Доклады Болгарской академии наук
Comptes rendus de l'Académie bulgare des Sciences
Tome 33, N° 5, 1980

MATHÉMATIQUES
Logique mathématique

SIMPLE EXAMPLES OF INCOMPLETE LOGICS

D. I. Vakarelov

(Submitted by Academician B. Petkanchin on December 18, 1979)

The first examples of incomplete modal logics were constructed by Thomason [1] and Fine [2]. These examples are very complicated and the question is to construct simple ones. We show that in the case of modal logics which are based on a part of the classical logic, such simple examples of incomplete logics can easily be obtained.

1. Distributive Modal Logics. The language of distributive modal logics contains an infinite set V of propositional variables, parentheses and the following logical symbols: T (truth), F (falsity), $\wedge$ (conjunction), $\vee$ (disjunction) and $\Box$ (necessity). The notion of formula is the usual one. If A and B are formulas, then the expression $A\vdash B$ is called sequent. The minimal system based on this language is denoted by $\mathbf{K}^-$.

Axioms and rules for $\mathbf{K}^-$: $A\vdash A$, $\dfrac{A\vdash B,\ B\vdash C,}{A\vdash C}$ $F\vdash A$, $A\vdash T$,

$A\wedge B\vdash A$, $A\wedge B\vdash B$, $\dfrac{C\vdash A,\ C\vdash B}{C\vdash A\wedge B}$

$A\vdash A\vee B$, $B\vdash A\vee B$, $\dfrac{A\vdash C,\ B\vdash C,}{A\vee B\vdash C}$ $C\wedge(A\vee B)\vdash(C\wedge A)\vee(C\wedge B)$,

$\dfrac{A\vdash B}{\Box A\vdash\Box B}$, $\Box A\wedge\Box B\vdash\Box(A\wedge B)$, $T\vdash\Box T$.

A sequent $A\vdash B$ is provable in $\mathbf{K}^-$ if there exists a finite sequence $A_1\vdash B_1,\ldots, A_n\vdash B_n$, such that $A_n=A$, $B_n=B$ and $A_i\vdash B_i$ is, for any i $(1\leq i\leq n)$, either an axiom or is obtained from previous ones by means of some of the rules.

Other systems: $\mathbf{T}^-=\mathbf{K}^-+\Box A\vdash A$

$\mathbf{S4}^-=\mathbf{T}^-+\Box A\vdash\Box\Box A$

$\mathbf{L}_1=\mathbf{K}^-+\Box F\vdash F+A\vdash\Box A$

$\mathbf{L}_2=\mathbf{S4}^-+\Box(A\vee B)\vdash A\vee\Box B$

2. Classical Modal Semantics. A classical modal frame (for short a frame) is a pair $\mathbf{K}=(K, R)$, where $K\neq\emptyset$ and $R\leq K\times K$. The frame (K, R) is called reflexive (transitive), if R is a reflexive (transitive) relation in K. A model is a triple (K, R, v), where $v: V\to P(K)$ is a valuation. The relation $x\models_v A$ (the formula A is true in x $(x\in K)$ at the valuation v) is defined inductively as in the usual Kripke definition as follows: $x\models_v p$ iff $x\in v(p)$, $p\in V$, $x\models_v T$, $x\not\models_v F$, $x\models_v A\wedge B$ iff $x\models_v A$ and $x\models_v B$, $x\models_v A\vee B$ iff $x\models_v A$ or $x\models_v B$, $x\models_v\Box A$ iff

$(\forall y)(xRy \rightarrow y \models_v A)$. The sequent $A \vdash B$ is true in the model (K, R, v) if $(\forall x \in K)(x \models_v A \rightarrow x \models_v B)$; $A \vdash B$ is true in the frame (K, R) (or (K, R) is a frame for $A \vdash B$) if $A \vdash B$ is true in any model (K, R, v); a set of sequents Σ is true in a frame (K, R) (or (K, R) is a frame for Σ) if any member of Σ is true in (K, R); a set of sequents Σ is true in a class of frames Γ if Σ is true in any member of Γ. A class of frames Γ is called adequate for a given logic L if for any two formulas $A, B: A \vdash B$ is provable in L iff $A \vdash B$ is true in Γ. A logic L is called (classically) complete if it has an adequate class of frames. Otherwise L is called incomplete.

Theorem 1. (i) The class of all frames is adequate for $\mathbf{K}^-$.
(ii) The class of all reflexive frames is adequate for $\mathbf{T}^-$.
(iii) The class of all reflexive and transitive frames is adequate for $\mathbf{S4}^-$.

The proof, by using Henkin models, is similar to that of the ordinary modal logics, except that prime theories can be used instead of maximal theories. (A subset x of formulas is a theory in a given logic L if: 1) $T \in x$, 2) if $A \in x$ and $A \vdash B$ is provable in L then $B \in x$, 3) if $A \in x$ and $B \in x$, then $A \wedge B \in x$; x is called prime theory if for any A and B: if $A \vee B \in x$ then $A \in x$ or $B \in x$.)

Corollary. $\mathbf{K}^-$; $\mathbf{T}^-$, and $\mathbf{S4}^-$ are complete logics.

The main purpose of this paper is the following

Theorem 2. (i) $\mathbf{L}_1$ is incomplete.
(ii) $\mathbf{L}_2$ is incomplete.

The proof follows from the following lemmas.

Lemma 3. Any frame for $\mathbf{L}_1$ is a frame for $\Box p \vdash p$.

Proof. Suppose the contrary. Then there exists a model (K, R, v) for $\mathbf{L}_1$ and $x \in K$ such that $x \models_v \Box p$ but $x \not\models_v p$. So we have $x\bar{R}x$. Since $\mathbf{K}$ is a model for $\Box F \vdash F$ and $x \not\models_v F$, then $x \not\models_v \Box F$. So there exists $y \in K$ such that xRy and $y \not\models_v F$. Define $w(p) = \{z \in K / xRz\}$. We have $x \in w(p)$, $y \notin w(p)$, so $x \models_w p$ and $y \not\models_w p$, which by xRy implies $x \not\models_w \Box p$. By axiom $p \vdash \Box p$ and $x \not\models_w \Box p$ we have $x \not\models_w p$ — a contradiction.

Lemma 4. $\Box p \vdash p$ is not provable in $\mathbf{L}_1$.

The proof will be given in the next section.

Lemma 5. Any frame for $\mathbf{L}_2$ is a frame for $p \vdash \Box p$.

Proof. Suppose the contrary. Then there exists a model (K, R, v) for $\mathbf{L}_2$ and $x \in K$, such that $x \models_v p$ but $x \not\models_v \Box p$. Then there exists $y \in K$ such that xRy and $y \not\models_v p$. Define a valuation w as follows: $w(p) = \{s \in K / s \neq x\}$, $w(q) = \{s \in K / s \neq y\}$. We have $x \not\models_w p$, $y \not\models_w q$ and by xRy $x \not\models_w \Box q$, so $x \not\models_w p \vee \Box q$. Then, by axiom $\Box(p \vee q) \vdash p \vee \Box q$ we have $x \not\models_w \Box(p \vee q)$. So there exists $t \in K$, such that xRt, $t \not\models_w p$ and $t \not\models_w q$, i. e. $t = x$ and $t = y$, so $x = y$ and $x \not\models_w p$ — a contradiction.

Lemma 6. $p \vdash \Box p$ is not provable in $\mathbf{L}_2$.

The proof will be given in the next section. Note that in [3] Lewis proved that any (ordinary) modal logic without axioms with iterated modalities is complete. Theorem 2 (i) shows that this is not true in the case of distributive modal logic.

3. **Intuitionistic Modal Semantics.** Now we shall give another semantics for K^- and its extensions in which all discussed logics are complete.

An intuitionistic modal frame (I-frame) is a triple $(K, \leq, R)$, where $K \neq \emptyset$, $\leq$ is a reflexive and transitive relation in K and R is a binary relation in K satisfying the following condition: if xRy and $s \leq x$ and $y \leq t$, then sRt. Let $I(K)$ be the set of all increasing subsets of K (A is increasing if $(\forall xy)$ $(x \in A$ and $x \leq y$ implies $y \in A)$). An I-model is a quadruple $(K, \leq, R, v)$, where v is a valuation from V into $I(K)$. The definition of the relation $x \models_v A$ is the same as in the previous section.

Example 1. (Proof of lemma 4) Let $K = \{0, 1\}$ with the usual ordering $\leq$ and $0R_1 1$ and $1R_1 1$. Then it is easy to see that $(K, \leq, R_1)$ is an I-frame for L_1 but not for $\Box p \vdash p$. Indeed, define $v(p) = \{1\}$, then we have $0 \models_v \Box p$ but $0 \not\models_v p$. This shows that $\Box p \vdash p$ is not true in the model $(K, \leq, R_1, v)$.

Example 2. (Proof of the lemma 6). Let $K = \{0, 1\}$ with the usual ordering $\leq$ and for any $x, y \in K$ we have $xR_2 y$. Then $(K, \leq, R_2)$ is an I-frame for $\mathbf{L}_2$ but not for $p \vdash \Box p$. Define $v(p) = \{1\}$ we see that $1 \models_v p$, $1R_2 0$ and $0 \not\models_v p$, so $1 \not\models_v \Box p$. So $p \vdash \Box p$ is not true in the model $(K, \leq, R_2, v)$.

Theorem 7. (i) The class of all I-frames is adequate for $\mathbf{K}^-$.

(ii) The class of all reflexive I-frames is adequate for $\mathbf{T}^-$.

(iii) The class of all reflexive and transitive I-frames is adequate for $\mathbf{S4}^-$.

(iv) The class of all I-frames satisfying the conditions $(\forall x)(\exists y)xRy$ and $xRy \to x \leq y$ is adequate for $\mathbf{L}_1$.

(v) The class of all reflexive and transitive I-frames satisfying the condition $xRy \to (\exists z)xRz$ and $z \leq x$ and $z \leq y$ is adequate for $\mathbf{L}_2$.

The proof is similar to that of theorem 1. In the canonical Henkin model the relation $\leq$ is the set-inclusion between prime theories. The condition for $\Box(A \vee B) \vdash A \vee \Box B$ in the logic $\mathbf{L}_2$ presents some difficulties. To prove that the canonical model satisfies this condition, one has to apply the Zorn Lemma.

Corollary. The logics $\mathbf{K}^-$, $\mathbf{T}^-$, $\mathbf{S4}^-$, $\mathbf{L}_1$ and $\mathbf{L}_2$ are complete with respect to the intuitionistic modal semantics.

Problem. Are there extensions of $\mathbf{K}^-$ which are incomplete with respect to the intuitionistic modal semantics?

Institute of Mathematics
Bulgarian Academy of Sciences
Sofia, Bulgaria

REFERENCES

1 S. K. Thomason. Theoria 40, 1974, 30. 2 K. Fine. Ibid. 40, 1974, 23. 3 D. Lewis. The J. Phil. Logic. 3, 1974, 457.