Lecture 1

1 marca 2011
07:10

I Classical Part

a) GR formulated as an SU(2) gauge theory (Holst action, Ashtekar variables)

b) Quantization of systems with constraints

c) Observables for GR, physical phase space. (Relational framework)

d) BK-Model
DGKL-Model + add. matter coupled to GR

· Literature:
Book: C. Rovelli Quantum Gravity, Cambrige, 2004
T. Thiemann, Modern Canonical QG, — " —, 2007

- Ashetar, Lewandowski, gr-qc/0404018
T. Thiemann gr-qc/0210094
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A. Perez (Spin foams) gr-qc/04090061
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II Kinematics

a) Kinematical Hilbert space of LQG
(Representation of Holonomy-flux-algebra

b) Geometric Operators : Area Operator

(Johannes Brunnemann: volume operator)

c.) Implementation of the classical constraints as operator (Gauß-, Diffeo Constraint)
Solutions of Constraints

d.) LOST-Theorem

III Dynamics

a) Quantization of the Hamiltonian constraint (Thiemann's trick)

b.) Quantization of the BK- and the DGKL-Models

c.) Semiclassical LQG: Coherent States

d) Black holes in LQG

e.) Outlook & open problems in LQG

I. Classical theory

1. Canon. formulation, new variables

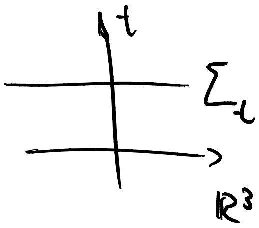
Aim: Canon. quantization of GR
$\rightsquigarrow$ need canonical formulation of GR $\longrightarrow$ Phase space, Canon. variables, Hamiltonian

$$\begin{array}{ccc} \text{Canon. formul.}_1 & \longrightarrow & \text{quantum theory}_1 \\ \cong & & \neq \\ \text{Canon. form.}_2 & \longrightarrow & \text{quantum theory}_2 \end{array}$$

Remarkably: $\exists$ formulation s.t. phase space is that of YM-theory!

Warm-up: Maxwell theory

$$S[A] = \frac{1}{4} \int_{\mathbb{M}^4} F_{\mu\nu} F^{\mu\nu} d^4x$$



Introduce equal time surfaces Σ_t

canonical coordinates: A_μ and momenta

$$P^\mu = \frac{\delta S}{\delta \dot{A}_\mu} \rightsquigarrow \quad P^a = E^a, \; P^0 = 0$$

$$S = \int dt \int_{\Sigma_t} d^3x \;\; E^a \dot{A}_a - \frac{1}{2}(B^2 + E^2) - A_0 \nabla E$$

Now: GR

Frame? („Tetrad") $e_I \equiv e_I^{\mu} \partial_\mu$: Basis of tangent space at each point. Also, map $\mathbb{R}^4 \to T_p M$

If there is metric on internal space $\rightsquigarrow$ define metric on M:

$$g^{\mu\nu} := e_I^{\mu} e_J^{\nu} \eta^{IJ}$$

Co-frame: $e^I = e^I_\mu dx^\mu$

$$S[\omega, e] = S_P[\omega, e] + S_H[\omega, e]$$

ω: SO(3,1) connection

$$S_P = \frac{1}{\cdot} \int \epsilon_{IJKL} \, e^I \wedge e^J \wedge F^{KL}(\omega)$$

$$S[\omega, e] = S_P[\omega, e] + S_H[\omega, e]$$

SO(3,1) connection

$$S_P = \frac{1}{4\kappa} \int \epsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL}(\omega)$$

$$S_H = -\frac{1}{2\kappa\beta} \int e^I \wedge e^J \wedge F_{IJ}(\omega)$$

$$\frac{\delta S}{\delta \omega} \overset{!}{=} 0 \iff D^{(\omega)} e = 0 \quad \rightsquigarrow \quad \omega \equiv \omega(e)$$

Insert into S

$$S_H[\omega(e), e] = 0 \qquad S_P[\omega(e), e] = EH$$

$\beta \in \mathbb{R}/\{0\}$ in the following.

Space-time decomposition:

time function t $\rightsquigarrow$ $t=$ const slices Σ_t

time vectorfield t^α: $\quad t^\alpha \partial_\alpha t \overset{!}{=} 1$

$$t^\alpha = N n^\alpha + N^\alpha$$

unit normal to Σ_t

Also: partial gauge fixing:

$$e^0_\mu \overset{!}{=} n_\mu$$

Thus only SO(3) remains as gauge group.

Decompose fields:

$$(e^I_\mu) = \begin{pmatrix} N & N^i \\ 0 & \boxed{e^i_a} \end{pmatrix}$$

co-triad.

Also break ω into parts: SO(3) connection on Σ_t

Γ, K

+ „the rest"

$$S = \frac{1}{\kappa\beta} \int E^a_i \dot{A}^i_a - (\omega^i_0 G_i + N^a C_a + N \cdot C)$$

$$S = \frac{1}{2\kappa\beta}\int dt \int_{\Sigma_t} E^a_i \dot{A}^i_a - \underbrace{\left(\omega^i_0 G_i + N^a C_a + N \cdot C\right)}_{h \text{ ("Hamiltonian")}}$$

$$A^i_a = \Gamma^i_a + \beta K^i_a \qquad E^a_i = \sqrt{\det q}\, e^a_i$$

$$(q_{ab} = e^i_a e^j_b \delta_{ij})$$

$$G_i = D^{(A)}_a E^a_i$$

$$C_a = E^b_i F^i_{ab} + (\;)^i_a G_i \qquad (H_a = C_a - G)$$

$$H = \frac{\beta}{2}\frac{1}{\sqrt{\det q}} E^a_i E^b_j \left[\epsilon^{ij}{}_k F^k_{ab} - 2(1+\beta^2) K^i_{[a} K^j_{b]}\right]$$

$$\{A^i_a(x), E^b_j(y)\} = 8\pi G\beta\, \delta_a{}^b\, \delta^i{}_j\, \delta(x,y)$$

Constraint equations $G_i = 0$ $H_a = 0$, $H = 0$, together with

$$\dot{A} = \{A, h\}, \quad \dot{E} = \{E, h\}$$

are equivalent to Einstein's equations.
Evolution is gauge:

$$\{A_a, \underbrace{G(\Lambda)}_{\int G_i \Lambda^i}\} = -D^{(A)}_a \Lambda = \frac{d}{d\epsilon}\Big|_{\epsilon=0} g_\epsilon A g_\epsilon^{-1} + g_\epsilon d g_\epsilon^{-1}$$

$$g_\epsilon = \exp(\epsilon\Lambda)$$

$$\{E^a, G(\Lambda)\} = [\Lambda, E^a_i]$$

$$\{A, \vec{H}(\vec{N})\} = \mathcal{L}_{\vec{N}}(A) \qquad \{E, \vec{H}(\vec{N})\} = \ldots$$

Action of $H(N)$ related to $\mathcal{L}_{n^a N}$.

Dirac algebra:

$\{G, G\} = G, \qquad \{\tilde{H}, \tilde{H}\} = \tilde{H}$

$\{G, \tilde{H}\} = G \qquad \{H, G\} = 0$

$\{H, \tilde{H}\} = H \qquad \{H, H\} = \tilde{H}$

Remarks: $\quad D+1$

1) $A^i \rightsquigarrow D(D-1)/2$ 1-forms

$E_i \to$ D vector fields

Matters only for $D=3$

$$\hat{H}(N)\,\Psi \stackrel{!}{=} 0$$

• Reduced Phase space Approach

(i) Solve the constraints already at the classical level (Observables)

• Observables are gauge-invariant

(ii) Dynamics of the observables (true physical evolution no gauge-transformation)

$g_{\alpha\beta} \rightsquigarrow q_{ab}$ 3+1 split Σ_t

• Notation: $(A, E) \rightsquigarrow (q_{ab}, P^{ab})$

$G_j, C_a, C \qquad C_a, C$

$\vec{C}(\vec{N}), \; C(N)$

• In the canonical

$\{C(N), O\} = \{\vec{C}(\vec{N}), O\} = 0$

$$\{C(N), O\} = \{\vec{C}(\vec{N}), O\} = 0$$

$\Leftrightarrow$ O is an observable (Dirac observables)

- $H_{can} = \vec{C}(\vec{N}) + C(N)$

$\Rightarrow$ if O is an observable we have

$$\{O, H_{can}\} = 0$$

This shows that the dynamics of observables cannot be generated by H_{can}.

- There must be a different generator for the dynamics of $\hat{O}$, physical Hamiltonian $H_{phys} \neq 0$ on the constraint hypersurface
- Idea of the reduced framework:

(i) Construct Dirac observables for (A, E) (q_{ab}, p^{ab})

(ii) Derive the associated H_{phys} then the dynamics

$$Q_{ab} := O_{q_{ab}} \quad , \quad P^{ab} := O_{p^{ab}}$$

$$\dot{Q}_{ab} = \{Q_{ab}, H_{phys}\}$$
$$\dot{P}^{ab} = \{P^{ab}, H_{phys}\}$$

gauge-invariant version of Einstein's equation

- this is the classical starting point for a reduced Quantization.

- <u>Construction of Dirac observables:</u>
- <u>Choice:</u> for each of the constraints in the system you choose one reference field, clocks

- there exist an general map for each gauge-variant quantity (q_{ab}, p_{ab}) to its corresponding Dirac observable

- Relational formalism (dynamics is formulated wrt the value that the reference fields take)

- Dirac observables:

$$Q_{ab}(\tau) = \sum_{n=0}^{\infty} \frac{(\tau - T)^n}{n!} \Big(\underbrace{\frac{1}{\{C, T\}} \chi_C}_{\chi_{C'}} \Big)^n \cdot q_{ab}$$

τ = physical time values

T = time reference field

χ_C = Hamiltonian VF of C

$= \exp(\chi_{C'}) \cdot q_{ab}$

$\chi_{C'} q_{ab} \; \{C, q_{ab}\}$

$\chi_C^n \cdot q_{ab} = \{\{q_{ab}, C\}, C\}, C\} \ldots\}$ up to order n

$\frac{1}{\{C, T\}}$ to ensure $\chi_{C'} T = 1 \Leftrightarrow \{C', T\} = 1$

We have $\{C, Q_{ab}\} = 0$

- We will construct Dirac observables in two particular models:

- GR: $4 \times \infty$ constraints $C(x) = 0$, $C_a(x) = 0$

$\longrightarrow$ $4 \times \infty$ many clocks in GR

- 4 scalar fields are appropiate

- 4 particular scalar fields

- BK-dust model (gravity + dust matter)

$S = S^{gravity}_{Einstein-Hilbert} + S^{dust}$

$$S = \underbrace{S^{gravity}}_{\substack{\text{Einstein-Hilbert} \\ \int_{\mathcal{M}} d^4x \sqrt{\det(g)}\, R}} + S^{dust}$$

$$S^{dust} = -\frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{\det(g)}\, \rho \left(g^{\alpha\beta} U_\alpha U_\beta + 1 \right)$$

$\rho \hat{=}$ energy density of the dust

$U_\alpha = -T_{,\alpha} + \omega_j S^j_{,\alpha}$ $\quad T, S^j, \omega_j \quad j = 1,2,3$

7 scalar fields

- ω_j is related to the momenta T, S^j
- 4 'independent' scalar fields, which we will use as reference fields

  $T \hat{=}$ reference field for time (Hamiltonian)

  $S^j \hat{=}$ reference field for spatial coord. (Diffeomorphism constraint)

- Assume: $\det(S^k_{,\alpha}) \neq 0 \quad S^k := (T, S^k)$
- Perform (3+1) decomposition

  $(T, P) \quad , \quad (S^j, P_j)$

first class constraints

$C^{tot} = C^g + C^{dust} = 0 \quad C^{dust} = -\sqrt{P^2 + q^{ab} (P T_{,a} + P_j S^j_{,a})(P T_{,b} + P_k S^k_{,b})}$

$C_a^{tot} = C_a^g + C_a^{dust} = 0$

$C_a^{dust} = P T_{,a} + P_j S^j_{,a}$

- Idea of BK:

$$C^{dust} = -\sqrt{P^2 + q^{ab} C_a^{dust} C_b^{dust}} \approx$$

$$\approx -\sqrt{P^2 + q^{ab} C_a^g C_b^g}$$

weakly equiv.

- Using $C^{tot} = 0$

$$\tilde{C}^{tot} = P + h(q,P) \, ,$$

$$h(p,q) = \sqrt{(C^q)^2 - q^{ab} C^q_a C^q_b}$$

- Same game for the diffeomorphism constraint

$S^j_{,a}$, S^a_j inverse

$$\tilde{C}^{tot}_j = P_j - h_j(S,T,q,p) \, , \quad S^j_{,a} S^a_k = \delta^j_k \, , \quad S^j_{,a} S^b_j = \delta^a_b$$

$$C_J := \left(\tilde{C}^{tot}, \tilde{C}^{tot}_j\right) \qquad h_j = S^a_j \left(P T_{,a} + C^q_a\right)$$

- Dirac Algebra for constraint

$$\{ C_J(x), C_K(y) \} = 0 \quad \text{mutually commute}$$

true Lie algebra for the constraints

- Now we use the C_K to construct Dirac observables.

by construction: $\{ S^K, C_J \} = \delta^K_J$

since $C^{tot} = P + h(p,q)$
Hamiltonian deparametrizes no T in $h(p,q)$

here no dust dof

- Let's apply the observable map

reduced $\tilde{C}^{tot}$

$$Q_{ab}(\tau, \sigma^j) = \exp\left(\{ h_\tau, \cdot \}\right)$$

$$\exp\left(\int_\Sigma d^3x \, \beta^j \{ C_j(x), \cdot \} \right) \cdot q_{ab} \Big|$$

phys. time

physical spatial coord.

reduced

$$h_\tau = \int_S d^3\sigma \, (\tau - T)\, h(s)$$ wrt C_a^{tot} $\beta_i =$ $\sigma^i - s^i$

- (Q_{ab}, P^{ab}) Dirac observables

- (ii) Dynamics

We want the associated phys. Hamiltonian

$$H_{phys} = \int_{S\ \text{dust space}} d^3 s \, H(s)$$ observable associated to $h(p,q)$

$$H(s) = \sqrt{(C^g)^2 + Q^{ab} C_a^g C_b^g} \quad \Phi_{h(p,q)} = H(Q, P)$$

- $H_{phys} \neq 0$ and it is gauge-invariant
- EOM:

$$\frac{dQ_{ab}}{d\tau} = \dot{Q}_{ab} = \{ Q_{ab}(x), H_{phys} \}$$
$$\frac{dP^{ab}}{d\tau} = \dot{P}^{ab} = \{ P^{ab}(x), H_{phys} \}$$

reduced phase formulation of GR with dust as clocks

- Classical starting point for quantization

- 2nd Model: DGKL-Model

Gravity + one K.G. scalar field

$$S = S^g + \frac{1}{2} \int_M d^4x \sqrt{\det(g)}\, g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}$$

- 3+1-Split (ϕ, π)
- Constraints:

$$C^{tot} = C^g + \frac{1}{2}\frac{\pi^2}{\sqrt{\det(q)}} + \frac{1}{2} q^{ab} \phi_{,a} \phi_{,b} \sqrt{\det(q)}$$
$$C_a^{tot} = C_a^g + \pi \phi_{,a}$$

- One reference field, so we will use C_a^{tot} to solve the Hamiltonian constraint for Π
- $$q^{ab}\phi_{,a}\phi_{,b} = \frac{q^{ab}C_a^{\phi}C_b^{\phi}}{\Pi^2} \approx \frac{q^{ab}C_a^{g}C_b^{g}}{\Pi^2}$$
- Reinsert this into C^{tot}:

$$0 = C^g + \frac{1}{2}\frac{\Pi^2}{\sqrt{\det(q)}} + \frac{1}{2}\sqrt{\det(q)}\,\frac{q^{ab}C_a^{g}C_b^{g}}{\Pi^2}$$

$$\Leftrightarrow \quad \Pi^4 + 2\sqrt{\det(q)}\,C^g\,\Pi^2 + \frac{1}{2}\sqrt{\det(q)}\,q^{ab}C_a^{g}C_b^{g} = 0$$

→ solving for Π

$$\Pi = \pm\sqrt{-\sqrt{\det(q)}\,C^g + \sqrt{\det(q)}\sqrt{(C^g)^2 - q^{ab}C_a^{g}C_b^{g}}}$$

- This allows us to rewrite (choose)

$$C^{tot} = \Pi - h(p,q) \qquad h$$