Loop quantum gravity and twisted geometries

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Outline

1. Motivations and overview
   Why do we need discrete geometries?
2. Twisted geometries
   Definition and relation to holonomy and fluxes
3. From spinors to twisted geometries
   Spinorial tools and derivation of the holonomy-flux algebra from harmonic oscillators
4. Applications
   polyhedra, new volume operators, cosmology, simplicity constraints, etc
5. Comments on the simplicity constraints
   The risks of relaxing them too much: bi-metric theories of gravity

Outline

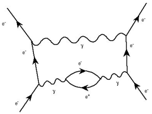
Motivations

Twisted geometries

From spinors to twisted geometries

Applications

Motivations: a paradigm shift

	kinematics	dynamics
QFT:	$\|n, p_i, h_i\rangle$	
	quanta: momenta, helicities, etc.	Feynman diagrams
	observables n: # of quantum particles	perturbative expansion degree of the graph $\Downarrow$ order of approximation desired

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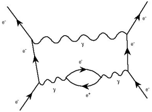
QFT:	$\|n, p_i, h_i\rangle$	
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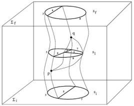
LQG:	$\|\Gamma, j_e, i_v\rangle$	
	quanta: areas and volumes	spin foams

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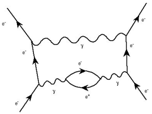
QFT:	$\|n, p_i, h_i\rangle$	
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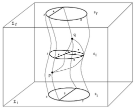
LQG:	$\|\Gamma, j_e, i_v\rangle$	
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	link to classical geometries? meaning of Γ?	what approximation?

Loop gravity and discrete geometries

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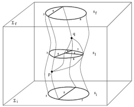

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spin foams

spin foams suggest
link with Regge geometries

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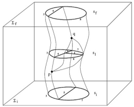

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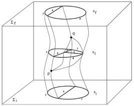

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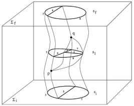

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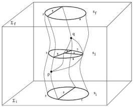

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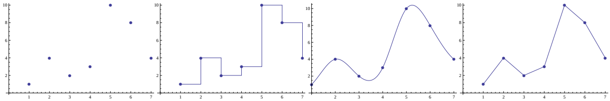
- Can they represent a *discrete* geometry, approximation of a smooth one?



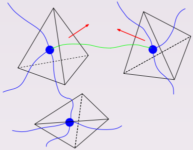
A convenient choice: twisted geometries

L. Freidel and SS, 1001.2748 and 1006.0199, C. Rovelli and SS, 1005.2927

For each point on the phase space at fixed graph, there are infinite continuous metrics that can correspond to it

Twisted geometries are a particular choice of interpolating geometry associated with a cellular decomposition of the manifold dual to Γ:

each classical holonomy-flux configuration on a fixed graph can be visualized as a collection of adjacent polyhedra with extrinsic curvature between them



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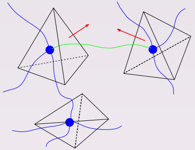

BUT: If we look at two neighbouring polyhedra, they induce two different geometries on the shared face: By construction, the area is the same, but the shape will differ in general.

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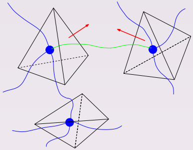

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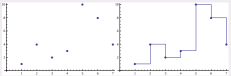
The geometries are twisted in the sense that they are well-defined locally (on each polyhedron), but are *discontinuous* at the intersections (the faces)



Outline

Motivations

Twisted geometries

From spinors to twisted geometries

Applications

Phase spaces of LQG

Hilbert space: $\mathcal{H} = \bigoplus_{\Gamma} \mathcal{H}_\Gamma$

- kinematical loop gravity $\Longrightarrow$ $\mathcal{H}_\Gamma = L_2(SU(2)^E)$

$\downarrow$ *Gauss law*

- gauge-inv. loop gravity $\Longrightarrow$ $\mathcal{H}_\Gamma = L_2(SU(2)^E/SU(2)^V)$

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Closed twisted geometries: a collection of polyhedra associated to the dual of the graph, describing discrete, possibly *discontinuous* geometries

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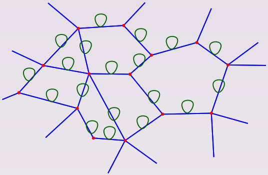
Focus first at the non gauge-inv. level:

- $L_2(SU(2))$ is the quantization of the classical phase space $T^*SU(2)$
- $\mathcal{H}_\Gamma$ is the quantization of the classical phase space $\times_e T^*SU(2)_e$

Phase space of loop gravity on a fixed graph

$$\times_e \, T^*SU(2)_e$$



A spinning top for each link of the graph

$T^*SU(2) = R^3 \times S^3$ $\longrightarrow$ Flux: $X_e = \int_{e^*} (gE^a)\hat{u}_a d^2S$

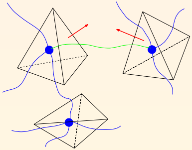
(X_e, g_e)

Holonomy: $g_e = \mathcal{P}e^{\int_e A}$

Change of parametrization: $(X, g) \leftrightarrow (j, N, \tilde{N}, \xi)$

Twisted geometries: definition

On each edge:



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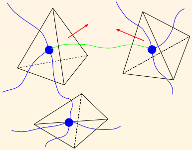

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$$(N, \tilde{N}, j, \xi) \Longrightarrow (X, g) \; : \qquad X \;=\; jN$$

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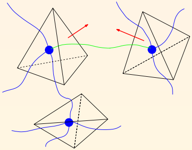

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$$(N, \tilde{N}, j, \xi) \Longrightarrow (X, g) \ : \qquad \begin{array}{rcl} X & = & jN \\ g & = & n e^{\xi \tau_3} \tilde{n}^{-1} \end{array}$$

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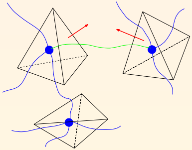

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Notice also that $\tilde{X} = j\tilde{N} = -g^{-1} X g$

Poisson brackets on the twisted geometries

- Poisson algebra of T^*SU(2)

$$\{X^i, X^j\} = \epsilon^{ij}{}_k X^k, \qquad \{X^i, \tilde{X}^j\} = 0 \qquad \{X^i, g\} = -\tau^i g, \qquad \{\tilde{X}^i, g\} = g\,\tau^i$$

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$$\Theta_{T^*SU(2)} = \mathrm{Tr}[X \mathrm{d}g g^{-1}]$$

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$$\Uparrow$$

Twisted geometries parametrization
(see also G.Immirzi '95)

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Twisted geometries parametrization
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$$\{\xi, j\} = 1, \qquad \{N^i, j\} = 0, \qquad \{\tilde{N}^i, j\} = 0,$$

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- $L : \mathcal{S}^2 \mapsto \mathbb{R}^3$ unique up to change of section. For the Hopf section, $L^i = (-\bar{z}, z, 1)$

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Gauge-invariance and polyhedra

Twisted geometries	$\Longleftrightarrow$	Loop gravity
$\underset{e}{\times}\left(T^*S^1 \times S^2 \times S^2\right)$		$\times_e\, T^*SU(2)$

Gauge-invariance and polyhedra

Twisted geometries $\Longleftrightarrow$ Loop gravity

$\underset{e}{\times}\left(T^*S^1 \times S^2 \times S^2\right)$ $\times_e T^*SU(2)$

$\downarrow$ *Gauss law reduction*

Gauge-inv. loop gravity

$\underset{e}{\times} T^*SU(2)//\underset{v}{\times}\ SU(2)$

Gauge-invariance and polyhedra

Twisted geometries $\underset{e}{\times} \left(T^*S^1 \times S^2 \times S^2\right)$	$\Longleftrightarrow$	Loop gravity $\times_e T^*SU(2)$
$\downarrow$ *closure reduction*		$\downarrow$ *Gauss law reduction*
Closed twisted geometries S_Γ	$\Longleftrightarrow$	Gauge-inv. loop gravity $\underset{e}{\times} T^*SU(2)//\underset{v}{\times} SU(2)$

Gauge-invariance and polyhedra

- on a vertex: $N_e \in \underset{e\in v}{\times} S^2_{j_e}$
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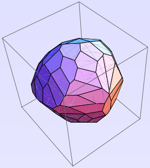
  [E.Bianchi,P.Doná,SS 1009.3402]



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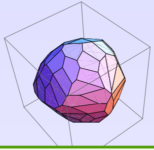

Just as the intertwiners are the building block of the Hilbert space, polyhedra are the building blocks of the classical phase space

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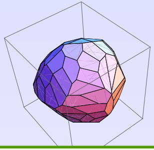

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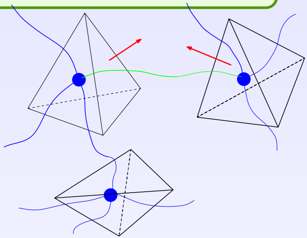
On the full graph: $S_\Gamma = \times_e T^* S^1 \times_v \mathcal{S}_F$



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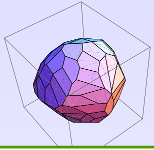
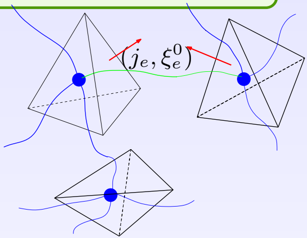
- Kapovich and Millson phase space: $\mathcal{S}_F = \{N_e \in \underset{e\in v}{\times} S^2_{j_e} \,|\, C=0\}/\mathrm{SO}(3)$
- Points in this phase space represent bounded convex flat polyhedra in $\mathbb{R}^3$
  [E.Bianchi,P.Doná,SS 1009.3402]



Just as the intertwiners are the building block of the Hilbert space, polyhedra are the building blocks of the classical phase space

On the full graph: $S_\Gamma = \times_e T^*S^1 \times_v \mathcal{S}_F$

Caveat: the j_e are gauge-invariant, but $\{C_v, \xi_e\} \neq 0$!
$\Rightarrow$ need to gauge-fix, $\xi_e \to \xi_e^0$



Gauge-invariance and polyhedra

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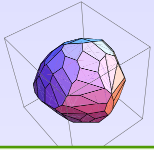

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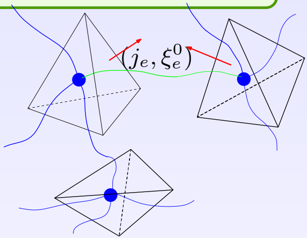

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Reduced parametrization?
Compare Dittrich and Ryan:
for triangulations, $\xi^{\mathrm{gauge-inv.}} =$ "pre"-dihedral angle
(It can not be just the dihedral angle because of the discontinuity!)

Gauge-invariance and polyhedra

Twisted geometries $\times_e \left(T^*S^1 \times S^2 \times S^2\right)$	$\Longleftrightarrow$	Loop gravity $\times_e \, T^*SU(2)$
↓ *closure reduction*		↓ *Gauss law reduction*
Closed twisted geometries S_Γ	$\Longleftrightarrow$	Gauge-inv. loop gravity $\times_e \, T^*SU(2) // \times_v \, SU(2)$

Gauge-invariance and polyhedra

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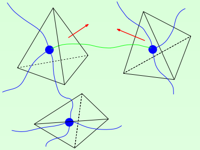
Gluing constraints



And the connection to Regge calculus?

Consider only 4-valent graphs, dual to triangulations

When closure conditions hold, a triangle acquires two geometries, one from each of the tetrahedra sharing it

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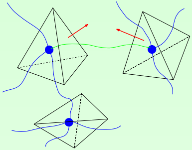

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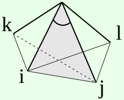
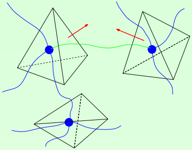
B.Dittrich and SS 0802.0864

$$F(\phi^v_{ee'}) = 0$$



among the scalar products
$N_e(v) \cdot N_{e'}(v) \equiv \cos \phi^v_{ee'}$
of the normals belonging to the two tetrahedra

Gluing constraints



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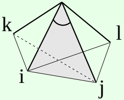

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When the gluing conditions hold, we recover Regge calculus

Overview

Twisted geometries	$\Longleftrightarrow$	Loop gravity
$\downarrow$ *closure reduction*		$\downarrow$ *Gauss law reduction*
Closed twisted geometries	$\Longleftrightarrow$	Gauge-inv. loop gravity

Overview

Twisted geometries	$\Longleftrightarrow$	Loop gravity
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$\downarrow$ *matching shapes reduction*		
Regge calculus		

Overview

Spinors

$\downarrow$ *matching area reduction*

Twisted geometries $\Longleftrightarrow$ Loop gravity

$\downarrow$ *closure reduction* $\downarrow$ *Gauss law reduction*

Closed twisted geometries $\Longleftrightarrow$ Gauge-inv. loop gravity

$\downarrow$ *matching shapes reduction*

Regge calculus

Outline

Motivations

Twisted geometries

From spinors to twisted geometries

Applications

Spinors

- $|\mathbf{z}\rangle = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \in \mathbb{C}^2$, complex structure $J|z\rangle = \begin{pmatrix} -\bar{z}_1 \\ \bar{z}_0 \end{pmatrix} = |z]$
- Hermitian inner product
$$\langle w|z\rangle = \bar{w}_0 z_0 + \bar{w}_1 z_1$$
- Antisymmetric bilinear form
$$[w|z\rangle = w_0 z_1 - w_1 z_0 = \epsilon^{ab} w_a z_b$$
- Geometrical meaning: null pole plus null flag: $|\mathbf{z}\rangle \mapsto (X^i, \phi)$
$$|\mathbf{z}\rangle\langle\mathbf{z}| = X^0 \mathbb{1} + X^i \sigma_i, \qquad \phi = \arg z_0 + \arg z_1$$
$$X^0 = \frac{1}{2}\langle\mathbf{z}|\mathbf{z}\rangle, \qquad X^i = \langle\mathbf{z}|\frac{\sigma^i}{2}|\mathbf{z}\rangle$$

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- Poisson brackets

$$\{z_a, \bar{z}_b\} = -i\delta_{ab}$$

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Edge phase space

Consider two spinors, $|\mathbf{z}\rangle$ and $|\tilde{\mathbf{z}}\rangle$, with canonical Poisson brackets:

$$(z_0, z_1, \tilde{z}_0, \tilde{z}_1) \in \mathbb{C}^4, \qquad \{z_a, \bar{z}_b\} = -i\delta_{ab}, \quad \{\tilde{z}_a, \bar{\tilde{z}}_b\} = -i\delta_{ab}$$

Claim: there is a phase space reduction s.t. $\mathbb{C}^4 : 8d \longrightarrow 6d : T^*SU(2)$

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$$\{H, z_A\} = \frac{i}{2} z_A, \qquad \{H, \tilde{z}_A\} = -\frac{i}{2}\tilde{z}_A, \qquad (|\mathbf{z}\rangle, |\tilde{\mathbf{z}}\rangle) \mapsto (e^{i\frac{\theta}{2}}|\mathbf{z}\rangle, e^{-i\frac{\theta}{2}}|\tilde{\mathbf{z}}\rangle),$$

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Symplectic reduction by $H = 0$ gives $T^*SU(2)$

H reduction

- initial Poisson brackets $\{z_a, \bar{z}_b\} = -i\delta_{ab}$ $|\mathbf{z}\rangle \mapsto (X^i, \phi)$

$$\{X^i, X^j\} = \epsilon^{ijk} X^k, \qquad \{X^0, \varphi\} = 1, \qquad \{X^3, \varphi\} = 0, \qquad \{X^\pm, \varphi\} = \frac{X^0}{X^\mp}$$

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- constraint $H = X^0 - \tilde{X}^0 = 0$
- reduced variables: Define

$$j \equiv \frac{1}{2}(X^0 + \tilde{X}^0), \qquad \xi_A \equiv i \left(\ln \frac{z_A}{\bar{z}_A} + \ln \frac{\tilde{z}_A}{\bar{\tilde{z}}_A} \right)$$

and evaluate

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- in terms of the standard holonomy-flux parametrization:

$$X^i(z_A) \equiv \langle\mathbf{z}|\frac{\sigma^i}{2}|\mathbf{z}\rangle, \qquad g(z_A, \tilde{z}_A) \equiv \frac{|\mathbf{z}\rangle[\tilde{\mathbf{z}}| - |\mathbf{z}]\langle\tilde{\mathbf{z}}|}{\sqrt{\langle\mathbf{z}|\mathbf{z}\rangle\langle\tilde{\mathbf{z}}|\tilde{\mathbf{z}}\rangle}}$$

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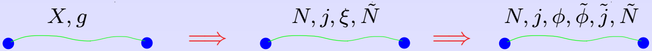
We obtain a spinorial parametrization of holonomies and fluxes:

$$\mathbb{C}^4 \ni (|\mathbf{z}\rangle, |\tilde{\mathbf{z}}\rangle) \stackrel{/H=0}{\longrightarrow} \big(X(\mathbf{z}), g(\mathbf{z})\big) \in T^*SU(2)$$

H interpretation

Interpretation of $\mathbb{C}^4_e$: twisted geometries with areas non matching:



Remark: from the two spinors I can define a twistor
$\Rightarrow H = 0$ is a condition that the twistor is null

Overview

Twistor space	$\Longrightarrow$	
$\downarrow$ *matching area reduction*		
Twisted geometries	$\Longleftrightarrow$	Loop gravity
$\downarrow$ *closure reduction*		$\downarrow$ *Gauss law reduction*
Closed twisted geometries	$\Longleftrightarrow$	Gauge-inv. loop gravity
$\downarrow$ *matching shapes reduction*		
Regge calculus		

Overview

Twistor space $\Longrightarrow$ $\times_v \left(\mathbb{C}^2\right)^{E(v)}$

$\downarrow$ *matching area reduction*

Twisted geometries $\Longleftrightarrow$ Loop gravity

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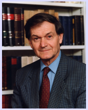
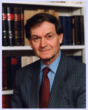
Regge calculus

Overview



Spin networks Twistors

Overview



Spin networks → Twisted geometries

Twistors → Twisted geometries

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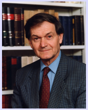

Spin networks   Twistors

Twisted geometries

↓

Regge geometries

Outline

Motivations

Twisted geometries

From spinors to twisted geometries

Applications

Applications

1. Geometry of polyhedra and volume operator
2. New coherent states and representation of the algebra
3. Parametrization of the gauge-invariant phase space
4. U(N) coherent states
5. Cosmological models
6. Simplicity constraints

Geometry of polyhedra

E. Bianchi, P. Doná and SS, 1009.3402

Explicit reconstruction procedure: $(j_e, N_e) \mapsto$ edge lengths, volume, adjacency matrix

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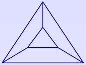
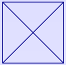
Dominant: 

Codimension 1: 

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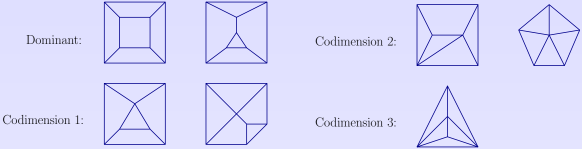

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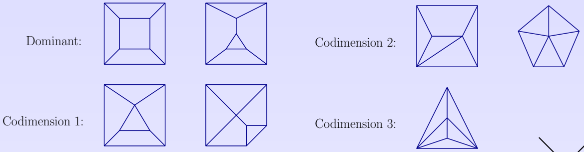
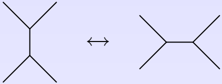


- The classes are all connected by 2-2 Pachner moves (they are all tessellations of the 2-sphere)



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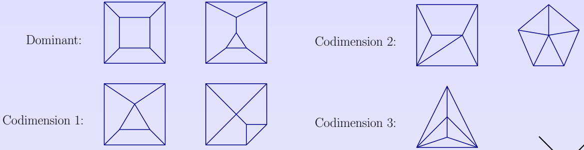

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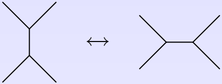


It is the configuration of normals to determine the class

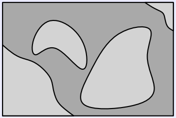
- The phase space $\mathcal{S}_F$ can be mapped in regions corresponding to different classes.

  – *Dominant classes* have all 3-valent vertices.

  [maximal n. of vertices, $V = 3(F-2)$, and edges, $E = 2(F-2)$]

  – *Subdominant classes* are special configurations with lesser edges and vertices, and span measure zero subspaces.

  [lowest-dimensional class for maximal number of triangular faces]



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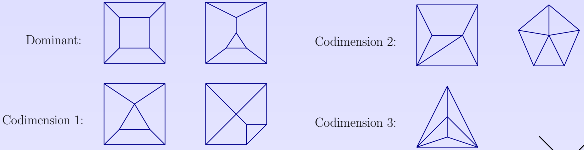

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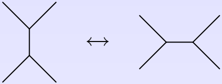

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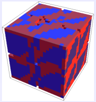

3d slice of $\mathcal{S}_6$, cuboids blue

A new volume operator

E. Bianchi, P. Doná and SS, 1009.3402

Use:

1. classical expression known from reconstruction algorithm $V(j_e, N_e)$ (for the moment only numerical for $F > 4$ – work in progress Hal Haggard)
2. coherent intertwiners labelled by N_e form an (over)-complete basis

$\Rightarrow$ define the operator on $\mathcal{H}_v$

$$\hat{V} = \int d\mu(N_e)\, V(j_e, N_e)\, ||j_e, N_e\rangle\langle j_e, N_e||\,.$$

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- Correct semiclassical limit by construction on vertices of any valency
- But not simply related to fundamental holonomy-flux operators

4-valent spectrum

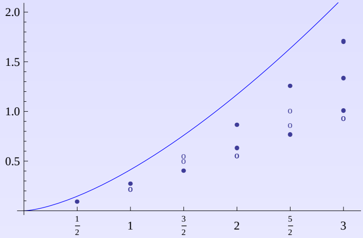

Figure: Some eigenvalues of $\hat{V}$. For comparison, the curve is the classical volume of an equilateral tetrahedron as a function of the area j (units $8\pi\gamma L_P^2 = 1$). The empty circles are single eigenvalues, the full circles have double degeneracy. The spectrum is gapped and bounded from the above by the classical maximal volume, which provides a large spin asymptote.

New coherent states

L. Freidel and SS, in (relaxed...) progress

- Heat-Kernel coherent states (Thiemann, Hall, Winkler, Sahlmann, Bahr)
- Edge factor: parametrize phase space via $SL(2,\mathbb{C}) \ni H = e^{iX} g$

$$\psi_H(g) = \sum_j d_j \, e^{-\frac{t}{2} j(j+1)} \, \chi_j(Hg^{-1}), \qquad \chi_j(Hg^{-1}) = \sum_{ab} D^{(j)}_{ab}(H) D^{(j)}_{ba}(g^{-1})$$

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- The resulting states provide an overcomplete basis with interesting minimization properties

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- Edge area-matching:

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Borja-Freidel-Garay-Livine 1010.5451

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$$S = \int dt \sum_v \sum_{e \in v} i \langle \mathbf{z}_{v,e} | \partial_t \mathbf{z}_{v,e} \rangle + \sum_e \lambda_e H_e + \sum_v \mu_v C_v$$

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$$S = \sum_t A_t \epsilon_t(\phi) + \sum_v \mu_v C_v + \sum_e \mu_{ee'} C_{ee'}$$

Compare with area-angle Regge calculus (Dittrich-SS 0802.0864)
and its canonical description (Dittrich-Ryan 1006.4295)

U(N) framework

E. Livine, F. Girelli, M. Dupuis, L. Freidel
Enrique Borja, Jacobo Diaz-Polo, Inaki Garay

N-valent vertex: $\mathcal{H}_{\{j_e\}} \equiv \text{Inv}\left[\bigotimes_{e\in v} V^{(j_e)}\right] \quad \longrightarrow \quad \mathcal{H}_J = \bigoplus_{\sum_e j_e = J} \mathcal{H}_{\{j_e\}}$

Each $\mathcal{H}_J$ carries an irrep of U(N)

- U(N) algebra (related to standard algebra)
  applications in cosmology: new characterization of isotropy and homogeneity
  E. Borja, J. Diaz-Polo, I. Garay, E. Livine, 1006.2451
- U(N) coherent states (related to coherent intertwiners)
  simpler formulas, more control on the properties of the states

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On the simplicity constraints

Plebanski action: $S(B, \omega, \Phi) = \int B_{IJ} \wedge F^{IJ}(\omega) + \Phi_{IJKL} B^{IJ} \wedge B^{KL}$

$$\delta_\Phi S = 0 \;\mapsto\; C = B^{IJ} \wedge B^{KL} - \frac{1}{12}\epsilon^{IJKL} < B, \star B >= 0 \;\mapsto\; B = e \wedge e$$

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Remark: same modification in the self-dual theory
$\Rightarrow$ NO extra degrees of freedom! Krasnov '07

- Why extra degrees of freedom in the non-chiral action?
- What is their physical interpretation?

Revisiting the simplicity constraints 1

- The role of the constraint is ***not*** to introduce a metric:
  a metric is already present in the formalism, through Urbantke's formula $g \sim BBB$
- The role of the constraints is to ***single out*** these (10) metric degrees of freedom out of the initial components of the B field

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SO(4): Two Urbantke metrics

$$\sqrt{g^{\mathrm{U}(\pm)}}\, g^{\mathrm{U}(\pm)}_{\mu\nu} = \frac{1}{12}\delta_{IN}\left(\delta_{JMKL} \pm \frac{1}{2}\epsilon_{JMKL}\right)\epsilon^{\alpha\beta\gamma\delta} B^{IJ}_{\mu\alpha} B^{KL}_{\beta\gamma} B^{MN}_{\delta\nu}$$
$$\Longrightarrow B^{IJ}_{\mu\nu} = B(g^{\mathrm{U}+}, g^{\mathrm{U}-}, b^+, b^-)$$

corresponding to the decomposition into self-dual and antiself-dual parts of SO(4)

Self-duality and metricity

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Urbantke's theorem: $B^i_{\mu\nu}$ is self-dual wrt the metric defined by itself

Self-duality and metricity

SU(2): Urbantke metric

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Plebanski's basis of self-dual 2-forms:

$$\Sigma^i(e) = e^0 \wedge e^i + \frac{1}{2}\epsilon^i{}_{jk} e^j \wedge e^k$$
$$\Longrightarrow B^i_{\mu\nu} = \sum_a b^i_a \Sigma^a_{\mu\nu}(e), \qquad \sqrt{g^{\mathrm{U}}}\, g^{\mathrm{U}}_{\mu\nu} = (\det\, b^i_a)\, e\, e^I_\mu e^J_\nu \delta_{IJ}$$

Take det $b^i_a = 1, \;\Rightarrow\; g^{\mathrm{U}}_{\mu\nu} = e^I_\mu e^J_\nu \delta_{IJ}$

Revisiting the simplicity constraints 2

The constraints

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can be decomposed into irreps:

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relaxing the constraints in the two formulations leads to very different theories

The 6 extra degrees of freedom

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Why 6 extra dofs in bi-metric theories?
Simplest counting: expand around "doubly flat" spacetime

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}, \qquad \bar{g}_{\mu\nu} = \delta_{\mu\nu} + \bar{h}_{\mu\nu}$$

and define

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$h^{(-)}_{\mu\nu}$ is <u>diffeo-invariant</u> $\Rightarrow$ masslessness no more protected by symmetry
It will generically acquire a mass term,

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$\Longrightarrow$ One massive spin-2 particle (5 dofs) and one massive scalar (1dof)

Caveat! The scalar is a ghost
[Fierz-Pauli '39, Boulware-Deser '72]

Unification playground

These type of generalized Plebanski theories are interesting for a number of reasons
One idea is to use them for grand unification schemes
[Smolin '08, Lisi, Smolin and SS '10]

- Enlarge the local gauge group, e.g $so(3,1) \mapsto so(N+3,1)$
- Spontaneously break the symmetry, e.g. $so(N) \mapsto \begin{pmatrix} so(3,1) & 4N \\ 4N & so(N) \end{pmatrix}$
- Perturbations around the symmetry-breaking vacuum give (modified) dynamics for
  - gravity
  - gauge fields
  - Higgs scalars from the off-diagonal sector

Moral...

All these is fun to play with... but the moral is:
do not mess with your constraints, unless you know what you are doing!

Outline

Motivations

Twisted geometries

From spinors to twisted geometries

Applications

Conclusions

- It is possible to visualize the truncation $\mathcal{H}_\Gamma$ as capturing a discretization of 3-geometries
- These are the assignment to each triangle of its oriented area, the two unit normals as seen from the two tetrahedra sharing it, and an additional angle related to the extrinsic curvature $(N, \tilde{N}, A, \xi) \Longleftrightarrow (X, g)$
- The 3-geometries are piecewise-flat but in general discontinuous
- At the saddle point of the EPRL model the shape-matching conditions are satisfied $\Rightarrow$ Regge action
- The twisted geometries can be easily derived from spinors associated to half-edges through the area-matching constraints $\Rightarrow$ introduction of spinorial techniques with potentially many applications

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