Numerical simulations of Causal Dynamical Triangulations 1

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Outline

1. General introduction
   - Path integral for Quantum Gravity
   - Basic assumptions of CDT
   - Regularization of a theory
   - Construction elements in 4d
   - Geometry of 3d states and a 4d configurations
2. Numerical setup
   - Objectives
   - Monte Carlo technique
   - Phase structure

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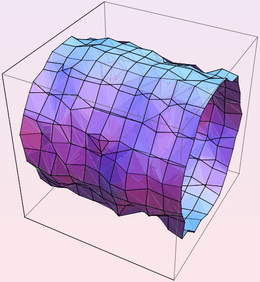
Path integral for Quantum Gravity

Quantum Gravity (without matter) - states of the system are defined as spatial geometries of the universe.
Example of the evolution of a one-dimensional closed universe:



Joining spatial geometries produces a **space-time geometry**. In this example the sum over trajectories becomes a (weighted) sum over all two-dimensional surfaces joining the in-state with the out-state.

Path integral for Quantum Gravity cont'd.

Our aim is to calculate the amplitude of a transition between two geometric states

$$G(\mathbf{g}_{\mathrm{i}}, \mathbf{g}_{\mathrm{f}}, t) := \sum\!\!\!\!\!\!\int_{\text{geometries: } \mathbf{g}_{\mathrm{i}} \to \mathbf{g}_{\mathrm{f}}} \mathrm{e}^{iS[\mathbf{g}_{\mu\nu}(t')]}$$

To define this path integral we have to specify the "measure" and the "domain of integration" - a class of admissible space-time geometries joining the in- and out- geometries.

Causal Dynamical Triangulations

- Using methods of QFT.
- Regularization of geometry follows the method of Dynamical Triangulations.
- New element: causality - Causal Dynamical Triangulations - additional restriction on the topology of space-time.

Very promising results of CDT

- Correct continuum limit.
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Basic concepts

- Path integral – amplitude of a quantum transition between in- and out- states can be written as a weighted sum (integral) over all possible trajectories.
- Possibility to perform analytic continuation in time – Wick rotation to imaginary time. In effect weights become real and positive and can be interpreted as probabilities.
- Lattice regularization – discretization of space-time provides a cut-off a.

In our approach (also in Dynamical Triangulations) we start with "Euclidean" formulation of space-time and then we eventually rotate back (or define) the time variable.

Wick rotation

Rotation to imaginary time $t \to it_4$ - the weight is formally real:

$$e^{iS[\mathbf{g}(t)]} \to e^{-S^E[\mathbf{g}(t_4)]}$$

After Wick rotation quantum amplitude becomes a weighted sum over geometric manifolds bounded by the in- and out-states.

The simplest form of the action – Hilbert–Einstein action

$$S[\mathbf{g}] = -1/G\,\mathrm{Curvature}(\mathbf{g}) + \lambda\,\mathrm{Volume}(\mathbf{g})$$

where G - gravitational constant, λ - cosmological constant (essential to suppress the entropy of quantum fluctuations).

This action used both by DT and CDT.

Measure and domain of integration in a path integral for QG

- **A.** Sum (integral) over diffeomorphism invariant equivalence classes of space-time metrics.
- **B.** Fixed topology of space-time.
- **C.** Suppressed formation of baby universes (fixed spatial topology).

- To suppress the divergent volume of the diffeomorphism group. Realized in the DT regularization.
- To suppress the divergence of the path integral coming from entropy. Realized in DT.
- Causality: it means the existence of a time foliation. For each time the topology of the universe is the same. Realized in CDT.

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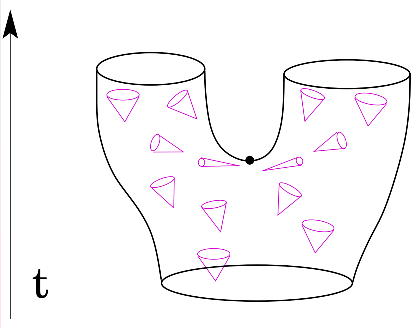
Difference between DT and CDT

Difference lies in the domain of integration over allowed space-time geometries. In DT one cannot avoid introducing causal singularities.

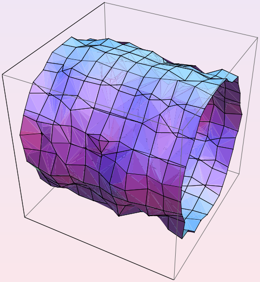


Example of a causal singularity, which leads to creation of baby universes. Creation of baby universes dominates the possible evolution.

Outline

Method of triangulations

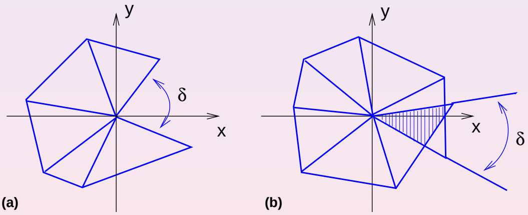
Counting equivalence classes of manifolds. Example in 2d.



Discretization: One of the the standard regularizations in QFT. Here: we replace a continuous space-time surface by a triangulated surface built from regular triangles with the edge a, serving as a cut-off. In the continuum limit $a \to 0$.

Consequence of introducing a triangulation

Example in 2d (Euclidean time):



In a triangulation a variable number of triangles can meet at each vertex. Deficit angle δ - (a) positive, (b) - negative. Curvature is localized in vertices. **In other points geometry is flat!**.

Three steps in regularization of a path integral

Regularization of a geometric state

One-dimensional state with a topology S^1 is built from links with length a.

Regularization of a space-time geometry (trajectory)

2d space-time surface built from equilateral triangles. Curvature localized in vertices.

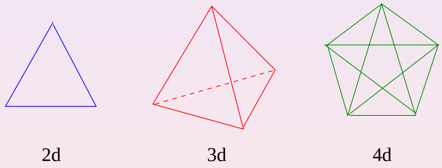
Regularization of a path integral

Integral over equivalence classes of metrics is replaced by a summation over all possible triangulations, belonging to some topological class.

Outline

Generalization to higher d

Method of Euclidean Dynamical Triangulations

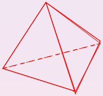


Replace 2d triangles by higher-dimensional simplices.

Outline

CDT: 3d geometric “states”

Spatial states are 3d geometries with a topology S^3.
Discretized states are constructed from 3d simplices - tetrahedra glued along triangular faces.

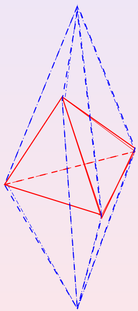


Regular tetrahedron (3-simplex) - a basic block to build 3d manifolds.

Space of states

There are many inequivalent ways of gluing tetrahedra. For N tetrahedra and a fixed topology this number **grows exponentially** $\sim \exp(\lambda N)$.

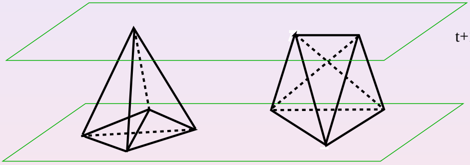
Connecting 3d states



In 4d each tetrahedron becomes a base of a pair of $\{4,1\}$ and $\{1,4\}$ simplices, pointing up or down in t. The lengths of edges in time direction are a_t (may be different than a_s).

Connecting 3d states cont'd'

We need two more types of simplices: $\{3, 2\}$ and $\{2, 3\}$.



Simplices $\{3, 2\}$ and $\{2, 3\}$ form a "layer" gluing together states at t and $t + 1$.

It takes at least 4 steps to connect two $\{4, 1\}$ simplices at times t and $t + 1$.

$$\{4, 1\} \to \{3, 2\} \to \{2, 3\} \to \{1, 4\} \to \{4, 1\}$$

Space-time manifolds in 4d (trajectories)

We build a 4d manifold with a topology $S_3 \times S_1$. Each manfold is characterized by a set of "global" numbers

- $N_4^{\{4,1\}}$ - number of $\{4,1\}$ and $\{1,4\}$ simplices.
- $N_4^{\{3,2\}}$ - number of $\{3,2\}$ and $\{2,3\}$ simplices.
- N_0 - number of vertices (0-simplices).
- T - time period.

Other "global" numbers depend on those above.
Each manifold is a specific way of gluing together geometric states at integer times t.
For a discretized manifold the Hilbert-Einstein action depends only on these global numbers.

Each 4d manifold is represented by a “local” information, describing how simplices are glued together. To do this we assign labels to vertices.

Definition

Manifolds are assumed to be simplicial manifolds:
Each (sub)simplex with a particular set of labels appears at most once.

Labels are analogues of coordinates. Relabelling is the analogue of a diffeomorphism transformation.

There is an exponentially large number of possible “local” realizations of geometry, corresponding to the same topology and the same set of “global” numbers.

Manifolds in 4d CDT: Summary

- Each "trajectory" is a sequence of T 3d geometric states with a topology S^3. These states are discretized: geometry is obtained by gluing together regular tetrahedra to form a closed S^3 simplicial manifold. Each state is characterized by an integer "time". 3-volume of a manifold is $\propto N_3(t)$ – number of tetrahedra.
- In 4d tetrahedra become bases of $\{4,1\}$ and $\{1,4\}$ simplices pointing up and down in "time" We have

$$\sum_t N_3(t) = N_4^{\{4,1\}}/2.$$

- To connect two states at t and $t+1$ we need a layer formed by $\{3,2\}$ and $\{2,3\}$ tetrahedra. This layer has no analogue in $d=2$ and $d=3$.

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Hilbert-Einstein action

For each space-time manifold we assign the action S_{HE} and a "probability" $\exp(-S_{HE})$.

$$S_{HE} = -(\kappa_0 + 6\Delta)N_0 + \kappa_4(N_4^{\{4,1\}} + N_4^{\{3,2\}}) + \Delta(2N_4^{\{4,1\}} + N_4^{\{3,2\}})$$

$\kappa_0, \ \kappa_4, \ \Delta$ - bare dimensionless coupling constants.

Discretization of a theory always leads to a dimensionless formulation. We will reintroduce physical dimensions later.

Analogy to Statistical Physics. Path integral $\rightarrow$ Ensemble of space-time discretized manifolds with a "partition function"

$$\mathcal{Z}(\kappa_0, \kappa_4, \Delta) = \sum_{\mathcal{T}} e^{-S_{HE}(\mathcal{T})}$$

Parameters of the H-E action

Physical properties of the system are determined by values of bare coupling constants

- $\kappa_4 - \kappa_4^{crit}(\kappa_0, \Delta)$ - related to the average "volume" $\langle N_4 \rangle$.
- κ_0 - related to the inverse of the bare gravitational constant.
- Δ - related to asymmetry between a_s and a_t.

$$\mathcal{Z}(\kappa_0, \kappa_4, \Delta) = \sum_{N_4} e^{-\kappa_4 N_4} Z_{N_4}(\kappa_0, \Delta)$$

where $N_4 = N_4^{\{4,1\}} + N_4^{\{3,2\}}$ - total number of simplices.

Outline

Objectives

Ideally we would like to be able not only to obtain the analytic formula for the partition function $\mathcal{Z}(\kappa_0, \kappa_4, \Delta)$, but also, using this function, to calculate arbitrary physical observables. Calculating (some of) these observables will be our objective.

$$\mathcal{Z}(\kappa_0, \kappa_4, \Delta) = \sum_{\mathcal{T}} e^{-S_{HE}(\mathcal{T})}$$

$$\langle \mathcal{A} \rangle = \frac{1}{\mathcal{Z}} \sum_{\mathcal{T}} \mathcal{A}(\mathcal{T}) e^{-S_{HE}(\mathcal{T})}$$

There is in general much more information in $\langle \mathcal{A} \rangle$ than in $\mathcal{Z}$.
$\mathcal{T}$ - triangulations $\equiv$ space-time configurations $\equiv$ trajectories.

Grand-canonical and canonical ensembles

Partition function $\mathcal{Z}(\kappa_0, \kappa_4, \Delta)$ from a statistical point of view defines a grand-canonical ensemble

$$\mathcal{Z}(\kappa_0, \kappa_4, \Delta) = \sum_{N_4} e^{-\kappa_4 N_4} \mathcal{Z}_{N_4}(\kappa_0, \Delta)$$

$\mathcal{Z}_{N_4}(\kappa_0, \Delta)$ defines a “canonical” ensemble with fixed four-volume N_4.
If a regularized theory should be finite – the sum in $\mathcal{Z}$ should be convergent. It follows that $\mathcal{Z}_{N_4}$ can grow at most exponentially with N_4 (restriction on a global topology).

$$Z_{N_4}(\kappa_0, \Delta) \approx \exp(\kappa_4^{crit}(\kappa_0, \Delta) N_4)$$

Observables

- Observables $\langle \mathcal{A} \rangle$ can be decomposed as

$$\langle \mathcal{A} \rangle = \sum_{N_4} \mathcal{P}(N_4) \langle \mathcal{A} \rangle_{N_4}$$

- In particular

$$\langle N_4 \rangle \sim 1/(\kappa_4 - \kappa_4^{crit})$$

"Canonical" averages are much easier to calculate (at least numerically).

$$\langle \mathcal{A} \rangle_{N_4} = \sum_{\mathcal{T}_{N_4}} P(\mathcal{T}) \mathcal{A}(\mathcal{T})$$

Canonical averages, infinite volume limit and continuum limit

- For a finite N_4 summation is over a finite (but exponentially large) set of configurations. Different configurations give contributions, depending on $P(\mathcal{T})$. Exact summation is practically impossible - we have to restrict ourselves to numerical estimates.
- Numerical estimate based on a smaller sample of “important” configurations.
- “Typical” (important) configurations - those with large probabilities (or large entropy, i.e. many different configurations with the same probability and similar physical properties)

Layout of a numerical experiment

For a set $\{\kappa_0, \Delta\}$ of bare coupling constants we perform numerical experiments at a sequence of volumes N_4. Each experiment means generating a large but finite sample of “important” configurations.

- These configurations are generated using the Monte Carlo technique.
- We calculate numerical estimates of the observable $\langle \mathcal{A} \rangle_{N_4}$
- We perform a finite size scaling analysis, i.e. we determine the scaling of the observable as a function of N_4 in the infinite volume limit $N_4 \to \infty$.
- We try to interpret this limit as a continuum limit by reintroducing physical dimensions.

Outline

Monte Carlo

In the space of configurations $\{\mathcal{M}\}$ we define a Markov process (a random walk in the configuration space) by choosing a probability $\mathcal{W}(\mathcal{M}_a \to \mathcal{M}_b)$ of a move from $\mathcal{M}_a$ to $\mathcal{M}_b$. Fictitious (discrete) time τ numbers the steps of a random walk. At each step we have a (normalized) distribution of probabilities $P_\tau(\mathcal{M}_i)$ with a recurrence relation

$$P_{\tau+1}(\mathcal{M}_j) = \sum_{\mathcal{M}_i} P_\tau(\mathcal{M}_i)\mathcal{W}(\mathcal{M}_i \to \mathcal{M}_j)$$

Monte Carlo cont'd'

Choosing transition probabilities

It is possible to choose $\mathcal{W}(\mathcal{M}_a \to \mathcal{M}_b)$ in such a way that the Markov process has a **unique** limiting distribution

$$P_\infty(\mathcal{M}_i) \propto \exp(-S(\mathcal{M}_i))$$

Detailed balance condition

$$\exp(-S(\mathcal{M}_a))\mathcal{W}(\mathcal{M}_a \to \mathcal{M}_b) = \exp(-S(\mathcal{M}_b))\mathcal{W}(\mathcal{M}_b \to \mathcal{M}_a)$$

There are infinitely many solutions of this condition.

Monte Carlo cont'd'

DB solution

We may have

$$\mathcal{W}(\mathcal{M}_a \to \mathcal{M}_b) = \mathcal{W}(\mathcal{M}_b \to \mathcal{M}_a) = 0$$

or

$$\frac{\mathcal{W}(\mathcal{M}_a \to \mathcal{M}_b)}{\mathcal{W}(\mathcal{M}_b \to \mathcal{M}_a)} = \exp\left(-(S(\mathcal{M}_b) - S(\mathcal{M}_a))\right)$$

- Non-zero transitions must satisfy ergodicity – all configurations can be reached by a random walk.
- They should connect configurations which are close - with small action difference (to be effective).

MC in numerical simulations

The numerical procedure is based on definitions presented above. On a computer we start the iterative process:

- Generate the initial configuration $\mathcal{M}_0$.
- Pick a (single) new configuration $\mathcal{M}_i$ with a probability given by $\mathcal{W}(\mathcal{M}_0 \to \mathcal{M}_i)$
- Pick a (single) new configuration $\mathcal{M}_j$ with a probability given by $\mathcal{W}(\mathcal{M}_i \to \mathcal{M}_j)$
- $\ldots$

If we perform sufficiently many steps and reach a particular configuration $\mathcal{M}_a$ we know that it will appear with a probability $\propto \exp(-S(\mathcal{M}_a))$.

MC in numerical simulations cont'd'

Configurations separated by many iteration steps are called statistically independent.
A set $\{\mathcal{M}_1, \mathcal{M}_2, \ldots \mathcal{M}_\mathcal{N}\}$ of independent configurations can be used to get the estimate

$$\langle \mathcal{A} \rangle \approx \frac{1}{\mathcal{N}} \sum_i^{\mathcal{N}} \mathcal{A}(\mathcal{M}_i)$$

Statistical error of the estimate depends on $\mathcal{N}$ and typically behaves as $1/\sqrt{\mathcal{N}}$.

Monte Carlo in CDT

We use this technique to obtain estimates in CDT.

Monte Carlo

- Finite set of local geometric moves, preserving topology.
- Detailed balance condition determining a probability to perform a particular change of geometry.

Local moves: “Alexander moves” – satisfy a condition of ergodicity.

Numerical setup in CDT

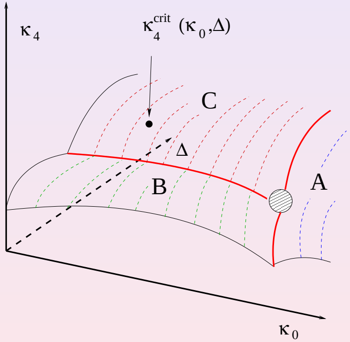
Alexander moves in general change the volume N_4. In our approach we either fix $N_4^{\{4,1\}} = 2\sum_t N_3(t)$ or we let it fluctuate with a Gaussian probability around $\langle N_4^{\{4,1\}}\rangle$.

- Physical properties of the system depend on κ_0 and Δ.
- We fine-tune $\kappa_4 \approx \kappa_4^{crit}$ to keep $\langle N_4^{\{4,1\}}\rangle$ stable.

In the Monte Carlo process we generate typically $10^7 - 10^8$ configurations. This is a finite sample representing typical configurations for a given set of $\{\kappa_0, \Delta\}$.

Outline

Approximate phase diagram of CDT

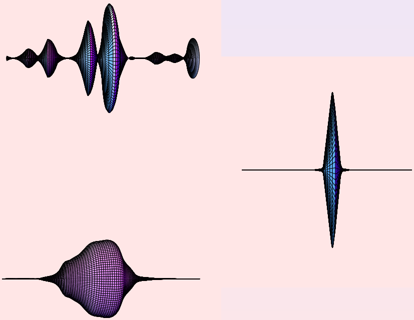


$\mathcal{Z}$ is defined for $\kappa_4 > \kappa_4^{crit}(\kappa_0, \Delta)$. Approaching a critical surface means taking an infinite volume limit. $\langle N_4 \rangle \sim 1/(\kappa_4 - \kappa_4^{crit})$.

Red lines - first order phase transitions. Perhaps a triple point.

Volume distribution in (imaginary) time

Different value of the critical exponent β: $\langle N_4^{\{3,2\}} \rangle_{N_4} \sim N_4^{\beta}$.



- **Phase A**. Not physical. Non-interacting 3d states. $\beta = 0$.
- **Phase B**. Not physical. Compactification into a 3d Euclidean DT. $0 < \beta < 1$, $d_H = \infty$.
- **Phase C**. Extended de Sitter phase. $\beta = 1$, $d_H = 4$

Topological effects

We formulated our model with a topology $S_3 \times S_1$, but the initial topology is dynamically modified.

- Among the observed phases only phase A has the unbroken symmetry of the translation in time. This phase is unphysical (no causal relation between different times).
- In phase B we observe a spontanous compactification of topology to that of Euclidean 3-sphere. The stalk is a lattice artefact and has a cut-off size.
- In phase C we also observe a spontanous compactification of topology to S_4 (to be discussed).