ON INTEGRATION THEORY FOR SPACES IN SPECTRAL DUALITY

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In this notes some aspects of non-commutative integration are generalized onto the spaces in spectral duality which have been introduced by Alfsen and Shultz [2].

Henceforth we shall assume that (V,K) is a base-norm space and (A,e) is the dual order-unit space (see [1]). Let $\underline{x}=\{x_E\}$ be an A^+-valued measure on a measurable space $(\Omega,\mathcal{A})$ which is σ-additive in w^*-topology and $x_\Omega=e$. Let $f:\Omega\to\mathbb{R}$ be a bounded measurable function. Then we can define the element $I(\underline{x},f)\in A$ by the equality

$$\langle I(\underline{x},f),\rho\rangle = \int_\Omega f\, d\langle\underline{x},\rho\rangle \qquad (\rho\in K).$$

The central place in Alfsen and Shultz non-commutative spectral theory belongs to the P-projection notion [2]. We shall not give the exact definition of P-projection here and only remark that if A is a self-adjoint part of von Neumann algebra M, then every P-projection P on A is of the form $Pa=pap$ for a projection $p\in M$. We denote the set of P-projections on A by $\mathcal{P}$ and $\mathcal{U}=\{Pe\colon P\in\mathcal{P}\}$. Assume next that the spaces (V,K) and (A,e) are in spectral duality [2]. Then $\mathcal{P}$ is an orthomodular lattice and for every $a\in A$ there exists the unique $\mathcal{U}$-valued measure $\underline{p}^a$ (spectral measure) on the Borel σ-algebra on $\sigma(a)$ - spectrum of a - such that $p^a_{\sigma(a)}=e$ and $a=I(\underline{p}^a,\gamma)$ where $\gamma(\lambda)=\lambda$ for $\lambda\in\sigma(a)$. Moreover, the map $\varphi\mapsto\varphi(a)=I(\underline{p}^a,\varphi)$ for the bounded Borel functions φ on $\sigma(a)$ satisfies the usual properties of the functional calculus.

Next we concider some inequalities for a faithfull trace τ (i.e. the element of K such that $(P+P')^*\tau=\tau$ for any $P\in\mathcal{P}$ and $\langle a,\tau\rangle>0$ for $a\in A^+$, $a\neq 0$) and convex continuous function $\varphi\colon[\alpha,\beta]\mapsto\mathbb{R}$ $(\alpha,\beta\in\mathbb{R})$.

<u>Theorem 1</u>. If a function $f\colon\Omega\mapsto[\alpha,\beta]$ is measurable, then

$$\langle\varphi(I(\underline{x},f)),\tau\rangle \le \langle I(\underline{x},\varphi\circ f),\tau\rangle . \qquad (*)$$

The crucial point in the proof of this theorem is the following fact which has been proved in von Neumann algebra case by Sherstnev ([3], prop. 12).

Lemma 2. For $a \in A$

$$\langle |a|, \tau \rangle = \inf \{ \langle a_1 + a_2, \tau \rangle : a_1, a_2 \in A \ ; \ a = a_1 - a_2 \} .$$

From this result it is easy to show that the equality $(*)$ is fulfiled for any convex piecewise linear function φ and then for any convex function.

Proposition 3. If $a \in A$, $\sigma(a) \subset [\alpha, \beta]$, then

$$\langle \varphi(a), \tau \rangle = \inf \sum \varphi(\lambda_i) \langle x_i, \tau \rangle ,$$

where inf is taken over all representations of a as a finite sum $a = \sum \lambda_i x_i$ with $x_i \in A^+$, $\sum x_i = e$ and $\lambda_i \in [\alpha, \beta]$.

Theorem 4. Let (W, K_w) and (B, e_b) be a base-norm space and an order-unit space in spectral duality, $\pi_i : B \mapsto A$ be positive linear maps, $b_i \in B$, $\sigma(b_i) \subset [\alpha, \beta]$ $(i=1, 2)$ and either a) $\pi_1(e_b) + \pi_2(e_b) = e$ or b) $\pi_1(e_b) + \pi_2(e_b) \le e$, $0 \in [\alpha, \beta]$, $\varphi(0) \le 0$ is held. Then

$$\langle \pi_1(\varphi(b_1)) + \pi_2(\varphi(b_2)), \tau \rangle \geq \langle \varphi(\pi_1(b_1) + \pi_2(b_2)), \tau \rangle .$$

Corollary 5. Let $P \in \mathcal{P}$, $0 \in [\alpha, \beta]$, $\varphi(0) \le 0$, $a \in A$ and $\sigma(a) \subset [\alpha, \beta]$. Then

$$\langle P \varphi(a), \tau \rangle \geq \langle \varphi(Pa), \tau \rangle .$$

Corollary 6. Let $a_i \in A$, $\sigma(a_i) \subset [\alpha, \beta]$ $(i=1,2)$, $0 \le \varkappa \le 1$. Then

$$\langle \varphi(\varkappa a_1 + (1-\varkappa) a_2), \tau \rangle \le \varkappa \langle \varphi(a_1), \tau \rangle + (1-\varkappa) \langle \varphi(a_2), \tau \rangle .$$

Corollary 7. For $1 \le p < \infty$ the function

$$a \mapsto \langle |a|^p, \tau \rangle^{1/p} = \left[\int |\lambda|^p \, d \langle \underline{p}^a, \tau \rangle \right]^{1/p}$$

is the norm on A.

Our next aim is to give the representation of the completions of A in the introduced norms (we shall denote these completions by $L_p(\tau)$).

We define $J_\tau = \{ \rho \in V : \exists \ \lambda \ge 0 \ (-\lambda\tau \le \rho \le \lambda\tau) \}$, $K_\tau = J_\tau \cap K$ and introduce norms on J_τ:

$$\|\rho\|_p = \inf \left\{ \left[\sum |\mu_i|^p \langle e, \rho_i \rangle \right]^{1/p} \right. \qquad (1 \le p < \infty),$$

where inf is taken over all representations of ρ as a finite sum $\rho = \sum \mu_i \rho_i$ with $\rho_i \in V^+$, $\sum \rho_i = \tau$, $\mu_i \in \mathbb{R}$ and

$$\|\rho\|_\infty = \inf \{ \lambda \ge 0 : -\lambda\tau \le \rho \le \lambda\tau \}$$

Theorem 8. For $1 \le p < \infty$ and $a \in A$

$$\|a\|_p = \sup \{ |\langle a,\rho\rangle| : \rho \in J_\tau , \ \|\rho\|_q \le 1 \} ,$$

where $p^{-1} + q^{-1} = 1$ for $p > 1$ and $q = \infty$ for $p = 1$.

In the proof of this theorem we essentially use the following lemma;

Lemma 9. If $P_1, \dots, P_n$ are mutually orthogonal P-projections, then $(P_1 \vee \dots \vee P_n)^* \tau = (P_1 + \dots + P_n)^* \tau$.

Corollary 10. Let

$$\hat{a}(\rho) = \lim_{n\to\infty} \langle a_n, \rho \rangle$$

for $\|.\|_p$-fundamental sequence $\{a_n\}$ in A. Then this formula defines correctly the injection of $L_p(\tau)$ into the space of linear functions on J_τ. Moreover,

$$\lim_{n\to\infty} \|a_n\|_p = \sup \{ |\hat{a}(\rho)| : \rho \in J_\tau , \ \|\rho\|_q \le 1 \}.$$

Thus $L_p(\tau)$ can be represented as a space of linear functions on J_τ. As J_τ is linearly generated by K_τ, then $L_p(\tau)$ can be represented also as a space of affine functions on K_τ.

Next we shall observe some order properties of the introduced spaces considering the order in $L_p(\tau)$ induced by the last representation.

Theorem 11. a) J_τ^+ is w^*-dence subcone of V^+ and hence it is dense in norm topology.

b) $L_p(\tau)^+$ is the closed generating cone of $L_p(\tau)$, $L_p(\tau)^+ \cap A = A^+$ and the closure of A^+ in $L_1(\tau)$ is equal to $L_1(\tau)^+$.

The following example shows that the properties of considered spaces may be quite different from those in von Neumann algebra case.

Example. Let X be an ordinary L_r-space ($1 < r < \infty$), X be its unit ball, $V = \mathbb{R} \oplus X$, $K = 1 \oplus X$, $Y = L_s$ ($r^{-1} + s^{-1} = 1$), $A = \mathbb{R} \oplus Y$, $A^+ = \{(\mu, y) : \mu \geq \|y\|\}$, $e = (1, 0)$. Then (V,K) and (A,e) are base-norm space and order-unit space in spectral duality [2] and $A = V^*$. The element $\tau = (1,0)$ in K is a faithful trace, the space $L_p(\tau)$ ($1 \le p < \infty$) is $\mathbb{R} \oplus Y$ provided with the norm $\|(\mu,y)\|_p = [\tfrac{1}{2} |\mu + \|y\||^p + \tfrac{1}{2} |\mu - \|y\||^p]^{1/p}$ and the dual space $L_p(\tau)^*$ is $\mathbb{R} \oplus X$ provided with the norm $\|(\lambda,x)\|_q = [\tfrac{1}{2} |\lambda + \|x\||^q + \tfrac{1}{2} |\lambda - \|x\||^q]^{1/q}$ for $1 < p < \infty$, $p^{-1} +$

$q^{-1}=1$ and $\|(\lambda, x)\|_{\infty} = |\lambda| + \|x\|$ for $p=1$. It is easy to show here that in contrast to von Neumann algebra case $L_1(\tau)$ is not isometrically isomorphic to V and $L_2(\tau)$ is not hilbertian if $r \neq 2$.

The proof of the most of adduced results can be found in [4], [5].

R e f e r e n c e s

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