(III) DeSitter spacetime

1) DeSitter Basics
2) Free field deSitter
3) Cosmic no hair
4) Interactions
5) Parametric rep. of deSitter correlators
6) IR - stability

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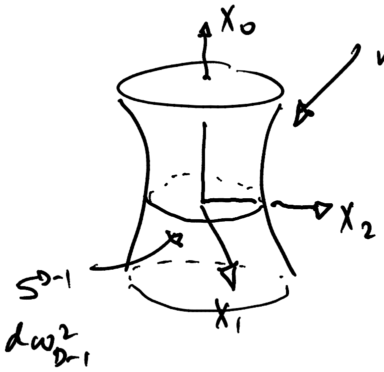
1) DeSitter basics:

DeSitter is a spacetime that:
- seems to describe early epoch of Universe
- seems to " recent " " ('accelerated expansion')

DeSitter space dS_D can be introduced as

$$dS_D = \{ X \in \mathbb{R}^{D+1} \mid -X_0^2 + X_1^2 + \dots + X_D^2 = H^{-2} \}$$



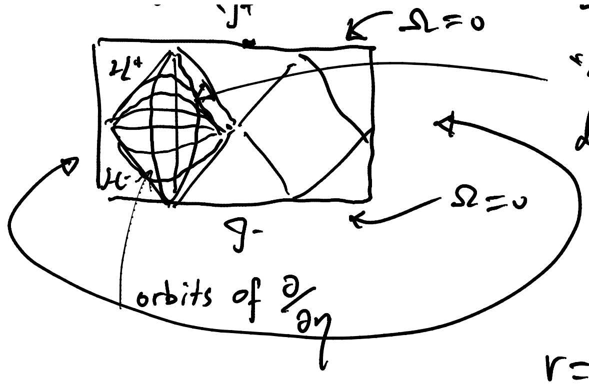
metric is that induced from $\mathbb{R}^{D,1}$

$$ds^2 = -d\tau^2 + H^{-2}\cosh^2 H\tau \, d\omega^2_{D-1}$$

H - Hubble constant

- DeSitter is a solution to Einstein eq.'s w/ positive cosm. const $\Lambda \propto H^2$
- DeSitter is a space of const. curvature $(\propto H^2)$
- Isometry group $O(D,1)$

A conformal diagram of dS can be obtained by writing metric as $ds^2 = \Omega^{-2}[-dt^2 + d\omega^2_{D-1}]$



"static region of dS"

$$ds^2 = -f\, d\eta^2 + f^{-1} dr^2 + r^2 d\omega^2_{D-2}$$

Identify

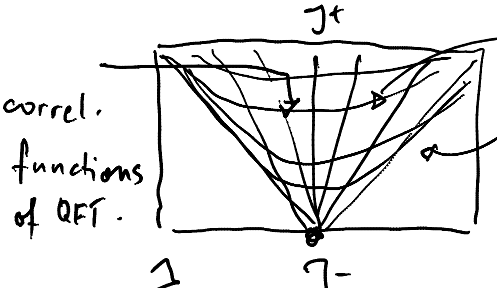
$$f(r) = 1 - H^2 r^2$$

$$r = \pm H^{-1} \text{ on } \mathcal{H}_\pm$$

In static chart $\partial/\partial\eta$ is a KVF

$\partial/\partial\eta$ is not globally time-like $\Rightarrow$ no analog of global "time-translations".

$\Rightarrow$
- No conserved energy for systems like e.g. $(\Box - m^2)\phi = 0$
- Notion of particle is problematic



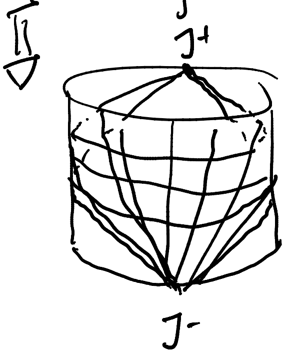
$\cong \mathbb{R}^{D-1} = \{t = \text{const.}\}$

horizon $\quad ds^2 = -dt^2 + e^{2Ht} d\vec{x}^2$

"cosmological chart"

dS_D = flat FRWL-spacetime

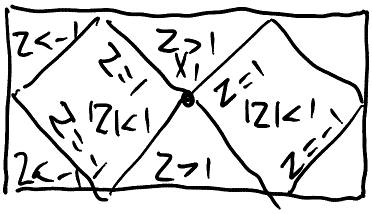
$(X_1, X_2 \in dS_D)$



Let $Z = H^2 X_1 \cdot X_2$

= 'point-pair invariant'

Z illustrates causal relationships in dS_D:



$$Z = \begin{cases} \cosh H\sqrt{-\sigma} & \text{timelike} \\ \cos H\sqrt{\sigma} & \text{spacelike} \end{cases}$$

σ = signed $\Box$'ed geodesic distance

2) The free field on dS

$$(\Box - m^2)\,\phi(x) = 0 \qquad (KG)$$

Want: $\underbrace{\text{States}}_{}$ (Gaussian) for such a quantum field

1) $\langle \phi(X_1)\,\phi(X_2)\rangle_\psi$ has to satisfy (KG) in both X_1, X_2
2) $$\langle [\phi(X_1), \phi(X_2)]\rangle_\psi = 2i\,\mathrm{Im}\,\langle \phi(X_1)\,\phi(X_2)\rangle$$ $$= i\Delta(X_1, X_2) \quad (= \Delta_A - \Delta_R)$$
3) Hadamard (OPE should hold)
4) positive: $\displaystyle\int_{X_1, X_2} \langle \phi(X_1)\,\phi(X_2)\rangle_\psi\, f(X_1)\,\overline{f(X_2)} \geq 0$ for any f.

It is also natural to consider invariant states:

$$\langle \phi(X_1)\,\phi(X_2)\rangle_\psi = \langle \phi(gX_1)\,\phi(gX_2)\rangle_\psi \qquad \forall g \in O(D,1)$$

It turns out that there is a unique invariant state for $m^2 > 0$:

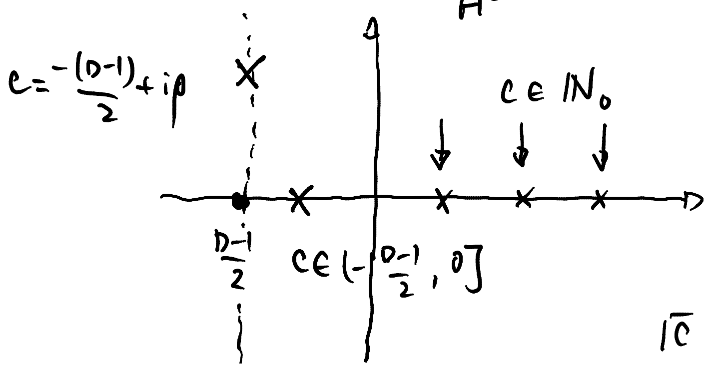
$$\langle \phi(X_1)\,\phi(X_2)\rangle_{\underbrace{BD}} = \text{cst}\cdot {}_1F_2\left(-c, D-1+c; \frac{D}{2}; \frac{1+Z}{2}\right)$$

Bunch-Davies $\qquad Z = H^2\, X_1 \cdot X_2$

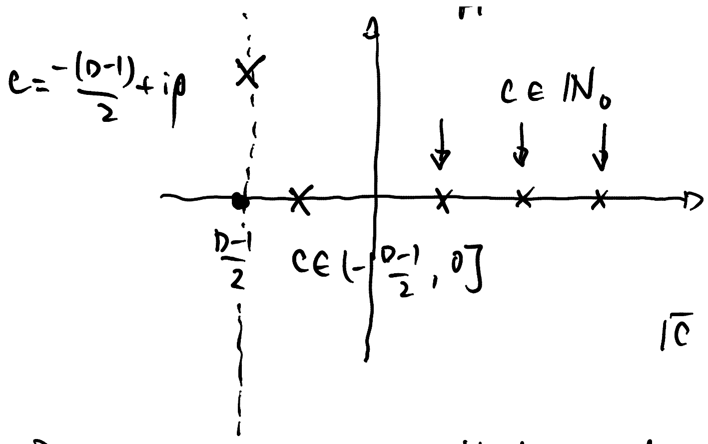
${}_1F_2$ hypergeom. fct.

$$c = -\frac{D-1}{2} + \sqrt{\left(\frac{D-1}{2}\right)^2 - \frac{m^2}{H^2}}$$

BD-state exists only for $m^2 > 0$



Relationship between values of c & representation theory of $SO(D,1)$



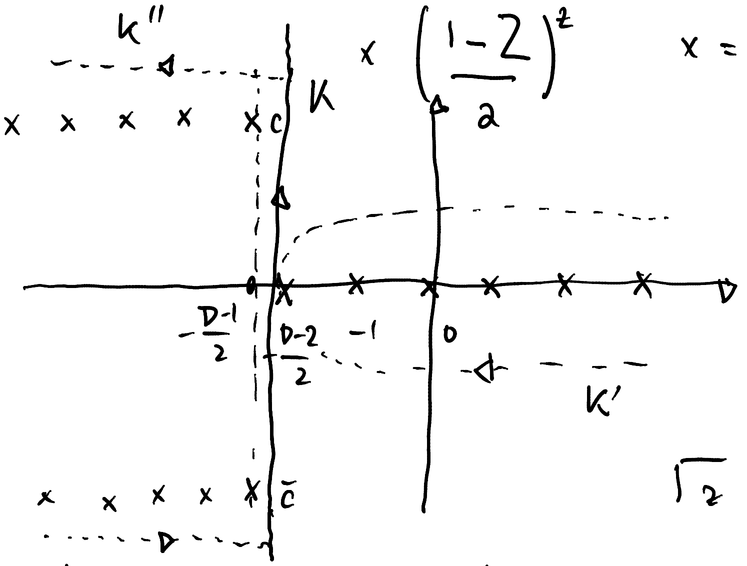
only for $m^2 > 0$

Relationship between values of c & representation theory of $SO(D,1)$

1) ✓ 2) ✓ 3) Hadamard condition:

$$\langle \phi(X_1)\phi(X_2)\rangle_{BD} = cst. \int_K \frac{\Gamma(D-1+c+z)\,\Gamma(-c+z)\,\Gamma(-z)}{\sin \pi z \quad \Gamma(\frac{D}{2}+z)} \times \left(\frac{1-Z}{2}\right)^z$$

$\times$ = poles of



IR-poles UV-poles

Using residue thm for K' $\Rightarrow$

$$\langle \phi(X_1)\,\phi(X_2)\rangle_{BD} \sim \frac{1}{(1-Z)^{\frac{D-2}{2}}} + \dots$$

$$\sim \frac{1}{\sigma^{\frac{D-2}{2}}} + \dots \quad \sigma \text{ - geodesic distance}^2$$

Hadamard cond. 3) ✓

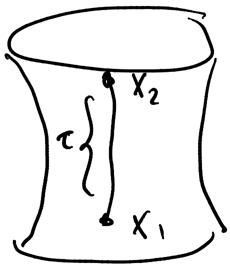
Alternatively, deforming $K \to K''$ & applying residue theorem

$$\langle \phi(X_1)\,\phi(X_2)\rangle \sim \frac{1}{\quad}$$

$$\langle \phi(X_1)\,\phi(X_2)\rangle_{BD} \sim \frac{1}{(1-Z)^{\frac{D-1}{2}+i\rho}} + \cdots$$

as $Z \to \infty \Rightarrow \langle \phi(X_1)\phi(X_2)\rangle_{BD} \sim e^{-H\left(\frac{D-1}{2}\right)\tau}$

$Z = \cosh H\tau$



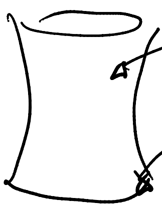
$\tau \to \infty$

A consequence of this IR behavior is that expect. value in arbitrary states decay exponentially in time

$$|\Psi\rangle = A|BD\rangle \qquad A = \int_{X_1 \ldots X_n} \overbrace{f(X_1,\ldots,X_n)}^{\text{comp. supp}}\,\phi(X_1)\cdots\phi(X_n)$$

$\Rightarrow \langle \phi(X)\rangle_\psi \sim O\left(e^{-H\left(\frac{D-1}{2}\right)\tau}\right)$ where

τ is the time coordinate of X



$\langle \phi(x)\rangle_\psi$ decays exponentially

fix $|\Psi\rangle$ down here

$\Rightarrow$ IR stability of deSitter spacetime

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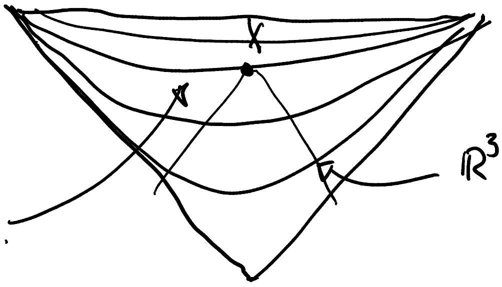
4) Interactions

Purposes: 1) Give concise & explicit formulas for

$\langle \phi(X_1)\cdots\phi(X_E)\rangle_{BD,\lambda}$ in a theory like

$$L = \left(\tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \lambda\phi^4\right) d\mu$$



$t = \text{const.}$

$\mathbb{R}^3$, $ds^2 = -dt^2 + e^{2Ht} d\vec{x}^2$

$\langle \phi(t, \vec{k}_1) \cdots \phi(t, \vec{k}_E) \rangle_{BD,\lambda}$ important observables in cosmology

i) $E = 2 \longleftrightarrow$ power spectrum of CMB

ii) $E = 3, \ldots \longleftrightarrow$ Non-Gaussianities in CMB

2) IR stability in on presence of $\lambda\phi^4$?

$\phi(X)$ = interacting field = perturbative formula in terms of retarded products

$$dS_D^{\mathbb{C}} = \{ X \in \mathbb{C}^{D+1} \mid X \cdot X = H^{-2} \}$$

contains both dS_D but also S^D = D-sphere

Shortcut: Do my calculations of $S^D \to$ analytic continuation to dS

Formula for correlation fcts.:

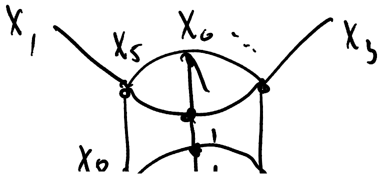
$$\langle \phi(X_1) \cdots \phi(X_E) \rangle_{BD,\lambda} = \underset{X_i \to dS_D}{\text{anal. cont.}} \left\{ \sum_{V=0}^{\infty} \frac{(-\lambda)^V}{V!} \times \right.$$

$$\left. \times \left\langle \phi(X_1) \cdots \phi(X_E) \left(\int_{S^D} \phi^4(Y)\, d\mu(Y) \right)^V \right\rangle_{BD,0} \right\}$$

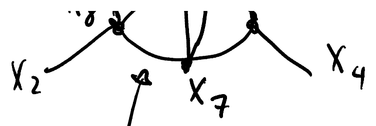
$$= \underset{X_i \to dS_D}{\text{anal. cont.}} \sum_{\text{graphs } G}{}' c_G \lambda^V \underline{I}_G(X_1, \ldots, X_E)$$

Aim: Calculate $\underline{I}_G$.

G



$X_5, X_6, \ldots$ integrated over S^D



$V = 8 \;, E = 4$

propagator $=$ cst. ${}_1F_2\left(-c, D-1+c; \frac{D}{2}; \frac{1+Z_{78}}{2}\right)$

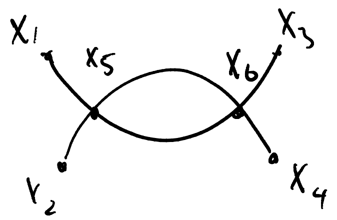
$$Z_{78} = H^2 X_1 \cdot X_2 \qquad (m^2 > 0)$$

$$I_G = \int_{\substack{\text{internal vertices} \\ \text{over } S^D}} \prod (\text{propagators})$$

$$I_G(X_1, \dots, X_E) = \left(\prod_{e \in EG} \int_K dz_e\right) \prod (\text{Gamma fct's of } z_e)$$
$$\left(\prod_{v=E+1}^{V+E} \int_{S^D} d\mu(X_v)\right) \prod_{e \in EG} \left[(X_{s(e)} - X_{t(e)})^2\right]^{z_e}$$

Need to do integrations over S^D

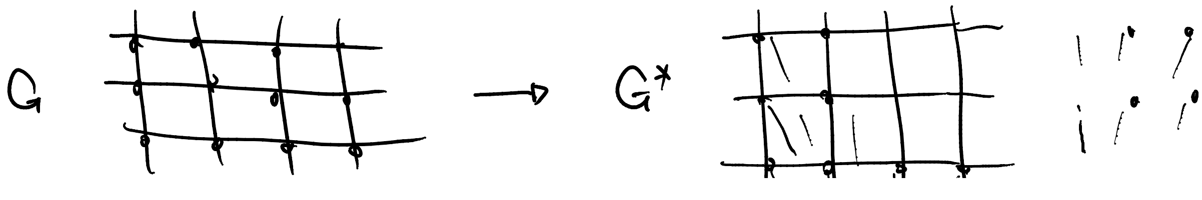
G

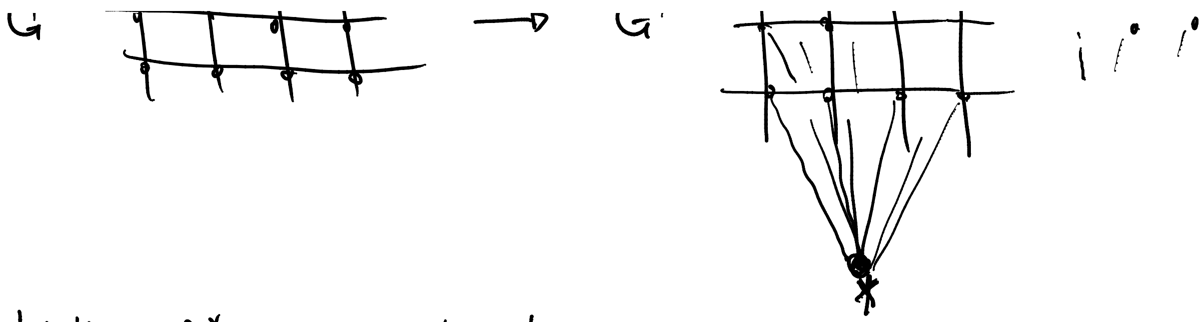


$$I_G(X_1, \dots, X_4)$$
$$\sim \int_K dz_{15} \int_K dz_{25} \cdots (\Gamma\text{'s})$$
$$\cdot \int_{S^D} d\mu(X_5) \int_{S^D} d\mu(X_6)$$
$$\cdot \left[(X_1 - X_5)^2\right]^{z_{15}} \left[(X_5 - X_6)^2\right]^{z_{56}} \cdots$$

$\longrightarrow$ renormalization

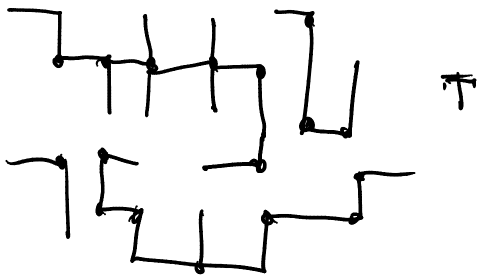
This is Long story but it turns out that renormalization can be done in a reasonably explicit form.





Within G^* you consider trees

$T \subset G^*$ is a subgraph without loops



Formula for $I_G(X_1, \dots, X_E)$

$$= \text{cst.} \int_{\vec{w}} \Gamma_G(\vec{w}) \prod_{1 \le r < s \le E} \left(\frac{1 - Z_{rs}}{2} \right)^{\sum\limits_{T \text{ conn. } r\, w.\, s} w_T}$$

- There is one contour integration over variable $w_T \in \mathbb{C}$ for each tree $T \subset G^*$, $\vec{w} = \{ w_T \}_{T \subset G^*}$
- $\Gamma_G(\vec{w})$ is a meromorphic fct of $\vec{w}$
- $Z_{rs} = H^2 X_r \cdot X_s$

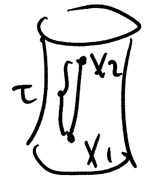
$$\Gamma_G(\vec{w}) = \frac{\Gamma\left(\frac{D+1}{2} + \sum_T w_T\right) \prod_T \Gamma(-w_T) \prod_{(ij) \in T} H\left(-\sum_{T \ni (ij)} w_F\right)}{\Gamma\left(\frac{D+1}{2} - \sum_{|T|=E} w_T\right) \prod_{(ij) \in \Phi} \Gamma\left(\frac{D+1}{2} + \sum_{T \not\ni (ij)} w_T\right) \prod \dots}$$

Φ particular tree

IR- stability of dS_D in case of an interacting

IR- stability of dS_D in case of an interacting field:

How does $\langle \phi(X_1)\phi(X_2)\rangle_{BD,\lambda}$ behave for Large time-Like separations



$\Longleftrightarrow$ How does $I_G(X_1, X_2)$ behave ... ?

$\Rightarrow$ analysis of poles of Γ_G shows that contours of $\vec{w}$ can be moved to the left to achieve

$$\left(\frac{1-Z_{12}}{2}\right)^{\cdots}_{Re(\ldots) < -\frac{D-1}{2}}$$

$$\Rightarrow I_G(X_1, X_2) \sim (Z_{12})^{-\frac{D-1}{2}} \text{ for } \underbrace{Z_{12}}_{e^{H\tau}} \to \infty$$

$$\Rightarrow \langle \phi(X_1)\phi(X_2)\rangle_{BD,\lambda} \sim \mathcal{O}\left(e^{-H\left(\frac{D-1}{2}\right)\tau}\right) \text{ as } \tau \to \infty$$

$$\Rightarrow \langle \phi(X)\rangle_{\psi,\lambda} \sim \mathcal{O}\left(e^{-H\left(\frac{D-1}{2}\right)\tau}\right)$$

$\Rightarrow$ exponential decay of 1-pt fct. in arbitrary states

$$\underset{\uparrow \text{observable}}{|C_{lm}|^2} \sim \int \underbrace{\langle \phi(t,\vec{k})\phi(t,\vec{0})\rangle_{BD,\lambda}}_{\text{needs}} \underbrace{K_{lm}(\vec{k})} \, d^3k$$

$$\langle T_{\mu\nu}(X)\rangle_{BD,\lambda} = 0$$

$$\langle T_{\mu\nu}(X)\rangle_{\psi,\lambda} = ?$$