19** (1975), 25–39.
4. On a Fourier expansion in continuous tensor products, *Publ. RIMS, Kyoto Univ.*, **11** (1976), 849–880.
5. The quasi-orbit space of continuous C^*-dynamical systems, *Trans. AMS*, **216** (1976), 105–113.
6. On the invariant $\Gamma(\alpha)$ and C^*-crossed products.

TAKEDA, Z.
1. On a theorem of R. Pallu de la Barrière, *Proc. Jap. Acad.*, **28** (1952), 558–563.
2. Perfection of measure space and W^*-algebras, *Kodai Math. Sem. Rep.*, **5** (1953), 23–26.
3. Conjugate spaces of operator algebras, *Proc. Jap. Acad.*, **30** (1954), 90–95.
4. On the representations of operator algebras, I, II, *Proc. Jap. Acad.*, **30** (1954), 299–304; *Tohoku Math. J.*, **6** (1954), 212–219.
5. Inductive limit and infinite direct product of operator algebras, *Tohoku Math. J.*, **7** (1955), 67–86.
6. On the extension of finite factors, II, *Proc. Jap. Acad.*, **35** (1959), 215–220.
7. On the extension theorem of the Galois theory for finite factors, *Proc. Jap. Acad.*, **37** (1961), 78–82.
8. On the normal basis theorem of the Galois theory for finite factors, *Proc. Jap. Acad.*, **37** (1961), 144–148.

TAKEDA, Z.; TURUMARU, T.
1. On the property "position p'", *Math. Jap.*, **2** (1952), 195–197.

TAKEHANA, H.
1. An automorphism on tensor product of operator algebras, *Math. Jap.*, **20** (1975), 99–100.

TAKEMOTO, H.
1. W^*-algebras with no non-separable cyclic representations, *Tohoku Math. J.*, **20** (1968), 567–576.
2. On the integral representation of some functional on a von Neumann algebra, *Tohoku Math. J.*, **21** (1969), 237–248.
3. On the homomorphisms of von Neumann algebras, *Tokohu Math. J.*, **21** (1969), 152–158.
4. A complement to "On the homomorphisms of von Neumann algebras", *Tohoku Math. J.*, **22** (1970), 210–211.
5. On the C_p-classes in the maximal ideal of a von Neumann algebra, *Tohoku Math. J.*, **23** (1971), 301–312.
6. On a characterization of AW^*-modules and a representation of Gelfand type of non-commutative operator algebras, *Michigan Math. J.*, **20** (1973), 115–127.
7. Decomposable operators in continuous fields of Hilbert spaces, *Tohoku Math. J.*, **27** (1975), 413–436.
8. On the weakly continuous field of Hilbert spaces and its applications to the reduction theory of W^*-algebras, *Tohoku Math. J.*, **28** (1976), 479–496.

9. On the algebraic reduction theory for countable direct summand C^*-algebras of separable C^*-algebras, *Japan-US Seminary*, 1977.
10. The algebraic reduction theory of a C^*-algebra, Sendai, 1977.

TAKEMOTO, H.; TOMIYAMA, J.
1. On the topological reduction of finite von Neumann algebras, *Tohoku Math. J.*, **25** (1973), 273–289.

TAKENOUCHI, O.
1. On the maximal Hilbert algebras, *Tohoku Math. J.*, **3** (1951), 123–131.
2. On the structure of maximal Hilbert algebras, *Math. J. Okayama Univ.*, **1** (1952), 1–31.
3. Sur une classe de fonctions continues de type positif sur un groupe localement compact, *Math. J. Okayama Univ.*, **4** (1955), 153–173.
4. Families of unitary operators defined on groups, *Math. J. Okayama Univ.*, **6** (1957), 171–179.
5. Sur la facteur-représentation d'un groupe de Lie de type (*E*), *Math. J. Okayama Univ.*, **7** (1957), 151–161.
6. Sur les algèbres de Hilbert, *CR Acad. Sci. Paris*, **250** (1960), 3436–3437.
7. Sur les sous-algèbres d'une algèbre de Hilbert, *Ann. Éc. Norm. Sup.*, **78** (1961), 211–240.
8. On type classification of factors constructed as infinite tensor product, *Publ. RIMS, Kyoto Univ.*, **4** (1968), 467–482.

TAKESAKI, M.
1. On the direct product of W^*-factors, *Tohoku Math. J.*, **10** (1958), 116–119.
2. On the conjugate space of operator algebras, *Tohoku Math. J.*, **10** (1958), 194–203.
3. A note on the cross-norm of the direct product of operator algebras, *Kodai Math. Sem. Rep.*, **10** (1958), 137–140.
4. A note on the direct product of operator algebras, *Kodai Math. Sem. Rep.*, **11** (1959), 178–181.
5. On the singularity of a positive linear functional on operator algebras, *Proc. Jap. Acad.*, **35** (1959), 365–366.
6. On the non-separability of singular representations of operator algebras, *Kodai Math. Sem. Rep.*, **12** (1960), 102–108.
7. On the Hahn-Banach type theorem and the Jordan decomposition of module linear mappings over some operator algebras, *Kodai Math. Sem. Rep.*, **12** (1960), 1–10.
8. On some representations of C^*-algebras, *Tohoku Math. J.*, **15** (1963), 79–95.
9. On the unitary equivalence among the components of decomposition of representations of involutive Banach algebras and the associated diagonal algebras, *Tohoku Math. J.*, **15** (1963), 365–393.
10. A complement to "On the unitary equivalence...", *Tohoku Math. J.*, **16** (1964), 226–227.
11. On the cross-norm on the direct product of C^*-algebras, *Tohoku Math. J.*, **16** (1964), 111–122.
12. Covariant representations of C^*-algebras and their locally compact automorphism groups, *Acta Math.*, **119** (1967), 273–302.
13. A duality in the representation theory of C^*-algebras, *Ann. Math.*, **85** (1967), 370–382.
14. Remarks on the reduction theory of von Neumann algebras, *Proc. AMS*, **20** (1969), 434–438.
15. A characterization of group algebras as a converse of Tanaka-Stinespring-Tatsuuma duality theorem, *Amer. J. Math.*, **91** (1969), 529–564.
16. A generalized commutation relation for the regular representation, *Bull. Soc. Math. France*, **97** (1969), 289–297.
17. *The theory of operator algebras*, Lecture Notes, UCLA, 1969–70.
18. *Tomita's theory of modular Hilbert algebras and its applications*, Lectures Notes in Math., No. 128, Springer-Verlag, 1970.
19. Disjointness of the KMS states of different temperatures, *Comm. Math. Phys.*, **17** (1970), 33–41.
20. Algebraic equivalence of locally normal representations, *Pacific J. Math.*, **34** (1970), 807–816.
21. A liminal crossed product of a uniformly hyperfinite C^*-algebra by a compact abelian automorphisms group, *J. Fnal Analysis*, **7** (1971), 140–146.

22. The quotient algebra of a finite von Neumann algebra, *Pacific J. Math.*, **36** (1971), 827–832.
23. Duality and von Neumann algebras, *Bull. AMS*, **77** (1971), 553–557; *Lecture Notes in Math.*, No. 247, p. 666–779, Springer-Verlag, 1972.
24. A short proof for the commutation theorem $(M_1 \otimes M_2)' = M_1' \otimes M_2'$, *Lecture Notes in Math.*, No. 247, p. 780–786, Springer-Verlag, 1972.
25. Conditional expectation in von Neumann algebra, *J. Fnal Analysis*, **9** (1972), 306–321.
26. States and automorphisms groups of operator algebras, *Proc. Symp. on mathematical aspects of statistical mechanics, SIAM-AMS*, **5** (1972), 77–84.
27. States and automorphisms of operator algebras. Standard representations and the Kubo-Martin-Schwinger boundary condition, *Lecture Notes on Phys.*, **20** (1973), 205–246.
28. Periodic and homogeneous states on a von Neumann algebra, I, II, III, *Bull. AMS*, **79** (1973), 202–206, 416–420, 559–563.
29. The structure of a von Neumann algebra with a homogeneous periodic state, *Acta Math.*, **131** (1973), 79–122.
30. Dualité dans les produits croisé des algèbres de von Neumann, *CR Acad. Sc. Paris*, **276** (1973), 41–43.
31. Algèbres de von Neumann proprement infinies et produits croisés, *CR Acad. Sc. Paris*, **276** (1973), 125–127.
32. Duality in crossed products and von Neumann algebras of type III, *Bull AMS*, **79** (1973), 1004–1005.
33. Duality for crossed products and the structure of von Neumann algebras of type III, *Acta Math.*, **131** (1973), 249–310.
34. Faithful states on a C^*-algebra, *Pacific J. Math.*, **52** (1974), 605–610.
35. Disintegration of locally normal states and algebraic equivalence of locally normal representations.
36. *Lecture Notes on operator algebras*, Los Angeles/Luminy, 1973/1974.
37. Relative commutant theorem in crossed products and the exact sequence for the automorphism group of a factor of type III, *Symposia Math.*, XX, 179–186, Academic Press, 1976.
38. Duality in properly infinite von Neumann algebras, *Trudy Mat. Inst. Akad. Nauk SSSR*, **136** (1975), 251–253.
39. Heisenberg commutant relations and a factor of type III, Bielefeld, 1976.
40. Factors of type III, *Lecture Notes in Math.*, **575** (1977), 139–156.

TAKESAKI, M.; TATSUUMA, N.
1. Duality and subgroups, I, II, *Ann. Math.*, **93** (1971), 344–364; *J. Fnal Analysis*, **11** (1972), 184–190.

TAKESAKI, M.; WINNINK, M.
1. Local normality in quantum statistical mechanics, *Comm. Math. Phys.*, **30** (1973), 129–152.

TAM, P.-K.
1. On the commutant of certain automorphism groups, *Can. J. Math.*, **25** (1973), 1165–1169.
2. On an ergodic abelian M-group, *Proc. Jap. Acad.*, **47** (1971), 456–457.

TANNAKA, T.
1. Dualität der nicht-kommutativen bikompakter Gruppen, *Tohoku Math. J.*, **53** (1938), 1–12.

TAREEVA, E. E.
1. On the C^*-algebraic approach in polymorphic phase transition (Russian), *Dokl. Akad. Nauk SSSR*, **219** : 3 (1974), 571–573.

TATSUUMA, N.
1. A duality theorem for locally compact groups, I, II, III, IV, *Proc. Jap. Acad.*, **41** (1965). 878–882; **42** (1966), 46–47, 87–90, 239–242; *J. Math. Kyoto Univ.*, **6** (1967), 187–295.
2. Invariancy of the Plancherel measure under the operation of Kronecker product, *Proc. Jap. Acad.*, **47** (1971), 252–256.
3. Prime ideals in the dual objects of locally compact groups, *Proc. Jap. Acad.*, **47** (1971) 249–251.

4. Plancherel formula for non-unimodular locally compact groups, *J. Math. Kyoto Univ.*, **12** (1972), 179–261.
5. An extension of AKTH-theory to locally compact groups.

TAUER, R. J.
1. Maximal abelian subalgebras in finite factors of type II, *Trans. AMS*, **114** (1965), 281–308.
2. Semi-regular maximal abelian subalgebras in hyperfinite factors, *Bull. AMS*, **71** (1965), 606–608.
3. *M*-semi regular subalgebras in hyperfinite factors, *Trans. AMS*, **129** (1967), 530–541.

TAYLOR, D. C.
1. The strict topology for double centralizer algebras, *Trans. AMS*, **150** (1970), 633–644.
2. A general Phillips theorem for C^*-algebras and some applications, *Pacific J. Math.*, **40** (1972), 477–488.
3. Interpolation in algebras of operator fields, *J. Fnal Analysis*, **10** (1972), 159–190.
4. A characterization of Banach algebras with approximate unit, *Bull. AMS*, **74** (1968), 761–766.
5. The strong bidual of $\Gamma(K)$, *Notices AMS*, **23**: 2 (1976), 732-B 8.
6. A general Hoffman-Wermer theorem for algebras of operator fields, *Proc. AMS*, **52** (1975), 212–216.

TAYLOR, J. L.
1. The Tomita decomposition of rings of operators, *Trans. AMS*, **113** (1964), 30–39.
2. The structure of convolution measure algebras, *Trans. AMS*, **119** (1965), 150–166.
3. Noncommutative convolution measure algebras, *Pacific J. Math.*, **31** (1969), 809–826.
4. A joint spectrum for several commuting operators, *J. Fnal Analysis*, **6** (1970), 172–191.
5. The analytic functional calculus for several commuting operators, *Acta Math.*, **125** (1970), 1–38.
6. The cohomology of the spectrum of a measure algebra, *Acta Math.*, **126** (1971), 195–226.
7. Homology and cohomology for topological algebras, *Adv. Math.*, **9** (1972), 137–182.
8. A general framework for a multi-operator functional calculus, *Adv. Math.*, **9** (1972), 183–252.
9. Functions of several noncommuting variables, *Bull. AMS*, **79** (1973), 1–34.
10. Banach algebras and topology, in *Algebras in Analysis*, (Ed. J. H. Williamson), Academic Press, 1975, 118–186.
11. Topological invariants of the maximal ideal space of a Banach algebra, *Adv. Math.*, **19** (1976), 149–206.

TAYLOR, K. F.
1. The structure of the regular representation of a locally compact group, *Math. Ann.*, **222** (1976), 211–224.

TELEMAN, S.
1. Sur la représentation linéaire des groupes topologiques, *Ann. Éc. Norm. Sup.*, **74** (1958), 319–339.
2. Sur les algèbres de J. von Neumann, *Bull. Sc. Math.*, **82** (1958), 117–126.
3. Harmonic analysis in regular algebras (Romanian), *St. Cerc. Mat.*, **18** (1966), 1421–1470.
4. Analyse harmonique dans les algèbres régulières, *Rev. Roum. Math. Pures Appl.*, **13** (1968), 641–750.
5. La représentation des anneaux tauberiens discrets par des faisceaux, *Rev. Roum. Math. Pures Appl.*, **14** (1969), 249–264.
6. La représentation des anneaux réguliers par des faisceaux, *Rev. Roum. Math. Pures Appl.*, **14** (1969), 703–717.
7. Représentation par faisceaux des modules sur les anneaux harmoniques, *CR Acad. Sci. Paris*, **269** (1969), 753–756.
8. Théorème de de Rham pour les algèbres harmoniques, *CR Acad. Sci. Paris*, **269** (1969), 1119–1121.
9. Sur les anneaux réguliers, *Rev. Roum. Math. Pures Appl.*, **15** (1970), 407–434.
10. On the regular rings of John von Neumann, *Rev. Roum. Math. Pures Appl.*, **15** (1970), 735–742.

11. La représentation des algèbres de von Neumann finies par les faisceaux, *Rev. Roum. Math. Pures Appl.*, **15** (1970), 143–151.
12. Representations of von Neumann algebras by sheaves, *Coll. Math. Soc. Bolyai*; 5. *Hilbert space operator*, Tihany (Hungary), 1970, 519–538.
13. La représentation par faisceaux des modules sur les algèbres harmoniques, *Rev. Roum. Math. Pures Appl.*, **16** (1971), 1247–1284.
14. On algebraic reduction theory, *Rev. Roum. Math. Pures Appl.*, **17** (1972), 287–307.
15. The theory of harmonic algebras, *Lecture Notes in Math.*, No. 248, 100–315, Springer-Verlag, 1972.
16. Algebraic reduction and dimension theory, *Rev. Roum. Math. Pures Appl.*, **17** (1972), 1265–1281.
17. The commutation theorem in the algebraic reduction theory, *Rev. Roum. Math. Pures Appl.*, **17** (1972), 1451–1472.
18. The theorem of de Rham for harmonic algebras, *J. Algebra*, **23** (1972), 271–290.
19. Extended von Neumann algebras and their representations by sheaves, *Rev. Roum. Math. Pures Appl.*, **18** (1973), 727–760.
20. Sur la réduction des algèbres de von Neumann, *Rev. Roum. Math. Pures Appl.*, **20** (1975), 693–742.
21. On reduction theory, *Rev. Roum. Math. Pures Appl.*, **21** (1976), 465–486.
22. Sur la réduction des algèbres de von Neumann, *CR Acad. Sc. Paris*, **280** (1975), 1685–1688.
23. On the irreducible disintegration of representations of C^*-algebras.
24. On the Baire probability measures which are pseudo-concentrate on the pure states set of a C^*-algebra, Bucharest, 1977.
25. On the Choquet and Bishop-de Leeuw theorems, X-th Banach Semester, Warsaw, 1977.

TESTARD, D.
1. Algèbre de covariance, *Ann. Inst. H. Poincaré*, **6** (1967), 267–297.
2. Type des représentations quasi-libres de l'algèbre de Clifford, *Ann. Inst. H. Poincaré*, **12** (1970), 329–342.
3. Asymptotic ratio set of von Neumann algebras generated by temperature states in statistical mechanics, *Rep. Math. Phys.*, **12** (1977), 115–118.

THOMA, E.
1. Zur Reduktionstheorie in separablen Hilbert-Räumen, *Math. Z.*, **67** (1957), 1–9.
2. Zur Reduktionstheorie in allgemein Hilbert-Räumen, *Math. Z.*, **68** (1957), 153–188.
3. Über unitäre Darstellungen abzählbarer diskreter Gruppen, *Math. Ann.*, **153** (1964), 111–138.
4. Zur harmonischen Analyse klassfiniter Gruppen, *Invent. Math.*, **3** (1967), 20–42.
5. Eine Characterisierung diskreter Gruppen von Typ I, *Invent. Math.*, **6** (1968), 190–196.
6. Eine Characterisierung kompakter Untergruppen lokalkompakter Gruppen, *Arch. Math.*, **20** (1969), 1–3.
7. Die unzerlegbaren, positiv-definiten Klassfunktionen der abzählbar unendlichen symmetrischen Gruppe, *Math. Z.*, **85** (1974), 40–61.

TIMMERMANN, W.
1. Ideals in algebras of unbounded operators, Dubna, 1977.
2. On an ideal in algebras of unbounded operators, Dubna, 1977.
3. Simple properties of some ideals of compact operators in algebras of unbounded operators, Dubna, 1977.
4. The uniform and the strong topology on realizations of the algebra of polynomials, Dubna, 1977.

TISCHER, J.
1. *Spurdarstellung von Erwartungsfunktionen und von Massen auf Projektoren*, Diplomarbeit, Erlangen, 1971.
2. *Ein Radon-Nikodym Satz für Gewichte auf von Neumann Algebren*, Thesis, Erlangen, 1974.

TISCHER, J.; WITTMER, H.
1. Invarianz gewisser Operatorenklassen unter nichtkommutativen ergodischen Mitteln, *Manuscripta Math.*, **13** (1974), 73–81.

TODOROV, T. S.

1. Discussion of a theorem of R. T. Powers, *Bulgarian Phys. J.*, **3** (1977), 266–269.

TOMITA, M.

1. Measure theory of complete Boolean algebras, *Mem. Fac. Sci. Kyushu Univ.*, **7** (1952), 51–60.
2. On rings of operators in non-separable Hilbert spaces, *Mem. Fac. Sci. Kyushu Univ.*, **7** (1953), 129–168.
3. On the regularly convex hull of a set in a conjugate Banach space, *Math. J. Okayama Univ.*, **3** (1954), 143–145.
4. Representations of operator algebras, *Math. J. Okayama Univ.*, **3** (1954), 147–173.
5. Banach algebra generated by a bounded linear operator, *Math. J. Okayama Univ.*, **4** (1955), 97–102.
6. Harmonic analysis of locally compact groups, *Math. J. Okayama Univ.*, **5** (1956), 133–193.
7. Composition of linear topological spaces, *Math. J. Okayama Univ.*, **6** (1957), 191–208.
8. Spectral theory of operator algebras, I, II, *Math. J. Okayama Univ.*, **9** (1959), 62–98; **10** (1960), 19–60.
9. The second dual of a C^*-algebra, *Mem. Fac. Sci. Kyushu Univ.*, **21** (1967), 185–193.
10. *Quasi-standard von Neumann algebras*, Mimeographed Notes, 1967.
11. Standard forms of von Neumann algebras, *The Vth functional analysis Symp. of the Math. Soc. of Japan*, Sendai, 1967.
12. Foundations of non-commutative Fourier Analysis, I, *Japan-US Seminary*, 1974.

TOMIUK, B. J.; WONG, P.-K.

1. The Arens product and duality in B^*-algebras, *Proc. AMS*, **25** (1970), 529–535.

TOMIYAMA, J.

1. On the projections of norm one in W^*-algebras, I, II, III, *Proc. Jap. Acad.*, **33** (1957), 608–612; *Tohoku Math. J.*, **10** (1958), 204–209; **11** (1959), 125–129.
2. A remark on the invariants of W^*-algebras, *Tohoku Math. J.*, **10** (1958), 37–41.
3. Generalized dimension function for W^*-algebras of infinite type, *Tohoku Math. J.*, **10** (1958), 121–129.
4. On the product projections of norm one in the direct product of operator algebras, *Tohoku Math. J.*, **11** (1959), 305–313.
5. Tensor products of commutative Banach algebras, *Tohoku Math. J.*, **12** (1960), 147–154.
6. Topological representations of C^*-algebras, *Tohoku Math. J.*, **14** (1962), 187–204.
7. A characterization of C^*-algebras whose conjugate spaces are separable, *Tohoku Math. J.*, **15** (1963), 96–102.
8. Applications of Fubini type theorem to tensor product of C^*-algebras, *Tohoku Math. J.*, **19** (1967), 213–226.
9. A remark on representations of CCR algebras, *Proc. AMS*, **19** (1968), 1506.
10. On the tensor products of von Neumann algebras, *Pacific J. Math.*, **30** (1969), 263–270.
11. *Tensor products and projections of norm one in von Neumann algebras*, Lecture Notes, Univ. Copenhagen, 1971.
12. *Applications of Fubini mappings to tensor product of Banach algebras*, Lecture Notes, Univ. Copenhagen, 1971.
13. Primitive ideals in tensor products of Banach algebras, *Math. Scand.*, **30** (1972), 257–272.
14. On some types of maximal abelian subalgebras. *J. Fnal Analysis*, **10** (1972), 373–386.
15. Applications de Fubini dans les produits tensoriels infinis d'algèbres de von Neumann, *Bull. Yamagata Univ.*, **8** (1974), 361–368.
16. The extension property and tensor products of C^*-algebras.
17. The extension property of von Neumann algebras and an associated class of C^*-algebras.
18. Derivations of C^*-algebras which are not determined by multipliers in any quotient algebra, *Proc. AMS*, **47** (1975), 265–267.
19. Derived algebras of C^*-algebras, *Lecture at Varrena Summer School*, 1973.
20. Tensor products and approximation problems of C^*-algebras, *Publ. RIMS, Kyoto Univ.*, **11** (1975), 163–183.
21. Fubini algebras and the commutation theorem of tensor product of C^*-algebras, in *Symposia Math.*, XX, 27–38, Academic Press, 1976.

22. On the Fubini product of von Neumann algebras, *Bull. Yamagata Univ.*, **9** (1976), 53–56.
23. Some aspects of the commutation theorem for tensor products of operator algebras, Marseille, 1977.

TOMIYAMA, J.; TAKESAKI, M.
1. Applications of fibre bundles to the certain class of C^*-algebras, *Tohoku Math. J.*, **13** (1961), 498–523.

TOPPING, D. M.
1. Vector lattices of selfadjoint operators, *Bull. AMS*, **69** (1963), 251–255; *Trans. AMS*, **115** (1965), 14–30.
2. Jordan algebras of selfadjoint operators, *Bull. AMS*, **71** (1965), 160–164; *Memoirs AMS*, No. 53, 1965.
3. An isomorphism invariant for spin factors, *J. Math. Mech.*, **15** (1966), 1055–1063.
4. Asymptoticity and semimodularity in projection lattices, *Pacific J. Math.*, **20** (1967), 317–325.
5. UHF algebras are single generated, *Math. Scand.*, **22** (1968), 224–226.
6. On linear combinations of special operators, *J. Algebra*, **10** (1968), 516–521.
7. Transcendental quasinilpotents in operator algebras, *J. Fnal Analysis*, **2** (1968), 342–351.
8. *Lectures on von Neumann algebras*, Van Nostrand, 1971.

TORRANCE, E.
1. Maximal C^*-subalgebras of a Banach algebra, *Proc. AMS*, **25** (1970), 622–624.

TRYCH-POHLMEYER, E. B.
1. *Equilibrium states of infinite quantum systems*, DESY, Hamburg, 1974.

TSUI, S.-K., J.
1. Decomposition of linear maps, *Trans. AMS*, **230** (1977), 87–112.

TSUJI, K.
1. Representation theorems of operator algebras and their applications, *Proc. Jap. Acad.*, **31** (1955), 272–277.
2. N^*-algebras and finite class groups, *Bull. Kyushu Inst. Tech.*, **1** (1955), 1–9.
3. Harmonic analysis on locally compact groups, *Bull. Kyushu Inst. Tech.*, **2** (1956), 16–32.
4. W^*-algebras and abstract L-spaces, *Bull. Kyushu Inst. Tech.*, **3** (1957), 11–13.
5. Annihilators of von Neumann algebras (annihilating spaces), *Bull. Kyushu Inst. Tech.*, **10** (1963), 25–39 (Corrections, ibid., **13** (1966), 19).
6. On construction of annihilating spaces, *Proc. Jap. Acad.*, **40** (1964), 662–663.
7. Tensor products of annihilating spaces, *Bull. Kyushu Inst. Tech.*, **13** (1966), 15–18.
8. The J. von Neumann commutation theorem, *Bull. Kyushu Inst. Tech.*, **13** (1966), 1–3.
9. On nest algebras of operators, *Proc. Jap. Acad.*, **46** (1970), 337–340.

TURUMARU, T.
1. On the commutativity of the C^*-algebra, *Kodai Math. Sem. Rep.*, **3** (1951), 51.
2. On the direct product of operator algebras, I, II, III, IV, *Tohoku Math. J.*, **4** (1952), 242–251; **5** (1953), 1–7; **6** (1954), 208–211; **8** (1956), 281–285.
3. Crossed products of operator algebras, *Tohoku Math. J.*, **10** (1958), 355–365.

UHLMANN, A.
1. Unitarily invariant functions on the space of type I and type III von Neumann algebras, *Rep. Math. Phys.*, **7** (1975), 449–456.
2. The "transition probability" in the state space of $*$-algebras, *Rep. Math. Phys.*, **9** (1976), 273–279.

UMEGAKI, H.
1. On some representation theorems in an operator algebra, I, II, III, *Proc. Jap. Acad.*, **27** (1951), 328–333; 501–505; **28** (1952), 29–31.
2. Operator algebras of finite class, I, II, *Kodai Math. Sem. Rep.*, **4** (1952), 123–129; **5** (1953), 61–63.
3. Decomposition theorems of operator algebras and their applications, *Japanese J. Math.* **22** (1952), 27–50.

4. Ergodic decomposition of stationary linear functional, *Proc. Jap. Acad.*, **30** (1954), 358–362.
5. Note on irreducible decomposition of a positive linear functional, *Kodai Math. Sem. Rep.*, **6** (1954), 25–32.
6. Conditional expectation in an operator algebra, I, II, III, IV, *Tohoku Math. J.*, **6** (1954), 177–181; **8** (1956), 86–100; *Kodai Math. Sem. Rep.*, **11** (1959), 51–74; **14** (1962), 59–85.
7. Positive definite functions and direct product of Hilbert space, *Tohoku Math. J.*, **7** (1955), 206–211.
8. Weak compactness in an operator space, *Kodai Math. Sem. Rep.*, **8** (1956), 145–151.
9. On information in operator algebras, *Proc. Jap. Acad.*, **37** (1961), 459–461.
10. Entropy functional in stationary channels, *Proc. Jap. Acad.*, **38** (1962), 668–672.
11. A functional method on amount of entropy, *Kodai Math. Sem. Rep.*, **15** (1963), 162–175.
12. Absolute continuity of information channels, *J. Multivariate Anal.*, **4** (1974), 382–400.

VAINERMANN, L. I.
1. Characterization of dual objects for locally compact groups (Russian), *Funkt. Analiz i pril.* **8** : 1 (1974), 75–76.

VAINERMAN, L. I.; KATZ, G. I.
1. Non commutative ring groups and Hopf-von Neumann algebras (Russian), *Dokl. Akad. Nauk SSSR*, **211**: 5 (1973), 1031–1034; *Matem. Sb.*, **94**: 2 (1974), 194–225, 335.

VANHEUVERZWIJN, P.
1. Generators for quasi-free completely positive semigroups, Leuven, 1977.

VARADARAJAN, V. S.
1. Groups of automorphisms of Borel space, *Trans. AMS*, **109** (1963), 191–220.

VARELA, J.
1. Duality of C^*-algebras, *Memoirs AMS*, **148** (1974), 97–108.
2. Sectional representation of Banach modules, *Math. Z.*, **139** (1974), 55–61.
3. Fields of automorphisms and derivations of C^*-algebras, *Pacific J. Math.*, **59** (1975), 277–294.

VAROPOULOS, N. TH.
1. Sur les formes positives d'une algèbre de Banach, *CR Acad. Sc. Paris*, **258** (1964), 2465–2467.

VASILESCU, F.-H.
1. The analysis of selfadjoint spectral measures (Romanian), *Stud. Cerc. Mat.*, **23** (1971), 129–146.
2. Positive forms on Lie algebras with involution, *Rev. Roum. Math. Pures Appl.*, **18** (1973), 951–957.

VASILESCU, F.-H.; ZSIDÓ, L.
1. Uniformly bounded groups in finite W^*-algebras, *Acta Sci. Math.*, **36** (1974), 189–192.

VASILIEV, N. B.
1. C^*-algebras with finite dimensional irreducible representations (Russian), *Uspekhi Mat. Nauk*, **21**: 1 (1966), 135–154.

VERBEURE, A.
1. Normal and locally normal quasi-free states of Fermi systems, *Cargese Lectures in Phys.*, **4** (1970), 349–362.

VERBEURE, A.; WEDER, R.
1. Stability in linear response and clustering property, *Comm. Math. Phys.*, **44** (1975), 101–105.

VERŠIK, A. M. (in Russian).
1. Nonmeasurable decompositions, orbit theory, algebras of operators, *Dokl. Akad. Nauk SSSR*, **199** (1971), 1004–1007.
2. Geometric theory of states, von Neumann boundary and duality for C^*-algebras, in "*Problems of the theory of probabilistic distributions*", *Zap. Nauč. Sem., LOMI*, vol. **29**, Izd. "Nauka", Leningrad, 1972; *J. Soviet Math.*, **3** (1975), 840–845.

3. Countable groups near to finite ones, Appendix to the russian translation of Greenleaf, F.P. [1].
4. Description of invariant measures for the actions of some infinite dimensional groups, *Dokl. Akad. Nauk, SSSR*, **218**: 4 (1974).

VERŠIK, A. M.; KEROV, S. V. (in Russian)
1. The asymptotics of the Plancherel measure of the symmetric groups and an asymptotic form of Young diagrams, *Dokl. Akad. Nauk, SSSR*, **233** (1977), 1024–1027.

VERŠIK, A. M.; ŠMIDT, A. A. (in Russian)
1. Symmetric groups of high degree, *Dokl. Akad. Nauk, SSSR*, **206** (1972), 269–272.
2. Limiting measures arising in the asymptotic theory of symmetric groups, I, II, *Teoria veroiatn. i prim., AN, SSSR*, **22**: 1 (1977), 72–88; **23**: 1 (1978).

VESTERSTRÖM, J.
1. Quotients of finite W^*-algebras, *Bull. AMS*, **77** (1971), 235–238; *J. Fnal Analysis*, **9** (1972), 322–335.
2. On the homomorphic image of the center of a C^*-algebra, *Math. Scand.*, **29** (1971), 134–136.
3. On open maps, compact convex sets and operator algebras, *J. London Math. Soc.*, **6** (1973), 289–297.
4. Positive linear extension operators for spaces of affine functions, *Israel J. Math.*, **16** (1973), 203–211.

VESTERSTRÖM, J.; WILS, W. I. M.
1. On point realization of L^∞-endomorphisms, *Math. Scand.*, **25** (1969), 178–180.
2. Direct integrals of Hilbert spaces, II, *Math. Scand.*, **26** (1970), 88–102.

VIDAV, I.
1. Sur un système d'axiomes caractérisant les algèbres C^*, *Glasnik mat. fis. astr.*, Drustvo, **16** (1961), 189–192.
2. Eine metrische Kennzeichnung der selbstadjungierten Operatoren, *Math. Z.*, **66** (1956), 121–128.

VOICULESCU, D.
1. Sur les sous-espaces parafermés invariants d'une algèbre de von Neumann, *Bull. Sc. Math.*, **96** (1972), 161–168.
2. Sur un théorème de Misonou, *Rev. Roum. Math. Pures Appl.*, **17** (1972), 309–310.
3. Some extensions of quasitriangularity, I, II, *Rev. Roum. Math. Pures Appl.*, **18** (1973), 1303–1320; 1439–1456.
4. Sur les représentations factorielles finies de $U(\infty)$ et autres groupes semblables, *CR Acad. Sc. Paris*, **279** (1974), 945–946.
5. Représentations factorielles de type II_1 de $U(\infty)$, *J. Math. Pures Appl.*, **55** (1976), 1–20.
6. Un théorème du type Weyl-von Neumann non-commutatif, *CR Acad. Sc. Paris*, **281** (1975), 735–736.
7. A non-commutative Weyl-von Neumann theorem, *Rev. Roum. Math. Pures Appl.*, **21** (1976), 97–113.
8. On a theorem of M. D. Choi and E. G. Effros, Bucharest, 1976.
9. Amenability and Katz algebras, Marseille, 1977.

VOLKMANN, P.
1. Operatorenalgebren mit einer endlichen Anzahl von maximalen Idealen, *Studia Math.*, **55** (1976), 151–156.

VOLKOVA, T. V.
1. On maximal abelian regular subalgebras of II_1 factors (Russian), *Uspekhi Mat. Nauk*, **30** : 6 (1975), 163–164.

VOWDEN, J.
1. On the Gelfand-Naimark theorem, *J. London Math. Soc.*, **42** (1967), 725–731.
2. A new proof in the spatial theory of von Neumann algebras, *J. London Math. Soc.*, **44** (1969), 429–432.

3. Normalcy in von Neumann algebras, *Proc. London Math. Soc.*, **27** (1973), 88–100.
4. C^*-norms and tensor products of C^*-algebras, *J. London Math. Soc.*, **7** (1974), 595–596.

WALTER, M. E.
1. Group duality and isomorphisms of Fourier and Fourier-Stieltjes algebras from a W^*-algebra point of view, *Bull. AMS*, **76** (1970), 1321–1325.
2. W^*-algebras and non-abelian harmonic analysis, *J. Fnal Analysis*, **11** (1972), 17–38.
3. The dual group of the Fourier-Stieltjes algebra, *Bull. AMS*, **78** (1972), 824–827.
4. A duality between locally compact groups and certain Banach algebras, *J. Fnal Analysis*, **17** (1974), 131–160.
5. On the structure of the Fourier-Stieltjes algebra, *Pacific J. Math.*, **58** (1975), 267–282.
6. Convolution on the reduced dual of a locally compact group, *Math. Scand.*, **37** (1975), 145–166.
7. Duality theory for non-abelian locally compact groups.

WANG, S. P.
1. On integrable representations, *Math. Z.*, **147** (1976), 201–203.

WARNER, G.
1. *Harmonic analysis on semi-simple Lie groups*, I, II, Springer Verlag, 1972.

WASSERMANN, S.
1. Tensor products of $*$-automorphisms of C^*-algebras, *Bull. London Math. Soc.*, **7** (1975), 65–70.
2. Extensions of normal functionals on W^*-tensor products, *Math. Proc. Cambridge Phil. Soc.*, **78** (1975), 301–307.
3. On the tensor products of certain group algebras, *J. Fnal Analysis*, **23** (1976), 239–254.
4. The slice map problem for C^*-algebras, *Proc. London Math. Soc.*, **32** (1976), 537–559.
5. Liftings in C^*-algebras: a counterexample, *Bull. London Math. Soc.*, **9** (1977), 201–202.
6. Injective W^*-algebras, *Math. Proc. Cambridge Phil. Soc.*, **82** (1977), 39–48.
7. A pathology in the ideal space of $L(H) \otimes L(H)$.

WEIL, A.
1. *L'intégration dans les groupes topologiques et ses applications*, Act. Sc. Ind. no. 1145, Hermann, Paris, 1953.
2. Sur certains groupes d'opérateurs unitaires, *Acta Math.*, **111** (1964), 143–211.

WEINLESS, M.
1. Existence and uniqueness of the vacuum for linear quantized fields, *J. Fnal Analysis*, **4** (1969), 350–379.

WEISS, G.
1. The Fuglede commutativity theorem modulo operator ideals.
2. The Fuglede commutativity theorem modulo the Hilbert Schmidt class and generating functions for matrix operators, I, II.
3. Partial unitary invariants for compact positive operators and trace class positive operators.

WELLES, B. B. JR.
1. Homomorphisms of $L^\infty(G)$ and liftings, *Indiana Univ. Math. J.*, **24** (1975), 987–994.

WENDEL, J. G.
1. Left centralizers and isomorphisms of group algebras, *Pacific J. Math.*, **2** (1952), 251–261.

WESTMANN, J. J.
1. Harmonic analysis on groupoids, *Pacific J. Math.*, **27** (1968), 621–632.
2. Non-transitive groupoid algebras, Irvine, 1967.
3. Ergodic groupoid algebras and their representations, Irvine, 1968.
4. Cohomology for the ergodic actions of countable groups, *Proc. AMS*, **30** (1970), 318–320.

WEYL, H.
1. *The classical groups. Their invariants and representations*. Princeton, 1939.
2. *The theory of groups and quantum mechanics*, Springer Verlag, 1928; Dover Publ., 1931.

WICHMANN, J.
1. Bounded approximate units and bounded approximate identities, *Proc. AMS*, **41** (1973), 547–550.
2. Hermitian $*$-algebras which are not symmetric, *J. London Math. Soc.*, **8** (1974), 109–112.
3. On commutative B^*-equivalent algebras, *Notices AMS*, **22** (1975), 720–46–19.
4. On the symmetry of matrix algebras, *Proc. AMS*, **54** (1976), 237–240.

WICKSTEAD, A. W.
1. The centralizer of $E \otimes_\lambda F$, *Pacific J. Math.*, **65** (1976), 563–571.

WIDOM, H.
1. Approximately finite algebras, *Trans. AMS*, **83** (1956), 170–178.
2. Embedding in algebras of type I, *Duke Math. J.*, **23** (1956), 309–324.
3. Non-isomorphic approximately finite factors, *Proc. AMS*, **8** (1957), 537–540.

WIERINGA, J. D.
1. *Thermodynamic limit and KMS states in quantum statistical mechanics. A C^*-algebraic approach*, Thesis, Groningen, 1970.

WIGHTMAN, A. S.
1. Quelques problèmes mathématiques de la théorie quantique relativiste, *Colloque Int.* (Lille, 1957), 1959, 1–38.
2. On the localisability of quantum mechanical systems, *Rev. Mod. Phys.*, **34** (1962), 845–872.
3. La théorie quantique locale et la théorie quantique de champs, *Ann. Inst. H. Poincaré*, **1** (1964), 403–420.

WIGHTMAN, A.; SCHWEBER, S.
1. Configuration space methods, *Phys. Rev.*, **98** (1955), 812–837.

WIGNER, E. P.
1. *Gruppentheorie und ihre Anwendungen*, Braunschweig, 1931.

WILDFOSEL, D.
1. Annihilator Banach $*$-algebras and the Peter-Weyl theorem, *J. Math. Anal. Appl.*, **55**, (1976), 509–524.

WILLIAMS, J. J.
1. Non-isomorphic tensor products of von Neumann algebras, *Canadian J. Math.*, **26** (1974), 492–512.
2. A representation for infinite tensor products of semi-finite factors.
3. Spatial product isomorphisms of tensor products of finite von Neumann algebras.

WILLIAMS, J. P.
1. Schwarz norms for operators, *Pacific J. Math.*, **24** (1968), 181–188.
2. Finite operators, *Proc. AMS*, **26** (1970), 129–136.
3. On the spectrum in the Calkin algebra.

WILLIG, P.
1. On trace class norm in factors of type II, *Comm. Pure Appl. Math.*, **21** (1968), 585–590.
2. Trace norms, global properties and direct integral decompositions of W^*-algebras, *Comm. Pure Appl. Math.*, **22** (1969), 839–862.
3. Properties Γ and L for type II_1 factors, *Proc. AMS*, **25** (1970), 836–837.
4. $B(H)$ is very non-commutative, *Proc. AMS*, **24** (1970), 204–205.
5. Property P and direct integral decompositions of W^*-algebras, *Proc. AMS*, **29** (1971), 494–498.
6. Property L and direct integral decompositions of W^*-algebras, *Proc. AMS*, **30** (1971), 87–91.
7. Type II W^*-algebras are not normal, *Proc. AMS*, **40** (1973), 115–119.
8. On hyperfinite W^*-algebras, *Proc. AMS*, **40** (1973), 120–122.
9. Central sequences, global properties and direct integral decompositions of W^*-algebras.
10. Generators and direct integral decompositions of W^*-algebras, *Tohoku Math. J.*, **26**, (1974), 35–38.

11. Continuous W^*-algebras are non normal, *Tohoku Math. J.*, **26** (1974), 483–486.

WILS, W. I. M.
1. *Stone-Cech compactifications and representations of operator algebras*, Thesis, Nijmegen, 1968.
2. Désintégration centrale des formes positives sur les C^*-algèbres, *CR Acad. Sc. Paris*, **267** (1968), 810–812.
3. On semigroups near the identity, *Proc. AMS*, **21** (1969), 762–763.
4. Désintégration centrale dans une partie convexe compacte d'un espace localement convexe, *CR Acad. Sc. Paris*, **269** (1969), 702–704.
5. Direct integrals of Hilbert spaces, I, *Math. Scand.*, **26** (1970), 73–88.
6. Two-sided ideals in W^*-algebras, *J. Reine Angew. Math.*, **244** (1970), 55–68.
7. The ideal center for partially ordered vector spaces, *Acta Math.*, **127** (1971), 41–78.
8. Central decompositions of C^*-algebras, *Proc. Fnal Analysis Week*, Aarhus Univ., 1969, 55–63.
9. Order bounded operators and central measures, *Lecture Notes in Phys.*, No. 29, Springer-Verlag, 1974, 54–66.

WINNINK, M.
1. *An application of C^*-algebras to quantum statistical mechanics of systems in equilibrium*, Thesis, Groningen Univ., 1968.
2. Algebraic aspects of the KMS condition, *Cargese Lectures in Phys.*, **4** (1970), 235–255.
3. Some general properties of thermodynamic states in an algebraic approach, in *Statistical Mechanics and Field Theory Lectures*, 1972, 311–333.
4. Almost equilibrium in an algebraic approach, in *Fundamental problems in statistical mechanics*, Vol. 3 (Ed. E. G. D. Cohen), Amsterdam, North-Holland Publ. Comp., 1975.

WITTMER, H.
1. *Darstellungs- und Implementierungssätze für von Neumann Algebren*, Thesis, Erlangen, 1974.

WOGEN, W.
1. On generators for von Neumann algebras, *Bull. AMS*, **75** (1969), 95–99.
2. On special generators for properly infinite von Neumann algebras, *Proc. AMS*, **28** (1971), 107–113.
3. Von Neumann algebras generated by operators similar to normal operators, *Pacific Math. J.*, **37** (1971), 539–543.
4. On some operators with cyclic vectors.

WOJTASZCZYK, P.
1. On linear properties of separable conjugate space of C^*-algebras, *Studia Math.*, **52** (1974), 143–147.
2. Linear topological properties of C^*-algebras, *Publ. Math. Res. Inst., Istanbul*, No. 1, 1974.

WOLFE, J. C.
1. Invariant states and conditional expectation of the anticommutation relations, *Comm. Math. Phys.*, **44** (1975), 53–72.
2. Free states and automorphisms of the Clifford algebra, *Comm. Math. Phys.*, **45** (1975), 53–58.

WOLFE, J. C.; EMCH, G. G.
1. C^*-algebraic formalism for coarse graining, I, II, III, *J. Math. Phys.*, **15** (1974), 1343–1347; 1348–1350; 1351–1365.

WOLFSON, K. G.
1. The algebra of bounded operators on Hilbert space, *Duke Math. J.*, **20** (1953), 533–538.
2. Baer rings of endomorphisms, *Math. Ann.*, **143** (1961), 19–28.
3. Baer subrings of the ring of linear transformations, *Math. Z.*, **75** (1961), 328–332.

WONG, PAK-KEN
1. Continuous complementors on B^*-algebras, *Pacific J. Math.*, **33** (1970), 255–260.
2. Modular annihilator A^*-algebras, *Pacific J. Math.*, **37** (1971), 825–834.
3. The Arens product and duality in B^*-algebras, II, *Proc. AMS*, **27** (1971), 535–538.

4. On the Arens product and annihilator algebras, *Proc. AMS*, **30** (1971), 79—83.
5. On the Arens product and commutative Banach algebras, *Proc. AMS*, **37** (1973), 111—113.
6. $*$-actions in A^*-algebras, *Pacific J. Math.*, **44** (1973), 775—779.
7. On the Arens product and certain Banach algebras, *Trans. AMS*, **180** (1973), 437—448.
8. The p-class in a dual B^*-algebra, *Trans. AMS*, **200** (1974), 355—368.
9. The second conjugates of certain Banach algebras, *Canadian J. Math.*, **27** (1975), 1028—1035.
10. A minimax formula for dual B^*-algebras, *Trans. AMS*, **224** (1976), 281—298.
11. On certain subalgebras of a dual B^*-algebra, *J. Austral. Math. Soc.*, **23** (1977), 105—111.

WOODS, E. J.
1. Continuity properties of the representations of the canonical commutation relations, *Comm. Math. Phys.*, **17** (1970), 1—20.
2. Topologies for test function spaces for representations of canonical commutation relations.
3. The classification of factors is not smooth, *Canadian J. Math.*, **25** (1973), 96—102.

WOODS, R. G.
1. Characterization of some C^*-embedded subspaces of βN, *Pacific J. Math.*, **65** (1976), 573—579.

WORONOWICZ, S. L.
1. On a theorem of Mackey, Stone and von Neumann, *Studia Math.*, **24** (1964), 101—105.
2. On the purification of factor states, *Comm. Math. Phys.*, **28** (1972), 221—235.
3. On the purification map, *Comm. Math. Phys.*, **30** (1973), 55—67.
4. Selfpolar forms and their applications to the C^*-algebra theory, *Rep. Math. Phys.*, **6** (1974), 487—495.
5. Ergodic mixtures of pure states, *Bull. Acad. Polon. Sci.*, **23** (1975), 253—258.
6. Positive maps of low dimensional matrix algebras, *Rep. Math. Phys.*, **10** (1976), 165—183.
7. The geometrical mean and the purification map, *Proc. Int. School of Math. Phys.*, Camerino, Sept. 30—Oct. 12, 1974, 439—451.
8. Nonextendible positive maps, *Comm. Math. Phys.*, **51** (1976), 243—283.
9. A remark on the polar decomposition of m-sectorial operators, *Letters Math. Phys.*
10. The polar decomposition of m-sectorial operators: Inequalities.
11. Operator systems and their applications to the Tomita-Takesaki theory.

WRIGHT, F. B.
1. A reduction theory for algebras of finite type, *Ann. Math.*, **60** (1954), 560—570.
2. The ideals in a factor, *Ann. Math.*, **68** (1958), 475—483.

WRIGHT, S.
1. Banach module valued derivations on C^*-algebras.
2. On orthogonalization of C^*-algebras.
3. Similarity orbits of approximately finite C^*-algebras.

WULFSOHN, A.
1. Produit tensoriel de C^*-algèbres, *Bull. Sc. Math.*, **87** (1963), 13—21.
2. Le produit tensoriel de certaines C^*-algèbres, *CR Acad. Sc. Paris*, **258** (1964), 6052—6054.
3. The primitive spectrum of a tensor product of C^*-algebras, *Proc. AMS*, **19** (1968), 1094—1096.
4. A compactification due to Fell, *Canadian Math. Bull.*, **15** (1972), 145—146.
5. Von Neumann factors are perfect, *Proc. AMS*, **41** (1973), 189—192.
6. Tensor products of Jordan algebras, *Canad. J. Math.*, **27** (1975), 60—74.

WYSEUR, R.
1. An "operational calculus" for weakly continuous one-parameter groups of weakly continuous isometries on a Banach space X.

WYSS, W.
1. The field algebra and its positive linear functionals, *Comm. Math. Phys.*, **27** (1972), 223—234.
2. Quantum field theory in terms of sesquilinear forms, *J. Math. Phys.*, **14** (1973), 1271—1276.
3. Algebraic methods in theoretical physics, in *Lectures in Theoretical Physics*, vol. XII-c, Gordon & Breach, 1971, 3—21.

YEADON, F. J.
1. A new proof of the existence of a trace in finite von Neumann algebras, *Bull. AMS*, **77** (1971), 257–260.
2. Convergence of measurable operators, *Proc. Cambridge Phil. Soc.*, **74** (1973), 257–268.
3. A note on the Mackey topology of von Neumann algebras, *J. Math. Anal. Appl.*, **45** (1974), 721–722.
4. Fixed points and amenability; a counterexample, *J. Math. Anal. Appl.*, **45** (1974), 718–720.
5. Non-commutative L_p-spaces, *Math. Proc. Cambridge Phil. Soc.*, **77** (1975), 91–102.
6. On a result of P. G. Dixon, *J. London Math. Soc.*, **9** (1975), 610–612.
7. Ergodic theorems for semifinite von Neumann algebras, I, *J. London Math. Soc.*, **16** (1977), 326–332.

YEN, TI
1. Trace on finite AW^*-algebras, *Bull. AMS*, **60** (1954), 364–365; *Duke Math. J.*, **22** (1955), 207–222.
2. Quotient-algebras of a finite AW^*-algebra, *Pacific J. Math.*, **6** (1956), 389–395.
3. Isomorphisms of unitary groups in AW^*-algebras, *Tohoku Math. J.*, **8** (1956), 275–280.
4. Isomorphisms of AW^*-algebras, *Proc. AMS*, **8** (1957), 345–349.

YLINEN, K.
1. Compact and finite dimensional elements of normed algebras, *Ann. Acad. Sci. Fennicae*, A **428** (1968).
2. The maximal ideal space of a Banach algebra of multipliers, *Math. Scand.*, **27** (1970), 166–180.
3. A note on the compact elements of C^*-algebras, *Proc. AMS*, **35** (1972), 305–306.
4. Vector space isomorphisms of C^*-algebras, *Studia Math.*, **46** (1973), 31–34.
5. Tensor products of complex L-spaces and convolution measure algebras, *Ann. Acad. Sci. Fennicae*, A **553** (1973).
6. Weakly completely continuous elements of C^*-algebras, *Proc. AMS*, **52** (1975), 323–326.
7. Dual C^*-algebras, weakly semi-completely continuous elements, and the extreme rays of the positive cone, *Ann. Acad. Sci. Fennicae*, A **599** (1975).
8. Fourier transforms of non-commutative analogues of vector measures and bimeasures with applications to stochastic processes, *Ann. Acad. Sc. Fennicae*, A, Math., **1** (1975), 355–385.
9. Characterizations of $B(G)$ and $B(G) \cap AP(G)$ for locally compact groups, *Proc. AMS*, **58** (1976), 151–157

YNGVASON, J.
1. On the algebra of test functions for field operators. Decomposition of linear functionals into positive ones, *Comm. Math. Phys.*, **34** (1973), 315–333.

YOOD, B.
1. Faithful $*$-representations of normed algebras, I, II, *Pacific J. Math.*, **10** (1960), 345–363; **14** (1964), 1475–1487.
2. Continuity for linear maps on Banach algebras, *Studia Math.*, **31** (1968), 263–266.
3. On Kadison's condition for extreme points of the unit ball in a B^*-algebra, *Proc. Edinburgh Math. Soc.*, **16** (1969), 240–250.
4. On algebras which are pre-Hilbert spaces, *Duke Math. J.*, **36** (1969), 261–271.
5. On axioms for B^*-algebras, *Bull. AMS*, **76** (1970), 80–82.
6. Incomplete normed algebras, *Bull. AMS*, **78** (1972), 50–52.
7. Closed prime ideals in topological rings, *Proc. London Math. Soc.*
8. Hilbert algebras as topological algebras, *Arkiv Math.*, **12** (1974), 131–151.
9. Commutativity properties in Banach $*$-algebras, *Pacific J. Math.*, **53** (1974), 307–317.
10. On non-negative spectrum in Banach algebras, *Proc. Edinburgh Math. Soc.*, **18** (1973), 295–298.
11. Decompositions of Hilbert algebras, *Notices AMS*, **23** : 3 (1976), 734–B 25.

YOSHINO, T.
1. Hyponormal operators in von Neumann algebras, *Tohoku Math. J.*, **28** (1976), 129–134.

YOSHIZAWA, H.
1. Unitary representations of locally compact groups. Reproduction of Gelfand-Naimark theorem, *Osaka Math. J.*, **1** (1949), 81–99.
2. On some types of convergence of positive definite functions, *Osaka Math. J.*, **1** (1949), 90–94.
3. A proof of the Plancherel theorem, *Proc. Jap. Acad.*, **30** (1954), 276–281.

YOSIDA, K.
1. On the duality of non-commutative compact groups, *Proc. Imp. Acad. Tokyo*, **19** (1943), 181–183.
2. On the unitary equivalence in general Euclid space, *Proc. Jap. Acad.*, **22** (1946), 242–245.
3. *Functional Analysis*, Springer-Verlag, 1965.

ZACHARY, W. W.
1. Non relativistic time-dependent scattering theory and von Neumann algebras, I, *J. Math. Phys.*, **13** (1972), 609–615.

ZELAZKO, W.
1. A characterization of multiplicative linear functionals in complex Banach algebras, *Studia Math.*, **30** (1968), 83–85.

ZELLER-MEIER, G.
1. Produits croisés d'une C^*-algèbre par un groupe d'automorphismes, *CR Acad. Sc. Paris*, **263** (1966), 20–23.
2. Représentations fidèles des produits croisés, *CR Acad. Sc. Paris*, **264** (1967), 679–682.
3. Sur les automorphismes des algèbres de Banach, *CR Acad. Sc. Paris*, **264** (1967), 1131–1132.
4. Produits croisés d'une C^*-algèbre par un groupe d'automorphismes, *J. Math. Pures Appl.*, **47** (1968), 101–239.
5. Deux autres facteurs de type II_1, *Invent. Math.*, **7** (1969), 235–242.

ZEMANEK, J.
1. A remark on transitivity of operator algebras, *Časopis pro pestovani matem.*, **100** (1975), 176–178.

Z'EP, D.-N.
1. Structure of the group C^*-algebra of affine transformations of a straight line (Russian), *Funkt. Analiz i pril.*, **9** : 1 (1975), 63–64.

ZIMMER, R. J.
1. Amenable ergodic group actions and an application to Poisson boundaries of random walks, *J. Fnal Analysis.*
2. On the von Neumann algebra of an ergodic group action, *Proc. AMS*, **66** (1977), 289–293.
3. Amenable ergodic actions, hyperfinite factors and Poincaré flows, *Bull. AMS.*
4. Hyperfinite factors and amenable ergodic actions, *Invent. Math.*, **41** (1977), 23–32.
5. Orbit spaces of unitary representations, ergodic theory, and simple Lie groups, *Ann. Math.* **106** (1977), 573–588.
6. Normal ergodic actions, *J. Fnal Analysis*, **25** (1977), 286–305.
7. Amenable pairs of groups and ergodic actions and the associated von Neumann algebras.
8. Ergodic actions and stochastic processes on groups and homogeneous spaces.

ZSIDÓ, L.
1. I. Cuculescu's proof for the commutant theorem, *Colloq. Math. Soc. "J. Bolyái", Hilbert space operators*, Tihany (Hungary), **5** (1970), 539–542.
2. The norm of a derivation in a W^*-algebra, *Proc. AMS*, **38** (1973), 147–150.
3. Reduction theory for W^*-algebras, in *Theory of operators and operator algebras* (Romanian), 131–262, Editura Academiei, 1973.
4. Topological decomposition of W^*-algebras, I, II (Romanian), *Stud. Cerc. Mat.*, **26** (1973), 859–945, 1037–1112.
5. Note on Dixmier's trace type sets in properly infinite W^*-algebras, *Rev. Roum. Math. Pures Appl.*, **19** (1974), 269–274.

6. A proof of Tomita's fundamental theorem in the theory of standard von Neumann algebras, *Rev. Roum. Math. Pures Appl.*, **20** (1975), 609–619.
7. The Weyl-von Neumann theorem in semifinite factors, *J. Fnal Analysis*, **18** (1975), 60–72.
8. Spectral and ergodic properties of the analytic generator, *J. Approx. Theory*, **20** (1977), 77–138.
9. On spectral subspaces associated to locally compact abelian groups, *Adv. Math.*
10. Topological decompositions of C^*-algebras, in *Global Analysis and Applications*, Int. Atomic Energy Agency, Vienna, 1974, vol. III, 305–308.
11. Analytic generators and the foundation of the Tomita-Takesaki theory of Hilbert algebras, *Proc. Int. School Math. Phys.*, Univ. Camerino, 1974, 182–267.
12. Hardy spaces associated to one-parameter groups (of $*$-automorphisms), *Symposia Math.*, XX, 169–178, Academic Press, 1976.
13. On the equality of two weights, *Rev. Roum. Math. Pures Appl.*, **23** (1978), 631–646.
14. Invariance and dual weights, Marseille, 1977.

ŽUK, V. A.; KATZ, G. I.
1. Algebras with canonical generators, and the structure of C^*-algebras of metabelian Lie groups (Russian), *Dokl. Akad. Nauk SSSR*, **229**:6 (1976), 1313–1316.

Added in proof:

AKEMANN, C. A.; JOHNSON, B. E.
1. Derivations of non-separable C^*-algebras.

BRATTELI, O.
10. When is a C^*-crossed product simple? *Japan-US Seminary*, 1977.
11. Crossed products of UHF algebras by product type actions.
12. Dynamical stability and the KMS condition in quantum statistical mechanics.

BRATTELI, O.; ROBINSON, D. W.
5. *Algebraic methods and mathematical physics.*

CHRISTENSEN, E.
8. Generators of semigroups of completely positive mappings.

CHRISTENSEN, E.; EVANS, D. E.
1. Cohomology of operator algebras and quantum dynamical semigroups.

CONNES, A.
29. Quelques aspects de la théorie des algèbres d'opérateurs, *Strasbourg, RCP 25*, **24** (1977), 13–87.

ELLIOTT, G. A.
23. On extending the trace as a linear functional, I, II, *Rep. Math. Phys.*
24. Universally weakly inner one-parameter automorphism groups of separable C^*-algebras.

GOLODEČ, V. YA.
38. Automorphisms of von Neumann algebras, *Dokl. Akad. Nauk SSSR*, **237**: 4 (1977), 770–772.

GREEN, P.
4. The structure of imprimitivity algebras.
5. Minimal primitive ideals of GCR C^*-algebras.
6. A primitive C^*-algebra with no non trivial projections.

LUNDBERG, L. E.
3. Vacuum polarization due to potential perturbations, *Comm. Math. Phys.*

MC ASEY, M.; MUHLY, P. S.; SAITO, K.-S.
1. Non self-adjoint crossed products.

PEDERSEN, G. K.
28. Remarks on the Connes spectrum for C^*-dynamical systems.

SAKAI, S.
39. *The theory of unbounded derivations in C^*-algebras*, Sections 1–4: Copenhagen, 1977; Sections 5–9: Newcastle upon Tyne, 1978.

TOMIYAMA, J.
24. Functional fields and module duals of operator algebras.
25. Completely positive maps in the tensor products of von Neumann algebras.
26. Approximation problems in C^*-algebras (Japanese), *Surikaisekikenkyûsho, Kôkyûsoku*, Nr. 265, 1976, 76–88.

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