II Quantum kinematics

1) Quantum algebra, AL rep.

Want: Rep of

$$[A_a^i(x), E_j^b(y)] = 8\pi\beta \overbrace{G\hbar}^{l_p^2} i \, \delta_a^b \, \delta_j^i \, \delta(x,y)$$

Need meaning!

For YM: Both, algebra and rep use background metric.

For GR: No background metric

$\longrightarrow$ LQG very unusual QFT

For Maxwell: Usually

$$\left[\int f^a A_a \sqrt{q} \, d^3x, \int f'_b E^b d^3x \right]$$

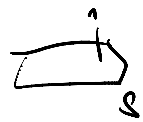
$$= i\hbar \int f^a f'_a \sqrt{q} \, d^3x$$

$a \Omega = 0$

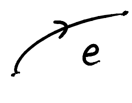
gas in box.

Can also do:

$$E(S) = \int_S E^a \epsilon_{abc} \, dx^a \wedge dx^b$$



$$A(e) = \int_e A$$



$$[A(e), E(S)] = i\hbar \, I(e,S) \cdot \mathbb{1}$$

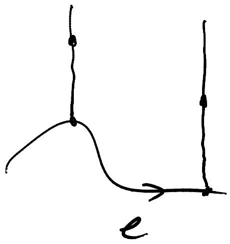
signed int. number.

$\rightsquigarrow$ metric dropped out!

Can do same for GR:

$$E_n(S) := \int_S n^i E^a_i \, \epsilon_{abc} \, dx^b \wedge dx^c$$

For A: analog of $\exp(i A(e))$



$$h_e[A] = \mathcal{P}\exp \int_e A \in SU(2)$$

$$= \mathbb{1} + \int_0^1 A_a(e(t))\, \dot{e}^a(t)\, dt + \int_0^1 dt_1 \int_{t_1}^1 dt_2 \; AA + \ldots$$

under gauge trafo: $g : \Sigma \to SU(2)$

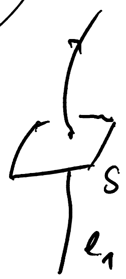
$$h_e \longmapsto g(s(e))\, h_e \, g(t(e))^{-1}$$

One finds

$$[E_n(S), h_e] = 0 \quad \text{if } S \cap e = 0$$

— " —

$$= h_{e_1} \tau_i n^i h_{e_2}$$



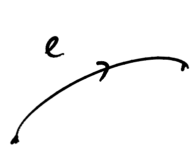
Slight generalization:

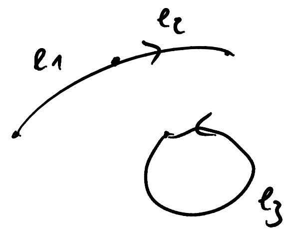
graph of paths $\gamma = \{e_1, e_2 \ldots e_n\}$
$f : SU(2)^n \to \mathbb{C}$
} $f_\gamma[A]$ „cylindrical function"

$$f_\gamma[A] = f(h_{e_1}[A], h_{e_2}[A], \ldots)$$

$f_\gamma \in Cyl_\gamma$

Note: 1) cylindrical function is cylindrical w.r.t. many graphs!





$\gamma = \{e\}$

$F[A] = f_\gamma[A] = f(h_e[A])$ $\qquad$ $\gamma = \{e_1, e_2, e_3\}$

$$F[A] = f(h_{e_1}[A] \cdot h_{e_2}[A])$$

2) given two cylindrical functions $f_\gamma, f'_{\gamma'}$

$\rightsquigarrow$ γ'' such that $f_\gamma, f'_{\gamma'}$ are $\in Cyl_{\gamma''}$

3) Prod. and sum of cyl. functions are cylindrical.

$\Rightarrow$ alg. Cyl of cyl. functions

$$[E_n(S), f_\gamma] = 8\pi\beta l_P^2 \, X_n(S) f_\gamma$$

$$= 4\pi\beta l_P^2 \sum_{v \in S \cap \gamma} n^i(v) \Big[\sum_{e \text{ at } v} \kappa(S,e)\, \hat{J}_i^{(v,e)} f\Big](h_{e_1}, \dots)$$

$$\kappa(S,e) = \begin{cases} 0 & \text{if } e \text{ tangential to } S \text{ at } v \\ 1 & \text{not tangential and above } S \\ -1 & \text{---''--- and below } S \end{cases}$$

$$\hat{J}_i^{(e,v)} = \mathbb{1} \otimes \dots \otimes \underbrace{\begin{Bmatrix} L_i \\ R_i \end{Bmatrix}}^{e} \otimes \mathbb{1} \otimes \dots$$

when $\begin{Bmatrix} e \text{ ingoing} \\ e \text{ outgoing} \end{Bmatrix}$ at v

Note: Commut. has jacobi-property, so

$$[f, [E, E']] = [X, X'](f) \neq 0$$

... to !

$\rightsquigarrow$ non-comm. spatial geometry!

Diffeos + Gauge act simply:

$$\left.\begin{aligned} \alpha_\phi(f_\gamma)(A) &= f_\gamma(\phi_* A) = f_{\phi(\gamma)}(A) \\ \alpha_\phi(E_n(S)) &= E_{\phi_* n}(\phi(S)) \end{aligned}\right\} \text{ autom. of alg. } \mathfrak{A}$$

<u>AL-rep.</u> (+ Rovelli and Smolin)

Cyl carries inner product: $f, f' \in Cyl_\gamma$

$$\langle f, f' \rangle := \int d\mu(g_1) \ldots d\mu(g_n) \, \overline{f(g_1 \ldots g_n)} \, f'(g_1 \ldots g_n)$$

Closure gives Hilbert space $\mathcal{H} = \mathcal{L}^2(\overline{\mathcal{A}}, d\mu_{AL})$

Rep:

$$\pi(f)\Psi[A] = f[A]\,\Psi[A]$$

$$\pi(E_n(S))\Psi[A] = 8\pi\beta i l_P^2 \, X_n(S)\Psi[A]$$

Properties:
- cycl. rep
- faithful
- Unitary rep of Diffeos / Gauge trafos:

$$U_\varphi h_e U_\varphi^{-1} = h_{\varphi(e)}$$

etc.

<u>Orthonormal basis of $\mathcal{H}_{AL}$:</u>

G compact Lie group. $\mathcal{H}_G = \mathcal{L}^2(G, d\mu)$

two reps of G:

$$(\rho_L(g')f)(g) = f(gg')$$

$$(\rho_R(g')f)(g) = f((g')^{-1}g)$$

Decompose into irreps: π some irrep

$$\pi \simeq V(\pi, m) = \text{span}\{\pi_{mn}(\cdot) \mid n = 1, \dots \dim\pi\}$$

left inv. by ρ_L

$$\bar{\pi} \simeq \bar{V}(\pi, n) = \text{span}\{\pi_{m,n}(\cdot) \mid m = 1, \dots \dim\pi\}$$

left inv. by ρ_R

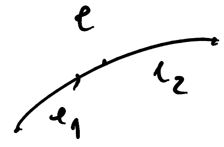
Peter-Weyl: $\{\sqrt{\dim\pi}\,\pi_{m,n}(\cdot) \mid \pi \text{ irreps } m, n = 1 \dots\}$

is ONB of $\mathcal{H}_0$

$$\mathcal{H}_\gamma = \overline{\text{Cyl}_\gamma}^{(\cdot,\cdot)} \simeq \mathcal{L}^2(SU(2)^n)$$

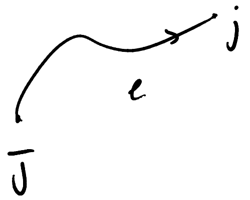
ONB: $\overset{j_1}{\pi}_{m_1 n_1}(h_{e_1}(A)) \cdot \overset{j_2}{\pi}_{m_2 n_2}(h_{e_2}) \dots$

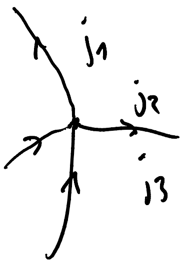
Note: $\mathcal{H}_\gamma \not\subset \mathcal{H}_{\gamma'}$



$$\pi_{mn}(g_e) = \sum_{m'} \pi_{mm'}(g_{e_1}) \cdot \pi_{m'n}(g_{e_2})$$

for $\overset{j}{\pi}(h_e)$:





Vertex $\rightsquigarrow$ tensor product

$$\bar{j}_1 \otimes \bar{j}_2 \otimes j_3 \otimes \dots$$

$-\pi$,

$= \bigoplus_{l} l$

$$\mathcal{H}_\gamma = \bigoplus_{\vec{j}} \mathcal{H}_{\gamma,\vec{j}} = \bigoplus_{\vec{j}\,\vec{l}} \mathcal{H}_{\gamma\vec{j}\vec{l}}$$

$$\mathcal{H}'_\gamma = \bigoplus_{\vec{j},\vec{l}'} \mathcal{H}_{\gamma\vec{j}\vec{l}}$$

no zero j_e
no zero l
at 2-valent vertex

$$\mathcal{H}_{AL} = \bigoplus_{\gamma} \mathcal{H}'_\gamma$$

- <u>Geometrical Operators</u>
- <u>Area - Operator</u>

Classical area (u_1, u_2) spatial

$$Ar(S) = \int_U d^2u \sqrt{\det(X^* q)}\,(u)$$

- X is an embedding $X: S \to \Sigma$,
  X^* pull-back, q_{ab} ADM metric
- Given the embedding X^a we can construct
  $X^a_{,u_1}$, $X^b_{,u_2}$
- Construct co-normal vector field
  $n_a X^a_{,u_i} \overset{!}{=} 0 \qquad i = 1,2$
- $n_a = \epsilon_{abc} X^b_{,u_1} X^c_{,u_2}$ $\underbrace{\qquad}_{q_{u_1 u_2}}$

- $n_a = \varepsilon_{abc} X^b_{,u_1} X^c_{,u_2}$
- $\det(X^*q) = \det(\underbrace{X^a_{,u_1} X^b_{,u_2} q_{ab}}_{q_{u_1u_2}})$

$$= q_{u_1u_1} q_{u_2u_2} - q_{u_2u_2} q_{u_1u_2}$$

- Aim: To express $\det(X^*q)$ in terms of A and E
- Project E^a_j onto co-normal-direction

$$n_a E^a_j = \varepsilon_{abc} X^b_{,u_1} X^c_{,u_2} E^a_j$$

- $q_{ab} = \frac{E^i_a E^j_b \delta_{ij}}{\det(E^i_a)}$, E^i_a are densitized co-triads
- General formula for inverse

$$E^a_j = \frac{1}{2} \frac{1}{\det(E^j_a)} \varepsilon^{abc} \varepsilon_{jkl} E^k_b E^l_c$$

$$(n_a E^a_j)(n_b E^b_k)\delta^{jk} = q_{u_1u_1} q_{u_2u_2} - q_{u_2u_1} q_{u_1u_2}$$

$$= \det(X^*q)$$

- Rewrite $Ar(S)$

$$Ar(S) = \int_U d^2u \sqrt{(n_a E^a_j)(n_b E^b_k)\delta^{jk}}\,(u)$$

- We need to regularize this expression
- Regularization:
- Choose a partition of U, $U = \bigcup_{n=0}^{N} U_n$

$$Ar(S) = \sum_{n \in U} \int_{U_n} d^2u \sqrt{(n_a E^a_i)(n_b E^b_j)\delta^{ij}}$$

$$\approx \sum_{U_n} \varepsilon^2 \sqrt{(n_a E^a_i)(n_b E^b_j)\delta^{ij}}\,(v)$$

$$= \sum_{U_n} \sqrt{\underbrace{\varepsilon^2 (n_a E^a_i)}\varepsilon^2 (n_b E^b_j)\delta^{ij}}\,(v)$$

$$= \sum \sqrt{E_i(S_n)\, E_j(S_n)\delta^{ij}}$$

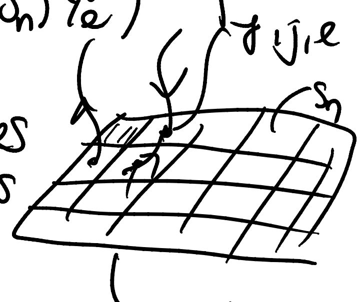
un $\nearrow$ fluxes for which operators exist

- We know the action of fluxes on SNF:

$$\hat{E}_j(S)\, T_{\gamma,j,e} = \frac{\beta \ell_p^2}{2} \sum_{e \in E(\gamma)} \varepsilon(e,S)\, Y_e^j\, [T_{\gamma,j,e}]$$

- Using this we get:

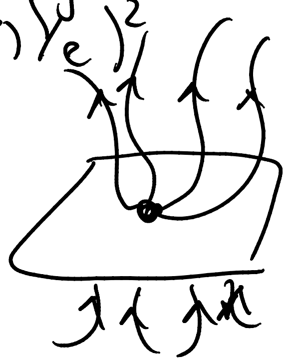
$$\hat{A}r(S) = \sum_{n} \frac{\ell_p^2}{4} \sqrt{\Big(\sum_{e\in E(\gamma)} \varepsilon(e,S_n)\, Y_e^j\Big)^2}\; T_{\gamma,j,e}$$



- $$\varepsilon(e,S) = \begin{cases} +1 & \text{for outgoing edges} \\ -1 & \text{for ingoing edges} \\ 0 & \text{for edges of type in or out} \end{cases}$$

- Therefore we can also sum over the intersection points and get

$$\hat{A}r(S) = \frac{\ell_p^2}{4}\beta \sum_{x\in P(S)} \sqrt{\Big(\sum_{e\in E(\gamma)} \varepsilon(e,S)\, Y_e^j\Big)^2}$$



- Spectrum of this operator:

$$\Big(\sum_{e\in E(\gamma)} \varepsilon(e,S)\, Y_e^j\Big)^2$$

$$= \Big(\sum_{\substack{e\in E(\gamma)\\ x\in b(e)}} Y^j_{e,up} - \sum_{\substack{e\in E(\gamma)\\ x\in f(e)}} Y^j_{e,down}\Big)^2$$

$$= \big(\textstyle\sum Y^j_{e,up}\big)^2 + \big(\sum Y^j_{e,down}\big)^2 - 2\big(\sum Y^j_{e,up}\big)\big(\sum Y^j_{e,down}\big)$$

$$= 2\big(\textstyle\sum Y^j_{e,up}\big)^2 + 2\big(\sum Y^j_{e,down}\big)^2 - \big(\sum Y^j_{e,up} + \sum Y^j_{e,down}\big)$$

- SNF is mapped into the abstract angular momentum space:

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes \dots \otimes |j_n m_n\rangle$$

- $$\mathrm{Spec}(Ar(S)) = \frac{\ell_p^2}{4}\beta\sqrt{2j_1(j_1+1) + 2j_2(j_2+1) - j_{12}(j_{12}+1)}$$

j_1 = total angular momentum of the outgoing edges

j_2 = ——— " ——— the ingoing edges

$j_{12} \hat{=}$ coupled angular momentum of the ingoing and outgoing edges

- Area: Minimal eigenvalue

$j_2 = 0$, $j_1 = \frac{1}{2}$ (vice versa)

$j_{12} = 0 + \frac{1}{2} = \frac{1}{2}$ coupled angular momentum

$$\lambda_0 = \frac{\ell_p^2 \beta}{4} \sqrt{2\left(\frac{1}{2}\left(\frac{1}{2}+1\right) + 0 - \frac{1}{2}\left(\frac{1}{2}+1\right)\right)}$$

$$= \frac{\ell_p^2 \beta \sqrt{3}}{8} \quad \text{area gap}$$

- Constraints in LQG:

- Gauß constraint

$$G(\Lambda) = \int_\Sigma d^3x \left(\Lambda^j D_a E^a_j\right)(x)$$

$$= - \int_\Sigma d^3x \left(D_a \Lambda^j\right) E^a_j(x)$$

- Quantization works similar to the regularization of the flux operator

$$\hat{G}(\Lambda) = \frac{i \hbar \ell_p^2}{2} \sum_{v \in V(\gamma)} \Lambda_j(v) \left(\sum_{\substack{e \in E(\gamma) \\ v = b(e)}} R^j_e - \sum_{\substack{e \in E(\gamma) \\ v = f(e)}} L^j_e \right)$$

- Diffeomorphism constraint

Classical expression

$$\vec{C}(\vec{N}) = 2 \int_\Sigma d^3x \left(N^a F^j_{ab} E^b_j\right)(x)$$

- Inner product is invariant under spatial diffeo's, hence finite diffeos are implemented unitarily

$$\hat{U}(\varphi) T_{\gamma,\pi,e} = T_{\varphi(\gamma),\pi,e} \quad \forall \varphi \in \mathrm{Diff}$$

$\hat{V}$

- <u>Question:</u> Does there exist an operator such that

$$\hat{U}_t = \hat{U}(\varphi_t) = \exp(it\hat{V})$$

$\hat{V}$ should be self-adjoint.

- <u>Def:</u> weakly continuous

$\hat{U}_t(\varphi)$ is weakly continuous if

$$\lim_{t\to 0} \langle T_{\gamma',\pi',e'}, \hat{U}_t(\varphi) T_{\gamma,\pi,e} \rangle = \langle T_{\gamma',\pi',e'}, T_{\gamma,\pi,e} \rangle$$

for $T_{\gamma,\pi,e} \in \mathcal{H}_{kin}$

- Let φ^v_t be a one-parameter family of diffeom. generated by a VF v which is non-vanishing. We choose γ in the support of the VF and then find $\varepsilon > 0$ so that $\varphi^v_t(\gamma) \neq \gamma$ for all $0 < t < \varepsilon$

Furthermore we choose $T_{\gamma',\pi',e'} = T_{\gamma,\pi,e}$

$$\lim_{t\to 0} \langle T_{\gamma,\pi,e}, \hat{U}_t(\varphi^v) T_{\gamma,\pi,I} \rangle$$

$$= \lim_{t\to 0} \langle T_{\gamma,\pi,e}, T_{\varphi^v_t(\gamma),\pi,e} \rangle = 0$$

$\#$

but RHS $\langle T_{\gamma,\pi,e}, T_{\gamma,\pi,e} \rangle = 1$

- $\hat{U}(\varphi)$ are not weakly continuous and we can not implement $\hat{C}(N)$ as operator on $\mathcal{H}_{kin}$.
- Problem? No because we have finite diffeomorphisms and then we require for physical states

$$\hat{U}(\varphi)\Psi_{phys} = \Psi_{phys}$$

$$\hat{U}(\varphi)\,\Psi_{phys} = \Psi_{phys}$$

- <u>Solution of the constraint operators in LQG:</u>

- <u>Gauß constraint</u>

$$\hat{G}(\Lambda) = \frac{i\beta\ell_p^2}{2}\sum_{v\in V(\gamma)}\Lambda_j(v)\left(\sum_{\substack{e\in E(\gamma)\\ v=b(e)}} R_e^j - \sum_{\substack{e\in E(\gamma)\\ v=f(e)}} L_e^j\right)$$

- Kinematical Hilbert space

$$\mathcal{H}_{kin} = \bigoplus_{\gamma}\mathcal{H}'_{\gamma} = \bigoplus_{\gamma}\bigoplus_{\pi,\ell}\mathcal{H}_{\gamma,\pi,\ell}$$

- We are looking for states

$$\hat{G}(\Lambda)\,\Psi_{phys} = 0$$

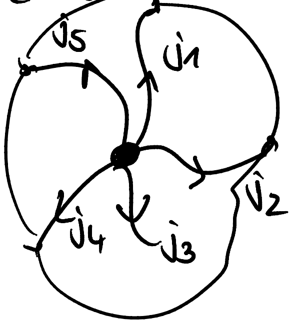
- The gauge-transformations act on vertices and a given SNF is transformed under the representation associated with the vertex.

- So choosing the trivial representation at the vertices provides a gauge-invariant SNF

$$\mathcal{H}^{G}_{inv} = \bigoplus_{\gamma,\pi,\ell}\mathcal{H}_{\gamma,\pi,\ell=0} \subset \mathcal{H}_{AL}$$

here live solutions of $\hat{G}(\Lambda)$

Example:



$$|j_1\rangle\otimes|j_2\rangle\otimes|j_3\rangle\otimes|j_4\rangle\otimes|j_5\rangle$$

$$J^{tot} = 0$$

Diffeomorphism constraint.

- <u>Diffeomorphism constraint:</u>

Here we are looking for states

$$\hat{U}(\varphi)\,\Psi_{diff} = \Psi_{diff}$$

- In constrast to the Gauß constraint we will not be able to find solutions within the kinematical HS $\mathcal{H}_{AL}$

- <u>Warm up:</u> QM

Momentum operator $\hat{p}\,\Psi_\lambda = \lambda\,\Psi_\lambda$

$$-i\hbar \frac{d}{dx}\Psi_\lambda(x) = \lambda\,\Psi_\lambda(x)$$

solution is $\Psi_\lambda(x) = e^{-ikx}$, $k = \frac{\lambda}{\hbar} \in \mathbb{R}$

- $\mathcal{H}_{QM} = L_2(\mathbb{R}, dx)$

$$\langle \Psi_\lambda, \Psi_\lambda \rangle = \int_{-\infty}^{+\infty} dx\, e^{ikx}\, e^{-ikx} = \infty$$

$\Rightarrow \Psi_\lambda \in \mathcal{H}_{QM}$

Ψ_λ generalized eigenvectors which live larger space than $\mathcal{H}_{QM}$

- <u>Refined Algebraic Quantization:</u>

- If we work with unbounded operators there are not defined on the entire Hilbert space but only on a dense subset

In our case $\mathcal{D}_{kin} \subset \mathcal{H}_{AL}$

- (algebraic) dual of $\mathcal{D}_{kin}$ denoted by $\mathcal{D}^*_{kin}$

$\mathcal{D}^*_{kin}$ - all not necessarily continuous

• D^*_{kin} = [illegible] linear functionals that act on D_{kin}

functional $\ell : D_{kin} \rightarrow \mathbb{C}$

• We have:

$$D_{kin} \subset \mathcal{H}_{AL} \subset D^*_{kin}$$

We are looking for solution in this set

• So far we have only defined the action of operators on states living in $\mathcal{H}_{AL}$, so we need to extend their action to D^*_{kin}

• Denote the dual action with $\hat{O}'$

$$[\hat{O}'\ell](f) := \ell(\hat{O}^\dagger f) \quad \text{f.a. } f \in D_{kin}$$

• Reason for the occurence of the adjoint:
If we consider an $\ell \in \mathcal{H}_{AL} \subset D^*_{kin}$ then by Riesz-representations theorem we can find a unique $f_\ell \in \mathcal{H}_{AL}$ such

$$\ell = \langle f_\ell, \cdot \rangle_{\mathcal{H}_{AL}}$$

• Compute the dual action for such an ℓ:

$$[\hat{O}'\ell](f) = \ell(\hat{O}^\dagger f) = \langle f_\ell, \hat{O}^\dagger f \rangle = \langle \hat{O} f_\ell, f \rangle.$$

• Dual action can be used to define the requierement for diff-invariant states

$$[\hat{U}'(\varphi)\ell](f) = \ell(\hat{U}^\dagger(\varphi) f) \stackrel{!}{=} \ell(f) \quad \text{for all } f \in D_{kin}$$

• Denote the set of solutions by D^*_{phys}

• And then we have a similar relation at the physical level

$$D_{phys} \subset \mathcal{H}_{phys} \subset D^*_{phys}$$

↑ Physical operators are defined here

- In order to construct the physical inner product we will need a rigging map

$$\eta : \mathcal{D}_{kin} \longrightarrow \mathcal{D}^*_{phys} \quad , \; f \mapsto \eta(f)$$

- Requirements on η:

(i) Dual action should preserve the space of solution

$$\hat{O}'\eta(f) = \eta(\hat{O} f) \quad \forall \; f \in \mathcal{D}_{kin}$$

(ii) $[\eta(\tilde{f})](f)$ needs to be a sesquilinear form for all $\tilde{f}, f \in \mathcal{D}_{kin}$

- Then we can define an inner product as:

$$\langle \Psi, \tilde{\Psi} \rangle_{phys} = \langle \eta(f), \eta(\tilde{f}) \rangle_{phys} := [\eta(\tilde{f})](f)$$

- Once a rigging map exist we can construct our inner product and the $\mathcal{H}_{phys}$.

- Example: Group averaging

- Suppose first class constraint $\hat{C}_I$, $[\hat{C}_I, \hat{C}_J] = f^{k}_{IJ} \hat{C}_k$ let C_I be self-adjoint then we define

$$\hat{U}(g) = \exp\left(i \sum_I t^I C_I\right) \qquad g \in G$$

- $\hat{U}(g)$ defines a unitary representation of G

- Rigging map:

$$\eta : \mathcal{D}_{kin} \longrightarrow \mathcal{D}^*_{phys}$$

$$f \longmapsto \eta(f) = \int_G d\mu_H(g) \langle \hat{U}(g) f, \cdot \rangle_{kin}$$

- Let's cross check that is indeed a solution

$$[\hat{U}'(g)\eta(f)](\tilde{f}) = \eta(f)(\hat{U}^+(g)\tilde{f})$$

$$= \int_G d\mu(\tilde{g}) \langle \hat{u}(\tilde{g})f, \tilde{u}^+(g)\tilde{f}\rangle$$

$$= \int_G d\mu(\tilde{g}) \langle \hat{u}(g)\hat{u}(\tilde{g})f, \tilde{f}\rangle$$

$$= \int_G d\mu(\tilde{g}) \langle \hat{u}(g\tilde{g})f, \tilde{f}\rangle \quad , \quad \bar{g} := g\tilde{g}$$

$$= \int_G d\mu(g^{-1}\bar{g}) \langle \hat{u}(\bar{g})f, \tilde{f}\rangle \quad \mu_H \text{ is 'left-invariant}$$

$$= \int_G d\mu(\bar{g}) \langle \hat{u}(\bar{g})f, \tilde{f}\rangle$$

$$= [\eta(f)](\tilde{f})$$

$\Rightarrow$ that $\eta(f)$ is invariant

- We will now use this framework in order to solve the diffeo:

Solutions: $[\hat{u}'(\varphi)\ell](f) = \ell(\hat{u}^+(\varphi)f) \stackrel{!}{=} \ell(f)$

- Since the SNF lie dense in $\mathcal{H}_{AL}$ it sufficient to require this for $T_{\gamma,\pi,\ell}$

$$\ell(\hat{u}^+(\varphi)T_{\gamma,\vec{j},\ell}) \stackrel{!}{=} \ell(T_{\gamma,\vec{j},\ell})$$

- multilabel $s = \{\gamma, \pi_{\vec{j}}, e\}$ T_s

- An idea motivated from the group-avering example is

$$T_S \longmapsto \sum_{\varphi \in Diff} \langle \hat{u}(\varphi)T_S, \cdot \rangle$$

it is diff-invariant

- Problem: One can find uncountably many diffeos that leave the SNF T_S invarient and then this map is ill-defined

- The construction of the solution is done in two steps:

(i) Average over the group of graph-symmetries

$TDiff_\gamma$ subgroup of Diff which maps γ to itself and also preserves all edges

$Diff_\gamma$ subgroup of Diff which preserves the graph γ but edges can be permuted

- Quotient $Diff_\gamma / TDiff_\gamma = GS_\gamma$

GS_γ = group of graph symmetries of γ

- Group averaging for GS_γ:
Projector which projects on subspace of $\mathcal{H}'_\gamma$, which is invariant $\hat{GS}_\gamma$

$$\hat{P}_{diff,\gamma} T_s = \frac{1}{N_\gamma} \sum_{\varphi \in GS_\gamma} \hat{U}(\varphi) T_s$$

number of elements in GS_γ

(i) Group-averaging for GS_γ:

- Projector:

$$\hat{P}_{diff,\gamma} T_s = \frac{1}{N_\gamma} \sum_{\varphi \in GS_\gamma} \hat{U}(\varphi) \overline{T_s}$$

N_γ : # of elements in GS_γ

solution of this average lie in subspace of $\mathcal{H}'_\gamma$ that is preserved by GS_γ

(ii) Average wrt to the remaining diffeos that move the graph γ to obtain $\check{\eta}$

$$\eta(T_s) = \frac{1}{N_\gamma} \sum_{\varphi \in \mathrm{Diff}/\mathrm{Diff}_\gamma} \langle \hat{U}(\varphi) P_{diff,\gamma} T_s, \cdot \rangle$$

- We can now define the diff-invariant inner product:

$$\langle \Psi, \tilde{\Psi} \rangle_{diff} = \langle \eta(f), \eta(\tilde{f}) \rangle_{diff} = [\eta(f)](\tilde{f})$$

- Cauchy completion wrt $\langle \cdot, \cdot \rangle_{diff}$ provides $\mathcal{H}_{diff}$
- Inner product has the property that operators on $\mathcal{H}_{diff}$ are self-adjoint if and only if they are self-adjoint wrt $\langle \cdot, \cdot \rangle_{kin}$
- So it is possible to obtain gauge-invariant and diff invariant solution that live in $\mathcal{H}^G_{diff}$.

More on reps of $\mathfrak{A}$

Why ask?

- generator of diffeo
- spectrum of geom operators ?

Natural requirements for „fundamental" rep

- cyclic $\left(\{ \pi(a)\Omega \mid a \in \mathfrak{A} \} \text{ dense} \right)$
- diffeo invariance

$$\varphi \longmapsto U_\varphi \quad \text{unitary}$$

$$U_\varphi \Omega = \Omega$$

Basic fact: (LOST, Fleischhack) There is only one such representation of $\mathfrak{A}$ (precisely defined)

$\exists$ other reps, that violate some of the premises.

Varadarajan: slightly admissible, diff invariant

Ackermann, Michael: Diff invariant + cyclic but larger algebra

Jurek Koslowski: Rep. with background (comp. w. thermal states in QFT)

AL ground state $\Omega \hat{=} \delta_{E,0}$ in momentum rep

$$\Omega_{E^{(0)}} \hat{=} \delta_{E,E^{(0)}}$$

$E^{(0)}$ classical triad field.

How to do?

$$\mathcal{H}_{E^{(0)}} = \mathcal{H}_{AL} \qquad \pi(h) = h$$

$$\pi(E_n(S)) = X_n(S) + E_n^{(0)}(S)\,\mathbb{1}$$

$$E_n^{(0)} = \int_S * E_i^{(0)} n^i \, d^2x$$

Properties: · Cyclic

Properties: · Cyclic

· Can define quan. operators

$$\hat{A}_S = \hat{A}_{AL} + A(S, E^{(0)})$$
$$\hat{V}_R = \hat{V}_{AL} + V(R, E^{(0)})$$

Only symmetries of $E^{(0)}$ are unitarily implemented

Can be remedied:

$$|T\rangle_{E^{(0)}} \equiv |T, E^{(0)}\rangle \in \mathcal{H}^{E^{(0)}}$$

$$\mathcal{H}_{[E^{(0)}]} = \bigoplus_{\bar{E}^{(0)} \in [E^{(0)}]} \mathcal{H}^{\bar{E}^{(0)}}$$

$$\bar{E}^{(0)} \in [E^{(0)}] \Leftrightarrow \exists\, g, \varphi : \bar{E}^{(0)} = \mathrm{ad}_g(\phi_* E^{(0)})$$

on $\mathcal{H}_{[E^{(0)}]}$: Unitary imp. ✓

not cyclic.

Implementation of diffeos:

$\mathrm{Diff}_{(\gamma, E^{(0)})}$ = Group of diffeos that
· map γ onto γ
· are symmetries of $E^{(0)}$

$\mathrm{TDiff}_{(\gamma, E^{(0)})}$ = —— " ——
· map edges of γ onto themselves

· as symmetries of $E^{(0)}$

$$GS_{(\gamma, E^{(0)})} = \mathrm{Diff}(\gamma, E^{(0)}) / \mathrm{TDiff}(\gamma, E^{(0)})$$