Lecture on GFT

7 marca 2011
17:59

1- Introduction & Reminders (Thomas' talk)

2- GFT's 'holonomy', 'spin' and 'metric' representation

(3- Simplicity contraints & GFT model for gravity)

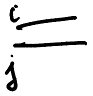
4- 'Diffeomorphisms' in GFT

1- Intro

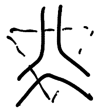
1.1 Higher-d generalization of matrix models

• Graphology

MM:



Matrix propagator

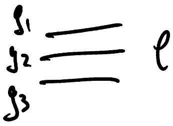


Matrix vertex

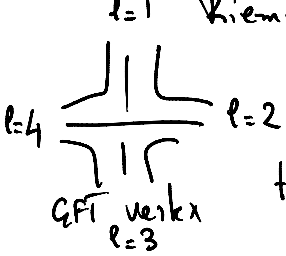
MM Feynman diagrams = Ribbon graphs

Riemann surfaces

GFT: ℓ



GFT propagator



GFT vertex

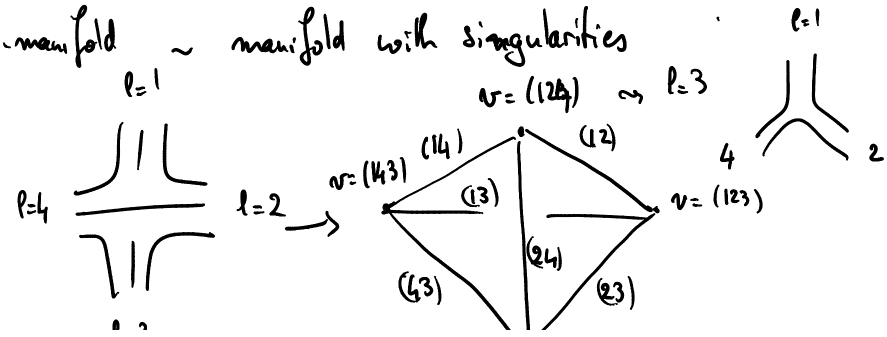
tetrahedron

(colored) d-GFT Feynman diagrams = d-strande graphs

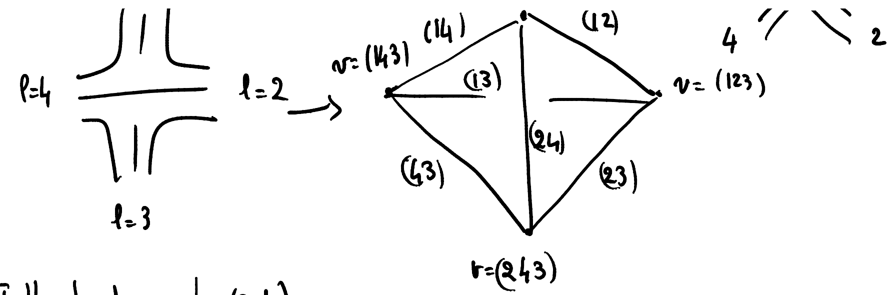
Fact. Colored GFT graphs are canonically associated to triangulated (pseudo)-manifolds

Pseudo-manifold ~ manifold with singularities

Ex:



Ex:



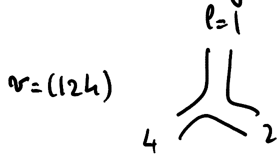
Full closed graph (3d)

Vertices $v \in \mathcal{G}$ $\hookrightarrow$ Tetrahedra $\tau \in \Delta$

(3-stranded) lines $L \in \mathcal{G}$ $\hookrightarrow$ Triangles $t \in \Delta$

Loops of strands (or face) $f \in \mathcal{G}$ $\hookrightarrow$ Edges $e \in \Delta$

3-bubble of $\mathcal{G}$ $\hookrightarrow$ Vertices $v \in \Delta$



attaching these subgraphs together

$\rightarrow$ Ribbon graph R

Connected component = 3-bubble $v \in \Delta$

Theorem (Manifold v.s Pseudo-manifolds)

A GFT graph is dual to a tr. manifold iif all 3-bubbles are planar Ribbon graphs.

Colored GFT: Gurau 0907.2582

Ben Geloun, Magnen, Rivasseau 0911.1719

• Graph amplitudes $A_\mathcal{G}$

$A_\mathcal{G}$ define simplicial path integrals (BF or gravity)

$$A_\mathcal{G} = \int_\Delta dg_\Delta \, e^{iS_\Delta(g_\Delta)} = \text{spinfoam model}$$

GFT's universal structure behind SF

(Reisenberger Rovelli '00)

Hope in GFT:

$$\sum' \int \mathcal{D}\varphi \, e^{iS(\varphi)} \sim \sum' \int \mathcal{D}g_\Delta \, e^{iS_\Delta(g_\Delta)}$$

$$\sum_{\substack{\text{Topologies}\\ \mathcal{M}}} \int_{\mathcal{M}} Dg\, e^{iS(g)} \longrightarrow \sum_{\text{triangulations}} \int_{\Delta} Dg_\Delta\, e^{iS_\Delta(g_\Delta)}$$

$$\longrightarrow \sum_{\mathcal{G}} \lambda^{|\mathcal{G}|} A_{\mathcal{G}} = \int D\phi\, e^{iS_{GFT}}$$

1.2 Spin foam models (3d gravity)

SF = discrete version of a functional integral

Ex : 3D gravity Riemanian signature

3d Manifold $\mathcal{M}$ $G = SU(2)$ or $SO(3)$

Variables: e triad frame field, A connection } 1-forms valued in $\mathfrak{g} = su(2)$

$$S(e,A) = \int \mathrm{Tr}\, e \wedge F(A)$$

$F(A) = dA + A \wedge A$ curvature 2-form

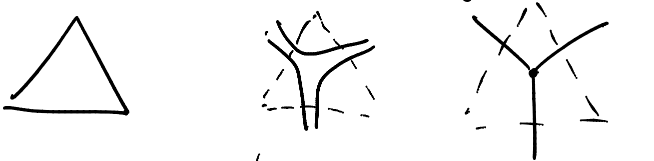
Discrete structures :

- Triangulation Δ of $\mathcal{M}$
- 'Poincaré dual complex' Δ^* (or 'dual polyhedral decomposition') of Δ

k-cells of Δ^* $\longleftrightarrow$ 3-k simplices of Δ (d)

[so 3 types of discrete structures : $\mathcal{G}$, Δ, Δ^*

2d :



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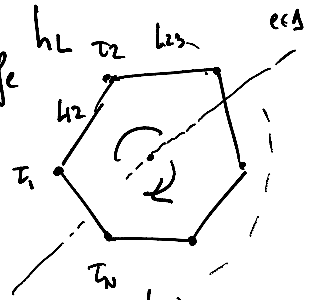
Discretization of action :

$e^I_\mu(x) \longrightarrow x_e \in su(2)$ for each edge $e \in \Delta$

$A_\mu^{IJ}(x) \longrightarrow h_L \in SO(3)$ for each line $L \in \Delta^*$

h_L holonomy from $\tau \to \tau'$

$F(A) \longrightarrow H_e(h_L) = \overrightarrow{\prod_{L\in\partial f_e}} h_L$



$$S_\Delta(x_e,h_L) = \sum_{e\in\Delta} \mathrm{Tr}\, x_e H_e(h_L)$$

1) $$Z_\Delta^{simp} = \int \prod_L dh_L \int \prod_e d^3x_e \, e^{iS_\Delta(x_e,h_e)}$$

Simplicial ⟷ holonomy ⟷ spin foam

1) 2) 3)

2) $$\int d^3x \, e^{i\mathrm{Tr}\, xg} = \delta_{SU(2)}(g) + \delta_{SU(2)}(-g) = \delta_{SO(3)}(g)$$

$$Z_\Delta^{hol} = \underbrace{\int \prod_L dh_L \prod_e \delta\Big(\prod_{L\in\partial f_e} h_L\Big)}_{\text{Measure on space of flat connections}}$$

3) $$\delta_{SO(3)}(g) = \sum_{j\in\mathbb{N}} (2j+1)\,\mathrm{Tr}\, D^j(g)$$ + recoupling theory

$$Z_\Delta^{spin} = \sum_{\{j_e\}} \prod_e (2j_e+1) \prod_\tau \begin{Bmatrix} j_1^\tau & j_2^\tau & j_3^\tau \\ j_4^\tau & j_5^\tau & j_6^\tau \end{Bmatrix}$$

Ponzano-Regge model

1.3 Outline

objective: To deepen duality btw GFT's and simplicial path integrals

- 3 formulations of GFT Z_Δ^{simp}, Z_Δ^{hol}, Z_Δ^{spin}

New 'metric' rep of GFT

- Two byproducts : $\begin{cases} 1\text{- Impose simplicity const in GFT action} \\ 2\text{- Diffeo in GFT} \end{cases}$

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2 - $(\text{GFT's})^{\text{hol, spin, }\overline{\text{metric}}}$

2.1 'Holonomy' representation

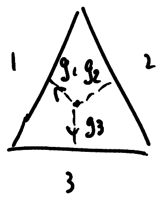
$(\varphi_\ell)_{\ell=1..4}$ complex fields on $SO(3)^{\otimes 3}$ (Colored Boulatov model 3d SF)

$\forall h \in SO(3) \qquad \varphi_\ell(hg_1, hg_2, hg_3) = \varphi_\ell(g_1, g_2, g_3)$

$$S[\varphi] = S_{kin}[\varphi] + S_{int}[\varphi]$$

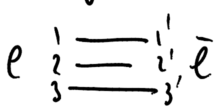
$$S_{kin}[\varphi] = \int \prod_i dg_i \sum_{\ell=1}^{4} \varphi_\ell(g_1 g_2 g_3)\, \bar{\varphi}_\ell(g_1 g_2 g_3)$$

$$S_{int}[\varphi] = \lambda \int [dg_i]^6\, \varphi_1(g_1 g_2 g_3)\, \varphi_2(g_3 g_4 g_5)\, \varphi_3(g_5, g_2, g_6)\, \varphi_4(g_6, g_4, g_1)$$

$\varphi_\ell =$ 

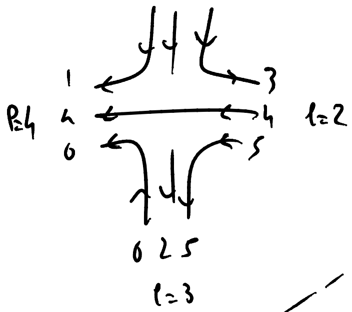
g_i = holonomy from center triangle to edge i

Feynman rules :

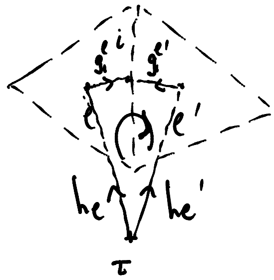


$$P_\ell(g, g')_{\ell=1} = \int dh \prod_{i=1}^{3} \delta(g_i^{-1} h\, g'_i)$$

1 2 3



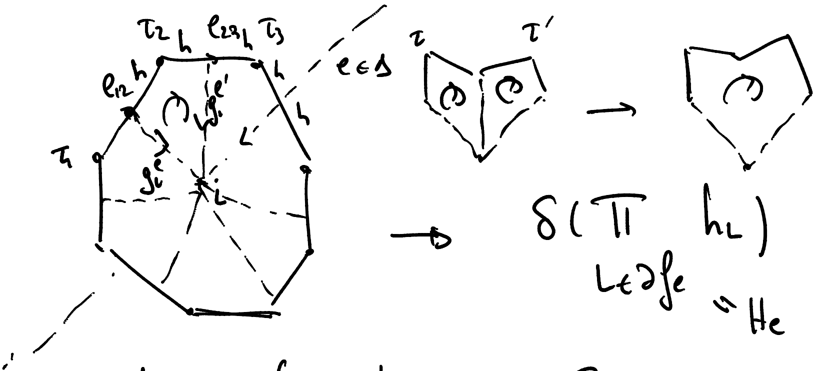
$$V(g_i, g_i') = \int \prod_{\ell=1}^{4} dh_\ell \prod_{i=1}^{6} \delta\left((g_i^\ell)^{-1} h_e h_{e'}^{-1} g_i^{e'} \right)$$



Plaquette of Δ^* are flat :

$$g_i^{e} g_i^{e'-1} h_{e'}^{-1} h_e = 1$$

in buiding up $\mathcal{G}$ (and Δ, and Δ^*) :



$$\rightarrow \delta\Big(\prod_{L \in \partial f_e} h_L \Big) \quad \text{“} H_e$$

$$A_{\mathcal{G}} = \int \prod_L dh_L \prod_e \delta(H_e) = Z_\Delta^{Hol}$$

2.2 'Spin' representation of GFT

obtained harmonic analysis on G $\quad L^2(G) \simeq \bigoplus_{j \in \mathrm{Irrep}} V^j \otimes V^{j*}$

Peter-Weyl expansion of gauge invariant field

$$\phi(g_1, g_2, g_3) = \sum_{j,m}{}' \underbrace{C^{j_1 j_2 j_3}_{m_1 m_2 m_3}} \phi^{j_1 j_2 j_3}_{n_1 n_2 n_3} \prod_i D^{j_i}_{m_i n_i}(g_i)$$

SO(3) - invariant tensor in $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$

Wigner 3j-symbol

- Orthogonality of Wigner matrices

$$\int dg \, D^{j}_{mn}(g) \, \overline{D^{j'}_{m'n'}}(g) = \frac{1}{d_j} \delta_{jj'} \delta_{mm'} \delta_{nn'}$$

- $\overline{D^{j}}(g) = D^{j}_{\ \ }(g)$

- Def Racah-Wigner 6j-symbol:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = C^{j_1 j_2 j_3}_{m_1 m_2 m_3}\, C^{j_3 j_4 j_5}_{-m_3 m_4 m_5}\, C^{j_5 j_2 j_6}_{-m_5 -m_2 m_6}\, C^{j_6 j_4 j_1}_{-m_6 -m_4 -m_1}$$

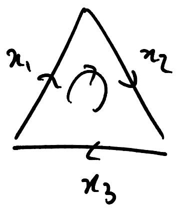
$\rightarrow$ GFT vertex = 6j-symbol

$$\mathcal{A}_g = \sum_{\{j_e\}}{}' \prod_e (2j_e+1) \prod_\tau \begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix} = Z_\Delta^{spin}$$

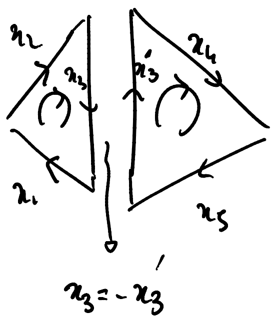
2.3 Metric representation gen Z_Δ^{simp}

Preliminary game : what would you propose

$x_i \in \mathbb{R}^3$



$$\phi(x_1,x_2,x_3) \propto \delta^3(x_1+x_2+x_3) * \psi(x_1,x_2,x_3)$$



$$\phi_1(x_1,x_2,x_3) * \phi_2(-x_3,x_4,x_5)$$

$$\phi_3(-x_5,-x_2,x_6) * \phi_4(-x_6,-x_4,-x_1)$$

Fourier transform on $\mathbb{R}^3$ $\forall a$ $\tilde{\phi}(p_1,p_2,p_3) = \hat{\phi}(a+p_1, a+p_2, a+p_3)$

Vertex : $\tilde{\phi}_1(p_1,p_2,p_3)\, \tilde{\phi}_2(p_3,p_4,p_5)\, \tilde{\phi}_3(p_5,p_4,p_6)$
$\tilde{\phi}_4(p_6,p_2,p_1)$

Similar story for standard GFT :

- 'group Fourier transform' $G = SO(3)$
- Non comm $SU(2)$ $\hookrightarrow$ $\star$-product on $F(x)$

1-variable 2.3.1 Group FT:

$$\mathcal{F}(G) \longrightarrow \mathcal{F}(su(2) \sim \mathbb{R}^3)$$

$$\hat{\varphi}(x) = \underbrace{\int dg}_{\text{haar meas on } G} \varphi(g)\, \underbrace{e_g(x)}_{e^{i \operatorname{Tr} xg}}$$

- 'star-product' on space of $\hat{\varphi}(x)$ dual to convolution product on G:

$$\hat{\varphi} \star \hat{\psi} = \widehat{\varphi \circ \psi} \qquad \varphi \circ \psi(h) = \int dg\, \varphi(g)\, \psi(g^{-1}h)$$

- $e_g \star e_{g'} = e_{gg'}$
- Inversion formula: $\varphi(g) = \int_{su(2)\sim\mathbb{R}^3} d^3x\, (\hat{\varphi} \star e_{g^{-1}})(x)$

Back to GFT:

$$\hat{\varphi}_e(x_1, x_2, x_3) = \int [dg_i]^3\, \varphi_e(g_1, g_2, g_3)\, e_{g_1}(x_1)\, e_{g_2}(x_2)\, e_{g_3}(x_3)$$

star product $(e_{g_1}\; e_{g_2}\; e_{g_3}) \star (e_{g_1'}\; e_{g_2'}\; e_{g_3'})$

$$= e_{g_1 g_1'}\; e_{g_2 g_2'}\; e_{g_3 g_3'}$$

2.3.2 Gauge invariance

$\forall h \quad \varphi_e(g_1, g_2, g_3) = \varphi_e(hg_1, hg_2, hg_3)$

$g_1 = hg_1$

$\Leftrightarrow$

$$\forall h \quad \hat{\varphi}_e(x_1, x_2, x_3) = \int [dg_i]\, \varphi_e(g_1, g_2, g_3)\, e_{hg_1}(x_1)\, e_{hg_2}(x_2)\, e_{hg_3}(x_3)$$

$$= e_h(x_1)\, e_h(x_2)\, e_h(x_3) \star \hat{\varphi}_e(x_1, x_2, x_3)$$

$$= e_h(x_1 + x_2 + x_3) \star \hat{\varphi}_e(x_1, x_2, x_3)$$

$$= e_h(x_1+x_2+x_3) \star \hat{\varphi}_e(x_1,x_2,x_3)$$

$$\hat{\varphi}_e(x_1,x_2,x_3) = \underbrace{\int dh\, e_h(x_1+x_2+x_3)}_{\delta_0(x_1+x_2+x_3)} \star \hat{\varphi}_e$$

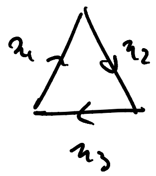
$$\boxed{\delta_x(y) = \int dg\; \underset{\underset{e_g(-x)}{\shortparallel}}{e_g^{-1}(x)}\, e_g(y)} \left(= \int dg\; e_g(x-y) \right)$$

Dirac distribution for $\star$-product:

$$\int d^3y\, (\delta_x \star f)(y) = f(x) = \int d^3y\, (f \star \delta_x)(y)$$

$$\delta_0(x_1+x_2+x_3) = \text{closure constraints}$$

$$\delta_0 \star \delta_0 = \delta_0$$

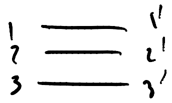
$\Rightarrow$ Invariant dual field  $\{x_i\}$ edge metric

GFT action: $\int dg\, \varphi(g)\, \psi(g) = \int d^3x\; \hat{\varphi}(x) \star \hat{\psi}(-x)$

$$S[\hat{\varphi}] = \prod_{i=1}^{3} \int d^3x_i \sum_e \hat{\varphi}_e(x_1,x_2,x_3)\; \hat{\varphi}_e(-x_1,-x_2,-x_3)$$

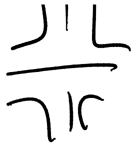
$$+ \lambda \int [d^3x_i]^6 \quad \hat{\varphi}_1(x_1,x_2,x_3)\, \hat{\varphi}_2(-x_3,x_4,x_5) \cdots \hat{\varphi}_4(\quad)$$

Feynman rules:



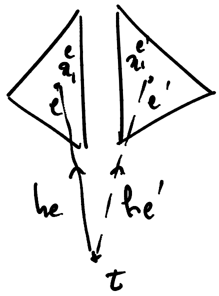
$$P_e(x_i, x_i') = \int dh \prod_{i=1}^{3} (\delta_{-x_i} \star e_h)(x_i')$$

$e=1 \quad 1\,2\,3$



$$V(x, x') = \prod_{i=1} \left(\delta_{-x_i^e} * e_{h_e h_{e'}^{-1}}\right)(x_i^{e'})$$

Geometrically :



Identification of edge metric on i in two frames e, e' related by holonomies $h_e h_{e'}^{-1}$.

δ property : $(\delta_x * e_h)(y) = (e_h * \underbrace{\delta_{hxh^{-1}})(y)}_{y = h \triangleright x}$

Full graph : strands joined using $*$-product

$\int \prod_l dh_e \quad \prod_f$