Numerical simulations of Causal Dynamical Triangulations 1

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1 General introduction
   - Path integral for Quantum Gravity
   - Basic assumptions of CDT
   - Regularization of a theory
   - Construction elements in 4d
   - Geometry of 3d states and a 4d configurations

2 Numerical setup
   - Objectives
   - Monte Carlo technique
   - Phase structure
Outline

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Quantum Gravity (without matter) - states of the system are defined as spatial geometries of the universe. Example of the evolution of a one-dimensional closed universe:

Joining spatial geometries produces a space-time geometry. In this example the sum over trajectories becomes a (weighted) sum over all two-dimensional surfaces joining the in-state with the out-state.
Our aim is to calculate the amplitude of a transition between two geometric states

\[ G(g_i, g_f, t) := \sum \int_{\text{geometries: } g_i \rightarrow g_f} e^{iS[g_{\mu\nu}(t')]} \] 

To define this path integral we have to specify the “measure” and the “domain of integration” - a class of admissible space-time geometries joining the in- and out- geometries.
Causal Dynamical Triangulations

- Using methods of QFT.
- Regularization of geometry follows the method of Dynamical Triangulations.
- New element: causality - Causal Dynamical Triangulations - additional restriction on the topology of space-time.

Very promising results of CDT

- Correct continuum limit.
- Information about quantum effects on the Planck scale.

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Basic concepts

- Path integral – amplitude of a quantum transition between in- and out- states can be written as a weighted sum (integral) over all possible trajectories.
- Possibility to perform analytic continuation in time – Wick rotation to imaginary time. In effect weights become real and positive and can be interpreted as probabilities.
- Lattice regularization – discretization of space-time provides a cut-off $a$.

In our approach (also in Dynamical Triangulations) we start with “Euclidean” formulation of space-time and then we eventually rotate back (or define) the time variable.
Wick rotation

Rotation to imaginary time $t \rightarrow it_4$ - the weight is formally real:

$$e^{iS[g(t)]} \rightarrow e^{-S_E[g(t_4)]}$$

After Wick rotation quantum amplitude becomes a weighted sum over geometric manifolds bounded by the in- and out-states.

The simplest form of the action – Hilbert–Einstein action

$$S[g] = -1/G \text{Curvature}(g) + \lambda \text{Volume}(g)$$

where $G$ - gravitational constant, $\lambda$ - cosmological constant (essential to suppress the entropy of quantum fluctuations).

This action used both by DT and CDT.
Measure and domain of integration in a path integral for QG

- **A.** Sum (integral) over diffeomorphism invariant equivalence classes of space-time metrics.
- **B.** Fixed topology of space-time.
- **C.** Suppressed formation of baby universes (fixed spatial topology).

- To suppress the divergent volume of the diffeomorphism group. Realized in the DT regularization.
- To suppress the divergence of the path integral coming from entropy. Realized in DT.
- Causality: it means the existence of a time foliation. For each time the topology of the universe is the same. Realized in CDT.
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Difference between DT and CDT

Difference lies in the domain of integration over allowed space-time geometries. In DT one cannot avoid introducing causal singularities.

Example of a causal singularity, which leads to creation of baby universes. Creation of baby universes dominates the possible evolution.
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Method of triangulations

Counting equivalence classes of manifolds. Example in 2d.

Discretization: One of the standard regularizations in QFT. Here: we replace a continuous space-time surface by a triangulated surface built from regular triangles with the edge $a$, serving as a cut-off. In the continuum limit $a \to 0$. 

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Example in 2d (Euclidean time):

In a triangulation a variable number of triangles can meet at each vertex. Deficit angle $\delta$ - (a) positive, (b) - negative. Curvature is localized in vertices. In other points geometry is flat!
Three steps in regularization of a path integral

Regularization of a geometric state
One-dimensional state with a topology $S^1$ is built from links with length $a$.

Regularization of a space-time geometry (trajectory)
2d space-time surface built from equilateral triangles. Curvature localized in vertices.

Regularization of a path integral
Integral over equivalence classes of metrics is replaced by a summation over all possible triangulations, belonging to some topological class.
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Generalization to higher d

Method of Euclidean Dynamical Triangulations

Replace 2d triangles by higher-dimensional simplices.

2d

3d

4d
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CDT: 3d geometric “states”

Spatial states are 3d geometries with a topology $S^3$. Discretized states are constructed from 3d simplices - tetrahedra glued along triangular faces.

Regular tetrahedron (3-simplex) - a basic block to build 3d manifolds.

Space of states

There are many inequivalent ways of gluing tetrahedra. For $N$ tetrahedra and a fixed topology this number grows exponentially $\sim \exp(\lambda N)$. 
Connecting 3d states

In 4d each tetrahedron becomes a base of a pair of \{4, 1\} and \{1, 4\} simplices, pointing up or down in $t$. The lengths of edges in time direction are $a_t$ (may be different than $a_s$).
We need two more types of simplices: \(\{3, 2\}\) and \(\{2, 3\}\).

Simplices \(\{3, 2\}\) and \(\{2, 3\}\) form a “layer” gluing together states at \(t\) and \(t + 1\).

It takes at least 4 steps to connect two \(\{4, 1\}\) simplices at times \(t\) and \(t + 1\).

\[
\{4, 1\} \rightarrow \{3, 2\} \rightarrow \{2, 3\} \rightarrow \{1, 4\} \rightarrow \{4, 1\}
\]
Space-time manifolds in 4d (trajectories)

We build a 4d manifold with a topology $S_3 \times S_1$. Each manifold is characterized by a set of “global” numbers

- $N_4^{\{4,1\}}$ - number of $\{4, 1\}$ and $\{1, 4\}$ simplices.
- $N_4^{\{3,2\}}$ - number of $\{3, 2\}$ and $\{2, 3\}$ simplices.
- $N_0$ - number of vertices (0-simplices).
- $T$ - time period.

Other “global” numbers depend on those above. Each manifold is a specific way of gluing together geometric states at integer times $t$. For a discretized manifold the Hilbert-Einstein action depends only on these global numbers.
Each 4d manifold is represented by a “local” information, describing how simplices are glued together. To do this we assign labels to vertices.

**Definition**

Manifolds are assumed to be simplicial manifolds: Each (sub)simplex with a particular set of labels appears at most once.

Labels are analogues of coordinates. Relabelling is the analogue of a diffeomorphism transformation.

There is an exponentially large number of possible “local” realizations of geometry, corresponding to the same topology and the same set of “global” numbers.
Manifolds in 4d CDT: Summary

- Each “trajectory” is a sequence of $T$ 3d geometric states with a topology $S^3$. These states are discretized: geometry is obtained by gluing together regular tetrahedra to form a closed $S^3$ simplicial manifold. Each state is characterized by an integer “time”. 3-volume of a manifold is $\propto N_3(t)$ – number of tetrahedra.

- In 4d tetrahedra become bases of $\{4,1\}$ and $\{1,4\}$ simplices pointing up and down in “time”. We have
  \[ \sum_t N_3(t) = N_4^{\{4,1\}} / 2. \]

- To connect two states at $t$ and $t + 1$ we need a layer formed by $\{3,2\}$ and $\{2,3\}$ tetrahedra. This layer has no analogue in $d = 2$ and $d = 3$. 

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Hilbert-Einstein action

For each space-time manifold we assign the action $S_{HE}$ and a "probability" $\exp(-S_{HE})$.

$$S_{HE} = -(\kappa_0 + 6\Delta)N_0 + \kappa_4(N_4^{4,1} + N_4^{3,2}) + \Delta(2N_4^{4,1} + N_4^{3,2})$$

$\kappa_0$, $\kappa_4$, $\Delta$ - bare dimensionless coupling constants.

Discretization of a theory always leads to a dimensionless formulation. We will reintroduce physical dimensions later.

Analogy to Statistical Physics. Path integral $\rightarrow$ Ensemble of space-time discretized manifolds with a "partition function"

$$\mathcal{Z}(\kappa_0, \kappa_4, \Delta) = \sum_{\mathcal{T}} e^{-S_{HE}(\mathcal{T})}$$
Parameters of the H-E action

Physical properties of the system are determined by values of bare coupling constants

- $\kappa_4 - \kappa_4^{\text{crit}}(\kappa_0, \Delta)$ - related to the average ”volume” $\langle N_4 \rangle$.
- $\kappa_0$ - related to the inverse of the bare gravitational constant.
- $\Delta$ - related to asymmetry between $a_s$ and $a_t$.

$$Z(\kappa_0, \kappa_4, \Delta) = \sum_{N_4} e^{-\kappa_4 N_4} Z_{N_4}(\kappa_0, \Delta)$$

where $N_4 = N_4^{4,1} + N_4^{3,2}$ - total number of simplices.
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Objectives

Ideally we would like to be able not only to obtain the analytic formula for the partition function $\mathcal{Z}(\kappa_0, \kappa_4, \Delta)$, but also, using this function, to calculate arbitrary physical observables. Calculating (some of) these observables will be our objective.

$$
\mathcal{Z}(\kappa_0, \kappa_4, \Delta) = \sum_{\mathcal{T}} e^{-S_{HE}(\mathcal{T})}
$$

$$
\langle A \rangle = \frac{1}{\mathcal{Z}} \sum_{\mathcal{T}} A(\mathcal{T}) e^{-S_{HE}(\mathcal{T})}
$$

There is in general much more information in $\langle A \rangle$ than in $\mathcal{Z}$. 

$\mathcal{T}$ - triangulations $\equiv$ space-time configurations $\equiv$ trajectories.
Grand-canonical and canonical ensembles

Partition function $\mathcal{Z}(\kappa_0, \kappa_4, \Delta)$ from a statistical point of view defines a grand-canonical ensemble

$$\mathcal{Z}(\kappa_0, \kappa_4, \Delta) = \sum_{N_4} e^{-\kappa_4 N_4} Z_{N_4}(\kappa_0, \Delta)$$

$Z_{N_4}(\kappa_0, \Delta)$ defines a “canonical” ensemble with fixed four-volume $N_4$.

If a regularized theory should be finite – the sum in $\mathcal{Z}$ should be convergent. It follows that $Z_{N_4}$ can grow at most exponentially with $N_4$ (restriction on a global topology).

$$Z_{N_4}(\kappa_0, \Delta) \approx \exp(\kappa_4^{\text{crit}}(\kappa_0, \Delta) N_4)$$
Observables

- Observables $\langle A \rangle$ can be decomposed as

$$\langle A \rangle = \sum_{N_4} P(N_4) \langle A \rangle_{N_4}$$

- In particular

$$\langle N_4 \rangle \sim \frac{1}{(\kappa_4 - \kappa_4^{\text{crit}})}$$

“Canonical” averages are much easier to calculate (at least numerically).

$$\langle A \rangle_{N_4} = \sum_{T_{N_4}} P(T) A(T)$$
Canonical averages, infinite volume limit and continuum limit

- For a finite $N_4$ summation is over a finite (but exponentially large) set of configurations. Different configurations give contributions, depending on $P(T)$. Exact summation is practically impossible - we have to restrict ourselves to numerical estimates.

- Numerical estimate based on a smaller sample of “important” configurations.

- “Typical” (important) configurations - those with large probabilities (or large entropy, i.e. many different configurations with the same probability and similar physical properties)
For a set \( \{ \kappa_0, \Delta \} \) of bare coupling constants we perform numerical experiments at a sequence of volumes \( N_4 \). Each experiment means generating a large but finite sample of “important” configurations.

- These configurations are generated using the Monte Carlo technique.
- We calculate numerical estimates of the observable \( \langle A \rangle_{N_4} \).
- We perform a finite size scaling analysis, i.e. we determine the scaling of the observable as a function of \( N_4 \) in the infinite volume limit \( N_4 \to \infty \).
- We try to interpret this limit as a continuum limit by reintroducing physical dimensions.
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In the space of configurations $\{\mathcal{M}\}$ we define a Markov process (a random walk in the configuration space) by choosing a probability $\mathcal{W}(\mathcal{M}_a \rightarrow \mathcal{M}_b)$ of a move from $\mathcal{M}_a$ to $\mathcal{M}_b$. Fictitious (discrete) time $\tau$ numbers the steps of a random walk. At each step we have a (normalized) distribution of probabilities $P_\tau(\mathcal{M}_i)$ with a recurrence relation

$$P_{\tau+1}(\mathcal{M}_j) = \sum_{\mathcal{M}_i} P_\tau(\mathcal{M}_i) \mathcal{W}(\mathcal{M}_i \rightarrow \mathcal{M}_j)$$
Choosing transition probabilities

It is possible to choose $\mathcal{W}(M_a \rightarrow M_b)$ in such a way that the Markov process has a **unique** limiting distribution

$$P_\infty(M_i) \propto \exp(-S(M_i))$$

Detailed balance condition

$$\exp(-S(M_a))\mathcal{W}(M_a \rightarrow M_b) = \exp(-S(M_b))\mathcal{W}(M_b \rightarrow M_a)$$

There are infinitely many solutions of this condition.
Monte Carlo cont’d’

DB solution

We may have

\[ \mathcal{W}(\mathcal{M}_a \rightarrow \mathcal{M}_b) = \mathcal{W}(\mathcal{M}_b \rightarrow \mathcal{M}_a) = 0 \]

or

\[ \frac{\mathcal{W}(\mathcal{M}_a \rightarrow \mathcal{M}_b)}{\mathcal{W}(\mathcal{M}_b \rightarrow \mathcal{M}_a)} = \exp \left( - (S(\mathcal{M}_b) - S(\mathcal{M}_a)) \right) \]

- Non-zero transitions must satisfy **ergodicity** – all configurations can be reached by a random walk.
- They should connect configurations which are **close** - with small action difference (to be effective).
MC in numerical simulations

The numerical procedure is based on definitions presented above. On a computer we start the iterative process:

- Generate the initial configuration $M_0$.
- Pick a (single) new configuration $M_i$ with a probability given by $W(M_0 \rightarrow M_i)$
- Pick a (single) new configuration $M_j$ with a probability given by $W(M_i \rightarrow M_j)$
- ...

If we perform sufficiently many steps and reach a particular configuration $M_a$ we know that it will appear with a probability $\propto \exp(-S(M_a))$. 
MC in numerical simulations cont’d’

Configurations separated by many iteration steps are called statistically independent.
A set \( \{M_1, M_2, \ldots, M_N\} \) of independent configurations can be used to get the estimate

\[
\langle A \rangle \approx \frac{1}{N} \sum_{i=1}^{N} A(M_i)
\]

Statistical error of the estimate depends on \( N \) and typically behaves as \( 1/\sqrt{N} \).
We use this technique to obtain estimates in CDT.

**Monte Carlo**

- Finite set of local geometric *moves*, preserving topology.
- Detailed balance condition determining a probability to perform a particular change of geometry.

Local moves: “Alexander moves” – satisfy a condition of *ergodicity*. 
Alexander moves in general change the volume $N_4$. In our approach we either fix $N_4^{\{4,1\}} = 2 \sum_t N_3(t)$ or we let it fluctuate with a Gaussian probability around $\langle N_4^{\{4,1\}} \rangle$.

- Physical properties of the system depend on $\kappa_0$ and $\Delta$.
- We fine-tune $\kappa_4 \approx \kappa_4^{\text{crit}}$ to keep $\langle N_4^{\{4,1\}} \rangle$ stable.

In the Monte Carlo process we generate typically $10^7 – 10^8$ configurations. This is a finite sample representing typical configurations for a given set of $\{\kappa_0, \Delta\}$. 
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Approximate phase diagram of CDT

- $\mathcal{Z}$ is defined for $\kappa_4 > \kappa_4^{crit}(\kappa_0, \Delta)$.
- Approaching a critical surface means taking an infinite volume limit.
  - $\langle N_4 \rangle \sim 1/(\kappa_4 - \kappa_4^{crit})$.

Red lines - first order phase transitions. Perhaps a triple point.
Volume distribution in (imaginary) time

Different value of the critical exponent $\beta$: $\langle N_{4}^{(3,2)} \rangle_{N_4} \sim N_{4}^{\beta}$.

- **Phase A.** Not physical. Non-interacting 3d states. $\beta = 0$.
- **Phase B.** Not physical. Compactification into a 3d Euclidean DT. $0 < \beta < 1$, $d_{H} = \infty$.
- **Phase C.** Extended de Sitter phase. $\beta = 1$, $d_{H} = 4$.
We formulated our model with a topology $S_3 \times S_1$, but the initial topology is dynamically modified.

- Among the observed phases only phase A has the unbroken symmetry of the translation in time. This phase is unphysical (no causal relation between different times).
- In phase B we observe a spontaneous compactification of topology to that of Euclidean 3-sphere. The stalk is a lattice artefact and has a cut-off size.
- In phase C we also observe a spontaneous compactification of topology to $S_4$ (to be discussed).