Standard forms of von Neumann algebras

By Minoru TOMITA

Introduction

As a histrical origination of the theory of von Neumann algebra, studying of the correlation between a von Neumann algebra and its commutant has been the most fundamental problem in the field. One central result has been described that every semifinite von Neumann algebra is algebraically isomorphic to a standard algebra which has been defined as an algebra spatially isomorphic to the extended left regular representation of a certain Hilbert algebra. (Cor. Prop 9. p. 98 [1]).

Unfortunately, such a standard algebra is not the general standard form of a von Neumann algebra unless we ignore type III von Neumann algebras. To study general correlation problem we need to formulate more general standardizations of von Neumann algebras. For this purpose we notice that every von Neumann algebra M has at least a generalized normal strictly positive functional (Theorem 1.3.4), and such a functional determines an algebraic isomorphism of M onto a certain type of von Neumann algebra which we call a modular standard algebra (Theorem 2.3.1).

We may define a generalized Hilbert algebra whose extended left regular representation is spatially isomorphic to a given modular standard algebra. Then the general correlation problem is reduced to the investigation of generalized Hilbert algebras.

The central purpose of this paper is to show the existence of modular operators related with generalized Hilbert algebras, and to verify ultimately that every generalized Hilbert algebra is equivalent to a certain modular Hilbert algebra. (Theorem 2.4.1). Theorem 2.4.1 shall be applied to verify that every modular standard algebra is antiisomorphic to its commutant (Theorem 2.4.2) and to establish the commutation theorem (Theorem 2.4.4) $(M \otimes N)' = M' \otimes N'$ for general von Neumann algebras M and N.

Section 1 is a preparation from a general theory of semicyclic *-algebras. Such a preparation simplifies the discussion of generalized Hilbert algebras. Semicyclic mapping is an important suppprementary concept in the elementary theory of von Neumann algebra when we treat unbounded positive functionals. We shall establish a generalized von Neumann density theorem (Theorem 1.5.4) and a generalized Kaplansky density theorem (Theorem 1.5.5)

§1. *-algebras with semicyclic mappings

1.1. Semicyclic *-algebras

Let $\mathfrak{H}$ be a Hilbert space, $\mathfrak{M}$ be a linear subset of $\mathfrak{H}$ and L be a *-algebra of bounded operators on $\mathfrak{H}$. We call L simply a *-algebra on $\mathfrak{H}$. L is a normed *-algebra by the operator norm $\| \quad \|$, however, it is not generally uniformly closed. We shall use the following notations. L' and L'' are the commutant and the second commutant of L. $U(L)$ is the unit ball of L. L^2 is the least *-algebra which contains all the multiplications AB of elements A and B of L. $L\mathfrak{M}$ is the least linear subset of $\mathfrak{H}$ which contains all Ax such that $A \in L$ and $x \in \mathfrak{M}$. L has an approximate identity in the following sense (cf [5]). Let ε be a number > 0 and $(A_1, \ldots A_n)$ be a finite subset of L. Then $U(L)$ contains a certain element A satisfying $\|A_i - AA_i\| < \varepsilon$ and $\|A_i - AA_i\| < \varepsilon$. A *-algebra L on $\mathfrak{H}$ is said nondegenerate if the set $L\mathfrak{H}$ is uniformly dense in $\mathfrak{H}$. The following three conditions are mutually equivalent (cf. [1], [2]). (1). L is nondegenerate. (2). If x is an element of $\mathfrak{H}$ such that $Ax = 0$ for $A \in L$, then $x = 0$. (3). $U(L)$ is strongly dense in $U(L'')$.

Let L be a *-algebra on a Hilbert space $\mathfrak{H}$. A linear mapping λ of L into $\mathfrak{H}$ is called semicyclic if

1.1.1. $\qquad \lambda(AB) = A\lambda(B) \qquad$ for $A \in L$ and $B \in L$.

Suppose that L has a semicyclic mapping λ. An element K of L' is said integrable by λ if the mappings $A \longrightarrow K\lambda(A)$ and $A \longrightarrow K^*\lambda(A)$ are uniformly continuous on L. We let L^c denote the totality of integrable elements of L'. If necessary, we shall use the notation $L^c(\lambda)$, instead of L^c.

Lemma 1.1.1 Let $\mathfrak{H}$ be a *-algebra on a Hilbert space $\mathfrak{H}$ which has a semicyclic mapping λ. An element K of L' is integrable by λ if and only if $\mathfrak{H}$ contains certain elements $\lambda'(K)$ and $\lambda'(K^*)$ satisfying

1.1.2. $$K\lambda(A) = A\lambda'(K),\ K^*\lambda(A) = A\lambda'(K^*) \quad \text{for } A \in L$$

Proof. If K is an element of L' for which such $\lambda'(K)$ and $\lambda'(K^*)$ are defined, then $A \longrightarrow K\lambda(A)$ and $A \longrightarrow K^*\lambda(A)$ are uniformly continuous on L, and K belongs to L^c. Conversely, let K be an element of L^c. There is a constant γ such that $\|K\lambda(A)\| \leq \gamma\|A\|$. For each number $\varepsilon > 0$ and each finite subset $\Delta = (A_1, \ldots A_n)$ of L, we consider the set $U(\Delta, \varepsilon)$ of all elements x of $\mathfrak{H}$ such that $\|x\| \leq \gamma$ and $\|K\lambda(A_i) - A_i x\| \leq \varepsilon$ for $1 \leq i \leq n$. $U(\Delta, \varepsilon)$ is a weakly compact subset of $\mathfrak{H}$ as a bounded uniformly closed convex set. The algebra L has an approximate identity, and U(L) contains a certain element A such that $\|A_i - A_iA\| \leq \gamma^{-1}\varepsilon$ for $1 \leq i \leq n$. $K\lambda(A)$ belongs to $U(\Delta, \varepsilon)$ since we have $\|K\lambda(A)\| \leq \gamma$ and

$$\|K\lambda(A_i) - A_iK\lambda(A)\| \leq \|K\lambda(A_i - A_iA)\| \leq \varepsilon.$$

The intersection of the directed system of all such nonempty weakly compact sets $U(\Delta, \varepsilon)$ is nonempty and contains an element $\lambda'(K)$ satisfying 1.1.2. The existence of $\lambda'(K^*)$ is similarly proved.

Lemma 1.1.2. Let L be a *-algebra on a Hilbert space $\mathfrak{h}$, and λ be a semicyclic mapping of L.

(a). If an element K of L' is noted in the form $K = X_1B_1 = B_2X_2$ for certain elements $B_1 \in L^c$ and $X_1 \in L'$, then K belongs to L^c and one of $\lambda'(K)$ is

1.1.3. $\qquad \lambda'(K) \equiv X_1 \quad \lambda'(B_1).$

(b). L^c is a *-subalgebra of L'.

(c). If L is nondegenerate, then the element $\lambda'(K)$ defined by 1.1.2. is unique for each $K \in L^c$. λ' is a semicyclic mapping of L^c.

Proof. (a). If K is such an element of L'' then from

$$K\lambda(A) = AX_1 \ \lambda'(B_1), \ K^*\lambda(A) = AX_2{}^* \lambda'(B_2{}^*),$$

we find that K belongs to L^c, and one of $\lambda'(K)$ is defined by 1.1.3. (b). L^c is a *-algebra since L is an involutive linear subset of L' such that the multiply KL of any two elements K and L of L^c belongs to L^c. (c). If L^c is nondegenerate, then $A\lambda'(K) = 0$ for all $A \in L$ implies $\lambda'(K) = 0$, and the element $\lambda'(K)$ defined by 1.1.2. is unique. By 1.1.3 the mapping λ' is semicyclic on L^c.

The pair (L, λ) of a $*$-algebra L on a Hilbert space $\mathfrak{h}$ and a semicyclic mapping λ of L is called a semicyclic $*$-algebra on $\mathfrak{h}$ if L and L^c are nondegenerate. If (L, λ) is a semicyclic $*$-algebra, the pair (L^c, λ') of the algebra L^c and the mapping λ' is called the integrable commutant of (L, λ). Let (L_1, λ_1) and (L_2, λ_2) be semicyclic $*$-algebras on a Hilbert space $\mathfrak{h}$. If L_2 is a $*$-subalgebra of L_1, and λ_2 is the restriction of λ_1 on L_2, then we note $(L_1, \lambda_1) \supseteq (L_2, \lambda_2)$, and we say that (L_2, λ_2) is a semicyclic $*$-subalgebra of (L_1, λ_1).

Theorem 1.1.1. Let (L, λ) be a semicyclic $*$-algebra on a Hilbert space $\mathfrak{h}$. Then (L^c, λ') and (L^{cc}, λ'') are semicyclic $*$-algebras on $\mathfrak{h}$ such that $(L^c)'' = L'$. We have $(L^{cc}, \lambda'') \supseteq (L, \lambda)$ and $(L^{ccc}, \lambda''') = (L^c, \lambda')$.

Proof. L^{cc} is a nondegenerate $*$-algebra which contains a nondegenerate $*$-algebra L. Then (L^c, λ') is semicyclic, and so is (L^{cc}, λ''). The relation $(L^{cc}, \lambda'') \supseteq (L, \lambda)$ implies $((L^c)^{cc}, (\lambda')^{cc}) \supset (L^c, \lambda')$ and $((L^{cc})^c, (\lambda'')') \subseteq (L^c, \lambda')$ since $(L_1, \lambda_1) \supseteq (L_2, \lambda_2)$ implies $(L_1^c, \lambda'_1) \subseteq (L_2^c, \lambda'_2)$. Hence we have $(L^{ccc}, \lambda''') = (L^c, \lambda')$. If K is an element of L', then by Lemma 1.1.2 the mapping $L \longrightarrow LKL$ carries $U(L^c)$ into L^c, and when L tends to the identity strongly, LKL tends to K strongly, so that K belongs to $L^{c''}$. $L^{c''} = L'$ is thus verified.

A semicyclic *-algebra (L,λ) is called maximal if it satisfies $(L,\lambda) = (L^{cc}, \lambda'')$. If (L,λ) is a semicyclic *-algebra, (L^{cc}, λ'') is called the maximal extension of (L, λ).

Theorem 1.1.2. Let (L,λ) be a semicyclic *-algebra whose L is a C*-algebra. Then the mapping λ is determined by an element g of the underlying space $\mathfrak{H}$ satisfying $\lambda(A) = Ag$ for $A \in L$.

Proof. Let $\{A_n\}$ be a sequence in L such that $A_n \longrightarrow A$ and $\lambda(A_n) \longrightarrow x$ for certain $A \in L$ and $x \in \mathfrak{H}$. Then for every $K \in L^c$ we have $K\lambda(A_n) = A_n\lambda'(K)$ and $Kx = A\lambda'(K) = K\lambda(A)$. Since L^c is nondegenerate we have $x = \lambda(A)$. As a closed linear mapping on the Banach space L, $A \longrightarrow 1\lambda(A)$ is uniformly continuous on L, and L^c contains the identity 1. Hence $\mathfrak{H}$ contains an element $g = \lambda'(1)$ such that $\lambda(A) = Ag$.

Lemma 1.1.3. Let (L,λ) be a semicyclic *-algebra on a Hilbert space $\mathfrak{H}$ and g be an element of $\mathfrak{H}$. Then L has a semicyclic mapping $\lambda + g$ which is defined by

1.1.4. $$(\lambda + g)(A) = \lambda(A) + Ag.$$

$(L, \lambda + g)$ is a semicyclic *-algebra whose integrable commutant is the algebra $(L^c, \lambda' + g)$.

Proof. $\lambda + g$ is obviously semicyclic on L, and if $K \in L$ is integrable by λ, then from

$$K(\lambda + g)(A) = A(\lambda' + g)(K),\ K^*(\lambda + g)(A) = A(\lambda' + g)(K),$$

K is integrable by $\lambda + g$. Then setting $L^c = L^c(\lambda) = L^c(\lambda + g)$, $(L^c, \lambda' + g)$ is the integrable commutant of $(L, \lambda + g)$.

Let L be a *-algebra on a Hilbert space $\mathfrak{H}$ which has a semicyclic mapping λ, and let $\lambda(L)$ denote the range of λ. The projection operator C_λ on $\mathfrak{H}$ whose range is the uniform closure of $\lambda(L)$ is called the carrier of λ. A semicyclic mapping λ of L is called cyclic if the range $\lambda(L)$ is everywhere dense in $\mathfrak{H}$. A semicyclic *-algebra (L, λ) is said a cyclic *-algebra if λ is a cyclic mapping. A cyclic *-algebra (L, λ) is said a separating cyclic *-algebra if (L^c, λ') is also a cyclic *-algebra.

Lemma 1.1.4. Let (L, λ) be a semicyclic *-algebra and (L^{cc}, λ'') be its maximal extension. Then the mappings λ' and λ'' have the same carriers $C_\lambda = C_{\lambda''}$. $1 - C_\lambda$ is an element of L^c such that $\lambda'(1 - C_\lambda) = 0$.

Proof. If A belongs to L, then from $A\,\lambda(L) \subseteq \lambda(L)$ and $A^*\,\lambda(L) \subseteq \lambda(L)$ we have $AC_\lambda = C_\lambda A$ and $(1 - C_\lambda)\lambda(A) = 0$. Then $1 - C_\lambda$ is an element of L^c satisfying $\lambda'(1 - C_\lambda) = 0$. If X is an element of L^{cc} then from

$$(1 - C_\lambda)\lambda''(X) = X\,\lambda'(1 - C_\lambda) = 0$$

$\lambda''(X)$ is contained in the uniform closure of $\lambda(L)$. Hence $\lambda(L)$ and $\lambda''(L^{cc})$ have the same uniform closures, and therefore the carriers of λ and λ'' are identical.

Lemma 1.1.5. Let L be a *-algebra on a Hilbert space $\mathfrak{H}$ which has a cyclic mapping λ. Then the algebra L^c is the totality of bounded operators K on $\mathfrak{H}$ such that $\mathfrak{H}$ contains elements $\lambda'(K)$ and $\lambda'(K^*)$ satisfying

1.1.5. $K\lambda(A) = A\lambda'(K)$, $K^*\lambda(A) = A\lambda'(K^*)$ for $A \in L$.

Proof. If K is an element of L^c then $\mathfrak{H}$ contains such $\lambda'(K)$ and $\lambda'(K)$. Conversely let K be a bounded operator in $\mathfrak{H}$ for which such $\lambda'(K)$ and $\lambda'(K^*)$ are defined. K commutes with every $A \in L$ since

$$KA\lambda(B) = K\lambda(AB) = AB\lambda'(K) = AK\lambda(B).$$

Therefore K belongs to L^c.

Lemma 1.1.6. Let L be a *-algebra on a Hilbert space which has a semicyclic mapping λ. (a). If the algebra L^c is nondegenerate, then (L,λ) is a cyclic *-algebra. (b). If L is nondegenerate and if $\lambda'(L^c)$ is everywhere dense in $\mathfrak{H}$, then (L,λ) is a separating cyclic *-algebra.

Proof. (a). L^c is nondegenerate and λ is cyclic. Then the sets $L^c\mathfrak{H}$ and $\lambda(L)$ are everywhere dense in $\mathfrak{H}$. From $K\lambda(A) = A\lambda'(K)$, $L\mathfrak{H}$ contains the set $L^c\lambda(L)$ which is everywhere dense in $\mathfrak{H}$. Then L is also nondegenerate, and (L,λ) is a cyclic *-algebra. (b). L^{cc} is a nondegenerate *-algebra since it contains a nondegenerate *-algebra L. Since λ' is cyclic on L^c, by (a), (L^c,λ') is a cyclic *-algebra and L^c is a nondegenerate *-algebra. Hence (L,λ) is a separating cyclic *-algebra.

Lemma 1.1.7. Let (L, λ) be a semicyclic *-algebra on a Hilbert spce $\mathfrak{H}$. Then the following three conditions are equivalent. (1). (L, λ) is a cyclic *-algebra. (2). The mapping λ' is one-to-one on L^c. (3). (L^{cc}, λ'') is a cyclic *-algebra.

Proof. The carriers of λ and λ'' are identical. Then (L, λ) is cyclic if and only if (L^{cc}, λ'') is cyclic. Let (L, λ) be a cyclic *-algebra, and K be an element of L^c. From $K\lambda(A) = A\lambda'(K)$ for $A \in L$ we find that $\lambda'(K) = 0$ implies $K = 0$, and that $K \longrightarrow \lambda'(K)$ is a one-to-one mapping. Conversely assume that λ' is one-to-one. From $\lambda'(1 - C_\lambda) = 0$ we have $C_\lambda = 1$ and $\lambda(L)$ is dense in $\mathfrak{H}$. Then λ is a cyclic mapping.

Lemma 1.1.8. A cyclic *-algebra (L, λ) is separating if and only if the mapping λ'' of L^{cc} is one-to-one.

Proof. The algebra (L^c, λ') is cyclic if and only if λ'' is one-to-one on L^{cc}. Then we obtain Lemma 1.1.8.

1.2. Elementary operations.

Let L be a nondegenerate *-algebra on a Hilbert space $\mathfrak{H}$, E be a projection operator in $\mathfrak{H}$, and $E\mathfrak{H}$ be the range of E. If A is an element of L, then the reduced operator A_E is defined as the restriction of the operator EAE on the space $E\mathfrak{H}$. We let L_E denote the totality of A_E such that $A \in L$.

If E is a projection in L', then $A \longrightarrow A_E$ is a $*$-homomorphism of L onto a nondegenerate $*$-algebra L_E which we call the induction of L. The algebra L_E is called the induced algebra of L. If E is a projection in L'' such that $A \longrightarrow EAE$ carries L into L, then the mapping $A \longrightarrow A_E$ is called the reduction of L. L_E is a nondegenerate $*$-algebra on $E\mathfrak{H}$ which we call the reduced algebra of L. A projection E in a von Neumann algebra M is said generating in M if the set $ME\mathfrak{H}$ is uniformly dense in $\mathfrak{H}$. The following Lemma 1.2.1 is well-known (cf. [1], [2]).

Lemma 1.2.1. Let M be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ and Z be the center of M. Let E, F, Z and $W = 1 - Z$ be projections such that $E \in M$, $F \in M'$ and $Z \in Z$. Then (a). M_{EF} and M'_{EF} are von Neumann algebras such that $(M_{EF})' = M'_{EF}$. (b). If we set $G = F_E$ and $H = E_F$, then $M_{EF} = (M_E)_G = (M_F)_H$. (c). The induction $A \longrightarrow A_F$ of M is an isomorphism if and only if F is generating in M'. (d). M is spatially isomorphic to the direct sum $M_Z \oplus M_W$.

Lemma 1.2.2. Let (L, λ) be a semicyclic $*$-algebra and E be a projection in L' such that $K \longrightarrow EKE$ carries L^c into L^c. Then the induced algebra L_E has a semicyclic mapping λ_E which is defined by

1.2.1. $$\lambda_E(A_E) = E\lambda(A) \quad \text{for } A \in L,$$

and the reduced algebra $L^c{}_E$ has a semicyclic mapping $(\lambda')_E$

which is defined by

1.2.2. $\quad (\lambda')_E(K_E) = \lambda'(EKE) \quad$ for $K \in L^c$.

(L_E, λ_E) is a semicyclic $*$-algebra and $((L^c)_E, (\lambda')_E)$ is its integrable commutant.

Proof. If A is an element of L such that $A_E = 0$, then $E\lambda(A) = 0$ follows from $EK\lambda(A) = AE\lambda'(K) = 0$ for $K \in L^c$. Then $\lambda_E(A_E)$ is uniquely defined for each $A \in L$, and λ_E is a semicyclic mapping of L_E. Similarly we find that $(\lambda')_E$ is a semicyclic mapping of $(L^c)_E$. $K \longrightarrow K_E$ carries L^c into $(L_E)^c$ since, if K is an element of L^c, then from

$$K_E\lambda_E(A_E) = A_E\lambda'(EKE), \quad K_E{}^*\lambda_E(A_E) = A_E\lambda'(EK^*E),$$

we find that K_E is an element of $(L_E)^c$ satisfying

$$(\lambda_E)'(K_E) = (\lambda')_E(K_E).$$

L_E is nondegenerate as the induced algebra of a nondegenerate algebra. $(L_E)^c$ is a nondegenerate algebra which contains the nondegenerate algebra $(L^c)_E$. Then (L_E, λ_E) is a semicyclic $*$-algebra. We verify that $(L_E)^c = (L^c)_E$. If K is an element of $(L_E)^c$, then, remarking $(L_E)^c \subseteq (L_E)' = (L')_E$; K is a reduced operator $K = L_E$ of a certain $L \in L'$ such that $L = ELE$. From

$$L\lambda(A) = A(\lambda_E)'(K), \quad L^*\lambda(A) = A(\lambda_E)'(K)$$

we find that L belongs to L^c and K belongs to $(L^c)_E$. Since $(L^c)_E = (L_E)^c$ is thus proved, $((L^c)_E, (\lambda')_E)$ is the integrable commutant of (L_E, λ_E).

Let (L,λ) be a semicyclic *-algebra. A projection E in L' is said normal with respect to (L,λ) if $K \longrightarrow EKE$ carries L^c into L^c and if $((L^{cc})_E, (\lambda'')_E)$ is maximal.

Lemma 1.2.3. Let (L,λ) be a semicyclic *-algebra on a Hilbert space. Then every projection Z in the center of L'' is normal with respect to (L,λ). If (L,λ) is a separating cyclic *-algebra, then (L_z, λ_z) is a separating cyclic *-algebra.

Proof. Notice that $ZKZ = KZ = ZK$ for $K \in L^c$. Then by Lemma 1.1.2 the mapping $K \longrightarrow ZKZ$ carries L^c into L^c, and (L_z, λ_z) is a semicyclic *-algebra whose integrable commutant is $((L^c)_z, (\lambda')_z)$. Similarly the mapping $A \longrightarrow ZAZ$ carries L^{cc} into L^{cc}. Then $((L^{cc})_z, (\lambda'')_z)$ is the integrable commutant of $((L^c)_z, (\lambda')_z)$ and is the maximal extension of (L_z, λ_z). Z is therefore normal. If (L,λ) is a separating cyclic *-algebra then $\lambda(L)$ and $\lambda'(L^c)$ are everywhere dense in $\mathfrak{H}$. The sets $\lambda_z(L_z)$ and $(\lambda')_z(L^c{}_z)$ are equal to $Z\lambda(L)$ and $Z\lambda'(L^c)$ respectively, and are everywhere dense in $Z\mathfrak{H}$. Then (L_z,λ_z) is a separating cyclic *-algebra.

Lemma 1.2.4. Let (L,λ) be a semicyclic *-algebra. Then the carrier E of the mapping λ is normal with respect to (L,λ). L^{cc} is the totality of $A \in L''$ such that A_E belongs to $(L_E)^{cc}$.

Proof. The mapping $K \longrightarrow EKE$ carries L^c into L^c since $1 - E$ belongs to L^c and since we have

$$EKE = K - (1 - E)K - K(1 - E) + (1 - E)K(1 - E).$$

Then (L_E, λ_E) is a semicyclic *-algebra and $((L^c)_E, (\lambda')_E)$ is its integrable commutant. $((L^{cc})_E, (\lambda'')_E)$ is then contained in $((L_E)^{cc}, (\lambda'')_E)$. Let X be an element of L'' such that $X_E \in (L_E)^{cc}$ and K be an element of L^c. Let $KE = UB$ be the polarization of KE. Then $B = U^*(K - K(1 - E)) = (K - K(1 - E))^*U$ and by Lemma 1.1.2 B is an element of L^c satisfying $B = EBE$ and $\lambda'(K) = U\lambda'(B) = U(\lambda')_E(B_E)$. From the identities

$$X\lambda'(K) = UX_E\lambda'_E(B_E) = K\lambda_E(X_E).$$

and

$$X^*\lambda'(K) = K\lambda_E(X^*_E),$$

we find that X is an element of L^{cc}, and L^{cc} is the totality of $X \in L''$ such that $X_E \in (L_E)^{cc}$. Hence we have $((L^{cc})_E, (\lambda'')_E) = ((L_E)^{cc}, (\lambda_E)'')$ and E is normal.

1.3. Generalized normal positive functionals.

Let M be a von Neumann algebra and M^+ denote the totality of Hermitian elements ≥ 0 of M. An extended real functional p on M^+ such that $0 \leq p(A) \leq +\infty$ is called an extended positive functional on M if it satisfies

1.3.1. $p(A + B) = p(A) + p(B)$,

1.3.2. $p(\alpha A) = \alpha p(A)$ for $\alpha \geq 0$, where we assume $0(+\infty) = 0$. Let p be an extended positive functional on M. An element A of M is said integrable by p if $p(A^*A) < \infty$ and $p(AA^*) < \infty$. The totality $M(p)$ of integrable elements of M is called the integrable part of M related with p. An extended positive functional p is called a generalized normal positive functional if (1). The integrable part $M(p)$ of M is a nondegenerate $*$-algebra, and (2). There is a certain system Λ of normal positive functionals on M such that

1.3.3. $$p(A) = \sup_{\omega \in \Lambda} \omega(A) \qquad \text{for } A \in M^+.$$

Theorem 1.3.1. Let M be a von Neumann algebra and π be a normal representation of M onto the von Neuman extension L'' of a certain semicyclic $*$-algebra (L, λ). Then there is a certain generalized normal positive functional p on M such that (1). $A \in M$ is integrable by p if and only if $\pi(A)$ is contained in L^{cc}, and (2). If A is an element of $M(p)$ then

1.3.4. $$p(A^*A) = \| \lambda''(\pi(A)) \|^2.$$

Proof. For every $K \in U(L^c)$ we define a normal positive functional ω_K on M by

$$\omega_K(A) = (\pi(A) \lambda'(K), \lambda'(K)),$$

and consider the set Λ of all ω_K such that $K \in U(L^c)$. We show that the functional p on M^+ which is defined by $p(A) = \sup_{\omega \in \Lambda} \omega(A)$ is the desired functional in the Theorem. Remark that if $A \in M$ then

$$p(A^*A) = \sup(\|\pi(A) \lambda'(K)\|^2 : K \in U(L^c)).$$

$\pi(A)$ belongs to L^{cc} if and only if A is integrable by p, and, if A is an element of $M(p)$, 1.3.4 is satisfied. $M(p)$ is a nondegenerate strongly dense $*$-subalgebra of M as the complete inverse immage of the algebra L^{cc} by the σ-weakly continuous mapping π. Then to see that p is a generalized normal positive functional, it is sufficient to verify the additivity of the functional p. Since 1.3.3. has been satisfied, it is easily shown that of $A \in M^+$ and $B \in M^+$ then

$$p(A) + p(B) \geq p(A + B) \geq \max(p(A), p(B)).$$

The additivity of p is verified if we show $p(A + B) \geq p(A) + p(B)$ in the case that $p(A) < \infty$ and $p(B) < \infty$. In this case $\pi(A^{\frac{1}{2}})$ and $\pi(B^{\frac{1}{2}})$ are elements of L^{cc} and we obtain $p(A) = \|\lambda(\pi(A^{\frac{1}{2}}))\|^2$ and $p(B) = \|\lambda(\pi(B^{\frac{1}{2}}))\|^2$. Let K be any element of $U(L^c)$. Then we have

$$\|K\lambda(\pi(A^{\frac{1}{2}})\|^2 + \|K\lambda(\pi(B^{\frac{1}{2}}))\|^2$$

$$= (\pi(A + B)\lambda'(K), \lambda'(K)) \leq p(A + B).$$

Let K strongly tend to the identity. Then from

$$K\lambda(\pi(A^{\frac{1}{2}})) \longrightarrow \lambda(\pi(A^{\frac{1}{2}})) \text{ and } K\lambda(\pi(B^{\frac{1}{2}})) \longrightarrow \lambda(\pi(B^{\frac{1}{2}})),$$

we obtain the inequality $p(A) + p(B) \leq p(A) + p(B)$. The additivity of p is thus proved, and p is a desired generalized normal positive functional on M. Let M be a von Neumann algebra and p be a generalized normal positive functional on M.

If π is a normal representation of M on the von Neumann extension of a certain cyclic *-algebra and if p and π are in the relation of Theorem 1.3.1. then such a representation π is uniquely determined up to the unitary equivalence. We call π the extended normal cyclic representation of M which corresponds to p.

Theorem 1.3.2. Every generalized normal positive functional p on a von Neumann algebra M corresponds to a certain extended normal cyclic representation of M.

Proof. Let M_0^+ denote the totality of $A \in M^+$ such that $p(A) < \infty$, and M_0 the linear span of M_0^+. If A is an element of M_0 which is noted as $A = \sum_{n=0}^{3} i^n A_n = \sum_{i=0}^{3} i^n B_n$ for certain elements A_n and B_n of M_0^+, then the identity

1.3.5. $\sum_{n=0}^{3} i^n p(A_n) = \sum_{n=0}^{3} i^n p(B_n)$

follows from $A_1 + B_3 = A_3 + B$, and $A_0 + B_2 = A_2 + B_0$. We let $q(A)$ denote the left side of 1.3.5. and extend the functional p to a linear functional q on M_0. The totality S of $X \in M$ such that $p(X^*X) < \infty$ is a left ideal of M. For, let L be a complex number, X and Y be elements of S and A be an element of M. Then LX, $X + Y$ and AX are contained in S since we have $(\alpha X)^*(\alpha X) = |\alpha|^2 X^*X$, $(X + Y)^*(X + Y) \leq 2X^*X + 2Y^*Y$ and $(AX)^*(AX) \leq \|A\|^2 X^*X$. If X and Y are elements of S then from $4Y^*X = \sum_{i=0}^{3} i^n(X + i^nY)^*(X + i^nY)$ we find that Y^*X is an element of M_0. The functional $q(Y^*X)$ which is defined for $X \in S$ and $Y \in S$ is a positive sequilinear form, and we find a certain Hilbert space $\mathfrak{H}$ and a linear mapping σ of S into a dense subset of $\mathfrak{H}$ such that $q(Y^*X) = ((X), \sigma(Y))$ and $\| \sigma(AX) \| \leq \| A \| \ \| (X) \|$ for $X \in S$, $Y \in S$ and $A \in M$. M is represented on a certain $*$-algebra $\Pi(M)$ of bounded operators on $\mathfrak{H}$ by a $*$-homomorphism π which is defined by $\pi(A) \sigma(X) = \sigma(AX)$ for $A \in M$ and $X \in S$. We show that π is a normal homomorphism. Let Λ be a system of normal positive functionals which satisfies 1.3.3. For every $\omega \in \Lambda$ and $X \in S$ we define positive functionals q_X and ω_X on M by $q_X(A) = q(X^*AX)$ and $\omega_X(A) = \omega(X^*AX)$. Then $q_X - \omega_X$ is a positive functional on M such that

$$\| q_X - \omega_X \| = q_X(1) - \omega_X(1) = p(X^*X) - \omega(X^*X),$$

when $\| \ , \|$ denotes the functional norm. The space of all normal positive functionals on M is uniformly closed, and Λ contains a sequence $\{\omega_n\}$ such that $\omega_n(X^*X) \longrightarrow p(X^*X)$. Then q is a normal positive functional on M such that $q(A) = (\Pi(A)\sigma(X), \sigma(X))$. Since the set $\sigma(S)$ is everywhere dense in $\mathfrak{H}$, π is a normal representation and $\pi(M)$ is a von Neumann algebra on $\mathfrak{H}$. From the assumption that $M(p)$ is nondegenerate, we find that $M(p)$ is strongly dense in M. For, $M(p)$ is the totality of $X \in S$ such that $X^* \in S$. If A is an element of M then the mapping $X \longrightarrow XAX$ carries the set $U(M(p))$ into $M(p)$ and, when X tends to the identity strongly, XAX tends to A strongly, and A is contained in the strong closure of $M(p)$.

Now let L be the represented algebra of $M(p)$ by π. Then L is strongly dense in $\pi(M)$. We notice that if $\pi(A) = 0$ then $\sigma(A) = 0$. If $\pi(A) = 0$ then for all $\omega \in \Lambda$ and $X \in S$ we have $\omega(X^*A^*AX) \leq p(X^*A^*AX) = \| \pi(A)\sigma(X)\|^2 = 0$ and $\omega(A^*A) = 0$, which means $p(A^*A) = 0$ and $\sigma(A) = 0$. We define a semicyclic mapping λ on L by $\lambda(\pi(A)) = \sigma(A)$ for $A \in M(p)$. Let A be an element of S and $A = UB$ be its polarization. B is an element of $M(p)$ such that $p(B^2) = p(A^*A) < \infty$ and $\sigma(A) = \pi(U)\lambda(\pi(B))$ is contained in $L''\lambda(L)$ and in the uniform closure of $L\lambda(L) \subseteq \lambda(L)$. Then the uniform closure of $\lambda(L)$ contains $\sigma(S)$ and is identical to $\mathfrak{H}$. Hence λ is a cyclic mapping of L. Finally we verify that L^c is non-degenerate. Let ω be a positive functional in Λ. Then a sequilinear form $\omega(Y^*X)$ is defined for $X \in S$ and $Y \in S$. By $0 \leq \omega(X^*X) \subseteq \| (X) \|^2$, an Hermitian operator $T_\omega \geq 0$ is defined by $\omega(Y^*X) = (T_\omega \sigma(X), \sigma(Y))$. T_ω commutes with every $\pi(A) \in \pi(M)$ since

$$\omega(X^*AY) = (T_\omega \pi(A) \sigma(Y) \sigma(X)) = (\pi(A)T_\omega\sigma(Y), \sigma(X)).$$

$T_\omega^{\frac{1}{2}}$ is an element of L' such that

$$\| T_\omega^{\frac{1}{2}} \lambda(\pi(A))\|^2 = \omega(A^*A) \leq \omega(1) \| \pi(A)\|^2$$

and it belongs to $U(L^c)$. From $(1 - T_\omega^{\frac{1}{2}})^2 \leq 1 - T_\omega$, we find that

$$\|\lambda(\pi(A)) - T_\omega^{\frac{1}{2}} \lambda(\pi(A))\|^2 \leq p(A^*A) - \omega(A^*A).$$

Λ contains a sequence $\{\omega_n\}$ such that $\omega_n(A^*A) \longrightarrow p(A^*A)$. Then the uniform closure of the set $L^c\mathfrak{H}$ contains $\lambda(L)$ and is identical to $\mathfrak{H}$. L^c is therefore nondegenerate and (L^c, λ) is a cyclic *-algebra on $\mathfrak{H}$, and hence π is the extended cyclic representation of M which corresponds with p.

Lemma 1.3.1. Let p be a generalized normal positive functional on a von Neumann algebra M, and π be the corresponding normal representation of M onto the von Neumann extension of a certain cyclic *-algebra (L, λ). If r is another generalized normal positive functional on M such that $p(A) \geq r(A)$ for all $A \in M^+$, then a certain Hermitian operator K in L' is defined by

$$1.3.6. \qquad r(A^*A) = r(K\lambda(\pi(A)), \lambda(\pi(A))).$$

Proof. We use the same notations in the proof of Theorem 1.3.2. The value of $r(A)$ is finite if $A \in M_0^+$, and r is extended to a linear functional r_0 on the space M_0. Consider a positive sequilinear form $r_0(Y^*X)$ which is defined for $X \in S$ and $Y \in S$. From

$$0 \leq \quad r_0(X^*X) \leq \quad \|\sigma(X)\|^2. \qquad \text{for} \quad X \in S,$$

we can define an Hermitian operator $K \geq 0$ on $\mathfrak{H}$ by $r_0(Y^*X) = (K\sigma(X)$, $\sigma(Y))$, to which we apply the proof of $T\omega \in \pi(M)'$, and we find that $K \in \pi(M)' = L'$.

A generalized normal positive functional p on a von Neumann algebra M is said strictly positive if $p(A) > 0$ for all $0 \neq A \in M^+$.

Theorem 1.3.3. Let p be a generalized normal positive functional on a von Neumann algebra M, and π be the corresponding normal representation of M on the von Neumann extension of a maximal cyclic *-algebra (L, λ). Then p is strictly positive if and only if (L, λ) is separating cyclic and π is an algebraic isomorphism.

Proof. π carries $M(p)$ onto L and satisfies $p(A^*A) = \|\lambda(\pi(A)\|^2$ for $A \in M(p)$. If p is strictly positive, then the mapping $A \longrightarrow \lambda(\pi(A))$ is one-to-one. π is an algebraic isomorphism since $A \in M$ and $\pi(A) = 0$ implies $p(A^*A) = 0$ and $A = 0$. Also the mapping λ is one-to-one on L, and (L, λ) is a separating cyclic *-algebra. Conversely, if π is an algebraic isomorphism and (L, λ) is separating, then the mapping $A \longrightarrow \lambda(\pi(A))$ is one-to-one on $M(p)$. If A is an element of M^+ such that $p(A) = 0$ then $p(A) = \|\lambda(\pi(A^{\frac{1}{2}}))\|^2 = 0$ implies $A = 0$. Therefore p is strictly positive.

Theorem 1.3.4. Let M be a von Neumann algebra and E be a projection in M. Then every generalized normal positive functional q on the reduced algebra M_E is extended to a generalized normal strictly positive functional q_0 on M such that $q(A_E) = q(EAE)$ and $q(EAE) \leq q(A)$ for all $A \in M$.

Proof. Let $\{E_\alpha : \alpha \in \Lambda\}$ be a maximal orthogonal system of cyclic projections E_α in M' which are orthogonal to the projection E. The range $E_\alpha \mathfrak{H}$ of E_α contains a everywhere dense subset $M'g_\alpha = (Ag_\alpha : A \in M')$ which is generated by a certain element g_α of $E_\alpha \mathfrak{H}$, and by the Zorn's principle we find that $I = E + \sum_\alpha E_\alpha$. Now we shall verify that a functional q_0 in the Lemma may be defined by

1.3.1. $\quad q_0(A) = q(A_E) + \sum_\alpha (Ag_\alpha, g_\alpha).$

Such q_0 is obviously an extended positive functional on M^+. The integrable part $M(q_0)$ of M contains the system $\{E_\alpha : \alpha \in \Lambda\}$ and the totality of EAE such that $A_E \in M(q)$. Then the set $M(q_0)\mathfrak{H}$ is uniformly dense in $\mathfrak{H}$, and $M(q_0)$ is non degenerate. Let Λ be a system of normal positive functionals on M_E such that $q(A) = \sup_{\omega \in \Lambda} \omega(A)$ for $A \in M_E$, and for each $A \in M$ define a normal positive functional ω_E on M by $\omega_E(A) = \omega(A_E)$. Also for each finite subsystem $\Delta = (\alpha_1, \dots \alpha_n)$ of Λ define a positive functional ψ_Δ by $\psi_\Delta(A) = \sum (Ag_{\alpha_i}, g_{\alpha_i})$ Then we have

$$q_0(A) = \sup (\omega_E(A) + \psi_\Delta(A) : \omega \in \Lambda, \Delta \subset \Lambda)$$

Therefore q_0 is a generalized normal positive functional on M. We show that q_0 is strictly positive. If A is an element of M^+ such that $q_0(A) = 0$, then $q(A_E) = 0$ and $(Ag_\alpha, g_\alpha) = 0$ for all $\in \Lambda$, which means $EAE = 0$ and $E_\alpha A E_\alpha = 0$ for all $\alpha \in \Lambda$. From this we find easily that $A^{\frac{1}{2}} = 0$ and $A = 0$.

1.4. Unbounded operators and cyclic *-algebras.

Let X be a closed operator which is defined in a everywhere dense subset of a Hilbert space $\mathfrak{H}$. We let $\mathcal{D}(X)$ denote the domain of X, and we consider that $\mathcal{D}(X)$ is a Hilbert space whose norm $\|x\|_X$ is defined by

1.4.1. $\|x\|_X^2 = \|x\|^2 + \|Xx\|^2.$

X is noted in the form

1.4.2. $X = UC = BU,$

where U is a partially isometric operator in $\mathfrak{H}$ whose range is the uniform closure of the range of B, and $B = (XX^*)^{\frac{1}{2}}$ and $C = (X^*X)^{\frac{1}{2}}$ are selfadjoint operators ≥ 0. Such the representation of X is called the polarization of X (cf. [1][2][3]). $D(X)$ and $D(B)$ are identical as Hilbert spaces.

Lemma 1.4.1. Let X be a closed operator which is defined in a dense subset of a Hilbert space $\mathfrak{H}$. A subset $\mathfrak{M}$ of $D(X)$ is everywhere dense in it if and only if the set $(1 + (X^*X)^{\frac{1}{2}})\mathfrak{M}$ is everywhere dense in $\mathfrak{H}$.

Proof. We note $(X^*X)^{\frac{1}{2}}$ by C and let $X = UC$ be the polarization of X. Then $W = (1 + C^2)^{\frac{1}{2}}(1 + C)^{-1}$ is an invertibe Hermitian operator in $\mathfrak{H}$. Let x be an element of $D(X)$ and set $y = (1 + C)x$. Remarking $\|Xx\| = \|Cx\|$ we have

$$\|x\|_X^2 = \|(1 + C)^{-1}y\|^2 + \|C(1 + C)^{-1}y\|^2$$
$$= \|Wy\|^2 = \|W(1 + C)x\|^2.$$

$x \longrightarrow W(1 + C)x$ is an isometry, and $x \longrightarrow (1 + C)x$ is a homeomorphism, of the Hilbert spac $D(X)$ onto $\mathfrak{H}$. Then a subset $\mathfrak{M}$ of $D(X)$ is dense in it if and only if $(1 + C)\mathfrak{M}$ is dense in $\mathfrak{H}$.

We consider a not generally closed operator Y in $\mathfrak{H}$ such that the domains of Y and Y^* are everywhere dense in $\mathfrak{H}$. Then Y^{**} is defined as a closed extension of Y. If A is a bounded operator in $\mathfrak{H}$ such that $YA \supseteq AY$, then we have $Y^*A^* \supseteq A^*Y^*$ and $Y^{**}A \supset AY^{**}$. We say that Y commutes with a $*$-algebra L on $\mathfrak{H}$ if $YA \supseteq AY$ holds for all $A \in L$. In partiqular, let M be a von Neumann algebra and X be a closed operator whose domain is everywhere dense in $\mathfrak{H}$. We use the J.V. Neumann's notation $X \eta M$ if X commutes with the algebra M'.

Lemma 1.4.2. Let L be a nondegenerate $*$-algebra on a Hilbert space $\mathfrak{H}$, and X be an operator such that the domains of X and X^* are everywhere dense in $\mathfrak{H}$. If X commutes with L then we have $X^* \eta L'$ and $X^{**} \eta L'$.

Proof. As X commutes with L, so do X^* and X^{**}. To see $X^* \eta L'$ it is sufficient to show that X commutes with L''. Consider elements A of L'' and x of $D(X^*)$. L is strongly dense in L'' and contains a sequence $\{A_n\}$ such that $A_n x \longrightarrow Ax$ and $A_n X^* x \longrightarrow AX^*x$. X^* is a closed operator and $\{A_n x\}$ is a sequence in $D(X^*)$ such that $X^*A_n x \longrightarrow AX^*x$. Then Ax is an element of $D(X^*)$ such that $X^*Ax = AX^*x$. We obtain therefore $X^*A \supset AX^*$, $X^* \eta L'$ and $X^{**} \eta L$.

Let M be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ and X be a closed operator ηM, and $X = BU = UC$ be the polarization of X. It is verified that $U \in M$, $B \eta M$ and $C \eta M$ (cf [2]. Lemma 4.4.1. p.33). We define a $*$-algebra $C_0(X)$ incidental to X

Let $(0,\infty)$ be the set of all positive real numbers, and $C_0(0,\infty)$ be the linear algebra of all continuous functions on $(0,\infty)$ with compact carriers. $C_0(X)$ denotes the *-algebra of all operators $f(B)$ such that $f \in C_0(0,\infty)$. $C_0(X)$ is an abelian *-subalgebra of M, and the mapping $f(B) \longrightarrow f(B)X$ carries $C_0(X)$ into M, and we have $f(B)X = Xf(C)$.

Let (L,λ) be a cyclic *-algebra on a Hilbert space $\mathfrak{h}$, and X be an operator defined in a subset of $\mathfrak{h}$. We say that $X\eta(L^c,\lambda')$ if X satisfies the following three conditions ; (1). The domains of X and X^* contains $\lambda(L)$. (2). X is a closed operator which $\eta L'$, (3). $\mathfrak{h}$ contains certain elements $\lambda'(X)$ and $\lambda'(X^*)$ such that

1.4.3. $$X\lambda(A) = A\lambda'(X), \quad X^*\lambda(A) = A\lambda'(X^*).$$

Theorem 1.4.1. Let (L,λ) be a cyclic *-algebra in a Hilbert space $\mathfrak{h}$. X be a closed operator which $\eta(L^c,\lambda')$ and $C_0(X)$ be the incidental *-algebra. Then

(a). $C_0(X)$ is an abelian *-subalgebra of L^c.

(b). If $K \in C_0(X)$, then $KX \in L^c$ and $(\lambda'(K^*KX)^*), \lambda'(X^*)) \geq 0$.

(c). $C_0(X)$ contains a sequence $\{K_n\}$ such that

1.4.4. $$\lambda'(K_n{}^2X) \longrightarrow \lambda'(X); \quad \lambda'((K_n{}^2X)^*) \longrightarrow \lambda'(X^*).$$

(d). Suppose that there is a constant γ such that

1.4.5. $$(\lambda'(K^*KX)^*, \lambda'(X^*)) \leq \gamma^2\|\lambda'(K)\|^2.$$

for all $K \in C_0(X)$. Then X is an element of L^c such that $\|X\| \leq \gamma$.

Proof of (a). Let $X = BU = UC$ and $X^* = CU^* = U^*B$ be the polarizations of X and X^*. As the mapping $f(t) \longrightarrow tf(t)$ is invertible and linear on $C_0(0,\infty)$, the mapping $f(B) \longrightarrow Bf(B)$ is invertible and linear on $C_0(X)$. To prove (a) it is sufficient to find $Bf(B) \in L^c$ for all $f \in C_0(0,\infty)$. We notice that

$$Bf(B)\lambda(A) = Uf(C)X^*\lambda(A) = A(Uf(C)\lambda(X^*))$$

and

$$Bf(B)^*\lambda(A) = A(Uf^*(C)\lambda(X^*)).$$

Then $Bf(B)$ is an element of L^c such that

1.4.6. $$\lambda'(Bf(B)) = Uf(C)\lambda'(X^*).$$

Proof of (b). From the identities

$$f(B)X\lambda(A) = A(f(B)\lambda'(X))$$

and

$$(f(B)X^* \lambda(A) = f(C)^*X^*\lambda(A) = A(f(C)^*\lambda'(X^*)),$$

$f(B)X$ is an element of L^c such that

1.4.7. $$\lambda'(f(B)X) = F(B)\lambda'(X),\ \lambda'((F(B)X)^*) = f(C)^*\lambda(X^*)$$

and

$$(\lambda'((f(B)^* f(B)X)^*), \lambda'(X^*)) = \| f(C)\lambda'(X^*)\|^2 \geq 0.$$

Proof of (c). L is nondegenerate and contains a sequence $\{A_n\}$ such that $A_n\lambda'(X) \longrightarrow \lambda'(X)$; $A_n\lambda'(X)$ is identical to $X\lambda(A_n)$ and is contained in the range of B. Then $\lambda'(X)$ is contained in the uniform closure of the range of B. Similarly $\lambda'(X^*)$ is contained in the uniform closure of the range of C.

Let (f_n) be a sequence in $C_0(X)$ such that $0 \leq f_1 \leq f_2 \leq$ and $f_n(t) \longrightarrow 1$ for $0 < t < \infty$. Then $f_n{}^2(B)\lambda'(X) \longrightarrow \lambda'(X)$ and $f_n{}^2(C)\lambda'(X^*) \longrightarrow \lambda'(X^*)$. Setting $K_n = f_n(B)$. by 1.4.6 we have $\lambda'(K_n{}^2X) = f_n(B)^2\lambda'(X)$ and $\lambda'((K_n{}^2X)^*) = f_n(C)^2\lambda'(X^*)$. Then (K_n) is a desiered sequence in (c).

Proof of (d). The assumption 1.4.5 may be noted in the following form :

1.4.5.' $\quad \lambda'((f(B)^*f(B)X)^*, \lambda(X^*)) \leq \gamma^2 \| f(B) \|^2.$

Reffering 1.4.6 and 1.4.7, the left side of 1.4.5 is estimated as

$$(\lambda'(f(B)^* f(B)X)^*), \lambda(X^*) = (f(C)^*f(C)\lambda(X^*), \lambda(X^*))$$
$$= \|f(C)\lambda(X^*)\|^2 \geq \|Uf(C)\lambda(X^*)\|^2 = \|\lambda'(B\ f(B))\|^2,$$

and we obtain the inequality

1.4.5." $\quad \|\lambda'(B\ f(B)) \leq \gamma \|\lambda'(f(B))\|.$ for $f \in C_0(0,\infty)$.

Let f be a function in $C_0(0,\infty)$ whose carrier is contained in the extended open interval (γ, ∞). Then there is a certain constant $\delta > \gamma$ such that $\delta f(t) \leq tf(t)$ for $0 < t < \infty$. Let f and g be functions ≥ 0 in $C_0(0,\infty)$ such that $g(t) = t$ and $h(t)^2 + \delta^2 = t^2$ for all t in the carrier of f. Then $f(B)$, $g(B)$ and $h(B)$ are elements of L^c such that $Bf(B) = g(B)f(B)$ and

$$B^2f(B) = g(B)^2f(B) = (h(B)^2 + \delta^2)f(B).$$

We reffer 1.5.9 to the following inequality

$$\|\lambda'(Bf(B))\|^2 = \|\lambda'(g(B)f(B))\|^2$$
$$= \delta^2 \|\lambda'(f(B))\|^2 + \|\lambda'(h(B)f(B))\|^2$$
$$\geq \delta^2 \|\lambda'(f(B))\|^2$$

and obtain $\lambda'(f(B)) = 0$. Since λ' is one-to-one we have $f(B) = 0$. Since $f(B)$ vanishes whenever the carrier of $f \in C_0(0, \infty)$ is contained in (γ, ∞), B and $X = BU$ are elements of L^c such that $\|X\| = \|B\| \leq \gamma$. Theorem 1.4.1 is thus proved.

Let (L, λ) be a cyclic *-algebra on a Hilbert space $\mathfrak{H}$, and x be an element of $\mathfrak{H}$. An element x^s of $\mathfrak{H}$ is called the adjoint of x if it satisfies

1.4.8. $(Ax, \lambda(B)) = (\lambda(A), Bx^s)$ for $A \in L$, $B \in L$,

or the equivalent identity

1.4.9. $(\lambda(A), x) = (x^s, \lambda(A^*))$. for $A \in L^2$.

An element x of $\mathfrak{H}$ is said adjointive if it has an adjoint. We let $\mathfrak{H}^s$ denote the totality of adjointive elements of $\mathfrak{H}$. Then for each $x \in \mathfrak{H}^s$ its adjoint x^s is uniquely defined. We define an innerproduct $(x, y)_s$ in $\mathfrak{H}^s$ by

1.4.10. $(x, y)_s = (x, y) + (y^s, x^s)$

and the norm $\|x\|_s$ by $\|x\|_s^2 = (x, x)_s$. It is easily verified that $x \longrightarrow x^s$ is a closed operator in $\mathfrak{H}$. Then we have.

Lemma 1.4.3. Let (L, λ) be a cyclic *-algebra on a Hilbert space $\mathfrak{H}$, and $\mathfrak{H}^s$ be the totality of adjointive elements of $\mathfrak{H}$. Then $\mathfrak{H}^s$ is a Hilbert space by the innerproduct 1.4.9. $x \longrightarrow x^s$ is a reflexive cojugatelinear isometry of $\mathfrak{H}^s$.

Theorem 1.4.2. Let (L, λ) be a cyclic *-algebra on a Hilbert space $\mathfrak{H}$, and x be an adjointive element of $\mathfrak{H}$. Then there is a certain closed operator $W \eta (L^c, \lambda')$ such that $x = \lambda'(W)$ and $x^s = \lambda'(W^*)$.

Proof. From the identity $(Ax, \lambda(B)) = (\lambda(A), Bx^s)$ follows that if $\lambda(A) = 0$ then $Ax = 0$ and $Ax^s = 0$. We define linear operators W_0 and W_1 on the space $\lambda(L)$ by

1.4.11. $\qquad W_0 \lambda(A) = Ax, \quad W_1 \lambda(A) = Ax^s \qquad$ for $\quad A \in L$.

Then W_0 and W_1 commutes with the algebra L and we have $W_0^* \supseteq W_1$. $W = W_0^{**}$ is a closed operator which satisfies $W \eta L'$, $W \supset W_0$ and $W^* = W_0^*$. Then we obtain $W \eta (L^c, \lambda')$, $\lambda'(W) = x$ and $\lambda'(W^*) = x^s$.

Lemma 1.4.4. Let (L, λ) be a cyclic *-algebra on a Hilbert space $\mathfrak{H}$. Then the set $\lambda'(L^{c2})$ is everywhere dense in the Hilbert space $\mathfrak{H}^s$. The identity 1.4.9. is extended to the following identity

1.4.12. $\qquad (\lambda''(X), x) = (x^s, \lambda''(X^*)) \qquad$ for $x \in \mathfrak{H}^s$ and $X \in L^{cc}$.

Proof. Let x be an element of $\mathfrak{H}^s$ and consider an operator W such that $W \eta (L^c, \lambda)$, $x = \lambda'(W)$ and $x^s = \lambda'(W^*)$. By (c) of Theorem 1.4.1 $C_0(W)$ contains a sequence $\{K_n\}$ such that

$\lambda'(K_n{}^2 W) \longrightarrow x$ and $\lambda'((K_n{}^2 W)^*) \longrightarrow x^s$ uniformly in $\mathfrak{h}$. $K_n{}^2 W$ is an element of L^{c2} such that $(\lambda'((K_n{}^2 W)^*)) = (\lambda'(K_n{}^2 W))^s$. Then we have $\| \lambda'(K_n{}^2 W) - x \|_s \longrightarrow 0$, and the set $\lambda'(L^{c2})$ is everywhere dense in the Hilbert space $\mathfrak{h}^s$. Now let X be an element of L^{cc} and K and L be elements of L^c. Then we find that

$$\begin{aligned}(\lambda''(X), \lambda'(KL)) &= (\lambda'((L^*K)^*), \lambda''(X^*)) \\ &= ((\lambda(KL))^s, \lambda''(X^*)).\end{aligned}$$

Since $\lambda'(L^{c2})$ is everywhere dense in $\mathfrak{h}^s$ we have

$$(\lambda''(X), x) = (x^s, \lambda''(X^*)) \quad \text{for} \quad x \in \mathfrak{h}^s.$$

§1.5. Density theorems for semicyclic *-algebras.

Let (L, λ) be a maximal semicyclic *-algebra. The restriction λ_K of the mapping λ on a *-sabalgebra K of L is a semicyclic mapping of K. If K is nondegenerate then (K, λ_K) is a semicyclic *-algebra since K^c is a nondegenerate *-algebra which contains the nondegenerate algebra L^c. We define the cyclic seminorm $|\lambda|$ on L by

1.5.1 $$|\lambda|(A) = (\| \lambda(A)\|^2 + \| \lambda(A^*)\|^2)^{\frac{1}{2}}.$$

The topology of L which is defined by the cyclic semi-norm $|\lambda|$ is called the cyclic topology of L. In the succeeding sections we shall mainly use the cyclic topologies to apply the next Theorem 1.5.1.

Theorem 1.5.1 Let (L,λ) be a maximal cyclic *-algebra and K be a *-subalgebra of L. (K, λ_K) is a semicyclic *-subalgebra with the maximal extension (L,λ) if and only if K is cyclicly dense in L.

To prove Theorem 1.5.1 we need to prepare Lemma 1.5.1 and 1.5.2.

Let $\mathfrak{H}$ be a Hilbert space and $\mathfrak{M}$ be a real linear subset of H. Elements x and y of $\mathfrak{H}$ are called mutually real orthogonal if $\mathrm{Re}(x, y) = 0$. We let $\mathfrak{M}^{\perp}$ denote the totality of elements of $\mathfrak{H}$ which are real orthogonal to $\mathfrak{M}$, and call it the real orthogonal complement of $\mathfrak{M}$. $\mathfrak{M}$ is everywhere dense in $\mathfrak{M}^{\perp\perp}$. If L is a *-algebra on H, we let L^H denote the real linear set of all Hermitian elements of L.

Lemma 1.5.1. Let (L,λ) be a maximal cyclic *-algebra on a Hilbert space $\mathfrak{H}$, and let K be a *-subalgebra of L.. K is cyclicly dense in L if and only if the set $\lambda(K^H)$ is everywhere dense in the set $\lambda(L^H)$.

Proof. The direct sum $H \oplus H$ is the cartesian product of the Helbert space H, and the cartesian product $\mathfrak{M} \oplus \mathfrak{N}$ of linear subsets $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathfrak{H}$ is a linear subset of $H \oplus H$. Notice that $A_1 + iA_2 \longrightarrow (A_1, A_2)$ is a one-to-one mapping of L onto the cartesian product $L^H \times L^H$, and the cyclic seminorm $|\lambda|$ is represented as

$$|\lambda|(A_1 + iA_2) = (2 \| \lambda(A_1)\|^2 + 2 \| \lambda(A_2)\|^2)^{\frac{1}{2}} .$$

The subalgebra K is cyclicly dense in L if and only if $\lambda(K^H) \oplus \lambda(K^H)$ is everywhere dense in $\lambda(L^H) \oplus \lambda(L^H)$. Then K is cyclicly dense if and only if $\lambda(K^H)$ is everywhere dense in $\lambda(L^H)$.

Lemma 1.5.2. Let (K,λ) be a cyclic *-algebra on a Hilbert space $\mathfrak{H}$, and $\mathfrak{H}^s$ be the Hilbert space of all adjointive elements of $\mathfrak{H}$ with respect to (K,λ). Then the real orthogonal complement of the set $\lambda(K^{2H})$ is the set $(x \in \mathfrak{H}^s : x = -x^s)$.

Proof. An element x of $\mathfrak{H}$ is real orthogonal to the set $\lambda(K^{2H})$ if and only if

$$(\lambda(A), x) = -(x, \lambda(A)) \qquad \text{for all} \quad A \in K^{2H}.$$

The above identity is equivalent to the identity

$$(\lambda(A), x) = -(x, \lambda(A^*)) \qquad \text{for all} \quad A \in K^2.$$

The second indentity is valid if and only if x is an adjointive element of $\mathfrak{H}$ such that $x = -x^s$.

Now Theorem 1.5.1 will be proved, devided to the next Lemma 1.5.3 and 1.5.4.

Lemma 1.5.3. Let (L,λ) be a maximal cyclic *-algebra on a Hilbert $\mathfrak{H}$. and K be a nondegenerate *-subalgebra of L such that $(L,\lambda) = (K^{cc}, \lambda''_K)$. Then K^2 is cyclicly dense in L.

Proof. By Lemma 1.1.7 (K, λ_K) is a cyclic *-subalgebra of (L, λ). To see that K^2 is cyclicly dense in L it is sufficient to show that $\lambda(L^H)$ is contained in $\lambda(K^{2H})^{\perp\perp}$.

Let $\mathfrak{H}^s$ be the Hilbert space of all adjointive elements of $\mathfrak{H}$ with respect to the algebra (K, λ_K). By Lemma 1.5.2 $\lambda(K^{2H})^\perp$ is the totality of $x \in \mathfrak{H}^s$ such that $x^s = -x$. If X is an element of L^H, then by Lemma 1.4.4 we have

$$(\lambda(X), x) = (x^s, \lambda(X^*)) = -(x, \lambda(X)) \quad \text{for} \quad x \in \lambda(K^{2H})^\perp .$$

Then $\lambda(X)$ is contained in $\lambda(K^{2H})^{\perp\perp}$, and $\lambda(K^{2H})$ is everywhere dense in $\lambda(L^H)$. Hence K^2 is cyclicly dense in L.

Lemma 1.5.4. Let (L, λ) be a maximal cyclic *-algebra on a Hilbert space $\mathfrak{H}$, and let K be a cyclicly dense *-subalgebra of L. Then (K, λ_K) is a cyclic *-subalgebra of (L, λ) such that $(L, \lambda) = (K^{cc}, \lambda_K'')$.

Proof. To simplify the notations we set $\mu = \lambda_K$. Since the mapping $A \longrightarrow \lambda(A)$ is cyclicly continuous on L, the set $\lambda(K)$ is everywhere dense in $\lambda(L)$ and in $\mathfrak{H}$. μ is therefore a cyclic mapping of K, and K^c is a nondegenerate *-algebra which contains L^c. By Lemma 1.1.6 (K, μ) is a cyclic *-algebra on $\mathfrak{H}$. Consider the elements X of L and K of L^c. As a cyclicly dense subset of L, K contains a certain sequence $\{A_n\}$ satisfying

$$\mu(A_n) \longrightarrow \lambda(X), \quad \mu(A_n{}^*) \longrightarrow \lambda(X^*).$$

$\mu'(K)$ is the adjoint of $\mu'(K^*)$ with respect to the algebra (K, μ). Then by Lemma 1.5.4 we have

$$(\mu'(K), \mu(A_n)) = (\mu(A_n{}^*), \mu'(K^*)).$$

Letting $n \longrightarrow \infty$, we have

$$(\mu'(K), \lambda(X)) = (\lambda(X^*), \mu'(K^*)$$

and for every $A \in K$

$$(K\lambda(X), \mu(A)) = (\lambda(X), K^*\mu(A)) = (\lambda(X), A\mu'(K^*))$$

$$= (\lambda(A^*X), \mu'(K^*)) = (\mu'(K), \lambda(X^*A)) = (X\mu'(K), \mu(A)).$$

Thus we obtain the relations $K\lambda(X) = X\mu'(K)$ and $K^*\lambda(X) = X\mu'(K^*)$. By Lemma 1.1.6 K is an element of L^c and hence we have $(L^c, \lambda') = (K^c, \mu')$ and $(L^c, \lambda) = (K^{cc}, \lambda''_K)$.

Corollary of Theorem 1.5.1. Let (L, λ) be a separating cyclic *-algebra on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{H}^s$ be the Hilbert space of all adjointive elements of $\mathfrak{H}$ with respect to (L, λ), and let K be a *-subalgebra of L^c. (K, λ'_K) is a separating cyclic *-algebra on $\mathfrak{H}$ with the maximal extension (L^c, λ') if and only if $\lambda'(K)$ is everywhere dense in $\mathfrak{H}^s$.

Proof. Let $\| \ , \|_s$ be the norm in $\mathfrak{H}^s$ and $|\lambda'|$ be the cyclic seminorm in L^c. By Lemma 1.4.4, $\lambda'(L^c)$ is a dense subset of $\mathfrak{H}^s$ satisfying

1.5.2. $\quad |\lambda'|(K) = \| \lambda'(K)\|_s \quad$ for $K \in L^c$.

Then K is cyclicly dense in L^c if and only if $\lambda'(K)$ is everywhere dense in $\mathfrak{H}^s$, and if K is cyclicly dense in L^c, (K, λ_K) is a separating cyclic *-algebra on $\mathfrak{H}$. Therefore we obtain the Corollary.

Let (L, λ) be a maximal semicyclic *-algebra. A norm $\| \ , \|_\lambda$ on I which is defined by

1.5.4. $\quad \|A\|_{\lambda} = \max(\|A\|, \quad \|\lambda(A)\|, \quad \|\lambda(A^*)\|),$

is called the cyclouniform norm of L.

Theorem 1.5.2. If (L,λ) is a maximal semicyclic *-algebra, then L is a Banach *-algebra by the cyclouniform norm. The unit ball $U(L,\lambda)$ of L defined by the cyclouniform norm is weakly compact.

Proof. Since L is obviously a Banach *-algebra we show that $U(L,\lambda)$ is weakly compact. Let $\mathfrak{H}$ be the underlying Hilbert space of (L,λ). Then $U(L,\lambda)$ is the totality of $A \in U(L'')$ satisfying, for $K \in L^c$ and $x \in \mathfrak{H}$,

$$\max(|(A\lambda'(K), x)|, |(A^*\lambda'(K), x)|) \leq \|K\| \|x\| .$$

The functionals $A \longrightarrow (A\lambda'(K), x)$ and $A \longrightarrow (A^*\lambda'(K), x)$ are weakly continuous on $U(L'')$. Then $U(L,\lambda)$ is weakly compact, as a closed subset of the weakly compact set $U(L'')$.

Consider a maximal cyclic *-algebra (L,λ) ona Hilbert space $\mathfrak{H}$ and the cyclic seminorm $|\lambda|$ on L . If $\{x_n\}$ is an arbitrary sequence in $\mathfrak{H}$ such that $\sum \|x_n\|^2 < \infty$, we define a seminorm $|\lambda|_x$ on L by

1.5.4. $\quad |\lambda|_x(A) = (|\lambda|(A)^2 + \sum \|Ax_n\|^2 + \sum \|A^*x_n\|^2)^{\frac{1}{2}}.$

The system of all these seminorms $|\lambda|_x$ defines a certain locally convex topology of L which we call the σ-cyclostrong topology of L.

Theorem 1.5.3. If (K, λ_K) is a semicyclic *-subalgebra of a maximal semicyclic *-algebra (L, λ) such that $(L, \lambda) = (K^{cc}, \lambda_K'')$, then K is σ-cyclostrongly dense in L.

Proof. Let $\{x_n\}$ be a sequence in $\mathfrak{H}$ such that $\sum_n \|x_n\|^2 < \infty$, and define the seminorm $|\lambda|_x$ by 1.5.4. Let M denote the von Neumann algebra L''. Define a normal positive functional ω_x on M by

$$\omega_x(A) = \sum_n (Ax_n, x_n),$$

and consider generalized normal positive functional p on M such that $M(p) = L$ and $p(A^*A) = \|\lambda(A)\|^2$ for $A \in L$. Then $q = \omega_x + p$ is a generalized normal positive functional on M such that $L = M(p) = M(q)$. Corresponding to the functional q, we consider a normal representation π of M on the von Neumann extension $\pi(M)$ of a certain maximal cyclic *-algebra $(\pi(L), \sigma)$, which carries L onto $\pi(L)$, and which satisfies $q(A^*A) = \|\sigma(\pi(A)\|^2$ and $|\sigma|(\pi(A))| = |\lambda|_x(A)$. Since $U(K)$ is strongly dense in $U(M)$, π carries K onto a certain strongly dense *-subalgebra $\pi(K)$ of $\pi(M)$. If the identity $(\pi(L), \sigma) = (\pi(K)^{cc}, (\sigma_{\pi(K)})'')$ is satisfied, then $\pi(K)$ is cyclicly dense in $\pi(L)$, and K is everywhere dense in L with respect to the semi-norm $|\lambda|_x$. Thus, K is cyclostrongly dense in L. Then Theorem 1.6.3 is reduced to prove $(\pi(L), \sigma) = (\pi(X)^{cc}, \sigma_{\pi(K)}'')$.

By Lemma 1.3.1 we take an Hermitian operator $T \geq 0$ in $\pi(L)'$ such that for every $A \in L$

$$p(A^*A) = \|\lambda(A)\|^2 = (T^2\sigma(\pi(A)), \sigma(\pi(A)))$$

and

$$\omega_x(A^*A) = ((1 - T^2)\ \sigma\ (\pi(A)), \sigma(\pi(A))).$$

Notice that

$$\|(1 - T)\sigma(\pi(A))\|^2 \leq \omega_x(A^*A) \leq \omega_x(1)\ \|\pi(A)\|^2$$

Then $X \longrightarrow (1 - T)\sigma(X)$ is uniformly continuous on $\pi(L)$, and $1 - T$ is an element of $\pi(L)^c$. Setting $g = \sigma'(1 - T)$ we have $T\sigma(X) = \sigma(X) - Xg$ for $X \in \pi(L)$. We let ρ denote the restriction of $T\sigma$ in $\pi(K)$. By Lemma 1.1.3 the identity $(\pi(K)^{cc}, \sigma_{\pi(K)}{}'') = (\pi(L), \sigma)$ is equivalent to $(\pi(K)^{cc}, \rho'') = (\pi(L), T\sigma)$. Let E and F be the carriers of λ and $T\sigma$, and let $\mathfrak{M}$ and $\mathfrak{N}$ be the ranges of E and F. The mapping $\lambda(A) \longrightarrow T\sigma(\pi(A))$ is extended to an isometry U of $\mathfrak{M}$ onto $\mathfrak{N}$ which determines a spatial isomorphism of the algebra (L_E, λ_E) onto $(\pi(L)_F, (T\sigma)_F)$. By Lemma 1.2.4. K_E is a nondegenerate $*$-subalgebra of L_E satisfying $((K_E)^{cc}, (\lambda_K)_E{}'') = (L_E, \lambda_E)$. Then $\pi(K)_F$ is a nondegenerate $*$-subalgebra of $\pi(L)$ satisfying $((\pi(K)_F)^{cc}, (\rho_F)'') = (\pi(L)_F, (T\sigma)_F)$. Since F is the carrier of ρ, by Lemma 1.2.4 $\pi(K)^{cc}$ is the totality of $A \in \pi(M)$ such that $A_E \in \pi(K)_F{}^{cc} = \pi(L)_F\}$ Then we find $(\pi(K)^{cc}, \rho'') = (\pi(L), T\sigma)$. Hence $(\pi(L), \sigma) = (\pi(K)^{cc}, \sigma_{\pi(L)}{}'')$ is proved, and K is cyclostrongly dense in L.

Theorem 1.5.4 If (K, λ_K) is a semicyclic $*$-subalgebra of a maximal semicyclic $*$-algebra (L, λ) such that $(K^{cc}, \lambda_K{}'') = (L, \lambda)$, then the unit ball $U(K, \lambda_K)$ is σ-cyclostrongly dense in $U(L,\lambda)$.

Proof. Let W be the totality of $A \in L$ such that $\max(\|\lambda(A)\|, \|\lambda(A^*)\|) < 1$. Since the mappings $A \longrightarrow \lambda(A)$ and $A \longrightarrow \lambda(A^*)$ are σ-strongly continuous on L, W is a σ-strongly open set in L, and $U(L,\lambda)$ is the σ-strong closure of the set $U(L) \wedge W$. If it is proved that $U(K)$ is σ-cyclostrongly dense in $U(L)$ then the σ-cyclostrong closure of $U(K, \lambda_K)$ contains the σ-syclostrong closures of $U(K) \cap W$ and then $U(L) \cap W$, and coinsides with $U(L, \lambda)$. Then the Theorem is reduced to verify that $U(K)$ is σ-cyclostrongly dense in $U(L)$. We consider the mapping $A \longrightarrow f(A)$ of L which is defined by

$$f(A) = 2(1 + AA^*)^{-1}A = 2A(1 + A^*A)^{-1}$$

and the mapping $A \longrightarrow \varphi(A)$ of $U(L)$ which is defined by

$$\varphi(A) = (1 + (1 - AA^*)^{\frac{1}{2}})^{-1}A = A(1 + (1 - A^*A)^{\frac{1}{2}})^{-1}.$$

By Lemma 1.1.2 f carries L into L and φ carries $U(L)$ into $U(L)$. Let A be an element of $U(L)$ and $A = BU$ be its polarization. Then we have, $f(A) = 2B(1 + B^2)^{-1}U$ and $\varphi(A) = B(1 + (1 - B^2)^{\frac{1}{2}})^{-1}U$. If y is a number in the closed interval $-1 \leq y \leq 1$, then the equation $y = 2x(1 + x^2)^{-1}$ has a solution $x = y(1 + (1 - y^2)^{\frac{1}{2}})^{-1}$. Then we have $f(\varphi(A)) = A = \varphi(f(A))$ for $A \in U(L)$, and f carries L onto $U(L)$. Since K is σ-cyclostrongly dense in L, $U(K)$ is σ-cyclostrongly dense in $U(L)$ if it is proved that f is σ-cyclostrongly continuous.

Now we shall use all the notations in the proof of Theorem 1.

5.4. The normal representation π satisfies $f(\pi(A)) = \pi(f(A))$ and $|\lambda| x(A) = |\sigma|(\pi(A))$. To see that f is σ-cyclostrongly continuous on L, it is sufficient to show that $f : \pi(L) \longrightarrow U(\pi(L))$ is continuous on $\pi(L)$ when the topology of the domain $\pi(L)$ of f is the σ-cyclostrong topology and the topology of the range $U(\pi(L))$ of f is the cyclic topology. Let X and Y be elements of $\pi(L)$. Then

$$\frac{1}{2}\sigma(f(X) - f(Y)) = (1 + YY^*)^{-1}\sigma(X - Y)$$
$$+ (1 + YY^*)^{-1}(YY^* - XX^*)(1 + XX^*)^{-1}\sigma(X),$$

where

$$YY^* - XX^* = Y(Y^* - X^*) + (Y - X)X^*.$$

We set $x_1 = \frac{1}{2}\sigma(f(X))$, $x_2 = \frac{1}{2}\sigma f(X^*)$, $x_3 = \sigma(X^*f(X))$ and $x_4 = \sigma(Xf(X^*))$. Then remarking that $(1 + YY^*)^{-1}$ and $2(1 + YY^*)^{-1}Y$ belong to $U(L'')$, we have

$$\|\sigma(f(X) - f(Y)\| \leq 2\ \|\sigma(X - Y\| + \|(Y^* - X^*)x_1\|$$
$$+ \|(Y - X)x_3\|$$

and

$$|\sigma|(f(X) - f(Y))$$
$$\leq \sqrt{3}\ (2|\sigma|(X - Y) + \sum_{i=1}^{4}\|(X - Y)x_i\|^2 + \sum_{i=1}^{4}\|(X^* - Y^*)x_i\|^2\ )^{\frac{1}{2}}.$$

The last inequality verifies that the mapping f on $\pi(L)$ is continuous. Theorem 1.5.5 is thus proved.

§ 2. Modular standard algebras.

§ 2.1. Generalized Hilbert algebras.

We consider a *-algebra $\mathfrak{A}$ which is a prehilbert space and whose innerproduct is denoted by (a, b). We say that $\mathfrak{A}$ is a prehilbert *-algebra if the following three conditions are satisfied.

(1). $(ab, c) = (b, a^*c)$

(2). For every $a \in \mathfrak{A}$ the mapping $b \longrightarrow ab$ is continuous on $\mathfrak{A}$.

(3). The algebra $\mathfrak{A}^2$ is everywhere dense in $\mathfrak{A}$.

Let $\mathfrak{A}$ be a prehilbert *-algebra. By the metrical completion $\mathfrak{A}$ is extended to a certain Hilbert space $\mathfrak{H}$ which we call the Hilbert extension of $\mathfrak{A}$. For every $a \in \mathfrak{A}$ we define a bounded operator L_a on $\mathfrak{H}$ by

2.1.1. $L_a b = ab$ for $b \in \mathfrak{A}$.

Then $a \longrightarrow L_a$ is a *-homomorphism of $\mathfrak{A}$ onto a certain *-algebra $L\mathfrak{A}$ on $\mathfrak{H}$ which we call the left regular representation of $\mathfrak{A}$. The von Neumann algebra $(L\mathfrak{A})''$ is called the extended left regular representation of $\mathfrak{A}$. $L\mathfrak{A}$ is a nondegenerate *-algebra since $L\mathfrak{A}\mathfrak{H}$ contains $\mathfrak{A}^2$ and is everywhere dense in $\mathfrak{H}$. An element k of $\mathfrak{H}$ is said commuting with $\mathfrak{A}$ if there is a certain bounded operator R_k and an element k^s of $\mathfrak{H}$ such that

2.1.2. $R_k a = L_a k, \quad (R_k)^* a = L_a k^s.$

The set $\mathfrak{A}^c$ of all elements of $\mathfrak{H}$ commuting with $\mathfrak{A}$ is called the commutant of $\mathfrak{A}$. Since $\mathfrak{A}$ is not an operator algebra, the above definition does not derive any confusion. For every $k \in \mathfrak{A}^c$ the operator R_k is uniquely defined. The set $R_{\mathfrak{A}^c}$ of all operators R_k such that $k \in \mathfrak{A}^c$ is called the right regular representation of $\mathfrak{A}^c$. The von Neumann algebra $(R_{\mathfrak{A}^c})''$ is called the extended right regular representation of $\mathfrak{A}^c$.

A prehilbert *-algebra $\mathfrak{A}$ is called a generalized Hilbert algebra if the commutant $\mathfrak{A}^c$ is everywhere dense in the Hilbert extension of $\mathfrak{A}$. Let $\mathfrak{A}$ be a generalized Hilbert algebra, $\mathfrak{A}^c$ be its commutant and $\mathfrak{H}$ be the Hilbert extension of $\mathfrak{A}$. An element a of $\mathfrak{H}$ is said commuting with $\mathfrak{A}^c$ if there is a certain bounded operator L_a on $\mathfrak{H}$ and an element a^* of $\mathfrak{H}$ such that for every $k \in \mathfrak{A}^c$

2.13. $\quad L_a k = R_k a, \ (L_a)^* k = R_k a^*.$

The set $\mathfrak{A}^{cc}$ of all elements of $\mathfrak{H}$ commuting with $\mathfrak{A}^c$ is called the second commutant of $\mathfrak{A}$. For each $a \in \mathfrak{A}^{cc}$ the operator L_a is uniquely defined. The set $L_{\mathfrak{A}^{cc}}$ of all operators L_a such that $a \in \mathfrak{A}^{cc}$ is called the left regular representation of $\mathfrak{A}^{cc}$. A generalized Hilbert algebra $\mathfrak{A}$ is called maximal if it satisfies $\mathfrak{A} = \mathfrak{A}^{cc}$.

Theorem 2.1.2. Let $\mathfrak{A}$ be a generalized Hilbert algebra, and $\mathfrak{H}$ be the Hilbert extension of $\mathfrak{A}$. Then $L_{\mathfrak{A}}$, $R_{\mathfrak{A}^c}$ and $L_{\mathfrak{A}^{cc}}$ are nondegenerate *-algebras on $\mathfrak{H}$ such that $L_{\mathfrak{A}}'' = (L_{\mathfrak{A}^{cc}})'' = (R_{\mathfrak{A}^c})'$. $a \longrightarrow L_a$ is an isomorphism of $\mathfrak{A}$ onto $L_{\mathfrak{A}}$. $(L_{\mathfrak{A}}, L_a \longrightarrow a)$ is a separating cyclic *-algebra which has the integrable commutant

$(R_{\mathfrak{A}^c}, R_k \longrightarrow k)$ and the maximal extension $(L_{\mathfrak{A}^{cc}}, L_a \longrightarrow a)$.

Proof. We take elements a, b of $\mathfrak{A}$ and k of $\mathfrak{A}^c$. Then

$$(L_b\, a, k) = (a, L_{b^*}\, k) = (a, R_k\, b^*)$$
$$= ((R_k)^* a, b^*) = (L_a\, k^s, b^*).$$

Since $L_{\mathfrak{A}}$ is nondegenerate, the above identity verifies that $L_a = 0$ implies $L_b\, a = 0$ for all $b \in \mathfrak{A}$ and $a = 0$. Then the *-homomophism $a \longrightarrow L_a$ is an isomorphism, and the mapping $L_a \longrightarrow a$ is a cyclic mapping of $L_{\mathfrak{A}}$. By Lemma 1.1.5 $(R_{\mathfrak{A}^c}, R_k \longrightarrow k)$ is the integrable commutant of $(L_{\mathfrak{A}}, L_a \longrightarrow a)$. Since $\mathfrak{A}^c$ is everywhere dense in $\mathfrak{H}$, $R_k \longrightarrow k$ is a cyclic mapping and by Lemma 1.1.6 $(L_{\mathfrak{A}}, L_a \longrightarrow a)$ is a separating cyclic *-algebra. We apply again Lemma 1.1.5 to the cyclic *-algebra $(R_{\mathfrak{A}^c}, R_k \longrightarrow k)$ and find that $(L_{\mathfrak{A}^{cc}}, L_a \longrightarrow a)$ is the integrable commutant of $(R_{\mathfrak{A}^c}, R_k \longrightarrow k)$. Hence we have $(L_{\mathfrak{A}})'' = (L_{\mathfrak{A}^{cc}})'' = (R_{\mathfrak{A}^c})'$.

Theorem 2.1.2. Let (L, λ) be a separating cyclic *-algebra on a Hilbert space $\mathfrak{H}$. Then $\mathfrak{H}$ contains a certain densely defined generalized Hilbert algebra $\mathfrak{A}$ satisfying $(L, \lambda) = (L_{\mathfrak{A}}, L_a \longrightarrow a)$.

Proof. We set $\mathfrak{A} = \lambda(L)$ and introduce *-algebraic operations in $\mathfrak{A}$ so that the mapping λ is an isomorphism of L onto $\mathfrak{A}$. Then the involution and the multiplication in $\mathfrak{A}$ are defined by

$$a^* = \lambda(A), \quad ab = \lambda(AB) \quad \text{for } a = \lambda(A), \quad b = \lambda(B).$$

Since $\mathfrak{A}^2 = L\lambda(L)$ is everywhere dense in $\mathfrak{H}$, $\mathfrak{A}$ is a pre-Hilbert *-algebra whose commutant $\mathfrak{A}^c = \lambda'(L^c)$ is everywhere dense in $\mathfrak{H}$. Then $\mathfrak{A}$ is a generalized Hilbert algebra satisfying

$(L, \lambda) = (L_{\mathfrak{A}}, L_a \longrightarrow a)$.

Let (L, λ) be a separating cyclic *-algebra. The generalized Hilbert algebra $\mathfrak{A}$ satisfying $(L, \lambda) = (L_{\mathfrak{A}}, L_a \longrightarrow a)$ is called the underlying generalized Hilbert algebra of (L, λ).

Theorem 2.1.3. If $\mathfrak{A}$ is a generalized Hilbert algebra, then $\mathfrak{A}^{cc}$ is a maximal generalized Hilbert algebra such that $\mathfrak{A}^{ccc} = \mathfrak{A}^{c}$.

Proof. $(L_{\mathfrak{A}^{cc}}, L_a \longrightarrow a)$ is a maximal separating cyclic *-algebra, and $\mathfrak{A}^{cc}$ is its underlying generalized Hilbert algebra. Since $(R_{\mathfrak{A}^{c}}, R_a \longrightarrow a)$ is the integrable commutant of $(L_{\mathfrak{A}^{cc}}, L_a \longrightarrow a)$ we have $\mathfrak{A}^{ccc} = \mathfrak{A}^{c}$. Then $\mathfrak{A}^{cc}$ is a maximal generalized Hilbert algebra.

If $\mathfrak{A}$ is a generalized Hilbert algebra, the maximal Hilbert algebra $\mathfrak{A}^{cc}$ is called the maximal extension of $\mathfrak{A}$.

§2.2. Modular Hilbert algebras.

Let $\mathfrak{A}$ be a prehilbert *-algebra, and K denote the complex number field. An automorphism group Γ of $\mathfrak{A}$ as a linear algebra is called a modular group of $\mathfrak{A}$ if the following six conditions are satisfied

(1). There is a certain homomorphism $\alpha \longrightarrow \Delta^{\alpha}$ of the additive group K onto Γ.

(2). $(\Delta^{\alpha} a)^* = \Delta^{-\bar{\alpha}} a^*$

(3). $(\Delta^{\alpha} a, b) = (a, \Delta^{\bar{\alpha}} b)$

(4). $((\Delta^{1} a)^*, b^*) = (b,a)$

(5). For every $a \in \mathfrak{A}$ and $b \in \mathfrak{A}$, $(\Delta^{\alpha} a, b)$ is an analytic function of the variable α on K.

(6). For every real number t, the set $(1 + \Delta^{t})\mathfrak{A}$ is everywhere dense in $\mathfrak{A}$.

A prehilbert *-algebra which has a certain modular group is called a modular Hilbert algebra. Let G be a locally compact group and $C_0(G)$ be the linear space of all continuous functions on G with compact carriers. If, f and g are elements of $C_0(G)$, we define the involution f^* of f by $f^*(a) = \Delta(a)\overline{f(a^{-1})}$, multiplication $f \circ g$ and the innerproduct (f, g) by

$$f \circ g = \int f(b)g(b^{-1}a)db, (f, g) = \int f(a)\overline{g(a)}da,$$

where da is a left invariant measure of G and $\Delta(a)$ is the related modular function on G such that $\int f(xa)dx = \Delta(a)\int f(x)dx$. It is well-known that $C_0(G)$ is a modular Hilbert algebra whose modular group $\Gamma = \{\Delta^{\alpha}\}$ is defined by

$$\Delta^{\alpha} : f(x) \longrightarrow \Delta^{\alpha}(x)f(x).$$

Lemma 2.2.1. Let $\mathfrak{A}$ be a modular Hilbert algebra which has a modular group $\Gamma = \{\Delta^{\alpha}\}$, and let $\mathfrak{H}$ be the Hilbert extension of $\mathfrak{A}$. Then the automorphism Δ^{1} of $\mathfrak{A}$ is extended to a certain self-adjoint operator $\bar{\Delta} \geq 0$ on $\mathfrak{H}$. If α is a complex number, then $\bar{\Delta}^{\alpha}$ is an extension of Δ^{α}, and $\mathfrak{A}$ is everywhere dense in the Hilbert space $\mathfrak{D}(\bar{\Delta}^{\alpha})$.

Proof. Let t be a real number. The set $(1 + \Delta^t)\mathfrak{A}$ is everywhere dense in $\mathfrak{H}$, and from $(\Delta^t a, a) \geq 0$ we find that $\|(1 + \Delta^t)a\| \geq \|a\|$ for $a \in \mathfrak{A}$. We define a bounded Hermitian operator $B_t \geq 0$ in $\mathfrak{H}$ by $B_t(1 + \Delta^t)a = a$. Then Δ^t is extended to a self-adjoint operator $U_t = B_t^{-1} - 1$ whose domain $\mathfrak{D}(U_t)$ contains $\mathfrak{A}$ as a dense subset. Notice that $\Delta^{t+s} = \Delta^t \Delta^s$ and $(U_t a, b)$ is continuous on the real line for fixed $a \in \mathfrak{A}$ and $b \in \mathfrak{A}$. Then we obtain $U_{t+s} = U_t U_s$, and setting $U_1 = \bar{\Delta}$ we find that $U_t = \bar{\Delta}^t \supseteq \Delta^t$. $(\Delta^\alpha a, b)$ and $(\bar{\Delta}^\alpha a, b)$ are regular analytic functions of the complex variable α which coincide on the real line. Then they coincide on K and we have $\bar{\Delta}^\alpha \supseteq \Delta^\alpha$ for all $\alpha \in K$. If t is the real part of a complex number α, then $\mathfrak{D}(\bar{\Delta}^\alpha)$, $\mathfrak{D}(\bar{\Delta}^t)$ and $\mathfrak{D}(U_t)$ are the same Hilbert space which contains $\mathfrak{A}$ as a dense subset.

We consider a modular Hilbert algebra which has a modular group $\Gamma = \{\Delta^\alpha\}$. Let $\mathfrak{H}$ be the Hilbert extension of $\mathfrak{A}$, and $\bar{\Delta}$ be the self-adjoint extension of Δ^1 in $\mathfrak{H}$. If t is a real number, then $a \longrightarrow (\Delta^t a)^*$ is an involution, i.e. a reflexive conjugate linear anti-automorphism, of the linear algebra $\mathfrak{A}$. We define two special involutions $a \longrightarrow a^z$ and $a \longrightarrow a^s$ by

2.2.1. $\quad a^z = (\Delta^{\frac{1}{2}} a)^*, \quad a^s = (\Delta a)^*.$

Then it is immediately shown that

2.2.2. $\quad (a, b) = (b^z, a^z),$

2.2.3. $\quad (a, b) = (b^s, a^*), \ (ab, c) = (a, cb^s).$

The involution $a \longrightarrow a^z$ is extended to a certain reflexive conjugate linear isometry J on $\mathfrak{h}$. If X is a bounded linear operator, we set $X^T = JX^*J$. X^T is called the transpose of X. If L is a $*$-algebra on $\mathfrak{h}$, the transpose $X \longrightarrow X^T$ induces an antiisomorphism of L onto a certain $*$-algebra L^T on $\mathfrak{h}$. L^T is called the transpose of the algebra L. From $(ab)^z = b^z a^z$ and $a^{z*} = a^{sz}$ we find that

2.2.4. $(L_a z)^{T*} = ba$, $(L_a z)^{Tb} = ba^s$,

Then every $a \in \mathfrak{A}$ is contained in $\mathfrak{A}^c$ and satisfies

2.2.5. $R_a = (L_a z)^{T*}$.

We let $R_{\mathfrak{A}}$ denote the totality of R_a such that $a \in \mathfrak{A}$ and call it the regular right representation of $\mathfrak{A}$.

Theorem 2.2.1. Let $\mathfrak{A}$ be a modular Hilbert algebra. Then $\mathfrak{A}$ is a generalized Hilbert algebra, and $R_{\mathfrak{A}}$ is a cyclicly dense $*$-subalgebra of $R_{\mathfrak{A}^c}$. The associated transpose $X \longrightarrow X^T$ carries $L_{\mathfrak{A}}$ onto $R_{\mathfrak{A}}$, $L_{\mathfrak{A}^{cc}}$ onto $R_{\mathfrak{A}^c}$ and $(L_{\mathfrak{A}})''$ onto $(R_{\mathfrak{A}^c})''$.

Proof. By 2.2.4 and 2.2.5 $R_{\mathfrak{A}}$ is the transpose $(L_{\mathfrak{A}})^T$ of $L_{\mathfrak{A}}$ and is a $*$-subalgebra of $R_{\mathfrak{A}^c}$. Since $\mathfrak{A}^c$ contains $\mathfrak{A}$ and is dense in $\mathfrak{h}$, $\mathfrak{A}$ is a generalized Hilbert algebra. Let $\mathfrak{h}^s$ be the Hilbert space of all adjointive elements of $\mathfrak{h}$ with respect to the separating cyclic $*$-algebra $(L_{\mathfrak{A}}, L_a \longrightarrow a)$. An element x of $\mathfrak{h}$ belongs to $\mathfrak{h}^s$ if and only if $\mathfrak{h}$ contains a certain element x^s satisfying

$$(Jx^s, a) = (x, a^{z*}) = (x, \Delta^{\frac{1}{2}} a) \quad \text{for } a \in \mathfrak{A}.$$

Then $x \in \mathfrak{h}$ is adjointive if and only if x belongs to $\mathcal{D}(\bar{\Delta}^{\frac{1}{2}})$, and the adjoint x^s of $x \in \mathfrak{h}^s$ is determined by

2.2.6. $x^s = J\,\bar{\Delta}^{\frac{1}{2}}\,x.$

$\mathfrak{h}^s$ and $\mathcal{D}(\bar{\Delta}^{\frac{1}{2}})$ are the same Hilbert space with the norm

$$\|x\|_s = (\|x\|^2 + \|x^s\|^2)^{\frac{1}{2}} = (\|x\|^2 + \|\bar{\Delta}^{\frac{1}{2}} x\|^2)^{\frac{1}{2}}.$$

Since $\mathfrak{A}$ is everywhere dense in $\mathfrak{h}^s$, by the Corollary of Theorem 1. 5.1. $R_{\mathfrak{A}}$ is a cyclicly dense subalgebra of $R_{\mathfrak{A}^c}$ satisfying $(R_{\mathfrak{A}})^{cc} = R_{\mathfrak{A}^c}$ and $(R_{\mathfrak{A}})'' = (R_{\mathfrak{A}^c})''$. Then the transpose of $(L_{\mathfrak{A}})''$ is $(R_{\mathfrak{A}^c}$ We show that the transpose of $L_{\mathfrak{A}^{cc}}$ is $R_{\mathfrak{A}^c}$. Let x be an eleme of $\mathfrak{A}^{cc}$ and a be an element of $\mathfrak{A}$. Reffering 2.2.5 we have

$$(L_x)^T a = J(L_{x^*}\, a^z) = J(R_{a^z}\, x^*) = L_a J x^*,$$

and

$$(L_x)^{T^*} a = L_a J x.$$

Then we fined that $(L_x)^T = R_{Jx^*}$ and $(L_{\mathfrak{A}^{cc}})^T \subseteq R_{\mathfrak{A}^c}$. Similarly we have $(R_{\mathfrak{A}^c})^T \subset L_{\mathfrak{A}^{cc}}$. Hence $(L_{\mathfrak{A}^{cc}})^T = R_{\mathfrak{A}^c}$ is proved.

Consider the tensor product $\mathfrak{h}_1 \otimes \mathfrak{h}_2$ of Hilbert spaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$. If $\mathfrak{M}$ and $\mathfrak{N}$ are linear subsets of $\mathfrak{h}_1$ and $\mathfrak{h}_2$, we let $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ denote the least linear set which contains the tensor products $x \otimes y$ of $x \in \mathfrak{M}$ and $y \in \mathfrak{N}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be modular Hilbert algebras and $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be their Hilbert extensions. Then $\mathfrak{A} \otimes \mathfrak{B}$ is a dense subset of $\mathfrak{h}_1 \otimes \mathfrak{h}_2$. $\mathfrak{A} \otimes \mathfrak{B}$ is considered as a *-algebra whose involution and multiplication are determined by

2.2.7. $(a \otimes b)^* = a^* \otimes b^*$, $(a \otimes b)(c \otimes d) = ac \otimes bd$.

The algebra $(\mathfrak{A} \otimes \mathfrak{B})^2$ is identical to $\mathfrak{A}^2 \otimes B^2$ and is everywhere dense in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, and, if $a \in \mathfrak{A}$ and $b \in B$, then the mapping $x \longrightarrow (a \otimes b)x$ is continuous on $\mathfrak{A} \otimes B$. Therefore $\mathfrak{A} \otimes B$ is a prehilbert *-algebra.

Theorem 2.2.2. The tensor product of two modular Hilbert algebras is a modular Hilbert algebra

Proof. Consider modular Hilbert algebras $\mathfrak{A}$ and B, their Hilbert extensions $\mathfrak{H}_1$, $\mathfrak{H}_2$, and the related modular groups $\Gamma_1 = \{ \Delta_1^\alpha \}$ and $\Gamma_2 = \{ \Delta_2^\alpha \}$. For every complex number α, let $\Delta_1^\alpha \otimes \Delta_2^\alpha$ be a linear-algebraic automorphism of $\mathfrak{A} \otimes B$ such that

2.2.8. $$(\Delta_1^\alpha \otimes \Delta_2^\alpha)(a \otimes b) = \Delta_1^\alpha a \otimes \Delta_2^\alpha b.$$

We show that the automorphism group $\Gamma_1 \otimes \Gamma_2 = \{\Delta_1^\alpha \otimes \Delta_2^\alpha\}$ is a modular group of $\mathfrak{A} \otimes B$. To see this, it is sufficient to verify that, for each real number t, the set $(1 + \Delta_1{}^t \otimes \Delta_2{}^t)(\mathfrak{A} \otimes B)$ is everywhere dense in $\mathfrak{A}_1 \otimes B_2$. Let $\bar{\Delta}_1$ and $\bar{\Delta}_2$ be the self-adjoint extensions of the automorphisms Δ_1^1 and Δ_2^1. If t is a real number, then $(1 + \bar{\Delta}_1{}^t)^{-1}$, $\bar{\Delta}_1{}^t(1 + \bar{\Delta}_1{}^t)^{-1}$, $(1 + \bar{\Delta}_2{}^t)^{-1}$ and $\bar{\Delta}_2{}^t(1 + \bar{\Delta}_2{}^t)^{-1}$ are bounded Hermitian operators ≥ 0 whose ranges are everywhere dense in $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively.

We set

$$W = (1 + \bar{\Delta}_1{}^t)^{-1} \otimes (1 + \bar{\Delta}_2{}^t)^{-1} + \bar{\Delta}_1{}^t(1 + \bar{\Delta}_1{}^t)^{-1} \otimes \bar{\Delta}_2{}^t(1 + \bar{\Delta}_2{}^t$$

Then W is a bounded Hermitian operator whose range is everywhere dense in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and which satisfies

$$W((1 + \Delta_1^t)\mathfrak{A}_1 \otimes (1 + \Delta_2^t)B') = (1 + \Delta_1^t \otimes \Delta_2^t)(\mathfrak{A} \otimes B).$$

Then the sets $(1 + \Delta_1^t)\, \mathfrak{A} \otimes (1 + \Delta_2^t)\, \mathfrak{B}$ and $(M\, \Delta_1^t \otimes \Delta_2^t)$ $(\mathfrak{A} \otimes \mathfrak{B})$ are everywhere dense in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. Hence $\mathfrak{A} \otimes \mathfrak{B}$ is a modular Hilbert algebra with the modular group $\Gamma_1 \otimes \Gamma_2$.

Lemma 2.2.2. If $\mathfrak{A}$ and $\mathfrak{B}$ are two modular Hilbert algebras. Then

$$(L_{\mathfrak{A}\otimes\mathfrak{B}})'' = (L_{\mathfrak{A}})'' \otimes (L_{\mathfrak{B}})'', \quad (R_{\mathfrak{A}\otimes\mathfrak{B}})'' = (R_{\mathfrak{A}})'' \otimes (R_{\mathfrak{B}})''$$

Proof. From

2.2.9. $L_{a \otimes b} = L_a \otimes L_b$ for $a \in \mathfrak{A}$, $b \in \mathfrak{B}$,

We find that $L_{\mathfrak{A}\otimes\mathfrak{B}}$ is the least linear algebra $L_{\mathfrak{A}} \otimes L_{\mathfrak{B}}$ which contains the tensor products $L_a \otimes L_b$ of elements of $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$. Then we have

$$(L_{\mathfrak{A}\otimes\mathfrak{B}})'' = (L_{\mathfrak{A}})'' \otimes (L_{\mathfrak{B}}), \text{ and similarly}$$

$$(R_{\mathfrak{A}\otimes\mathfrak{B}})'' = (R_{\mathfrak{A}}) \otimes (R_{\mathfrak{B}})''.$$

§ 2.3. Modular standard algebras.

A von Neumann algebra M is said modular standard if M is spatially isomorphic to the extended left regular representation of a certain generalized Hilbert algebra. By Theorem 2.1.1 and 2.1.2 we obtain the next Theorem 2.3.1.

Theorem 2.3.1. A von Neumann algebra is modular standard if and only if it is the von Neumann extension of a certain separating cyclic *-algebra.

Theorem 2.3.2. Every von Neumann algebra is algebraically isomorhic to a certain modular standard algebra.

Proof. Let M be a von Neumann algebra. Theorem 1.3.4 implies that M has at least a generalized normal strictly positive functional p, and by Theorem 1.3.3 the corresponding extended normal cyclic representation of M is an algebraic isomorphism of M onto a certain modular standard algebra.

Theorem 2.3.3. (a). The commutant of a modular standard algebra M is modular standard. (b). If M is modular standard and if Z is a projection in the center of M, then M_Z is modular standard.

Proof. We can suppose that M is the von Neumann extension of a separating cyclic *-algebra (L, λ). Then M' is the von Neumann extension of a separating cyclic *-algebra (L^c, λ'), and M_z is the von Neumann extension of a separating cyclic *-algebra (L_z, λ_z). Then M' and M_z are modular standard algebras.

A von Neumann algebra M is said of reduced type if it is spatially isomorphic to a reduced algebra of a certain modular standard algebra. M is said of induced type if it is spatially isomorphic to an induced algebra of a certain modular standard algebra.

Theorem 2.3.4. A von Neumann algebra is of reduced type if and only if its commutant is of induced type.

Proof. Let N be a modular standard algebra and E be a projection in N. A von Neumann algebra M is spatially isomorphic to N_E if and only if M' is spatially isomorphic to N'_E. From this we obtain the Theorem.

Theorem 2.3.5. A von Neumann algebra M is spatially isomorphic to a direct sum of two von Neumann algebras which are of induct type and of reduced type.

We shall prove Theorem 2.3.5 combining Theorem 2.3.2 to the comparability theorem of the algebraic isomorphism. Let π be a normal homomorphism of a von Neumann algebra M onto another von Neumann algebra N. We say that π is an induction if M' contains a certain projection E such that the induced algbra M_E has a spatial isomorphism π_E onto N satisfying $\pi(A) = \pi_E(A_E)$ for all $A \in M$. An induction π of M onto N is an isomorphism if and only if E is generating in M'. Let π be an algbraic isomorphism of a von Neumann algebra M onto a von Neumann algebra N, and Z be a projection in the center of M. Then an algebraic isomorphism π_Z of M_Z onto $N_{\pi(Z)}$ is defined by $\pi_Z(A_Z) = \pi(A)_{\pi(Z)}$ for all $A \in M$. We call π_Z the subisomorphism of π restricted on M_Z.

Theorem 2.3.6. Let π be an algebraic isomorphism of a von Neumann algebra M onto a von Neumann algebra N. Then the center of M contains certain projections Z and $W = 1 - Z$ such that

π_Z and $(\pi_W)^{-1}$ are inductions.

Proof. Let $\mathfrak{H}$ and $\mathfrak{K}$ be the underlying Hilbert spaces of M and N and consider the direct sum $\mathfrak{H} \oplus \mathfrak{K}$ of $\mathfrak{H}$ and $\mathfrak{K}$ of which elements are denoted by $\langle x_1, x_2 \rangle$. We regard $\mathfrak{H}$ and $\mathfrak{K}$ as subspaces of $\mathfrak{H} \oplus \mathfrak{K}$ identifying every $x \in \mathfrak{H}$ to $\langle x, 0 \rangle$ and every $y \in \mathfrak{K}$ to $\langle 0, y \rangle$. Let H and K be projections defined on $\mathfrak{H} \oplus \mathfrak{K}$ whose ranges are $\mathfrak{H}$ and for every $A \in M$ we define an operator $\tau(A)$ on $\mathfrak{H} \oplus \mathfrak{K}$ by $\tau(A)\langle x, y \rangle = \langle Ax, \pi(A)y \rangle$. τ is an algebraic isomorphism of M onto a certain von Neumann algebra $\tau(M)$ on $\mathfrak{H} \oplus \mathfrak{K}$. H and K are elements of $\tau(M)'$, and by the comparability theorem (cf.[1],p228,Theorem 1) we may find projections Z and $W = 1 - Z$ in the center of M such that $\tau(Z)K \precsim \tau(Z)H$ and $\tau(W)H \precsim \tau(W)K$. We take projections H_0 and K_0 in $\tau(M)'$ and partially isometric operators U and V in $\tau(M)'$ such that $U^*U = H_0 \subset \tau(Z)H$, $UU^* = \tau(Z)K$, $V^*V = K_0$ $\subset \tau(W)K$ and $VV^* = \tau(W)H$. Notice that $M' = \tau(M')_H$ and $Z = \tau(Z)_H$. Then by (b) of Lemma 1.2.1 we find that $\tau(M)'_{H\tau(Z)}$ $= M'_Z$ and $E = H_{\theta_{H\tau(Z)}} \in M'_Z$. Similarly we obtain $F = K_{\theta_{K\tau(W)}}$ $\in N'_{\pi(W)}$. Let $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{M}_1$ and $\mathfrak{M}_2$ denote the ranges of the projections $Z, W, \pi(Z), \pi(W)$, E and F. Then they are simultaneously the ranges of the projections $\tau(Z)H, \tau(W)H$, $\tau(Z)K, \tau(W)K$, H_0 and K_0 respectively. The operator U transposes $\mathfrak{M}_1$ isometricly onto $\mathfrak{K}_1$, and if $A \in M$ then we have $UAU^{-1}y = \pi(A)y$ for all $y \in K_1$. We define a spatial isomorphism φ of $(M_Z)_E$ onto $N_{\pi(Z)}$ by $\varphi(X_E) = \pi_Z(X)$ for $X \in M_Z$. Then π_Z is an induction of M_Z onto $N_{\pi(Z)}$. Similarly it is verified that $(\pi_W)^{-1}$ is an induction of $N_{\pi(W)}$ onto M_W.

Proof of Theorem 2.3.5. Let M be a von Neumann algebra and consider an algebraic isomorphism π of M onto a certain modular standard algebra N. The center of M contains projections Z and $W = 1 - Z$ such that the isomorphisms π_Z and $(\pi_W)^{-1}$ are inductions. We set $M_1 = M_Z$, $M_2 = M_W$, $N_1 = N_{\pi(Z)}$, $N_2 = N_{\pi(W)}$, $\varphi = \pi_Z$ and $\psi = (\pi_W)^{-1}$. M is spatially isomorphic to the direct sum $M_1 \oplus M_2$, and N_1 and N_2 are modular standard algebras. We can find a projection F in N_2' and a spatial isomorphism ψ_F of N_{2F} onto M_2 such that $\psi(X) = \psi_F(X_F)$ for all $x \in N_2$. Then M_2 is a von Neumann algebra of induced type. Similarly there are a projection E in M_1' and a spatial isomorphism φ_E of M_{1E} onto N_1 such that $\varphi(X) = \varphi_E(X_E)$ for all $X \in M$. Then M_{1E} is a modular standard algebra spatially isomorphic to N_1, and E is a nondegenerate projection in M'. If the next Lemma 2.3.1 is proved, then M_1 is of reduced type, and Theorem 2.3.5 is proved.

Lemma 2.3.1. A von Neumann algebra M is of reduced type if M' contains a certain generating projection E such that M_E is modular standard.

Proof. Let $\mathfrak{H}$ be the underlying Hilbert space of M. M_E and M'_E are modular standard algebras, and M'_E is the von Neumann extension L'' of a certain maximal separating cyclic *-algebra (L, λ) on the range $E\mathfrak{H}$ of E. M'_E has a generalized normal strictly positive functional p_0 which is defined by $p_0(A^*A) = \|\lambda(A)\|^2$ for $A \in L$, and L is the integrable part of M_E'.

By Theorem 1.3.4 we extend p_0 to a generalized normal strictly positive functional p on M' such that $p_0(A_E) = p(EAE)$ and $p(EAE) \leq p(A)$ for all $A \in M'$. Consider the algebraic isomorphism π of M' on the von Neumann extension K'' of a certain maximal separating cyclic *-algebra (K, μ) on a Hilbert space K which corresponds to p. Then π carries the integrable part $M'(p)$ onto K, and satisfies $p(A^*A) = \|\mu(\pi(A))\|^2$ for all $A \in M'(p)$. By Lemma 1.3.1 K' contains an Hermitian operator T^2 which represents the positive functional $p(EAE)$ in the form

$$p(EA^*AE) = (T^2\mu(\pi(A)),\ \mu(\pi(A))) \quad \text{for } A \in M'(p).$$

From the relation

$$p_0(A_E{}_*A_E) = p(EA^*EAE) \leq p(EA^*AE),$$

we find that the reduction $A \longrightarrow A_E$ carries $M'(p)$ onto L and we have for $A \in M'(p)$ and $B \in M'(p)$,

$$\|\lambda(A_E)\|^2 = \|T\mu(\pi(EA))\|^2$$

and

$$(\lambda(A_E),\ \lambda(B_E)) = (T\mu(\pi(EA)),\ T\mu(\pi(EB))).$$

Let A, B, X and Y be elements of $M'(p)$. From the identity $EB^*A\lambda(X_E) = \lambda((B^*AEX)_E)$ it is calculated that

$$(A\lambda(X_E),\ B\lambda(Y_E)) = (\lambda(B^*AEX)_E,\ \lambda(Y_E))$$
$$= (T\mu(\pi(AEX)),\ T\mu(\pi(BEY))).$$

The linear set $L\lambda(L_E)$ is uniformly dense in $\mathfrak{H}$ since $\lambda(L_E)$ is everywhere dense in $E\mathfrak{H}$ and since $L''E\mathfrak{H}$ is identical to $ME\mathfrak{H}$

and is uniformly dense in $\mathfrak{h}$. Then the mapping $A\lambda(X_E) \longrightarrow T_\mu(\pi(AEX))$ is extended to an isometry U which carries $\mathfrak{h}$ onto a certain closed subspace $\mathfrak{M}$ of K. Let F denote the projection of K onto $\mathfrak{M}$. F is an element of the algebr K' such that $\pi(A)_G = UAU^{-1}$ for all $A \in M$. M' is of induce type since it is spatially isomorphic to the induced algebra $\pi(M')_G$ of the modular standard algebra $\pi(M')$. So that M is an algebra of reduced type. Lemma 2.3.1 and Theorem 2.3.5 are thus proved

§ 2.4. The fundamental equivalence theorem.

Two generalized Hilbert algebras $\mathfrak{A}$ and $\mathfrak{B}$ are said isomorphic to each other if there is a certain isometric isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. Two generalized Hilbert algebras are said equivalent if their maximal extensions are isomorphic to each other. We call the next Theorem 2.4.1 the Fundamental Equivalence Theorem.

Theorem 2.4.1. Every generalized Hilbert algebra is equivalent to a certain modular Hilbert algebra.

Theorem 2.4.1 will be proved in the next section. In this subsection we describe some applications of Theorem 2.4.1 assuming the validity. Theorem 2.4.1 is obviously equivalent to the next Corollary 1.

Corollary 1. Every maximal generalized Hilbert algebra is the maximal extension of a certain modular Hilbert algebra.

By Corollary 1, the extended left regular representaion of a generalized Hilbert algebra is always the extended regular representation of a certain modular Hilbert algebra. Then we obtain

Corollary 2. A von Neumann algebra is modular standard if and only if M is spatially isomorphic to the extended left regular representation of a certain modular Hilbert algebra.

Let $\mathfrak{A}$ be a maximal generalized Hilbert algebra and $\mathfrak{B}$ be its modular Hilbert subalgebra whose maximal extension is $\mathfrak{A}$. Let $\Gamma = \{ \Delta^{\alpha} \}$ be the modular group of $\mathfrak{B}$, and $\mathfrak{H}$ be the Hilbert extension of $\mathfrak{A}$. The self adjoint extension $\overline{\Delta}$ of Δ' is called the modular operator associated with $\mathfrak{A}$. The involution J, and the transpose $X \longrightarrow X^T$, associated with $\mathfrak{B}$, is called the involution and the transpose associated with the algebra $\mathfrak{A}$.

Theorem 2.4.2. Let $\mathfrak{A}$ be a maximal generalized Hilbert algebra. Then the associated modular operator $\overline{\Delta}$, the involution J and the transpose $X \longrightarrow X^T$ are invariants of $\mathfrak{A}$. $X \longrightarrow X^T$ induces an antiisomorphism between $L_{\mathfrak{A}}$ and $R_{\mathfrak{A}^c}$, and between $(L_{\mathfrak{A}})''$ and $(R_{\mathfrak{A}^c})$.

Proof. Let $\mathfrak{H}$ be the Hilbert extension of $\mathfrak{A}$, and $\mathfrak{B}$ be a modular Hilbert subalgebra of $\mathfrak{A}$ whose maximal extension is $\mathfrak{A}$. Let $\Gamma = \{\Delta^{\alpha}\}$ be the modular group of $\mathfrak{B}$ and $\overline{\Delta}$ be the self adjoint extension of Δ'. We show that $\overline{\Delta}$ does not depend on the choice of $\mathfrak{B}$. Let $\mathfrak{M}$ be the set of all elements a of $\mathfrak{A}^c$ such that a^s belongs to $\mathfrak{A}$. Reffering the identities 2.2.1 and 2.2.6 every $x \in \mathfrak{M}$ and $a \in \mathfrak{B}'$ satisfies

$$(x^{s*}, a) = (x, a^{s*}) = (x, \Delta a)$$

and $x^{s*} = \overline{\Delta} x$.

Then $\bar{\Delta}$ is a unique self-adjoint operator in $\mathfrak{H}$ satisfying $\bar{\Delta} x = x^{s*}$ for $x \in \mathfrak{M}$. By 2.2.6 the involution J is determined by $Jx = \bar{\Delta}^{\frac{1}{2}} x^s$ for $x \in \mathfrak{A}^c$. The operators $\bar{\Delta}$, J and the transpose $X \longrightarrow X^T$ are therefore the invariants of $\mathfrak{A}$. Since $X \longrightarrow X^T$ transposes $L_{\mathfrak{A}}$ onto $R_{\mathfrak{A}^c}$ and $(L_{\mathfrak{A}})''$ onto $(R_{\mathfrak{A}^c})$, we obtain the Theorem. From Theorem 2.4.2 we obtain immediately:

Corollary of Theorem 2.4.2. A modular standard algebra is anti-isomorphic to its commutant.

Theorem 2.4.3. The tensor product $M \otimes N$ of modular standard algebras M and N is a modular standard algebra which satisfies the commutation relation $(M \otimes N)' = M' \otimes N'$.

Proof. We can assume that M and N are extended left regular representations of modular Hilbert algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then by Lemma 2.2.3 $M \otimes N$ and $M' \otimes N'$ are the extended left and right regular representation of $\mathfrak{A} \otimes \mathfrak{B}$. Then $M \otimes N$ is a modular standard algebra satisfying $(M \otimes N)' = M' \otimes N'$.

Theorem 2.4.4. If M and N are von Neumann algebras, then we have $(M \otimes N)' = M' \otimes N'$.

Proof. A von Neumann algebra is spatially isomorphic to a direct sum of von Neumann algebras of induced type and of reduced type. Then it is sufficient to verify the theorem in the following three cases. (a). M and N are of induced types. (b). M and N are of reduced types. (c). M is of induced type and N is of reduced type.

In the case (a), we can assume that M and N are induced algebras $M = M_{0E}$ and $N = N_{0F}$ of modular standard algebras M_0 and N_0. E and F are projections in M_0' and N_0'. Then $E \otimes F$ is a projection in $M_0' \otimes N_0' = (M_0 \otimes N_0)'$ such that

$$M \otimes N = (M_0 \otimes N_0)_{E \otimes F}$$

and

$$\begin{aligned} M' \otimes N' &= (M_0)'_E \otimes (N_0')_F = (M_0' \otimes N_0')_{E \otimes F} \\ &= (M_0 \otimes N_0)'_{E \otimes F} = ((M_0 \otimes N_0)_{E \otimes F})'. \\ &= (M \otimes N)', \end{aligned}$$

In the case (b), M' and N' are of induced type and it is reduced to the case (a). In the case (c). We can assume that M is a induced algebra M_{0E}, and N is a reduced algebra N_{0F}, of modular standard algebras M_0 and N_0. Since E is a projection in M_0', $E \otimes 1$ is a projection in $M_0' \otimes N_0' = (M_0 \otimes N_0)'$. Similarly F is a projection in N_0 and $1 \otimes F$ is a projection in $M_0 \otimes N_0$. Then remarking that $E \otimes F = (E \otimes 1)(1 \otimes F)$ we have $((M \otimes N)_{E \otimes F})' = (M \otimes N)'_{E \otimes F}$. Since $(M \otimes N)_{E \otimes F} = M \otimes N$ and $(M \otimes N)'_{E \otimes F} = M'_E \otimes N'_F = M' \otimes N'$ we have $(M \otimes N)' = M' \otimes N'$.

§3. Structures of generalized Hilbert algebras.

3.1. Extended multiplications and involutions.

Throughout this section we let $\mathfrak{A}$ denote a generalized Hilbert algebra, $\mathfrak{A}^c$ the commutant of $\mathfrak{A}$, and $\mathfrak{H}$ the Hilbert extension of $\mathfrak{A}$. We extend the multiplication in $\mathfrak{A}$ to a certain multiplication between elements of $\mathfrak{A}$, $\mathfrak{H}$ and $\mathfrak{A}^c$. Let a be an element of $\mathfrak{A}$, k be an element of $\mathfrak{A}^c$ and x be an element of $\mathfrak{H}$. Then ax and xk are defined by

3.1.1. $ax = L_a x, \qquad xk = R_k x.$

Since

3.1.2. $ak = L_a k = R_k a,$

two multiplications in 3.1.1 are identical whenever they are defined simultaneously.

Lemma 3.1.1. $\mathfrak{A}^c$ is a linear algebra with an involution $k \longrightarrow k^s$ if we define the multiplication in $\mathfrak{A}^c$ as the restriction of the multiplication 3.11 in $\mathfrak{A}^c$. $k \longrightarrow R_k$ is a *-algebraic anti-isomorphism of $\mathfrak{A}^c$ onto $R_{\mathfrak{A}^c}$.

Proof. Consider the underlying generalized Hilbert algebra $\widehat{\mathfrak{A}}^c$ of the separating cyclic *-algebra $(R_{\mathfrak{A}^c}, R_k \longrightarrow k)$. $\widehat{\mathfrak{A}}^c$ and $\mathfrak{A}^c$ are the same subspace of $\mathfrak{H}$. However, the multiplications in $\widehat{\mathfrak{A}}^c$ and $\mathfrak{A}^c$ are antiisomorphic since the multiplication $k \circ m$ in $\widehat{\mathfrak{A}}^c$ is defined by

3.1.3 $\qquad k \circ m = R_k m = mk.$

Then $k \longrightarrow R_k$ is an antiisomorphism of $\mathfrak{A}^c$ onto $R_{\mathfrak{A}^c}$.

Let a and b be elements of $\mathfrak{A}$, k and m be elements of $\mathfrak{A}^c$, and x and y be elements of $\mathfrak{H}$. The extended multiplication satisfies obviously the following relations.

3.1.4. ax is bilinear with respect to a and x. xk is bilinear with respect to x and k

3.1.5. $(ak)k = a(xk)$

3.1.6. $(ab)x = a(bx)$, $(xk)m = x(km)$

3.1.7. $(ax, y) = (x, a^*y)$, $(xk, y) = (x, yk^s)$.

3.1.8. $(a, k) = (k^s, a^*)$

3.1.9. $\|ax\| \leq \|L_a\| \ \|x\|$, $\|xk\| \leq \|x\| \|R_k\|$.

If an element x of $\mathfrak{H}$ is adjointive with respect to the algebra $(L_{\mathfrak{A}}, L_a \longrightarrow a)$, then x is said s-adjointive and the adjoint of x is called the s-adjoint of x. The s-adjoint of an element x of $\mathfrak{H}$ is determined as an element x of $\mathfrak{H}$ satisfying

3.1.10. $(x, a) = (a^*, x^s)$ for $a \in \mathfrak{A}$.

The set $\mathfrak{H}^s$ of all s-adjointive elements of $\mathfrak{H}$ is a Hilbert space whose innerproduct $(x, y)_s$ and the norm $\|x\|_s$ are defined by

3.1.11. $(x, y)_s = (y^s, x^s) = (x, y) = (y^s, x^s)$,

3.1.12. $\|x\|_s^2 = (x, x)_s = \|x\|^2 + \|x^s\|^2$.

The uniform topology of the Hilbert space $\mathfrak{H}^s$ is called the s-cyclic topology. $\mathfrak{A}^c$ is an s-cyclicly dense subset of $\mathfrak{H}^s$.

A linear algebra $\mathfrak{B}$ of $\mathfrak{A}^c$ is called an s-sübalgebra if it is invariant under the involution $x \longrightarrow x^s$. If $\mathfrak{B}$ is an s-subalgebra of $\mathfrak{A}^c$, then $R_{\mathfrak{B}} = (R_k : k \in \mathfrak{B})$ is a *-subalgebra of $R_{\mathfrak{A}^c}$. $R_{\mathfrak{B}}$ is cyclicly dense in $R_{\mathfrak{A}^c}$ if and only if $\mathfrak{B}$ is s-cyclicly dense in $\mathfrak{H}^s$. If an element x of $\mathfrak{H}$ is adjointive with respect to the algebra $(R_{\mathfrak{A}^c}, R_k \longrightarrow k)$, then x is said *-adjointive, and the adjoint of x is said the *-adjoint of x. The *-adjoint of an element x of $\mathfrak{H}$ is determined as an element x^* of $\mathfrak{H}$ satisfying

3.1.12. $\qquad (x, k) = (k^s, x^*) \qquad$ for $k \in \mathfrak{A}^c$.

The set $\mathfrak{H}^*$ of all *-adjointive elements of $\mathfrak{H}$ is a Hilbert space whose innerproduct $(x, y)_*$ and norm $\|x\|_*$ are defined by

3.1.13. $\qquad (x, y)_* = (y^*, x^*) = (x, y) + (y^*, x^*),$

3.1.14. $\qquad \|x\|_*^2 = (x, x)_* = \|x\|^2 + \|x^*\|^2.$

The uniform topology of the Hilbert space $\mathfrak{H}^*$ is called the *-cyclic topology. $\mathfrak{A}$ is a *-cyclicly dense subset of $\mathfrak{H}^*$.

Lemma 3.1.3. Let $\mathfrak{B}$ a *-subalgebra of $\mathfrak{A}$ which is uniformly dense in $\mathfrak{H}$. Then $\mathfrak{B}$ is a generalized Hilbert algebra. $\mathfrak{A}$ is the maximal extension of $\mathfrak{B}$ if and only if $\mathfrak{B}$ is *-cyclicly dense in $\mathfrak{A}$.

Proof. The left regular representation $L_{\mathfrak{B}} = (L_b : b \in \mathfrak{B})$ of $\mathfrak{B}$ is a *-subalgebra of $L_{\mathfrak{A}}$, which has a cyclic mapping $L_b \longrightarrow b$. since $(L_{\mathfrak{B}})^c$ is a nondegenerate *-algebra which contains $(L_{\mathfrak{A}})^c$, by Lemma 1.1.6 $(L_{\mathfrak{B}}, L_b \longrightarrow b)$ is a separating cyclic *-algebra whose underlying algebra is $\mathfrak{B}$. $\mathfrak{B}$ is *-cyclicly dense in $\mathfrak{H}^*$ if and only if $L_{\mathfrak{B}}$ is cyclicly dense in $L_{\mathfrak{A}}$... Therefore $\mathfrak{B}$ is *-cyclicly dense in $\mathfrak{A}$ if and only if the maximal extension $\mathfrak{B}^{cc}$ of $\mathfrak{B}$ is identical with $\mathfrak{A}$.

Lemma 3.1.4. An element x of $\mathfrak{H}^*$ belongs to $\mathfrak{A}$ if and only if there is a constant γ satisfying

3.1.15. $\quad \|xk\| \leq \gamma \|k\| \quad$ for $k \in \mathfrak{A}^c$.

An element x of $\mathfrak{H}^s$ belongs to ^{c} if and only if there is a constant γ satisfying

3.1.16. $\quad \|ax\| \leq \gamma \|a\| \quad$ for $a \in \mathfrak{A}$.

Proof. Let a be an element of $\mathfrak{H}^*$. If there is a constant γ satisfying 3.1.15. we find a bounded operator L_a on $\mathfrak{H}$ such that

$$L_x k = xk = R_k x \qquad \text{for } k \in \mathfrak{A}^c.$$

From

$$(xm, k) = (m, x^*k) \qquad \text{for } m \in \mathfrak{A}^c, \; k \in \mathfrak{A}^c,$$

We find that $(L_x{}^*)k = R_k x^*$. Then x is an element of $\mathfrak{A}$.

Let x be an element of $\mathfrak{H}^*$. By Theorem 1.5.2 there is a certain closed operator $L_x \eta (L_{\mathfrak{A}}, L_a \longrightarrow a)$ satisfying

$$L_x k = R_k x, \; (L_x)^* k = R_k x^* \qquad \text{for } k \in \mathfrak{A}^c.$$

Consider the incidental *-algebra $C_0(L_x)$ of L_x. By Theorem 1. 5.1. $C_0(x)$ is an ablian *-subalgebra of $L_{\mathfrak{A}}$, and is isomorphic to a certain *-subalgebra $\mathfrak{A}(x)$ of $\mathfrak{A}$ by the mapping $L_a \longrightarrow a$. We call $\mathfrak{A}(x)$ a *-algebra incidental to x.

Lemma 3.1.7. Let x be an element of $\mathfrak{z}^*$ and $\mathfrak{A}(x)$ be a *-algebra incidental to x. $\mathfrak{A}(x)$ is an ablian *-subalgebra of $\mathfrak{A}$. The mapping $a \longrightarrow ax$ carries $\mathfrak{A}(x)$ into $\mathfrak{A}$ and satisfies $((a^*ax)^*, x^*) \geq 0$. Suppose that there is a constant γ satisfying

3.1.17. $((a^*ax)^*, x^*) \leq \gamma \|a\|$ for $a \in \mathfrak{A}(x)$.

Then x is an element of $\mathfrak{A}$ such that $\|L_x\| \leq \gamma$.

Proof. By (b) of Theorem 1.5.1, $L_a \longrightarrow L_a L_x$ carries $C_0(L_x)$ into $L_{\mathfrak{A}}$. Since 1.4.7 implies $L_a L_x = L_{ax}$ for $a \in \mathfrak{A}(x)$, the mapping $a \longrightarrow ax$ carries $\mathfrak{A}(x)$ into $\mathfrak{A}$ and satisfies $((a^*ax)^*, x^*) \geq 0$ for $a \in \mathfrak{A}(x)$. If γ is a constant which satisfies 3.1.17, then by (d) of Theorem 1.5.1 L_x is an element of $L_{\mathfrak{A}}$ satisfying $\|L_x\| \leq \gamma$.

If x is an element of $\mathfrak{z}^s$, then we define a *-algebra $\mathfrak{A}^c(x)$ incidental to x which satisfies the next Lemma 3.1.8 as an analogy of Lemma 3.1.7.

Lemma 3.1.8. Let x be an element of $\mathfrak{z}^s$. Then $\mathfrak{A}^c$ contains an ablian *-algebra $\mathfrak{A}^c(x)$ which satisfies the following conditions. $k \longrightarrow xk$ carries $\mathfrak{A}^c(x)$ into $\mathfrak{A}^c$ and satisfies $((xkk^s)^s, x^s) \geq 0$ for $k \in \mathfrak{A}^c(x)$. Suppose that there is a constant γ such that

3.1.18. $((xk\,k^s)^s, x^s) \leq \gamma \|k\|$ for $k \in \mathfrak{A}^c(x)$.

Then x is an element of $\mathfrak{A}^c$ satisfying $\|R_x\| \leq \gamma$.

§3.2. The existense of the modular operator.

The modular operator and the involution which we have defined in Section 2 is based on the assumption of Theorem 2.4.1.

Since our present problem is to prove this theorem we have to define these operators in a different way.

Theorem 3.2.1. (a). Let Δ be an operator in $\mathfrak{H}$ such that

3.2.1. $\Delta x = x^{s*}$,

and let the domain of Δ be the totality of $x \in \mathfrak{H}$ such that x^{s*} is defined. Then Δ and Δ^{-1} are positive selfadjoint operators in $\mathfrak{H}$. The domain of Δ^{-1} is the totality of $x \in \mathfrak{H}$ such that x^{*s} is defined, and we have

3.2.2. $\Delta^{-1}x = x^{*s}$.

(b). There is a certain reflexive conjugate linear isometry J of $\mathfrak{H}$ satisfying

3.2.3. $x^* = \Delta^{\frac{1}{2}}Jx = J\,\Delta^{-\frac{1}{2}}x$ for $x \in \mathfrak{H}^*$,

3.2.4. $x^s = \Delta^{-\frac{1}{2}}Jx = J\,\Delta^{\frac{1}{2}}x$ for $x \in \mathfrak{H}^s$.

We call the operator Δ the modular operator, and J the involution, associated with $\mathfrak{A}$. To prove Theorem 3.2.1 we prepare the next lemma 3.2.1.

Lemma 3.2.1. If x is an element of $\mathfrak{H}^*$ and y is an element of $\mathfrak{H}^s$, then

3.2.5. $(x, y) = (y^s, x^*)$.

Proof. $\mathfrak{A}$ is $*$-cyclicly dense in $\mathfrak{H}^*$ and contains a sequence $\{a_n\}$ satisfying $a_n \longrightarrow x$ and $a_n^* \longrightarrow x^*$ in $\mathfrak{H}$. $\mathfrak{A}^c$ is s-cyclicly dense in $\mathfrak{H}^s$ and contains a sequence $\{k_n\}$ satisfying $k_n \longrightarrow y$ and $k_n^s \longrightarrow y^s$ in $\mathfrak{H}$. From the identity (a_n, k_n)

$= (k_n{}^s, a_n{}^*)$ we obtain 3.2.5.

Proof of Theorem 3.2.1. We reffer 3.1.11 which defines the innerproduct of $\mathfrak{H}^s$, and define an Hermitian operator $T \geq 0$ in the Hilbert space $\mathfrak{H}^s$ by

3.2.6. $\qquad (T x, y)_s = (x, y), \; ((1 - T)x, y) = (y^s, x^s).$

Then the involution $x \longrightarrow x^s$ in $\mathfrak{H}^s$ satisfies

3.2.7. $\qquad (T x^s)^s = (1 - T)x.$

Notice that $\| T^{\frac{1}{2}} x \|_s = \| x \|$, where the range of $T^{\frac{1}{2}}$ is s-cyclicly dense in $\mathfrak{H}^s$ and $\mathfrak{H}^s$ is uniformly dense in $\mathfrak{H}$. Then the mapping $x \longrightarrow T^{\frac{1}{2}} x$ is extended to a certain isometry U of the Hilbert space $\mathfrak{H}$ onto the Hilbert space $\mathfrak{H}^s$. We set $S = U^{-1}TU$. Then S is an Hermitian in H such that $S \geq T$ and $S^{\frac{1}{2}} = U \geq T^{\frac{1}{2}}$. We define an involution J on $\mathfrak{H}$ by

3.2.8. $\qquad Jx = U^{-1}(Ux)^s.$

We consider the transpose $X \longrightarrow X^T = JX^*J$ associated with J. Then $X \longrightarrow X^T$ is an antiautomorphism of the total operator algebra $B(\mathfrak{H})$ on $\mathfrak{H}$. Since 3.2.7 implies

3.2.9. $\qquad S^T = 1 - S,$

setting $\quad \Delta = S^{-1}(1 - S)$ we have

3.2.10. $\qquad \Delta^T = \Delta^{-1}, \quad (\Delta^{\frac{1}{2}})^T = \Delta^{-\frac{1}{2}}$.

and by 3.2.7 $x \in \mathfrak{H}^s$ implies

3.2.11. $\qquad x^s = S^{\frac{1}{2}} J S^{-\frac{1}{2}} x = \Delta^{-\frac{1}{2}} Jx = J \Delta^{\frac{1}{2}} x.$

$\mathfrak{H}^s$ is identical with $\mathcal{D}(\Delta^{\frac{1}{2}})$ as Hilbert spaces, since they are the range of $S^{\frac{1}{2}}$ and they satisfy

3.2.12. $\|x\|_s^2 = \|x\|^2 + \|\Delta^{\frac{1}{2}}x\|^2.$

By Lemma 3.2.1, an element x of $\mathfrak{H}$ has the *-adjoint x* if and only if x* is an element of $\mathfrak{H}$ satisfying

$$(Jx^*, y) = (x, (Jy)^s) = (x, \Delta^{-\frac{1}{2}} y).$$

for all $y \in \mathcal{D}(\Delta^{-\frac{1}{2}})$. Then $\mathfrak{H}^*$ is the domain of $\Delta^{-\frac{1}{2}}$. $\mathfrak{H}^*$ and $\mathcal{D}(\Delta^{-\frac{1}{2}})$ are identical as Hilbert spaces since

3.2.13. $\|x\|_*^2 = \|x\|^2 + \|\Delta^{-\frac{1}{2}}x\|^2$ for $x \in \mathfrak{H}^*$.

The *-adjoint x* is therefore determind by 3.2.3 If ω is a non-zero non-positive 1 complex number, we use the following notations.

3.2.14. $\gamma(\omega) = (2|\omega| - \omega - \overline{\omega})^{-\frac{1}{2}}$,

3.2.15. $B_\omega = (\Delta - \omega)^{-1}.$

Then we have $(B_\omega)^T = (\Delta^{-1} - \omega)^{-1}.$

Theorem 3.2.2. Let ω be a non-zero non-positive complex number. Then $a \longrightarrow B_\omega a$ carries $\mathfrak{A}$ into $\mathfrak{A}^c$, and setting $b = B_\omega a$ we have

3.2.16. $\|R_b\| \leq \gamma(\omega) \|L_a\|.$

$k \longrightarrow R_\omega^T k$ carries $\mathfrak{A}^c$ into $\mathfrak{A}$, and setting $m = B_\omega^T k$ we have

3.2.17. $\|L_m\| \leq \gamma(\omega) \|R_k\|$.

Proof. Let x be an element of $\mathcal{D}(\Delta)$, and consider the s-subalgebra $\mathfrak{A}^c(X)$ of $\mathfrak{A}^c$ incidental to x. For every $k \in \mathfrak{A}^c$ we have

$$\|((\Delta - \omega)x)k\|^2$$
$$= \|(\Delta x)k\|^2 + |\omega|^2\|xk\|^2 - 2\ \mathrm{Re}\ (\omega xk,\ (\Delta x)k).$$

By Lemma 3.1.8 we have

$$(xk,\ (\Delta x)k) = (xkk^s,\ x^{s*})$$
$$= (x^s,\ (xkk^s)^s) \geq 0.$$

Then we have

$$\|(\Delta x)k\|^2 + |\omega|^2\|xk\|^2 \geq 2|\omega|\ (xk,\ (\Delta x)k),$$

and

$$\|((\Delta - \omega)x)k\|^2 \geq (2|\omega| - \omega - \bar{\omega})(x^s,\ (xkk^s)^s).$$

Let a be an element of $\mathfrak{A}$ and set $b = R_\omega a$. Then from $a = (\Delta - \omega)b$. we have

$$(b^s,\ (bkk^s)^s) \leq \gamma(\omega)^2\ \|L_a\|^2\|k\|^2.$$

Then by Lemma 3.1.8 b is an element of $\mathfrak{A}^c$ satisfying 3.2.16.

3.3. Analytic forms of modular operators.

We consider the complex Riemann sphere K_0 which consists of the complex number field K and the infinity point, and let $[0, \infty]$ denote the extended positive half line $(z : 0 \leq z \leq \infty)$. An analytic function f which is defined in a certain open set U_f in K_0 containing $[0, \infty]$ is said an analytic function on $[0, \infty]$. Two analytic functions on $[0, \infty]$ are regarded to be identical if they are identical in a certain region containing $[0, \infty]$.

Then we can assume that U_f is a region containing $[0,\infty]$ whose boundary Γ_f is rectifiable , and that f is a continuous function on $\Gamma_f \cup U_f$. We let $A[0,\infty]$ denote the set of all analytic functions on $[0,\infty]$ which vanishes at ∞. Then every function f in $A[0,\infty]$ is represented by an integral

3.3.1 $$f(\lambda) = \frac{-1}{2\pi i}\int_{\Gamma_f}(\lambda - w)^{-1} f(w)\,dw,$$

where the integral surrounds U_f to the positive dirrection. The analytic form $f(\Delta)$ of Δ is defined by

3.3.2. $$f(\Delta) = \frac{-1}{2\pi i}\int_{\Gamma_f} f(w)\, B_w\, dw.$$

Theorem 3.3.1. Let f be a function in $A[0,\infty]$. Then $a \longrightarrow f(\Delta)a$ carries $\mathfrak{A}$ into $\mathfrak{A}^c$, and $k \longrightarrow f(\Delta^{-1})k$ carries $\mathfrak{A}^c$ into $\mathfrak{A}$.

Proof. If b is an element of $\mathfrak{A}$ then

$$b(f(\Delta)a) = \frac{-1}{2\pi i}\int_{\Gamma_f} f(w)\, b(B_w a)\, dw .$$

Let γ be the length of Γ_f, and c be the supremum of the function $\gamma(w)|f(w)|$ on Γ_f. Then

$$\| b(f(\Delta)a)\| \leq (2\pi)^{-1} c\, \Gamma \| b\| \quad \text{for} \quad b \in \mathfrak{A}.$$

$zf(z)$ is analytic on $[0,\infty]$ and $\Delta f(\Delta)$ a is defined. Then $f(\Delta)a$ is an element of $\mathfrak{H}^s = \mathfrak{D}(\Delta^{\frac{1}{2}})$, and hence by Lemma 3.1.4. $f(\Delta)a$ belongs to $\mathfrak{A}^c$. Analogously, we find that $k \in \mathfrak{A}^c$ implies $f(\Delta^{-1})k \in \mathfrak{A}$.

Theorem 3.3.2. Let $\mathfrak{A}^s$ be the totality of $a \in \mathfrak{A}$ such that $\Delta a \in \mathfrak{A}$. $\mathfrak{A}^s$ is an s-cyclicly dense s-subalgebra of $\mathfrak{A}^c$. Let

$\mathfrak{A}^{c*}$ be the totality of $k \in \mathfrak{A}^c$ such that $\Delta^{-1}k \in \mathfrak{A}^c$. Then $\mathfrak{A}^{c*}$ is a *-cyclicly dense *-subalgebra of $\mathfrak{A}$.

Proof. If a is an element of $\mathfrak{A}^s$, then $b = a + \Delta a$ belongs to $\mathfrak{A}$ and $a = B_{-1}b$ belongs to $\mathfrak{A}^c$. Hence $\mathfrak{A}^s$ is a subset of $\mathfrak{A}^c$. Let a and b be elements of $\mathfrak{A}^s$. From $a^s = (\Delta a)^*$ and $\Delta(a^s) = a^*$ we find that a^s and $\Delta(a^s)$ are elements of $\mathfrak{A}$, and a^s belongs to $\mathfrak{A}^s$. Also, $(ab)^{s*} = a^{s*}b^{s*}$ implies

3.3.3. $\Delta(ab) = (\Delta a)(\Delta b)$.

Since ab and $\Delta(ab)$ are elements of $\mathfrak{A}$, ab belongs to $\mathfrak{A}^s$. Therefore $\mathfrak{A}^s$ is an s-subalgebra of $\mathfrak{A}^c$. To see that $\mathfrak{A}^s$ is s-cyclicly dense in $\mathfrak{A}^c$ and in $\mathfrak{H}^s$, Remarking $\mathfrak{H}^s = \mathfrak{D}(\Delta^{\frac{1}{2}})$. and Lemma 1.4.1, it is sufficient to show that $(1 + \Delta^{\frac{1}{2}})\mathfrak{A}^s$ is uniformly dense in $\mathfrak{H}$. Consider the functions $z(1 + z^2)^{-1}$ and $(1 + z^2)^{-1}$ in $A[0,\infty]$, and take an element k of $\mathfrak{A}^c$. Since $\Delta^{-1}(\Delta^{-2} + 1)^{-1}k$ and $(\Delta^{-2} + 1)^{-1}k$ are elements of $\mathfrak{A}$, $\Delta^{-1}(\Delta^{-2} + 1)^{-1}k$ is an element of $\mathfrak{A}^s$. Then $\Delta^{-1}(\Delta^{-2} + 1)^{-1}\mathfrak{A}^c$ is a subset of $\mathfrak{A}^s$, and $(1 + \Delta^{\frac{1}{2}})\mathfrak{A}^s$ contains $(1 + \Delta^{\frac{1}{2}})\Delta^{-1}(\Delta^{-2} + 1)^{-1}\mathfrak{A}^c$. $(1 + \Delta^{\frac{1}{2}})\Delta^{-1}(\Delta^{-2} + 1)^{-1}$ is a bounded Hermitian operator whose range is dense in $\mathfrak{H}$, and $\mathfrak{A}^c$ is dense in $\mathfrak{H}$. Then $(1 + \Delta^{\frac{1}{2}})\Delta^{-1}(\Delta^{-2} + 1)^{-1}\mathfrak{A}^c$ and $(1 + \Delta^{\frac{1}{2}})\mathfrak{A}^s$ are everywhere dense in $\mathfrak{H}$, and $\mathfrak{A}^s$ is an s-cyclicly dense s-subalgebra of $\mathfrak{A}^c$. Analogously we find that $\mathfrak{A}^{c*}$ is a *-cyclicly dense *-subalgebra of $\mathfrak{A}$.

Lemma 3.3.1. If α is a complex number, then the intersection of $\mathfrak{A}^s$ and $\mathfrak{D}(\Delta^{\alpha})$ is everywhere dense in the Hilbert space $\mathfrak{D}(\Delta^{\alpha})$.

Proof. Let t be the real part of α, n be an integer such that $n \geq |t| + 1$ and f be the function $f(z) = z^n(z^{2n} + 1)^{-1}$. We show that $\Delta^{-1}f(\Delta^{-1})\mathfrak{A}^{c*}$ is contained in $\mathfrak{A}^{c*} \cap \mathcal{D}(\Delta^{\alpha})$ and is everywhere dense in $\mathcal{D}(\Delta^{\alpha})$. If k is an element of $\mathfrak{A}^{c*}$, then k and $\Delta^{-1}k$ belong to $\mathfrak{A}^{c}$, and $f(\Delta^{-1})k$ and $\Delta^{-1}f(\Delta^{-1})k$ belong to $\mathfrak{A}$. Then $\Delta^{-1}f(\Delta^{-1})k$ belongs to $\mathfrak{A}^{s}$, and $\Delta^{-1}f(\Delta^{-1})\mathfrak{A}^{c*}$ is contained in $\mathfrak{A}^{s}$. Notice that $(1 + \Delta^{t})\Delta^{-1}f(\Delta^{t})$ is a bounded Hermitian operator whose range is everywhere dense in $\mathfrak{h}$, and $\mathfrak{A}^{c*}$ is uniformly dense in $\mathfrak{h}$. Then $(1 + \Delta^{t})\Delta^{-1}f(\Delta^{-1})\mathfrak{A}^{c*}$, is everywhere dense in $\mathfrak{h}$. Hence $\mathfrak{A}^{s}\wedge\mathcal{D}(\Delta^{\alpha})$ is everywhere dense in $\mathcal{D}(\Delta^{\alpha})$.

Lemma 3.3.2. Let a be an element of $\mathfrak{A}^{s}$, x be an element of $\mathfrak{h}$, ω and η be numbers in $K_0 - [0,\infty]$ such that $\omega\eta$ belongs to $K_0 - [0,\infty]$. Then $B_{\omega}a$ is an element of $\mathfrak{A}^{s}$ satisfying

3.3.4. $$(B_{\omega}a)(B_{\eta}x) = R\omega\eta((\Delta B_{\omega}a)x + \eta a(B_{\eta}x)).$$

Proof. From

$$\Delta B_{\omega}a = \omega^{-1}(\Delta^{-1} - \omega^{-1})a, \quad B_{\omega}a = \omega^{-1}(a - \Delta B_{\omega}a),$$

we find that $B_{\omega}a$ belongs to a^{s}. Since $\mathfrak{A}^{s}$ is uniformly dense in $\mathfrak{h}$, it is sufficient to show 3.3.4 in the case that x belongs to $\mathfrak{A}^{s}$. We obtain

$$\Delta((B_{\omega}a)(B_{\eta}x)) = (\Delta B_{\omega}a)(\Delta B_{\eta}x)$$
$$= (a + \omega B_{\omega}a)(x + \eta B_{\eta}x),$$

and

$$(\Delta-\omega\eta)\ ((B_\omega a)(B_\eta x)) = (\Delta B_\omega a)x + \eta a(B_\eta x).$$

3.3.4 follows from the last identity.

Let γ be any number ≥ 1. We define a function φ_γ on $[0, \infty]$ by $\varphi_\gamma(x) = 1$ for $\frac{1}{\gamma} \leq x \leq \gamma$ and $\varphi_\gamma(x) = 0$ otherwise. Setting $E_\gamma = \varphi_\gamma\ (\Delta)$, we have $E_\gamma\Delta = \Delta E_\gamma$ $E_\gamma{}^T = E_\gamma$ and $\max(\|\Delta E_\gamma\|, \|\Delta^{-1}E_\gamma\|) \leq \gamma$, where $E_\gamma{}^T$ is the transpose $JE_\gamma{}^*J$ of E_γ. Let f be an analytic function on $[0, \infty]$. We consider the set $W(f) = (tz^{-1} : 0 \leq t \leq \infty,\ z \in \Gamma_f \cup U_f)$. If γ is a number ≥ 1, $[\frac{1}{\gamma}, \gamma]$ and W_f are disjoint compact subsets of K_0 we consider a certain rectifiable closed curve whose interior contains $[\frac{1}{\gamma}, \gamma]$, and whose exterior contains $W(f)$.

Lemma 3.3.1. Let x be an element of $\mathfrak{H}$, a be an element of $\mathfrak{A}^{c*}$, f be an analytic function on $[0, \infty]$, n be a number ≥ 1 and C be a rectifiable curve which devides W_f and $[\frac{1}{\gamma}, \gamma]$ to its exterior and interior and intersects with $[0, \infty]$ only at two points. Then $f(\Delta)a$ belongs to $\mathfrak{A}$ and

3.3.5. $$(f(\Delta)a)(E_\gamma x) = \frac{-1}{2\pi i} \int_C f(\eta^{-1}\Delta)(aB_\eta E_\gamma x)d\eta\ .$$

Proof. If η is not on the line $[0, \infty]$, then

$$B_\eta E_\gamma = (\Delta E_\gamma - \eta)^{-1} E_\gamma$$

and therefore

3.3.6. $$E_\gamma = \frac{-1}{2\pi i} \int_C B_\eta E_\gamma d\eta\ .$$

If the function f in the Lemma is a constant, then 3.3.5 follows immediately from 3.3.6. Any analytic function f on $[0,\infty]$ is a sum of a constant and a function in $A[0,\infty]$. Then it is sufficient to prove the Lemma supposing that f is an element of $A[0,\infty]$. In the identity 3.3.4, we suppose that ω is a fixed number in Γ_f and substitute x to $E_\gamma x$. Multiply $d\eta$ to the bothside of the identity and integrate along C to the positive direction. By 3.3.6 the left side of the identity is calculated by

$$(B\omega a)(E_\gamma x) = \frac{-1}{2\pi i} \int_C (B\omega a)(B\eta E_\gamma x)d\eta .$$

To calculate the right side we notice that $B\omega\eta$ is an analytic function of η on and in the interior of the curve C. Then we have

$$\frac{-1}{2\pi i} \int_C B\omega\eta ((\Delta B\omega a)(E_\gamma x))d\eta = 0$$

and

3.3.7. $$(B\omega a)(E_\gamma x) = \frac{-1}{2\pi i} \int_C B\omega\eta(\eta a B\eta E_\gamma x)d\eta$$

We multiply $f(\omega)d\omega$ to the both side of 3.3.7 and integrate the identity along the curve Γ_f. Then the left hand is calculated by

$$(f(\Delta)a)(E_\gamma x) = \frac{-1}{2\pi i} \int_{\Gamma_f} f(\omega)(B\omega a)(E_\gamma x)d\omega .$$

Notice that if η is a number in C, then

$$f(\eta^{-1}t) = \frac{-1}{2\pi i} \int_{\Gamma_f} f(\omega)\, \eta (t - \omega\eta)^{-1} d\omega \quad \text{for } t \in [0,\infty].$$

Therefore

$$f(\eta^{-1}\Delta) = \frac{-1}{2\pi i} \int_{\Gamma_f} f(\omega)\, \eta\, B\omega\eta\, d\omega .$$

Hence we obtain 3.3.5.

3.4. Formutation of a modular Hilbert subalgebra

Let W be the set $(-Z : Z \in [0, \infty])$, and for every complex number α we define an analytic function Z^{α} on $K_0 - W$ by

3.4.1. $Z^{\alpha} = \mathrm{Exp}(\alpha \log|Z| + i\,\alpha \arg Z)$.

where we assume $-\pi < \arg Z < \pi$. Then we have $Z^{\alpha+\beta} = Z^{\alpha} Z^{\beta}$. The normal operator Δ^{α} is naturally defined by the function Z^{α} restricted on $[0, \infty]$. If α is noted as $\alpha = t + is$, then $\Delta^{\alpha} = \Delta^{t} \Delta^{is}$. Δ^{t} is a positive selfadjoint operator and Δ^{is} is an unitary operator.

Theorem 3.4.1. Let α be a complex number, a be an element of $\mathfrak{A} \wedge \mathcal{D}(\Delta^{\alpha})$ and k be an element of $\mathfrak{A}^{c} \wedge \mathcal{D}(\Delta^{-\alpha})$. Then $a\Delta^{\alpha} k$ is an element of $\mathcal{D}(\Delta^{\alpha})$ satisfying

3.4.2. $\Delta^{\alpha}(a \Delta^{-\alpha} k) = (\Delta^{\alpha} a)k$.

Proof. Let α be a complex number, and for each number $1 \geq \delta \geq 0$ let $f_{\delta}(Z)$ denote a function $f_{\delta}(Z) = (\frac{Z+\delta}{1+\delta Z})^{\alpha}$. $f_0(Z)$ is the function Z^{α} which is defined in $K_0 - W$. If $1 \geq \delta > 0$, f_{δ} is analytic except on the segment $-\frac{1}{\delta} \leq Z \leq -\delta$. Then f_{δ} is an analytic function on $[0, \infty]$ and $W(f_{\delta})$ is contained in $\mathrm{Re}Z \leq 0$. We consider a fixed number $n \geq 1$ and a rectifiable curve which is contained in the half plane $\mathrm{Re}Z > 0$, whose interior contains $[\frac{1}{\gamma}, \gamma]$ and which intersects with $(0, \infty)$ only at two points. Let a and b be elements of $\mathfrak{A}^{s}$ and k be an element of $\mathfrak{A}^{c}$. Then $E_n k$ and $R_{\eta} k$ are elements of $\mathfrak{h}^{s} = \mathcal{D}(\Delta^{\frac{1}{2}})$.

Suppose that $\delta > 0$. Then from 3.3.5 we have

$$(f\delta(\Delta)a,\ b(E_\gamma k)^s)$$

$$= \frac{-1}{2\pi i}\int_C (a,\ (f_\delta(\bar{\eta}^{-1}\Delta)b)(B_\eta k)^s)\,d\eta\ .$$

Since $\mathfrak{A}^s$ is uniformly dense in $\mathfrak{h}$ the element a in the last identity can be taken from $\mathfrak{A}$. Then we have

3.4.2. $$(f\delta(\Delta)a,\ b(E_\gamma k)^s)$$

$$= \frac{-1}{2\pi i}\int_C (a(B\eta E_\gamma k),\ f_\delta(\bar{\eta}^{-1}\Delta)b)\,d\eta\ .$$

for $a \in \mathfrak{A}\wedge\mathcal{D}(\Delta^\alpha)$, $k \in \mathfrak{A}^c\wedge\mathcal{D}(\Delta^{-\alpha})$ and $b \in \mathfrak{A}^s\wedge\mathcal{D}(\Delta^\alpha)$.
We show that 3.4.2 is valid in the case that $\delta = 0$. Let α be noted as $\alpha = t + is$, and set

$$\gamma_1 = \inf(\mathrm{Re}\,\eta : \bar{\eta} \in C)$$

$$\gamma_2 = \sup(|\eta| : \bar{\eta} \in C)$$

$$\gamma_3 = \sup(|\arg\eta| : \bar{\eta} \in C),$$

$$C = \mathrm{Exp}\,(\gamma_3|s|).$$

We estimate the function $f_\delta(\eta^{-1}z)$ when $z \in [0,\infty]$. If $\eta \in C$, and $\lambda \in [0,\infty]$, and $0 \le \delta \le 1$ then reffering our definition of z^α we obtain

3.4.3. $$|f_\delta(\eta^{-1}\lambda)| = |\eta^{-1}\lambda + \delta|^t |1 + \eta^{-1}\delta\lambda|^{-t}$$

$$\times\ \mathrm{Exp}(\mp s\,\arg((\eta^{-1}\lambda + \delta)(1 + \eta^{-1}\delta\lambda)^{-1}).$$

Since $\mathrm{Re}\ \eta > 0$ we have

$$|\arg(1 + \eta^{-1}\delta\lambda) - \arg(\eta^{-1}\lambda + \delta)| \leq |\arg \eta|$$

and

$$\mathrm{Exp}(-s \arg((\eta^{-1}\lambda + \delta)(1 + \eta^{-1}\delta\lambda)^{t}\cdot)) \leq \mathrm{Exp}(\gamma_3 |s|) = C.$$

If $t > 0$, then from $|1 + \eta^{-1}\delta\lambda|^{t} \geq 1$ we find that

$$|\eta^{-1}\lambda + \delta|^{t} |1 + \eta^{-1}\delta\lambda|^{-t} \leq |\gamma_1\lambda + 1|^{t},$$

and

$$\| f_\delta(\eta^{-1}\Delta)b\| \leq C \|(\gamma_1^{-1}\Delta + \delta)^{t} b\| .$$

Similarly, if $t < 0$, then from $|1 + \eta\delta\lambda^{-1}|^{-t} \geq 1$ we have

$$|\eta^{-1}\lambda + \delta|^{t} |1 + \eta^{-1}\delta\lambda|^{-t} \leq |\gamma_2\lambda^{-1} + 1|^{-t}$$

and

$$\| f_\delta(\eta^{-1}\Delta)b\| \leq C \|(\gamma_2\Delta^{-1} + 1)^{-t} b\| .$$

If $t = 0$, then

$$\| f_\delta(\eta^{-1}\Delta)b\| \leq C\|b\| .$$

We obtain an analogous estimation in the case that $\eta = 1$. Then function $f_\delta(\eta^{-1}\lambda)$ of the variable η and λ, which is defined for $\eta \in C$ and $\lambda \in [\frac{1}{k}, k]$ where k is a number ≥ 1, is continuous and when $\delta \longrightarrow 0$ f_δ tends uniformly to f_0. From this we find that if $\delta \longrightarrow 0$ then.

$$\| f_\delta(\Delta)b - f_0(\Delta)b\| \longrightarrow 0$$

and

$$\sup_{\eta \in C} \| f_\delta(\eta^{-1}\Delta)b - f_0(\eta^{-1}\Delta)b \| \longrightarrow 0.$$

Hence as a limit form of 3.4.2 we obtain

3.4.3. $$(\Delta^\alpha a, b(E_\gamma k)^s)$$

$$= \frac{-1}{2\pi i} \int_c (a(B\eta E_\gamma k), (\bar{\eta}^{-1}\Delta)^\alpha b) d\eta .$$

Notice that

$$\frac{-1}{2\pi i} \int_c \eta^{-\alpha} B\eta E_\gamma k \, d\eta = \Delta^{-\alpha} E_\gamma k.$$

Then we have

$$(\Delta^\alpha a, b(E_\gamma k)^s) = (a\Delta^{-\alpha} E_\gamma k, \Delta^\alpha b).$$

Let $n \longrightarrow \infty$, then E_n tends to 1 strongly. Since $E_n{}^z = E_n$ we have $(E_n k)^s = E_n k^s \longrightarrow k^s$. Then we have

$$(\Delta^\alpha a, bk^s) = (a\Delta^{-\alpha} k, \Delta^\alpha b).$$

and

$$((\Delta^\alpha a)k, b) = (a\Delta^{-\alpha} k, \Delta^\alpha b).$$

b is an arbitrary element of $\mathfrak{A}^s \cap \mathcal{D}(\Delta^\alpha)$ which is dense in domain $\mathcal{D}(\Delta^\alpha)$. Then we have

$$\Delta^\alpha(a\Delta^{-\alpha} k) = (\Delta^\alpha a)k.$$

Theorem 3.4.1 is thus proved ..

Theorem 3.4.2. Let t be a real number. $a \longrightarrow \Delta^{it}a$ is an automorphism of $\mathfrak{A}$, and

3.4.3. $$L_{\Delta^{it}a} = \Delta^{it}L_a\,\Delta^{-it} \qquad \text{for } a \in \mathfrak{A}.$$

$k \longrightarrow \Delta^{it}k$ is an automorphism of $\mathfrak{A}^c$, and

3.4.4. $$R_{\Delta^{it}k} = \Delta^{it}R_k\,\Delta^{-it}.$$

Proof. Notice that $\mathcal{D}(\Delta^{it}) = \mathcal{D}(\Delta^{-it}) = \mathfrak{H}$. Then by Theorem 3.4.1 we have

3.4.5. $$\Delta^{it}(a(\Delta^{-it}k)) = (\Delta^{it}a)k \qquad \text{for } a \in \mathfrak{A},\ k \in \mathfrak{A}^c.$$

From this we obtain the Theorem.

Lemma 3.4.2. Let μ be a positive definite continuous function on the real number field. Then $a \longrightarrow \mu(\log\Delta)a$ carries $\mathfrak{A}$ into $\mathfrak{A}$, and we have

3.4.6. $$\| L_{\mu(\log\ )a} \| \leq \mu(0)\| L_a \| \quad \text{for } a \in \mathfrak{A}.$$

Proof. μ is a Fourier transform

3.4.7. $$\mu(\lambda) = \int_{-\infty}^{+\infty} \operatorname{Exp}(i\lambda t)\,d\hat{\mu}(t)$$

of a certain Borel measure $\hat{\mu} \geq 0$ on the real line with the total mass $\mu(0)$. Notice that $\mu(\log\Delta)$ is a strong integral

3.4.8. $$\mu(\log\Delta) = \int_{-\infty}^{+\infty} \Delta^{it}\,d\hat{\mu}(t)$$

and if $a \in \mathfrak{A}$ and $k \in \mathfrak{A}^c$ we have

$$(\mu(\log\Delta)a)k = \int_{-\infty}^{+\infty} (\Delta^{it}a)k\,d\hat{\mu}(t).$$

Since $(\mathcal{M}(\log \Delta)a)^* = \mathcal{M}(\log \Delta)a^*$, applying Lemma 3.1.4 a is an element of $\mathfrak{A}$ satisfying 3.4.6.

Theorem 3.4.3. $\mathfrak{A}$ contains a certain *-cyclicly dense *-subalgebra which is invariant under every transform $a \longrightarrow \Delta^{\alpha} a$ and which is everywhere dense in every Hilbert space $\mathcal{D}(\Delta^{\alpha})$.

Proof. Let R be the real number field and $C_0(R)$ be the linear space of all complex valued continuous functions on R. We let $\mathcal{E}$ denote the least linear space which contains all the convolutions $f * g$ of functions f and g in $C_0(R)$ where $f * g$ denote

$$f * g(\lambda) = \int_{-\infty}^{+\infty} f(t)g(\lambda - t)dt.$$

We define the adjoint $\tilde{f}$ of f by $\tilde{f}(\lambda) = \overline{f(-\lambda)}$. Then for every $f \in C_0(R)$ $f * \tilde{f}$ is a positive definite function and $\mathcal{E}$ is linearly spanned by these functions $f * \tilde{f}$. We notive that $f \longrightarrow \tilde{f}$ transposes $\mathcal{E}$ onto $\mathcal{E}$ and if α is a complex number $f(\lambda) \longrightarrow f(\lambda)\mathrm{Exp}(\alpha\lambda)$ transposes $\mathcal{E}$ onto $\mathcal{E}$. In fact let E^{α} denote the function $\mathrm{Exp}(\alpha\lambda)$. Then

$$E^{\alpha}(f * g) = (E^{\alpha} f) * (E^{\alpha} g).$$

We let $\mathfrak{B}$ denote the least linear subalgebra of $\mathfrak{A}$ which contains all $f(\log \Delta)a$ such that $a \in \mathfrak{A}$ and $f \in \mathcal{E}$. $\mathfrak{B}$ is a *-subalgebra since

$$(f(\log \Delta)a)^* = \tilde{f}(\log \Delta)a^*.$$

$\mathcal{B}$ is invariant under the transform $a \longrightarrow \Delta^{\alpha} a$ since

$$\Delta^{\alpha} f(\log \Delta)a = E^{\alpha}(\log \Delta) f(\log \Delta)a.$$

We show that $\mathcal{B}$ is dense in each Hilbert space $\mathcal{D}(\Delta^{\alpha})$. To see this we let $\alpha = t + is$, and consider a function f in $C_0(R)$ and an element a of $\mathfrak{A}$. Let X be the carrier of f and U be a bounded open set which contains X. As it is well-known, there is a sequence $\{f_n\}$ in $\mathcal{E}$ which converges uniformly to f and the carrier of each f_n is contained in U. Then we obtain

$$(1 + \Delta^{t})(f_n(\log \Delta) - f(\log \Delta)a) \longrightarrow 0 \quad (n \longrightarrow \infty)$$

uniformly in $\mathfrak{H}$. Notice that the Hilbert spaces $\mathcal{D}(\Delta^{\alpha})$ and $\mathcal{D}(\Delta^{t}$ are identical, and $\mathfrak{A}$ is uniformly dense in $\mathfrak{H}$. Then $\mathcal{B}$ is dense in each Hilbert space $\mathcal{D}(\Delta^{t}) = \mathcal{D}(\Delta^{\alpha})$, and in particular $\mathcal{B}$ is *-cyclicly dense in $\mathfrak{H}^{*}$.

The algebra $\mathcal{B}$ in Theorem 4.3.4 is contained in $\mathfrak{A}^{s}$ and is an s-subalgebra of $\mathfrak{A}^{c}$. Then we have

3.4.9. $\Delta^{\alpha}(ab) = \Delta^{\alpha} a \; \Delta^{\alpha} b \qquad$ for $\quad a \in \mathcal{B}, \quad b \in \mathcal{B}$.

From this we find that $\mathcal{B}$ is a *-syclicly dense modular Hilbert subalgebra of $\mathfrak{A}$. Hence we obtain Theorem 2.4.1 and the Corollary 1

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