The Volume Operator in Loop Quantum Gravity

Johannes Brunnemann

jbrunnem@math.upb.de

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Overview

0 Motivation

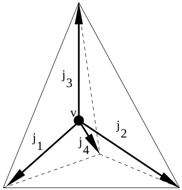
1 Construction / Regularization / Implementation on $\mathcal{H}_{kin}$

2 Evaluation of Matrix Elements (Rep'n. Theory / Combinatorics)

3 Spectral Properties.

4 To Do.

Gauge Invariant 4-Vertex



- Gauge Invariance

$$J_1 + J_2 + J_3 + J_4 \stackrel{!}{=} 0$$

- Matrix Element

$$< j_{12}|\hat{q}_{123}|j_{12} - 1 > \quad =$$

$$= \frac{1}{\sqrt{(2j_{12}-1)(2j_{12}+1)}} \Big[(j_1 + j_2 + j_{12} + 1)(-j_1 + j_2 + j_{12})(j_1 - j_2 + j_{12})(j_1 + j_2 - j_{12} + 1)$$

$$(j_3 + j_4 + j_{12} + 1)(-j_3 + j_4 + j_{12})(j_3 - j_4 + j_{12})(j_3 + j_4 - j_{12} + 1) \Big]^{\frac{1}{2}}$$

$$= \quad - \quad < j_{12} - 1|\hat{q}_{123}|j_{12} >$$

N-Vertex: Simplified Expression for the Matrix Element

$$
\begin{aligned}
< \vec{a}|\hat{q}_{IJK}|\vec{a}' > \; &= \\
&= \frac{1}{4}(-1)^{j_K+j_I+a_{I-1}+a_K}(-1)^{a_I-a'_I}(-1)^{\sum_{n=I+1}^{J-1} j_n}(-1)^{-\sum_{p=J+1}^{K-1} j_p} \times \\
&\quad \times X(j_I,j_J)^{\frac{1}{2}} X(j_J,j_K)^{\frac{1}{2}} \sqrt{(2a_I+1)(2a'_I+1)}\sqrt{(2a_J+1)(2a'_J+1)} \times \\
&\quad \times \left\{ \begin{array}{ccc} a_{I-1} & j_I & a_I \\ 1 & a'_I & j_I \end{array} \right\} \left[\prod_{n=I+1}^{J-1} \sqrt{(2a'_n+1)(2a_n+1)}(-1)^{a'_{n-1}+a_{n-1}+1} \left\{ \begin{array}{ccc} j_n & a'_{n-1} & a'_n \\ 1 & a_n & a_{n-1} \end{array} \right\} \right] \times \\
&\quad \times \left[\prod_{n=J+1}^{K-1} \sqrt{(2a'_n+1)(2a_n+1)}(-1)^{a'_{n-1}+a_{n-1}+1} \left\{ \begin{array}{ccc} j_n & a'_{n-1} & a'_n \\ 1 & a_n & a_{n-1} \end{array} \right\} \right] \left\{ \begin{array}{ccc} a_K & j_K & a_{K-1} \\ 1 & a'_{K-1} & j_K \end{array} \right\} \times \\
&\quad \times \Bigg[(-1)^{a'_J+a'_{J-1}} \left\{ \begin{array}{ccc} a_J & j_J & a'_{J-1} \\ 1 & a_{J-1} & j_J \end{array} \right\} \left\{ \begin{array}{ccc} a'_{J-1} & j_J & a'_J \\ 1 & a_J & j_J \end{array} \right\} \\
&\qquad - (-1)^{a_J+a_{J-1}} \left\{ \begin{array}{ccc} a'_J & j_J & a'_{J-1} \\ 1 & a_{J-1} & j_J \end{array} \right\} \left\{ \begin{array}{ccc} a_{J-1} & j_J & a'_J \\ 1 & a_J & j_J \end{array} \right\} \Bigg] \times \\
&\quad \times \prod_{n=2}^{I-1} \delta_{a_n a'_n} \prod_{n=K}^{N} \delta_{a_n a'_n}
\end{aligned}
$$

4-Vertex

Analytical Insights

- Special Form: only 1 antisymmetric, D-dim tridiagonal matrix, sign factor $\sigma(123)$ only gives overall scaling of the spectrum.

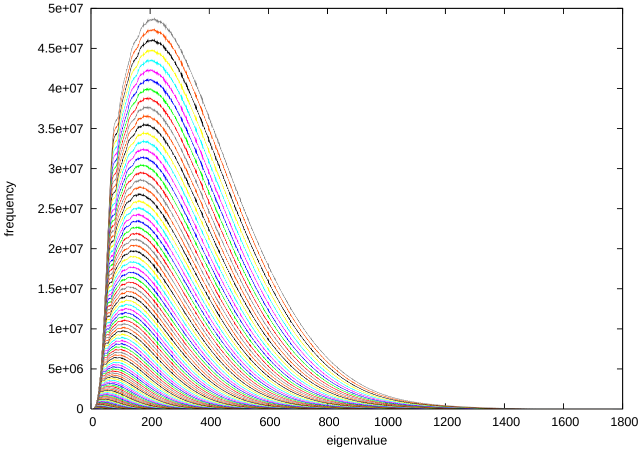
$$\widehat{q}_{123} = \begin{pmatrix} 0 & -q_1 & 0 & \cdots & 0 & 0 & 0 \\ q_1 & 0 & -q_2 & \cdots & 0 & 0 & 0 \\ 0 & q_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -q_{D-2} & 0 \\ 0 & 0 & 0 & \cdots & q_{D-2} & 0 & -q_{D-1} \\ 0 & 0 & 0 & \cdots & 0 & q_{D-1} & 0 \end{pmatrix}$$

where $q_k = q_k(j_1, j_2, j_3, j_4)$

Numerical Results.

Histograms for the generic (gauge invariant) 4-vertex

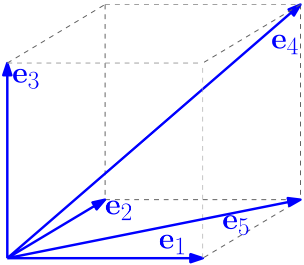
... up to $j_{\max} \leq 126/2$. (By ‘generic’ we mean excluding co-planar edges.)



Oriented Matroids

Motivation from Vectors I

$\mathbb{R}^3$, $\mathcal{M}$ vector config with sorted ground set $E = (\mathbf{e}_1, \ldots, \mathbf{e}_5)$.



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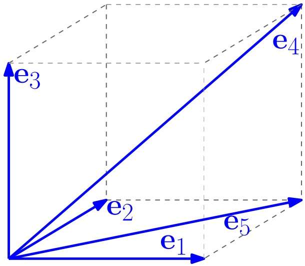

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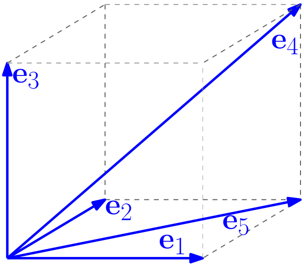

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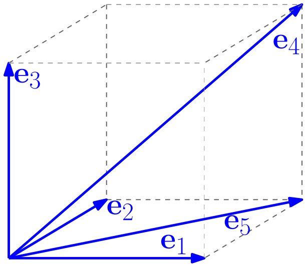
(ii) re-labelling



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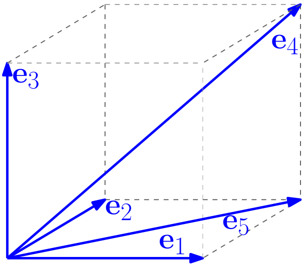

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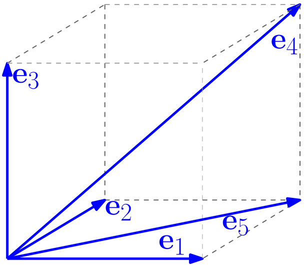

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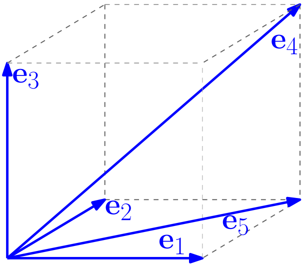

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  - $\chi_{\mathcal{B}}(S) = \begin{cases} \pm 1 & S \in \mathcal{B} \\ 0 & S \notin \mathcal{B} \end{cases}$

... in our example $\chi_{\mathcal{B}}(B) = \pm\,\mathrm{sgn}(\det B)$ (if $\chi_{\mathcal{B}}$ chirotope, then also $-\chi_{\mathcal{B}}$, depending of our notion of 'positive' orientation)

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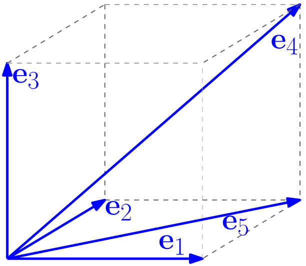

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B	123	124	125	134	135	145	234	235	245	345
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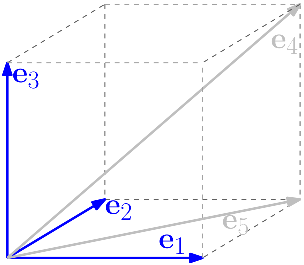

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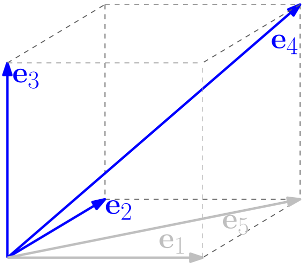

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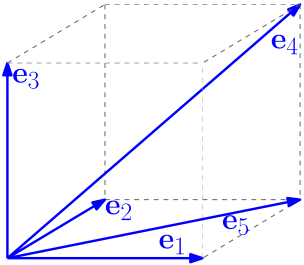

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$\chi_{\mathcal{B}}(B)$	$+$	$+$	0	$-$	$-$	$-$	$+$	$+$	$+$	0

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Re-labelling and reorientation act non-trivially on $\chi_{\mathcal{B}}(B)$. One finds in total 4 (1 uniform) equivalence classes of chirotopes for $D = 3$, $N = 5$:

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$\chi_{\mathcal{B},2}(B)$	+	+	+	+	+	+	+	+	+	0
$\chi_{\mathcal{B},3}(B)$	+	+	0	+	+	+	+	+	+	0
$\chi_{\mathcal{B},4}(B)$	+	+	+	+	+	+	0	0	0	0

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Our example

B	123	124	125	134	135	145	234	235	245	345
$\chi_{\mathcal{B}}(B)$	+	+	0	−	−	−	+	+	+	0

is contained in equiv. class 3 (set $\mathbf{e}_K \to -\mathbf{e}_K$ for $K = 1, 3, 4, 5$ and use properties of $\det$)

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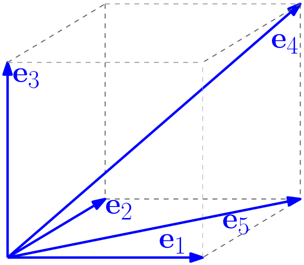

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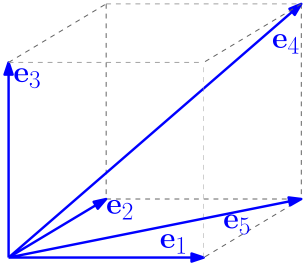

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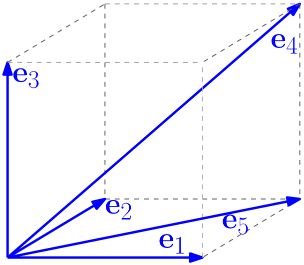

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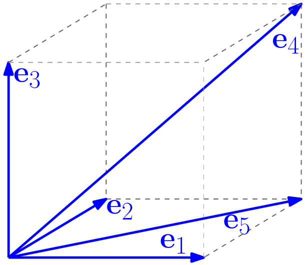

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- Signed Subsets, $S \subseteq E$
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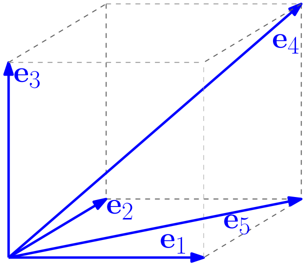

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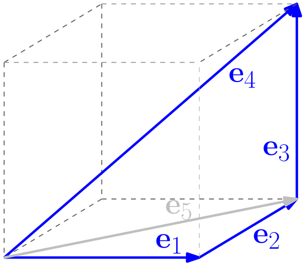

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	C_1		
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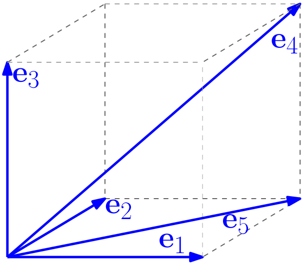

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	C_1	C_2	C_3
C^+	$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	$\{\mathbf{e}_1, \mathbf{e}_2\}$	$\{\mathbf{e}_3, \mathbf{e}_5\}$
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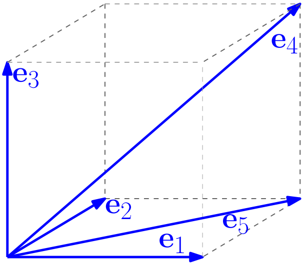

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  - Relative Sign

$$\mathrm{sgn}_C(\mathbf{e}_K) = \begin{cases} \pm 1 & \mathbf{e}_K \in C^\pm \\ 0 & \mathbf{e}_K \notin C \end{cases}$$

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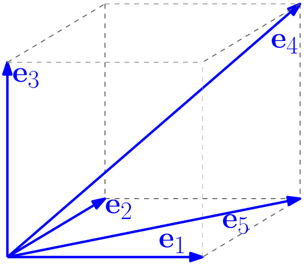
Motivation from Vectors II

$\mathbb{R}^3$, $\mathcal{M}$ vector config with sorted ground set $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_5\}$.



Signed Circuits $\mathcal{C}$

- Circuits
  - $\mathcal{C} = \{C \subseteq E : C$ min. lin. dep. $\}$
    Min. lin. dep. $0 = \sum_{k=1}^{N(c)} \lambda_K \mathbf{e}_K$
    ($\mathbf{e}_K \in C, \quad \lambda_K \in \mathbb{R}$)
- Signed Subsets, $S \subseteq E$
  - $C = \{C^+, C^-\}$ where $C^\pm = \{\mathbf{e}_K : \lambda_K \gtrless 0\}$
  - $(-C)^\pm = C^\mp$. Both, $C, -C$ contained in $\mathcal{C}$
  - Relative Sign

$$\operatorname{sgn}_C(\mathbf{e}_K) = \begin{cases} \pm 1 & \mathbf{e}_K \in C^\pm \\ 0 & \mathbf{e}_K \notin C \end{cases}$$

  - $\underline{C} =$ supp $C := C^+ \cup C^-$

Oriented Matroids

Motivation from Vectors III

Description of vector config $\mathcal{M}$ over ground set E in terms of $\mathcal{B}(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ equivalent.

- for every $B \in \mathcal{B}$ and for every $e \in E \backslash B$ there is a unique $\pm C \in \mathcal{C}$ such that

$$B \cup \{e\} \subseteq \underline{C}.$$

- Given two bases $B_1, B_2 \in \mathcal{B}$, $B_1 = (e, b_2, b_3)$, $B_2 = (f, b_2, b_3)$ we have $B_1 \cup \{f\} = B_2 \cup \{e\} \subseteq \underline{C}$ for one $\pm C \in \mathcal{C}$. It holds that

$$\mathrm{sgn}_C(e) \cdot \mathrm{sgn}_C(f) = \chi_{\mathcal{B}}(B_1) \cdot \chi_{\mathcal{B}}(B_2)$$

Oriented Matroids

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Can convert between the two equivalent descriptions!

Oriented Matroids

For Di-Graphs

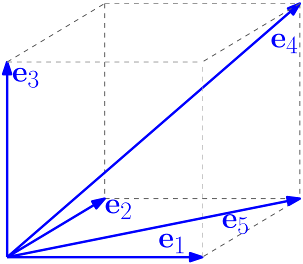
The same combinatorics contained in a directed graph:

Oriented Matroids

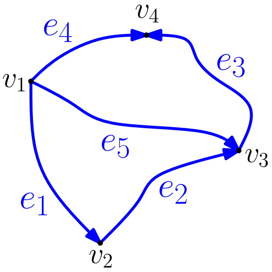
For Di-Graphs

The same combinatorics contained in a directed graph:

$\mathcal{M}_{\text{vector}}$



$\mathcal{M}_{\text{graph}}$



Oriented Matroids
For Di-Graphs

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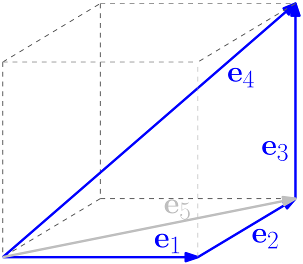
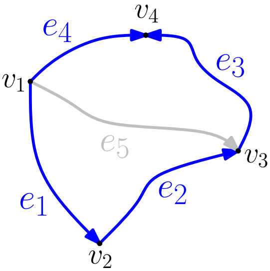
$\mathcal{M}_{\text{vector}}$



$\mathcal{M}_{\text{graph}}$



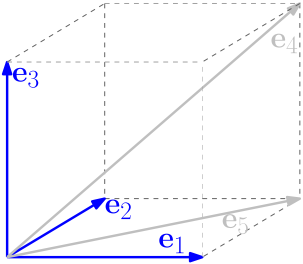
Signed Circuits $\mathcal{C}$ $\equiv$ Loops

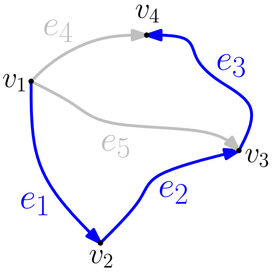
Oriented Matroids

For Di-Graphs

The same combinatorics contained in a directed graph:

$\mathcal{M}_{\text{vector}}$ $\mathcal{M}_{\text{graph}}$



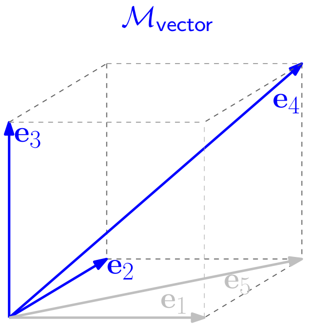


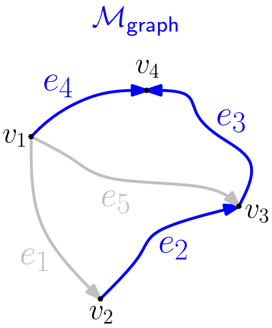
Signed Bases $\mathcal{C}$ $\equiv$ Spanning Trees

Oriented Matroids

For Di-Graphs

The same combinatorics contained in a directed graph:



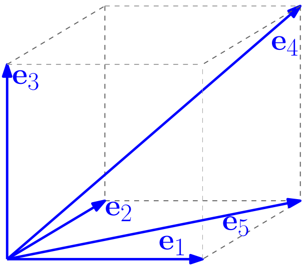


Signed Bases $\mathcal{C}$ $\equiv$ Spanning Trees

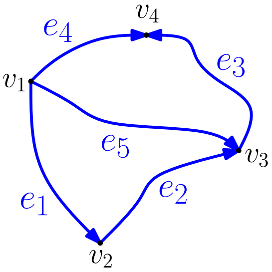
Oriented Matroids
For Di-Graphs

The same combinatorics contained in a directed graph:

$\mathcal{M}_{\text{vector}}$



$\mathcal{M}_{\text{graph}}$



Only two realizations of the more general combinatorial concept of an oriented matroid $\mathcal{M}$ of rank 3 over the ground set E in terms of its signed bases $\mathcal{M} = (E, \mathcal{B})$, respectively signed circuits $\mathcal{M} = (E, \mathcal{C})$.

Oriented Matroids

Axiomatic Definition: Signed Circuits

A family $\mathcal{C}$ of signed subsets of a finite set E is called the set of signed circuits of an oriented matroid $\mathcal{M} = (E, \mathcal{C})$ on E if

- (C0) Non-emptiness: $\emptyset \notin \mathcal{C}$
- (C1) Symmetry: $\mathcal{C} = -\mathcal{C}$, that is for every $C \in \mathcal{C}$ also its opposite $-C \in \mathcal{C}$.
- (C2) Incomparability: if $\underline{C_1} \subseteq \underline{C_2}$ then either $C_1 = C_2$ or $C_1 = -C_2$ $\forall C_1, C_2 \in \mathcal{C}$.
- (C3) Elimination: For all $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq -C_2$, if $e \in C_1^+ \cap C_2^-$ $\exists\, C_3 \in \mathcal{C}$ such that $C_3^\pm \subseteq (C_1^\pm \cup C_2^\pm) \setminus \{e\}$.

Equivalent formulation also in terms of $\mathcal{B}(\mathcal{M})$. Can be extended to infinite ground sets [Bruhn et al.].

More Difficult: Higher Valence

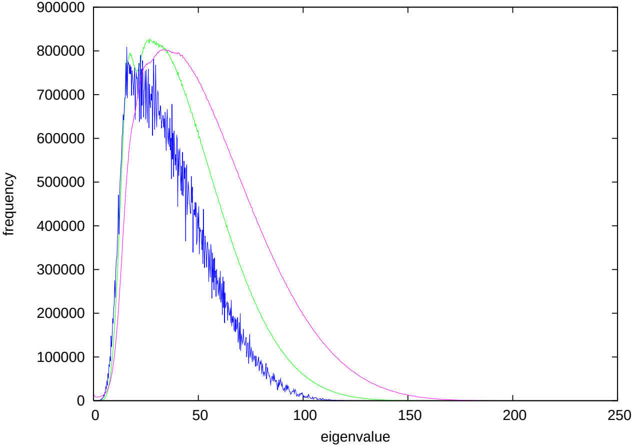
Sign factor combinatorics for 4–7-valent non-coplanar vertices

N_v	# triples	$\#\vec{\epsilon}(N_v)$ sprinkled	$\#\vec{\epsilon}$ perm. equiv. classes	# $\vec{\sigma}$ configs	# realizable reor. equiv. classes
3	1	2	1	1	1
4	4	16	3	3	1
5	10	384	4	4	1
6	20	23 808	41	39	4
7	35	3 486 720	706	673	11
8	56	$\geq$ 747 735 880	28 287		135
9	84	?	?		4 381

Numerical Results.

Histograms for each sigma configuration $\vec{\sigma}$ at the (gauge invariant) 5-vertex

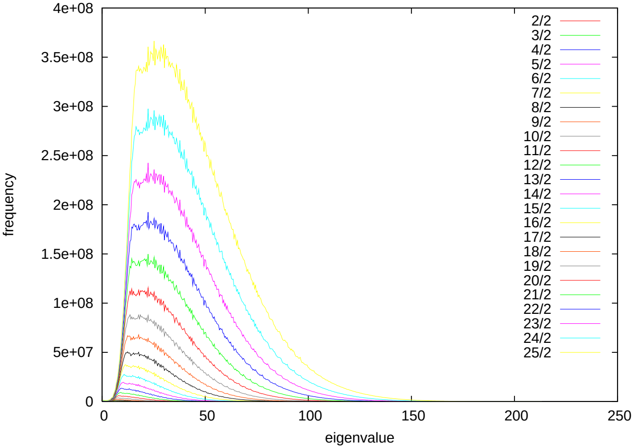
... up to $j_{\max} = 25/2$. The blue is for $\vec{\sigma} = (\sigma_{123}, \sigma_{124}, \sigma_{134}, \sigma_{234}) = (2, 0, 0, 0)$, the green for $\vec{\sigma} = (2, 2, 2, 0)$, and the purple for $\vec{\sigma} = (2, 2, 4, 0)$. Each histogram has 512 bins.



Numerical Results

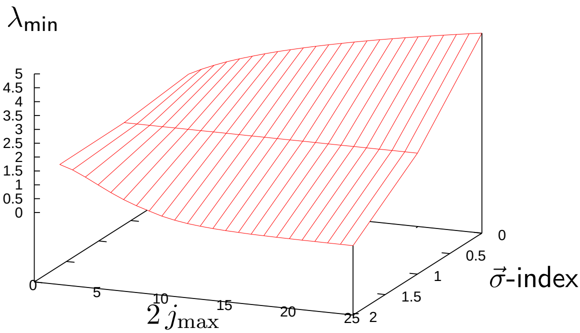
Histograms for the overall generic (gauge invariant) 5-vertex

... up to $j_{\max} \leq 25/2$. (By 'generic' we mean excluding co-planar edges.)
Each histogram has 512 bins.



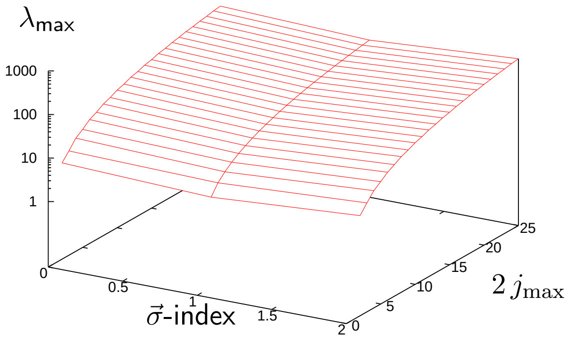
Numerical Results

Smallest non-zero eigenvalues λ_{min} at the (gauge invariant) 5-vertex



Numerical Results

Largest eigenvalues λ_{max} of the (gauge invariant) 5-vertex



Literature I

T. Thiemann.
Modern Canonical Quantum General Relativity.
Cambridge University Press, 2007.

A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler.
Oriented Matroids.
Cambridge University Press, second edition, 1999.2

C. Rovelli and L. Smolin, "Discreteness of area and volume in quantum gravity", Nucl. Phys. B **442** (1995) 593 [Erratum-ibid. B **456** (1995) 753]; [arXiv:gr-qc/9411005].

R. De Pietri, "Spin Networks and Recoupling in Loop Quantum Gravity", Nucl. Phys. Proc. Suppl. **57**, 251 (1997); [arXiv:gr-qc/9701041].

T. Thiemann, "Closed formula for the matrix elements of the volume operator in canonical quantum gravity",
J. Math. Phys. **39**, 3347 (1998); [arXiv: gr-qc/9606091].

A. Ashtekar and J. Lewandowski, "Quantum theory of geometry. II: Volume operators", Adv. Theor. Math. Phys. **1**, 388 (1998); [arXiv:gr-qc/9711031].

Literature II

- K. A. Meissner, "Eigenvalues of the volume operator in loop quantum gravity", Class. Quant. Grav. **23** (2006) 617; [arXiv:gr-qc/0509049].
- K. Giesel and T. Thiemann, "Consistency check on volume and triad operator quantisation in loop quantum gravity. I", Class. Quant. Grav. **23** (2006) 5667; [arXiv:gr-qc/0507036].
- Tim A. Koslowski, "Dynamical Quantum Geometry (DQG Programme)", arXiv:0709.3465v1 [gr-qc]
- Eugenio Bianchi, Hal M. Haggard, "Discreteness of the volume of space from Bohr-Sommerfeld quantization", arXiv:1102.5439v1 [gr-qc]
- J. Brunnemann and D. Rideout, "Properties of the Volume Operator in Loop Quantum Gravity I: Results", Class. Quant. Grav.**25**:065001,2008, [arXiv:0706.0469v1] [gr-qc].
- J. Brunnemann and D. Rideout, "Oriented Matroids – Combinatorial Structures Underlying Loop Quantum Gravity", Class.Quant.Grav.27:205008,2010 arXiv:1003.2348v2 [gr-qc] Johannes Brunnemann, David Rideout

Literature III

- C. Flori, T. Thiemann, "Semiclassical analysis of the Loop Quantum Gravity volume operator: I. Flux Coherent States", arXiv:0812.1537v1 [gr-qc]
- H. Bruhn, R. Diestel, M. Kriesell, R. Pendavingh and P. Wollan.
  Axioms for infinite matroids.
  [arXiv:1003.3919v2[math.CO]].
- L. Finschi and K. Fukuda.
  Generation of Oriented Matroids – A Graph Theoretical Approach.
  Discrete Comput Geom, 27: 117 – 136, 2002.