SPECTRAL THEORY FOR BASE-NORM SPACES WITH A TRACE

Oleg Tihonov

Kazan State University. Research Institute of Mathematics and Mecanics.
420008. USSR, Kazan, Lenin street, 18.

1. I n t r o d u c t i o n . The present paper is within the framework of Alfsen and Shultz non-commutative spectral theory. This theory deals with a couple of ordered normed spaces, an order-unit space A and a base-norm space V, which are assumed to be in separating order and norm duality. The main attention in [2] and [3] was concentrated on the conditions which imply the existence and uniqueness of integral representation of the elements of A similar to the spectral representation of self-adjoint operators in a Hilbert space. In this note we show that if A and V are in "projective" duality and V has a faithful trace then the elements of V possess a unique "spectral" representation with respect to the trace. The main idea of the proof is the dualization of Alfsen and Shultz approach. Thus, we replace the order unit from A by the trace from V, the set of projective units by the set of "projective traces" and so on. Such spectral representation with respect to the trace may be considered as the approach to generalization of Segal non-commutative integration theory that differs from [4].

For more detailed presentation see [5],[6].

2. P r e l i m i n a r i e s . Throutout this note we will assume that (V,K) is a complete b a s e - n o r m s p a c e, that is, V is a positively generated ordered Banach space, K is a distinguished base for the positive cone V^+ and the unit ball in V is equal to $\operatorname{conv}(K \cup -K)$ [1]. The dual of (V,K) is an o r d e r - u n i t s p a c e (A,e), that is, A is a positively generated ordered Banach space, the positive cone A^+ is the dual of V^+, the order unit e for A is a linear functional on V that is equal to 1 on K, the unit ball in A is equal to $\{a \in A \mid -e \leqslant a \leqslant e\}$ [1].

The central concept in Alfsen and Shultz non-commutative spectral theory is the notion of P-projection [2] which may be introduced as follows:

DEFINITION. Positive w*-continuous projection P on A of norm at most 1 is called P - p r o j e c t i o n if there exists a positive w^*-continuous projection P' on A of norm at most 1 such that

$$\ker^+ P = \mathrm{im}^+ P' , \qquad \mathrm{im}^+ P = \ker^+ P' ,$$

$$\ker^+ P^* = \mathrm{im}^+ P'^* , \qquad \mathrm{im}^+ P^* = \ker^+ P'^* ,$$

where P^* and P'^* are the dual projections on V, $\ker^+ P = A^+ \cap \ker P$, $\mathrm{im}^+ P' = A^+ \cap \mathrm{im}\, P'$ and so on.

For every P-projection P the projection P' with the properties stated in the previous definition is unique and it is also a P-projection.

In the important special case where A is the self-adjoint part of a von Neumann algebra, V is its predual and K is the set of normal states on A, then the P-projections are precisely the maps $a \mapsto pap$ with p a self-adjoint projection in A.

The set of all P-projections is denoted by $\mathcal{P}$. This set is endowed with a partial ordering:

$$P \leqslant Q \quad \text{if} \quad \mathrm{im}\, P \subset \mathrm{im}\, Q .$$

Two P-projections P and Q are said to be o r t h o g o n a l, in symbols $P \perp Q$, if $P \leqslant Q'$, and they are said to be c o m p a t i b l e if $PQ=QP$.

A face F of K is said to be n o r m - e x p o s e d if $F = \{\rho \in K \mid \langle a, \rho \rangle = 0\}$ for some $a \in A^+$ and F is said to be p r o j e c t i v e if it is of this form with $a=Pe$ for some $P \in \mathcal{P}$.

DEFINITION. We will say that (V,K) and (A,e) are in p r o j e c t i v e d u a l i t y if every norm-exposed face of K is projective.

THEOREM 1 [3]. IF (V,K) and (A,e) are in projective duality then $\mathcal{P}$ endowed with the partial ordering "$\leqslant$" and the operation of complementation $P \mapsto P'$ is a complete orthomodular lattice.

3. S u p p o r t p r o j e c t i o n s a n d c o m p a t a b i l i t y . In the sequel we will assume that (V,K) and (A,e) are in projective duality. Under this assumption for every $\rho \in V^+$ the set $\{P \in \mathcal{P} \mid P^* \rho = \rho\}$ has the smallest element (see [3]) which will be called the s u p p o r t p r o j e c t i o n of ρ and denoted by P_ρ. Two elements $\rho, \sigma \in V^+$ will be said to be o r t h o g o n a l, in symbols $\rho \perp \sigma$, if $P_\rho \perp Q_\sigma$.

By [3, prop. 1.3] every $\rho \in V$ admits the unique decomposition of the form $\rho = \rho^+ - \rho^-$ where $\rho^+, \rho^- \in V^+$ and $\|\rho\| = \|\rho^+\| + \|\rho^-\|$. It is easy to varify that $\rho^+ \perp \rho^-$ and, moreover, the conditions $\rho = \rho^+ - \rho^-$ and $\rho^+ \perp \rho^-$ determine ρ^+ and ρ^-.

DEFINITION. A P-projection P and an element ρ of V are said to be c o m p a t i b l e if $P^*\rho + P'^*\rho = \rho$. They are said to be **bicompatible** if P is compatible with ρ and with all P-projections compatible with ρ.

This definition is the dual version of the notion of compatability and bicompatability of a P-projection and an element of A [2]. Proceeding as in [2] we obtain the following theorem that is essencial in the treatment of the spectral theorem.

THEOREM 2. For every $\rho \in V$ there exists the unique compatible with ρ P-projection P_0 such that $\langle a,\rho\rangle > 0$ for $a \in \mathrm{im}^+ P_0 \setminus \{0\}$ and $\langle a,\rho\rangle \leqslant 0$ for $a \in \mathrm{im}^+ P_0'$. Moreover, this P-projection P_0 is equal to $P_{\rho+}$ and it is bicompatible with ρ.

DEFINITION. An element $\tau \in K$ is called a **trace** if it is compatible with any $P \in \mathcal{P}$ [3]. Similar to von Neumann algebra case we will say that τ is **faithful** if $\langle a,\tau\rangle > 0$ for any $a \in A^+ \setminus \{0\}$.

4. **Projective traces.** In the sequel we will assume a faithful trace τ being distinguished. Similar to the notion of projective units [2], the elements $P^*\tau$ with $P \in \mathcal{P}$ will be called **projective traces**, the set of all projective traces will be denoted by $\mathcal{T}$.

THEOREM 3. The set $\mathcal{T}$ endowed with the partial ordering relativized from V and the operation of complementation $P^*\tau \mapsto \tau - P^*\tau$ is a complete orthomodular lattice isomorphic to $\mathcal{P}$ by means of the map $P \mapsto P^*\tau$. Furthermore, within the previous notation, $P^*\tau \perp Q^*\tau$ iff $P \perp Q$, and P is compatible with $Q^*\tau$ iff P is compatible with Q $(P,Q \in \mathcal{P})$. If $\{P_\alpha^*\tau\}$ is an increasing net of projective traces then it converges to $(\bigvee P_\alpha)^*\tau$, and if $\{P_\alpha^*\tau\}$ is a decreasing net then it converges to $(\bigwedge P_\alpha)^*\tau$.

5. **V^+-valued measures and integration.** For a set function $\mathbb{Y} = \{Y(E)\}$ defined on a class $\mathcal{A}$ of sets of Ω with values in V and an element $a \in A$ we will denote by $\langle a,\mathbb{Y}\rangle$ the corresponding real-valued set function, that is, $\langle a,\mathbb{Y}\rangle(E) = \langle a, Y(E)\rangle$ for any $E \in \mathcal{A}$.

DEFINITION. An additive set function $\mathbb{X}$ defined on a σ-algebra $\mathcal{A}$ of sets of Ω with values in V^+ is said to be **V^+-valued measure** if it satisfies the following equivalent conditions:

(i) $\mathbb{X}$ is σ-additive with respect to the norm,

(ii) $\mathbb{X}$ is σ-additive with respect to the weak topology,

(iii) $\langle e,\mathbb{X}\rangle$ is σ-additive.

Note that if V^+-valued measure $\mathbb{T} = \{T(E)\}$ is actually $\mathcal{T}$-valued then the equality $T(E) = P(E)^*\tau$ correctly defines the set function $\mathbb{P} = \{P(E)\}$ on $\mathcal{A}$ with values in $\mathcal{P}$.

PROPOSITION 1. Let $N: \mathbb{R} \mapsto V$ be a bounded, monotone non-decreasing, right continuous function and $N(\lambda) \to 0$ as $\lambda \to -\infty$. Then there exists the unique V^+-valued measure $\mathcal{X}$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ such that $X((-\infty, \lambda])=N(\lambda)$ for any $\lambda \in \mathbb{R}$. Moreover,

a) if P-projection P is compatible with $N(\lambda)$ for any $\lambda \in \mathbb{R}$ then P is compatible with $X(E)$ for any $E \in \mathcal{B}(\mathbb{R})$;

b) if $N(\lambda)$ belongs to $\mathcal{T}$ for any $\lambda \in \mathbb{R}$ then the measure $\mathcal{X}$ is $\mathcal{T}$-valued;

c) if, under the hypothesis of point b), $N(\lambda)=Q_\lambda^*\tau$ with $Q_\lambda \in \mathcal{P}$, $\mathcal{P}$ denotes the $\mathcal{P}$-valued set function corresponding to $\mathcal{X}$ and Q_λ is compatible with $\rho \in V$ for any $\lambda \in \mathbb{R}$, then $P(E)$ is compatible with ρ for any $E \in \mathcal{B}(\mathbb{R})$.

PROPOSITION 2. Let $\mathcal{X}$ be a V^+-valued measure on a measurable space $(\Omega, \mathcal{A})$, φ be a measurable real-valued function on Ω, then the following are equivalent:

(i) φ is integrable with respect to $\mathcal{X}$,

(ii) φ is integrable with respect to $\langle a, \mathcal{X}\rangle$ for any $a \in A$,

(iii) φ is integrable with respect to $\langle e, \mathcal{X}\rangle$.

PROPOSITION 3. Let $\mathcal{X}$ be a V^+-valued measure on a measurable space $(\Omega, \mathcal{A})$, φ be an integrable real-valued function and $\rho = \int_\Omega \varphi \, d\mathcal{X}$. If $X(E)$ is compatible with $P \in \mathcal{P}$ for any $E \in \mathcal{A}$ then ρ is compatible with P.

6. Spectral theorem. Now we can state the main theorem and sketch the proof.

THEOREM 4. For every $\rho \in V$ there exists the unique $\mathcal{T}$-valued measure $\mathcal{T}^{(\rho)}=\{T^{(\rho)}(E)\}$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ such that $T^{(\rho)}(\mathbb{R})=\tau$ and $\rho = \int_{\mathbb{R}} \gamma \, d\mathcal{T}^{(\rho)}$ where $\gamma(\lambda)=\lambda$ for all $\lambda \in \mathbb{R}$. Moreover, $T^{(\rho)}(E)$ is bicompatible with ρ for any $E \in \mathcal{B}(\mathbb{R})$.

Proof. For $\lambda \in \mathbb{R}$ let P_λ denotes the support projection of $(\rho - \lambda\tau)^+$. By theorem 2 P_λ is bicompatible with ρ and $\langle a, \rho\rangle > \lambda\langle a, \tau\rangle$ for $a \in im^+P_\lambda \setminus \{0\}$, $\langle a, \rho\rangle \leq \langle a, \tau\rangle$ for $a \in im^+P_\lambda'$. Proceeding as in [2] we prove that $P_\lambda = \bigvee_{\mu > \lambda} P_\mu$ for all $\lambda \in \mathbb{R}$. Furthermore, $\bigvee_{\lambda \in \mathbb{R}} P_\lambda = I$ and $\bigwedge_{\lambda \in \mathbb{R}} P_\lambda = 0$.

The function $N(\lambda)=P_\lambda'^*\tau$ on $\mathbb{R}$ is monotone non-decreasing, right continuos, $N(\lambda) \to 0$ as $\lambda \to -\infty$ and $N(\lambda) \to \tau$ as $\lambda \to +\infty$. Let $\mathcal{T}^{(\rho)}$ be the $\mathcal{T}$-valued measure corresponding to $N(\lambda)$ by proposition 1. Then the customary integration arguments are applicable to prove that $\rho = \int_{\mathbb{R}} \gamma \, d\mathcal{T}^{(\rho)}$. By propositions 1 and

3 $T^{(\rho)}(E)$ is bicompatible with ρ for any $E \in \mathcal{B}(\mathbb{R})$.

The uniqueness of $\mathbb{T}^{(\rho)}$ follows from proposition 3 and theorem 2.

To show the usefulness of the theorem we adduce some statements whose proofs are based on it.

PROPOSITION 4. For every $\rho \in V^+$ there exists a non-decreasing sequence $\{\rho_n\}$ from V^+ converging to ρ and such that $\rho_n \leq n\tau$. Therefore, $\{\rho \in V \mid \exists \lambda \in \mathbb{R}\ (-\lambda\tau < \rho < \lambda\tau)\}$ is dense in V.

PROPOSITION 5. Let (A,e) be a factor, that is, the compatability of $P \in \mathcal{P}$ with all P-projections implies P is either 0 or I, and let τ be a trace from V. Then there are no other traces in V.

PROPOSITION 6. Let $1 \leq p < \infty$. Then the set $\{\rho \in V \mid \int_{\mathbb{R}} |\lambda|^p \, d\langle e, \mathbb{T}^{(\rho)}\rangle < \infty\}$ endowed with the norm $\|\rho\|_p = [\int_{\mathbb{R}} |\lambda|^p \, d\langle e, \mathbb{T}^{(\rho)}\rangle]^{1/p}$ is a Banach space.

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