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III Quantum dynamics

1) Quantum Ham. constraint

$H_E(N)$

$$H(N) = \frac{1}{2}\int_\Sigma N \, \frac{E^a_i E^b_j}{\sqrt{\det q}} \left[\epsilon^{ij}{}_k F^k_{ab} - (1+\beta^2) K^i_{[a} K^j_{b]}\right)$$

$T(N)$

General difficulties:

· Non-polynomial, ordering

Specific difficulties:

· inverse volume

· F? K?

Guiding principle: Dirac algebra, in part.

$$[\vec{H}(\vec{N}), H(N)] = H(\mathcal{L}_{\vec{N}} N)$$

$$[H(N), H(M)] = \vec{H}(\vec{S}) + G(\;)$$

$$S^a = (N \partial_b M - M \partial_b N) \frac{E^a_i E^{bi}}{\det q}$$

$$[H_E(N), H_E(M)] = \vec{H}(\vec{S})$$

Can get well defined operators that are anomaly-free in a certain sense!
Highly non-trivial.

Thiemann's quantization (with ideas from Rovelli, Thiemann's quantization (with ideas from Rovelli, Smolin, Lewandowski)

First idea: Poisson relations:

$$V = \int_\Sigma d^3x \sqrt{\det q}\,, \quad \bar{K} = \int d^3x\, K^i_a E^a_i$$

Then

$$\frac{E^a_i E^b_j\, \epsilon^{ijk}}{\sqrt{\det q}} = 4\, \epsilon^{abc} \{V, A^k_c\}$$

and

$$K^i_a = 2 \{\bar{K}, A^i_a\}$$

Can write

$$H_E(N) = c \int_\Sigma N\, \epsilon^{abc}\, \mathrm{tr}\left(F_{ab} \{A_c, V\}\right)$$

$$T(N) = \tilde{c} \int_\Sigma N\, \epsilon^{abc}\, \mathrm{tr}\left(\{A_a, \bar{K}\}\{A_b, \bar{K}\} \cdot \{A_c, V\}\right)$$

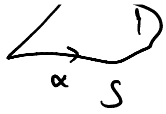
For $\bar{K}$:

$$\bar{K} = \{V, H_E(1)\}$$

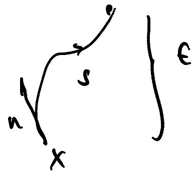
can use $\{\cdot,\cdot\} \longmapsto \frac{1}{i\hbar}[\cdot,\cdot]$

Second idea:

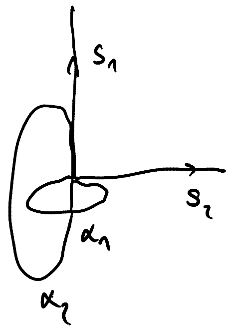
$$\int_S F = \frac{1}{2}\left(h_{\alpha^{-1}} - h_\alpha\right) + O(\epsilon^2)$$



$$\epsilon n^a \{A_a(x), V\} \simeq -h_s^{-1}\{h_s, V\}$$



$$H_E(N) \simeq c \sum_{\square} N(v_\square) \sum_{I=1}^{3} \mathrm{tr}\left(\left(h^{-1}_{\alpha_I} - h_{\alpha_I}\right) h^{-1}_{s_I}\{h_{s_I}, V\}\right)$$



Details don't matter classically for $\square \to \Sigma$

<u>Quantization</u>:

In principle: Pick family of regularizations. Pick h, $\hat{V}$, commutators, take limit $\square \to \Sigma$.

Problems:

1) Create ∞ many loops/vertices in limit
   $\rightsquigarrow$ Order $\hat{V}$ to the right

2) Still does not converge (typically
   $H_E^{(\square)}\Psi \not\to H_E^{(\square')}\Psi$

3) ambiguities.

For 2), 3): consider matrix elements.

diffeo invariant states, state dependent

regularization

Can achieve

$$\lim_{\square \to \Sigma} \left(\Psi \,\middle|\, H_E^{(\square)}(N) \,\middle|\, f \right\rangle \text{ well defined.}$$

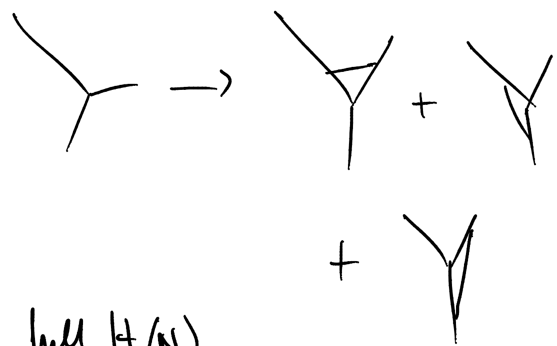
In fact, becomes ~~~~ constant at finite regulator!

Note: Defines operator

$$H_E^{+} : \mathcal{H}_{\mathrm{Diff}} \to \mathrm{Cyl}^*$$

Generic features:

- acts locally around vertices
- creates/annihilates edges/spins



Same for full $H(N)$.

<u>Solutions</u>: states Ψ in $\mathcal{H}_{\mathrm{Diff}}$ s.t.

$$\left(\Psi \,\middle|\, H(N) \,\middle|\, f \right\rangle = 0 \quad \forall f, N$$

Details matter.

<u>Relation to SF models</u> (Rovelli, Reisenberger):

Formal expansion of projector P_H on kernel of H:

$$P_H = \delta(H) = \int DN \, e^{iH(N)}$$

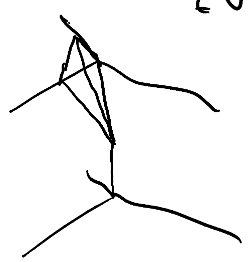
$$= 1 + i \int DN \int H(x) N(x) \, dx$$

$$+ \frac{i^2}{2} \int DN \int dx_1 dx_2 \, N(x_1) N(x_2) H(x_1) H(x_2)$$

$$+ \dots$$

$$(T_{\gamma_1} | P T_{\gamma_2})$$

$$= \sum_{N=0}^{\infty} \underbrace{\sum_{v_1} \dots \sum_{v_N}}_{\in V(\gamma_2)} \frac{i^N}{N!} (T_{\gamma_1} | H(v_1) \dots H(v_N) T_{\gamma_2})$$



$H \longleftrightarrow SF$

Remarks:

Anomaly-freeness: Want to look at

$$[H_E, H_E]$$

Problem: $H_E : \mathcal{H}_{diff} \to Cyl^*$

Solution: 1) Thiemann: Look at $[\,,\,]$ on $\mathcal{H}_{kin}$ before $D \to \Sigma$

$$[H^\square(N), H^\square(M)] = \text{Something} \neq 0$$

$$(\Psi | \text{Something} = 0$$

Sol2) Lew. Marolf „vertex smooth states"

$(\Psi| \in Cyl^*$ vertex-smooth if

$$(\Psi | U_\varphi f_\gamma)$$

depends only on vertices.

$(\Psi_{diff} | H(N)$ is vertex smooth, extends to vertex-smooth.

$$(\Psi_{VS} | [H, H] = 0$$

but quantization of RHS also vanishes on V.S. states. $\rightsquigarrow$ Inconclusive.

- <u>Quantization of the BK-Model</u>

Classical: (Q_{ab}, P^{ab}),

$$H_{phys} = \int_S d^3s \sqrt{(C^4)^2 - Q^{ab} C_a^4 C_b^4}$$

- <u>Quantize</u>: Ashetar variables

$$(A_a^j, E_j^a) \xrightarrow{\text{Observable}} (A_j^J, E_J^j)$$

$J = 1,2,3$ Lie algebra

j = spatial indices

- Gauß constraint

$$D_J E_J^j = G_J = 0$$

- Gauß constraint is solved at the QT
- $H_{phys}(E, A)$
- (ii) Discuss the algebra of observables

$$\{O_f, O_{f'}\} \cong O_{\{f,f'\}^*}$$

$$\{f,f'\}^* = \{f,f'\} - \sum_{J=0}^{3} \{f, C_J^{tot}\}\{f', S^J\} - f \leftrightarrow f'$$

$$C_J^{tot} = (C, C_j) \quad , \quad S^J = (T, S^j)$$

- In our case for purely gravitational dof $\{f,f'\}^* = \{f,f'\}$

$$\{A_J^j(s), E_k^k(s)\} \cong \delta_J^k \delta_j^k$$

- Hence we can use the representation of $\mathcal{H}_{4L}$ for $\mathcal{H}_{phys}$ in this model
- Choice!
- <u>Remark</u>: As a first step Choice!

$$H_{phys} = \int_S d^3s \sqrt{|(C^\varphi)^2 - Q^{ab} C_a C_b|^2}$$

- <u>Regularization</u>:

Works similar like for the Hamiltonian constraint

$$H_{phys}^{\square} = \sum_{\square \in \mathcal{P}_S} \sqrt{|(C^\varphi(\square))^2 - \delta^{IJ} C_I(\square) C_J(\square)|}$$

$$\text{where} \quad C(\square) = \int_\square d^3s\, C(s) \quad C_I(\square) \int_\square d^3s\, C_I(s) \quad C_I = -i G_I$$

- $B_I^j = \frac{1}{2} \varepsilon^{ijk} F_{kl}^I \quad , \quad B = B_I^j \tau_I e_j^k \tau_k$

• $(\mathrm{Tr}(\mathcal{B}))^2 = (C^g)^2$, $[\mathrm{Tr}(\mathcal{B}\tau_I)]^2_{\frac{1}{4}} = Q^{ij}C_iC_j$

• Operator is then operator:

$$\hat{H}_{phys} = \lim_{D \to S} \sum_{D \in S} \sqrt{|\hat{C}(D)\hat{C}^{\dagger}(D) - \delta^{IJ}\hat{C}_I(D)\hat{C}_J(D)|}$$

• Remark: We cannot quantize C_j but $Q^{ij}C_iC_j$

• $C_j \sim FE$, $Q^{ij} = \frac{EE}{\det(E)}$

$Q^{ij}C_iC_j \sim \frac{EE}{\det(E)}FEFE$, $C \sim \frac{EE}{\sqrt{\det(E)}}F$

• Thiemann's trick: $\tau_\mu = (\mathbb{1}, \tau_j)$ $\mu = 0,1,2,3$

$$\int_D d^3s\, \mathrm{Tr}(\mathcal{B}\tau_\mu) = \int_D d^3s\, \mathrm{Tr}(F \wedge e\,\tau_\mu)$$

$$= \frac{1}{k}\int_D d^3s\, \mathrm{Tr}(F \wedge \{V(D), A\}\tau_\mu \})$$

• Classical Symmetries:

$\{C_j, H_{phys}\} = \theta$, $\{H(s), H(s')\} = 0$

QT: "$[\hat{C}_j, H_{phys}] = 0$" $[\hat{H}(s), \hat{H}(s')] = 0$

Hphys should be spatially diff-inv.

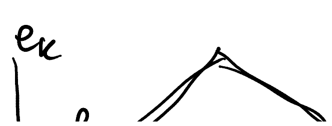
• We would like to work in $\mathcal{H}_{kin}$

• Spatially diff-inv. operators that are graph-changing cannot be implemented in the representation of $\mathcal{H}_{kin}$

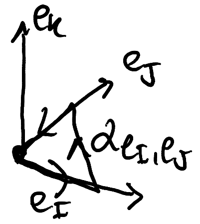
• Consequence: Quantize it graph-preserving

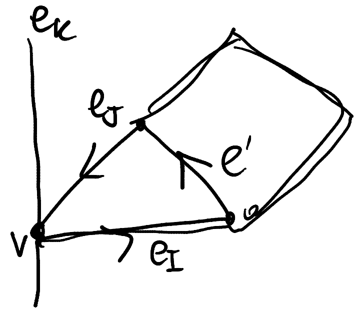
$\mathcal{H}_{AL} = \bigoplus \mathcal{H}_{\gamma}'$ preserves each γ could be a problem!

• Minimal loop:



- Minimal loop:





- Define the operator for each graph:

$$\hat{C}_{\mu,\gamma,v} = \frac{1}{\ell_p^2} \sum_{e_I, e_J, e_K} \varepsilon^{IJK} \frac{1}{|L|} \sum_{\substack{\alpha \in \text{Minimal} \\ \text{loops at } v}}$$

$$\mathrm{Tr}\left(\tilde{\tau}_\mu h_\alpha h(e_K)\left[h^{-1}(e_K), \hat{V}_\gamma\right]\right)$$

- In order to avoid that no edges are deleted we need to introduce a projector P_γ:

$$\hat{H}_{phys,\gamma} = \sum_{v \in V(\gamma)} \sqrt{|P_\gamma [\hat{C}_{0,\gamma,v} \hat{C}^\dagger_{0,\gamma,v} - \hat{C}_{I,\gamma,v} \hat{C}^\dagger_{I,\gamma,v}] P_\gamma|}$$

$$\hat{H}_{phys} = \bigoplus_\gamma H_{phys,\gamma}$$

- Semiclassical tools to test the QT

- DGKL-Model

gravity + 1 K.G. field

$\tilde{C} = \pi - h \qquad h = \sqrt{-\sqrt{q}\, C^g + \sqrt{q}\sqrt{(C^g)^2 - q^{ab} C_a C_b}}$

$C_a = 0$

$G_j = 0$

- Constraints are not reduced classically but in the QT:

- Hamiltonian:

$$(\hat{\pi} + \hat{h})\,\Psi(\phi, A) = 0 \quad , \quad \hat{\pi} = -i\hbar \frac{\delta}{\delta\phi(x)}$$

formal solution $\Psi(\phi, A) = e^{i\int d^3x\, \phi h}\, \varphi(A)$

formal solution $\Psi(\phi, A) = e^{i\int d^3x\, \phi h(A)} \varphi(A)$

- Quantum Dirac observables:

Def: A quantum Dirac observable is the restriction to the space of solutions to the constraints of any operator $\hat{O}$ that satisfies:

(i) $\hat{O}$ is gauge- and diff-invariant

(ii) $[\hat{O}, \hat{C}] = 0$

Expression:

$$O(\hat{L}) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[\hat{L}, \int d^3x\, \hat{\phi} \hat{h} \right]_{(n)}$$

$$\left[\hat{L}, \int d^3x\, \hat{\phi} \hat{h}\right]_{(n)} = \left[\hat{L}, \left[\hat{L}, \int d^3x\, \hat{\phi}\hat{h}\right]_{(n-1)}\right]$$

$$\hat{O}(\hat{L}) = e^{i\int d^3x\, \hat{\phi} h}\, \hat{L}\, e^{-i\int d^3x\, \hat{\phi} h}$$

- $\hat{O}(L)$ is only spatially diff- and gauge invariant if the operator $\hat{L}$ is

$\phi(x) = \phi_0 = \text{const}$ time-parameter

- Dynamics: Observables

$$\frac{d}{d\phi_0} O_{\phi_0}(\hat{L}) = -i \left[\hat{h}_{phys}, O_{\phi_0}(\hat{L})\right]$$

- In order formulate QT precisely we need to implement $\hat{h}_{phys}$

- (i) $\mathcal{H}^G_{diff}$ diff and gauge invariant

(ii) We need operators on $\mathcal{H}^G_{diff}$ that have a geometric interpretation from which we can construct Dirac observables

(iii) Define $\hat{h}_{phys}$ on $\mathcal{H}^G_{diff}$

- (i) Rigging map for Diff(M)

$$\widehat{h_{phys}} = \int d^3x \sqrt{-\sqrt{q}\,C^{gr} + \sqrt{q}\sqrt{(C^{gr})^2 - q^{ab}C_a^{gr}C_b^{gr}}}$$

- $q^{ab}C_aC_b$ should annihilate diff-inv. states
- Choice: $h_{phys} = \int d^3x \sqrt{-2\sqrt{q}\,C^{gr}}$
- (iv) $\hat{\sqrt{q}}(x)\,\hat{C}^{gr} < 0$

(v) $\{h(x), h(y)\} = 0 \xrightarrow{QT} [\hat{h}(x), \hat{h}(y)] = 0$

- $\mathcal{H}_{diff,x}$ $\quad$ $Diff_{,x}$

OUTLOOK (IMPORTANT (OPEN) PROBLEMS)

- Relation: Canonical $\longleftrightarrow$ Covariant
- Dynamics
  Relational formalism
  $\to$ Physics out of the models
- Semiclassical improvement
- Matter
- Black hole