The LQG black hole

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Black hole thermodynamics:

- Mass is energy
- Hawking temperature is temperature
- Area is entropy?

Entropy of what? What are the microstates?

In loop quantum gravity (LQG):

- Area is entropy of quantum isolated horizon
- Microstates = states of SU(2) Chern-Simons theory with punctures
  ≈ quantum shapes of horizon

But: Treatment not entirely from first principles.
The picture in LQG:

- Polymeric excitations endow horizon with area
- Bulk theory and CS theory coupled at punctures
- Number of surface & CS states for fixed area $\sim \exp \frac{A}{2}$

Long story: Rovelli, Krasnov ('96)
Ashtekar + Baez + Corichi + Krasnov ('98)
Domagala + Lewandowski, Meissner ('04)
Engle + Noui + Perez ('10)

and many more.
2. Isolated horizons

Local notion of black hole horizon? Isolated horizon (Ashtekar Beetle Fairhurst (‘98), specialization of trapping horizon (Hayward (‘94))

Something like a local Killing horizon:

• Null surface \( H = S^2 \times \mathbb{R} \)
• Foliation of \( H \) with corresponding null normal non-expanding.
• Pull-back of connection time independent on \( H \).

⇒ Quasi-local expression for horizon mass (angular momentum, charge...)
⇒ Laws of BH mechanics
  ▶ KH is IH,
  ▶ IH inside/coincides with EH,
  ▶ IH + assumptions ⇒ \( \exists \) EH.

For rest of talk: spherically symmetric IH (“Type I”)
Ashtekar-Barbero variables (Engle, Noui, Perez ('10))

IH boundary condition: \[ F^I (A^\gamma) = \frac{(1 - \beta^2) \pi}{a_H} \Sigma^I \]

with

\[ \Sigma^I := \epsilon^I_{JK} e^J \wedge e^K = \delta^{IJ} \epsilon_{abc} E^a_J \ dx^b \wedge dx^c \]

Presymplectic structure:

\[ \kappa \gamma \Omega (\delta_1, \delta_2) = \int_\Sigma 2 \delta [1 \Sigma^I \wedge \delta_2] A_I - \frac{a_H}{(1 - \beta^2) \pi} \int_\Delta \delta_1 A^I \wedge \delta_2 A_I \]

Appearance of Chern-Simons term on boundary suggests separate quantization of boundary DOF. More later.
3. Quantum theory

Bulk theory as before, but now spin-net edges can end on the horizon. Fix graph. Standard LQG-results give

\[ \mathcal{H} = (\otimes_i j_{p_i}) \otimes \mathcal{H}_{\text{bulk}} \quad \exists \, \psi = |\{j_{p_i}, m_{p_i}\}, \ldots\rangle \]

\[
[\hat{J}^I(p), \hat{J}^J(p)] = \epsilon^{IJ}_K \hat{J}^K(p)
\]

\[
\hat{E}^I(p) = 16\pi G\beta \sum_{p_i} \delta(p, p_i) \hat{J}^I(p)
\]

\[
\hat{a}_H = 8\pi \beta l_p^2 |\hat{J}(p)|
\]
Boundary theory must satisfy:

\[-\frac{a_H}{\pi(1 - \beta^2)} \epsilon^{ab} \hat{F}_{ab}(p) = \frac{1}{16\pi G \beta} \sum \delta(p, p_i) \hat{J}^i(p)\]

Again, follow ENP: Take SU(2) CS theory with particles,

\[S[A, \Lambda_i] = \frac{k}{4\pi} \int \text{tr}[A \wedge dA + \frac{2}{3} A \wedge A \wedge A] + \sum_i \int_{c_i} \text{tr}[\tau_3 (\Lambda_i^{-1} d\Lambda_i + \Lambda_i^{-1} A\Lambda_i)]\]

Canonical analysis:

\((\lambda_i, S_i) : \{S^I_i, S^J_i\} = \epsilon^{IJ}_K \hat{S}^K_i, \{S^I_i, \Lambda_i\} = -\tau^I \Lambda_i\)

plus three constraints per particle. Further constraint:

\[\frac{k}{4\pi} \epsilon^{ab} \hat{F}_{ab}(p) = \sum_i \delta(p, p_i) \hat{S}_i\]
This suggests:

\[ k = \frac{a_H}{4\pi l_P^2 \beta (1 - \beta^2)} \]

\[ \hat{S}_i = \hat{J}(p_i) \]

\[ \mathcal{H} = \mathcal{H}_{CS}(j_1, j_2 \ldots) \otimes \mathcal{H}_{bulk} \]

Note furthermore: Because \( \Delta \) is simply connected

\[ \mathcal{H}_{CS}(j_1, j_2, \ldots) \subset \text{Inv}(j_1, j_2, \ldots) \]

In fact, for \( k \) large, it can be shown that

\[ \mathcal{H}_{CS}(j_1, j_2, \ldots) = \text{Inv}(j_1, j_2, \ldots) \]

for large level \( k \) (large BH).

Finally: Constraints don’t change the picture.
4. State counting

Count: All sequences \( j_1, j_2, \ldots \) such that

\[
4\pi\beta l_P^2 \sum_i \sqrt{j_i(j_i + 1)} \leq a_H
\]

with multiplicity \( \dim \text{Inv}(j_1 \otimes j_2 \otimes \ldots) \)

Agullo, Barbero, Borja, Diaz-Polo, Villasenor (‘10) find:

\[
N_{\leq}(a) = \frac{2}{(2\pi)^2 i} \int_0^{2\pi} \int_{x_0-i\infty}^{x_0+i\infty} \frac{\sin^2 \omega}{s} \left(1 - \sum_{k=1}^{\infty} \frac{\sin(k+1)\omega}{\sin \omega} e^{-s\sqrt{k(k+2)}} \right)^{-1} e^{as} \, ds \, d\omega
\]
Analysis of the **pole structure** of the integrand gives:

\[
\ln N_\leq(a) = \frac{\tilde{\beta}}{2\pi\beta} \frac{a_h}{4l_P^2} - \frac{3}{2} \ln(a_H/l_P^2) + O(a_H^0)
\]

with \( \tilde{\beta} \) determined by

\[
\sum_{k=1}^{\infty} (k + 1) e^{-\tilde{\beta}\sqrt{k(k+2)}/2} = 1
\]

This leads to the choice

\[
\beta = \frac{\tilde{\beta}}{2\pi} \approx 0.274067
\]

which reproduces the Bekenstein-Hawking entropy law.
5. Remarks

1. History:

- Rovelli, Krasnov (‘96): Counting punctures

- Ashtekar + Baez + Corichi + Krasnov (‘98):
  - Use of isolated horizon condition
  - Chern-Simons on boundary
  - gauge fixing to U(1) on boundary
  - state counting only approximate

- Domagala + Lewandowski, Meissner (‘04): correct combinatorial formulation, counting, statistics.

- Engle + Noui + Perez (‘10): Treatment without extraneous gauge fixing
2. Relation to quasinormal modes (QNM):

QNM in this context: ringing modes of fields (metric perturbations) on BH spacetimes

Complex frequencies. For scalar modes on Schwarzschild:

\[ \lim_{n \to \infty} \text{Re}(M\omega_n) \approx 0.04371235 \]

(image from: Dreyer ‘02)

In fact, limit of real part
Hod’s prediction (‘98):

\[ \lim_{n \to \infty} \text{Re}(M \omega_n) = \frac{\ln 3}{8\pi} \]

Proven by Motl (‘03). Why is this interesting?

Bekenstein (‘73): Area quantum

\[ \Delta A = 4 \ln(k) l_P^2 \]

Hod’s reasoning: With k=3:

\[ \Delta A = 32\pi M \Delta M = 32\pi \hbar \omega_{\text{QNM}} \]

QNM spectrum = emission spectrum of quantum BH??
Situation in LQG:
Area spec much more complicated, but there is minimal nonzero eigenvalue.

Dreyer (‘03): Can get $\Delta A = 4 \ln(3) l_P^2$ for gauge group $SO(3)$ in LQG

Uses (incorrect)

$$\beta^{ABCK} = \frac{\ln 3}{2\pi \sqrt{2}}$$

Domagala, Meissner, Lewandowski: Does not work for correct counting.

Possible way out: Different ordering in area operator gives equidistant area spectrum in LQG. Dreyer’s argument then seems to work again.

But: Extension to charged, rotating case unclear (Perez, Sahlmann, Sudarsky (‘04))
3. Entropy quantization

Area spectrum in LQG not equidistant. But...

A. Corichi, J. Diaz-Polo and E. Fernandez-Borja (‘07):

With

$$\Delta a = \beta \chi l_P^2, \quad \text{with} \quad \chi \approx 8.80 \doteq 8 \ln 3$$
Fully analytic investigation using tools from number theory in progress: Agullo, Barbero, Borja, Diaz-Polo, Villasenor (‘10).

Structure seems to go away for large black holes.
4. Connection to convex polyhedra, polymers

Beautiful recent work by Bianchi, Dona, Speziale (’10):

- derivation of area-entropy relation from polymer physics
- Horizon DOF as DOF of quantum convex polyhedron?

Given uniquely by set of $N$ normalized vectors $\vec{n}_i$ and $N$ numbers $A_i$, with closure relation

$$\sum_i \vec{n}_i A_i = 0$$

Quantization of symplectic space
6. Summary / Outlook

In LQG:

- BH area is entropy of quantum isolated horizon
- microstates = states of SU(2) Chern-Simons theory with punctures
  ≈ quantum shapes of horizon

Things fit very nicely together. Seems the LQG picture captures at least part of the truth.
But much more to understand:

- Charged, rotating black holes
- Dynamical situation
- Holography?
- How does the thermodynamics come in?
- ...
2. LQG basics

Variables: (Ashtekar, Barbero)

\[ S[\omega, e] = \frac{1}{4k} \int_M \left[ \epsilon_{ijkl} e^i \wedge e^j \wedge \Omega^{kl} - \frac{2}{\beta} e^i \wedge e^j \wedge \Omega_{ij} \right] \]

\[ = \frac{1}{k\gamma} \int dt \int_{\Sigma_t} E^a_i A^I_a - \omega^I_0 G_I + N^a C_a + NC \]

\[ A^I_a = \Gamma^I_a + \beta K^I_a \quad \quad E^a_I = \sqrt{\det q} e^a_I \]

\[ i, j, \ldots : SO(3,1) \quad \quad I, J, \ldots : SO(3)(\text{or } SU(2)) \]

\[ \{ A^I_a(x), E^J_b(y) \} = 8\pi \beta G \delta^b_a \delta^I_J \delta(x,y) \]