Non-commutative geometry and matrix models II

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outline part II:

embedded NC spaces, matrix models, and emergent gravity

- noncommutative gauge theory
- Yang-Mills matrix models
- general geometry in matrix models (embedded NC spaces, curvature)
- nonabelian gauge fields, fermions, SUSY
- quantization of M.M: heat kernel expansion, UV/IR mixing
- aspects of (emergent) gravity, outlook

Yang-Mills matrix models

dynamical embedded NC spaces $\leftrightarrow$ gravity
well suited for quantization

$$Z = \int dX^a e^{-S[X]}$$

$$S[X] = \mathrm{Tr}[X^a, X^b][X^{a'}, X^{b'}]\delta_{aa'}\delta_{bb'} + (matter)$$

note:

- matrix configuration X^a ... matrix geometry ("background")
- integration over space of geometries
  $\rightarrow$ "emergent" (dominant, effective) geometry
- very closely related to NC gauge theory
- $D = 10$, add Majorana-Weyl fermions $\rightarrow$ IKKT model
  (=dim-red. of $D = 10$ SYM) "nonperturb. def. of IIB string theory"
  Ishibashi, Kawai, Kitazawa and Tsuchiya hep-th/9612115
- more generally: $\exists$ intersecting spaces, stacks, etc.

Gauge theory on $\mathbb{R}^4_\theta$

Let $[\bar{X}^\mu, \bar{X}^\nu] = i\bar{\theta}^{\mu\nu}$, $\bar{X}^\mu \in \mathcal{L}(\mathcal{H})$ (Moyal-Weyl) consider fluctuations around $\mathbb{R}^4_\theta$:

$$X^\mu = \bar{X}^\mu - \bar{\theta}^{\mu\nu} A_\nu$$

recall $[\bar{X}^\mu, \phi] = i\theta^{\mu\nu}\partial_\nu\phi \rightarrow$

$$\begin{aligned} [X^\mu, X^\nu] &= i\bar{\theta}^{\mu\nu} + i\bar{\theta}^{\mu\mu'}\bar{\theta}^{\nu\nu'}(\partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'} + i[A_{\mu'}, A_{\nu'}]) \\ &= i\bar{\theta}^{\mu\nu} + i\bar{\theta}^{\mu\mu'}\bar{\theta}^{\nu\nu'} F_{\mu'\nu'} \\ &= i\bar{\theta}^{\mu\mu'}\bar{\theta}^{\nu\nu'}(\bar{\theta}^{-1}_{\mu\nu} + F_{\mu'\nu'}) \end{aligned}$$

$F_{\mu\nu}(x)$ ... $\mathfrak{u}(1)$ field strength

gauge transformations:

$$\begin{aligned} X^\mu \rightarrow UX^\mu U^{-1} &= U(\bar{X}^\mu - \bar{\theta}^{\mu\nu}A_\nu)U^{-1} = \bar{X}^\mu + U[\bar{X}^\mu, U^{-1}] + \bar{\theta}^{\mu\nu}UA_\nu U^{-1} \\ &= \bar{X}^\mu + \bar{\theta}^{\mu\nu}(U\partial_\nu U^{-1} + UA_\nu U^{-1}) \end{aligned}$$

infinites: $U = e^{i\Lambda(X)}$, $\delta A_\mu = i\partial_\mu\Lambda(X) + i[\Lambda(X), A_\mu]$

Yang-Mills action:

$$S_{YM}[X] = \mathrm{Tr}[X^\mu, X^\nu][X^{\mu'}, X^{\nu'}]\delta_{\mu\mu'}\delta_{\nu\nu'}$$
$$= \rho \int d^4x (F_{\mu\nu} + i\bar{\theta}^{-1}_{\mu\nu})(F_{\mu'\nu'} + i\bar{\theta}^{-1}_{\mu'\nu'})\bar{G}^{\mu\mu'}\bar{G}^{\nu\nu'}$$

or

$$\mathrm{Tr}([X^\mu, X^\nu] - i\bar{\theta}^{\mu\nu})([X^{\mu'}, X^{\nu'}] - i\bar{\theta}^{\mu'\nu'})\delta_{\mu\mu'}\delta_{\nu\nu'} = \rho \int d^4x F_{\mu\nu} F_{\mu'\nu'} \bar{G}^{\mu\mu'}\bar{G}^{\nu\nu'}$$

(same up to surface term $\mathrm{Tr}[X, X] = \int F \to 0$)

... NC $U(1)$ gauge theory on $\mathbb{R}^4_\theta$,

effective metric

$$\bar{G}^{\mu\nu} = \bar{\theta}^{\mu\mu'}\bar{\theta}^{\nu\nu'}\delta_{\mu'\nu'}, \qquad \rho = |\bar{\theta}^{-1}_{\mu\nu}|^{1/2}$$

reduces to usual $U(1)$ gauge theory on $\mathbb{R}^4$ (as classical F.T.!!)

invariant under gauge trafo

$$X^\mu \to UX^\mu U^{-1},$$
$$F_{\mu\nu} \to UF_{\mu\nu}U^{-1} \sim \text{symplectomorphism}$$

no “local” observables ! (need trace)

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or

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no “local” observables ! (need trace)

coupling to scalar fields:

consider

$$S[X,\phi^i] = \mathrm{Tr}\left([X^\mu,X^\nu][X^{\mu'},X^{\nu'}]\delta_{\mu\mu'}\delta_{\nu\nu'} + [X^\mu,\phi^i][X^{\mu'},\phi^i]\delta_{\mu\mu'}\right)$$
$$= \rho\int d^4x\left(F_{\mu\nu}F_{\mu'\nu'}\,\bar{G}^{\mu\mu'}\bar{G}^{\nu\nu'} + D_\mu\phi^i D_\nu\phi^i\,\bar{G}^{\mu\nu}\right)$$

$$[X^\mu,\phi] = i\bar{\theta}^{\mu\nu}(\partial_\nu + i[A_\mu,.])\phi =: i\bar{\theta}^{\mu\nu}D_\mu\phi$$

(dropping surface terms)
gauge transformation

$$\phi^i \to U\phi^i U^{-1} \qquad \text{(adjoint)}$$

same form as

$$S[X] = \mathrm{Tr}[X^a,X^b][X^{a'},X^{b'}]\delta_{aa'}\delta_{bb'}, \qquad a = 1,...,4+k$$

note:

- extremely simple *origin of gauge fields*:
  arbitrary fluctuations $X^\mu \to X^\mu + \mathcal{A}^\mu$ $\quad (\mathcal{A}^\mu = -\theta^{\mu\nu} A_\nu)$

  configuration space = $\{4$ hermitian matrices $X^a\}$

  works only on NC spaces!
- matrix models $\mathrm{Tr}[X,X][X,X]$ $\sim$ gauge-invariant YM action
- generalized easily to $U(n)$ theories but
  $U(1)$ sector does not decouple from $SU(n)$ sector
- one-loop: UV/IR mixing $\to$ not QED, problem
  except in $\mathcal{N}=4$ SUSY case: finite (!?)

... nevertheless phys. wrong for $U(1)$ sector:
proper interpretation in terms of (emergent) geometry, gravity.

try something similar for fuzzy sphere:

$$S[X] = \frac{1}{g^2}\mathrm{Tr}\left([X^a, X^b][X_a, X_b] - 4i\varepsilon_{abc}X^aX^bX^c - 2X^aX_a\right)$$
$$= \frac{1}{g^2}\mathrm{Tr}\left([X^a, X^b] - i\varepsilon^{abc}X_c)([X_a, X_b] - i\varepsilon_{abc}X^c\right)$$
$$= \frac{1}{g^2}\mathrm{Tr}\, F^{ab}F_{ab} \quad \geq 0$$

where $X^a \in \mathrm{Mat}(\mathrm{N}, \mathbb{C}), \quad \mathrm{a} = 1, 2, 3$ and

$$F^{ab} := [X^a, X^b] - i\varepsilon^{abc}X_c$$ field strength

solutions (minima!):

$$F^{ab} = 0 \quad \Leftrightarrow \quad [X^a, X^b] = i\varepsilon^{abc}X_c$$

$$X^a = \lambda^a,$$ λ^a ... rep. of $\mathfrak{su}(2)$

any rep. of $su(2)$ is a solution! $X^a = \begin{pmatrix} \lambda^a_{(M_1)} & 0 & \dots & 0 \\ 0 & \lambda^a_{(M_2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^a_{(M_k)} \end{pmatrix}$

concentric fuzzy spheres $S^2_{M_i}$!

geometry & topology dynamical !

try something similar for fuzzy sphere:

$$
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S[X] &= \frac{1}{g^2}\mathrm{Tr}\left([X^a,X^b][X_a,X_b] - 4i\varepsilon_{abc}X^aX^bX^c - 2X^aX_a\right)\\
&= \frac{1}{g^2}\mathrm{Tr}\left([X^a,X^b] - i\varepsilon^{abc}X_c)([X_a,X_b] - i\varepsilon_{abc}X^c\right)\\
&= \frac{1}{g^2}\mathrm{Tr}\,F^{ab}F_{ab} \qquad \geq 0
\end{aligned}
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F^{ab} &= 0 \quad \Leftrightarrow \quad [X^a,X^b] = i\varepsilon^{abc}X_c\\
X^a &= \lambda^a, \qquad \lambda^a \ldots \text{rep. of } \mathfrak{su}(2)
\end{aligned}
$$

any rep. of $su(2)$ is a solution! $X^a = \begin{pmatrix} \lambda^a_{(M_1)} & 0 & \ldots & 0\\ 0 & \lambda^a_{(M_2)} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \lambda^a_{(M_k)} \end{pmatrix}$

concentric fuzzy spheres $S^2_{M_i}$!
geometry & topology dynamical !

expand around solution:

$$X^a = \lambda^a + A^a \qquad \in \mathrm{Mat}(N, \mathbb{C})$$

$$\begin{aligned} F^{ab} &= [\lambda^a, A^b] - [\lambda^b, A^a] - i\varepsilon^{abc} A_c + [A^a, A^b] \\ F &= F^{ab}\xi^a\xi^b = dA + AA \end{aligned}$$

can be interpreted in terms of

$$\begin{cases} U(1) \text{ gauge theory on } S^2_N \text{ (tang. fluct. if) } \lambda^a A_a = 0 \\ \text{coupled to scalar field } D_\mu \phi D^\mu \phi \text{ (radial fluctuations) } X^a = \lambda^a (1 + \phi) \end{cases}$$

can fix geometry, suppress radial field by adding constraint

$$\tilde{S}[X] = \mathrm{Tr}\left(([X^a, X^b] - i\varepsilon^{abc} X_c)([X_a, X_b] - i\varepsilon_{abc} X^c) + (X^a X_a - C_N)^2\right)$$

$\Rightarrow$ indeed deformed Maxwell theory on S^2_N, as *classical* F.T.

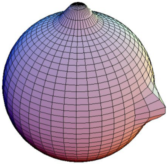
however:

recall fuzzy sphere: near north pole $x^a = (0, 0, 1)$

$$X^3 = \sqrt{1 - (X^1)^2 - (X^2)^2}$$

expect:

radial deformation $X^3 = \lambda^3 + A^3 = \phi(X^1, X^2) \quad \ldots$



deformation of embedding, geometry!

geometry $\leftrightarrow$ NC gauge theory ???

$\Rightarrow$ consider deformed fuzzy spaces, effective geometry

Yang-Mills Matrix Models, reconsidered

let $g_{ab} = \delta_{ab}$ ... $SO(D)$ (resp. $g_{ab} = g_{ab}$ ... $SO(n,m)$)

$$S = -Tr\left([X^a, X^b][X^{a'}, X^{b'}]g_{aa'}g_{bb'} + \text{fermions}\right)$$

$$X^a = X^{a\dagger} \in Mat(\infty, \mathbb{C}), \qquad a = 1, ..., D$$

gauge symmetry $X^a \to UX^aU^{-1}$, or

$$S = -Tr\left(([X^a, X^b] - i\theta^{ab}\mathbf{1})([X^{a'}, X^{b'}] - i\theta^{a'b'}\mathbf{1})g_{aa'}g_{bb'} + ...\right)$$

(up to boundary terms $Tr[X, X]$)

- pre-geometric; { NC space(-time), metric (gravity) } solutions (emergent)
- { nonabelian gauge fields, "gravitons" } ... fluctuations of NC space
- $D = 10$: quantization well-defined (?!)

Space-time & geometry from matrix models:

e.o.m.: $\delta S = 0 \Rightarrow [X^a, [X^{a'}, X^{b'}]] g_{aa'} = 0$

solutions: ($\to$ NC spaces)

- 1) prototype (d=4):

$[X^a, X^b] = i\theta^{ab}\,\mathbb{1}$, rank $\theta^{ab} = 4$

split $X^a = (\bar{X}^\mu, \Phi^i)$, $\mu = 1, ..., 4$

$$\left.\begin{array}{rcl} [\bar{X}^\mu, \bar{X}^\nu] &=& i\bar{\theta}^{\mu\nu}\,\mathbb{1} \\ \Phi^i &=& 0 \end{array}\right\} \quad \ldots \quad \mathbb{R}^4_\theta$$

interpretation:

$X^a: \quad \mathbb{R}^4_\theta \hookrightarrow \mathbb{R}^{10}$ ..."embedded quantum plane"

fluctuations $X^a = \bar{X}^a + \delta X^a \to$ propagating fields on $\mathbb{R}^4_\theta$

Noncommutative spaces and Poisson structure

$(\mathcal{M}, \theta^{\mu\nu}(x))$ ... $2n$-dimensional manifold with Poisson structure

Its quantization $\mathcal{M}_\theta$ is NC algebra such that

$$\begin{aligned} \mathcal{I}:\ \mathcal{C}(\mathcal{M}) &\to \mathcal{A} \cong Mat(\infty, \mathbb{C}) \\ f(x) &\mapsto \hat{f}(X) \\ x^a &\mapsto X^a, \qquad e^{ikx} \mapsto e^{ikX} \end{aligned}$$

such that $$[\hat{f}(X), \hat{g}(X)] = \mathcal{I}(i\{f(x), g(x)\}) + O(\theta^2)$$

("nice") $\Phi \in Mat(\infty, \mathbb{C}) \quad \leftrightarrow \quad$ quantized function on $\mathcal{M}$

furthermore:

$$\begin{aligned} (2\pi)^2\, Tr(\phi(X)) &\sim \int d^4x\, \rho(x)\, \phi(x) \\ \rho(x) &= \mathrm{Pfaff}\,(\theta^{-1}_{\mu\nu}) \ \ldots \end{aligned}$$ symplectic volume

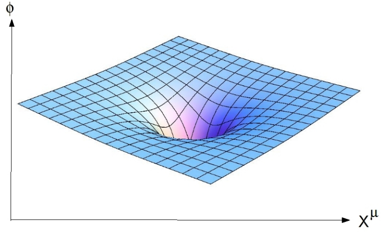
(cf. Bohr-Sommerfeld quantization)

- 2) generic solutions (d=4): deformations of $\mathbb{R}^4_\theta \subset \mathbb{R}^{10}$

$X^a = (X^\mu, \Phi^i(X^\mu)), \qquad \mu = 1, ..., 4;$

$\mathcal{A} \cong Mat(\infty, \mathbb{C})$ generated by X^μ

$[X^\mu, X^\nu] \sim i\{x^\mu, x^\nu\} = i\theta^{\mu\nu}(x),$ generic NC space $\subset \mathbb{R}^D$



interpretation:

$$X^a \sim x^a : \quad \mathcal{M}^4 \hookrightarrow \mathbb{R}^{10}$$

... 4-dim (or 3+1-dim.) space(time), ”brane“

quantized Poisson-MF $(\mathcal{M}, \theta^{\mu\nu}(x))$

Effective geometry of NC brane:

consider scalar field coupled to Matrix Model ("test particle")

use $[f,\varphi] \sim i\theta^{\mu\nu}(x)\partial_\mu f \partial_\nu \varphi \qquad \Rightarrow$

$$
\begin{aligned}
S[\varphi] &= -Tr\,[X^a,\varphi][X^b,\varphi]\,g_{ab} \qquad (U(\mathcal{H}) \quad \text{gauge inv.!}) \\
&\sim \int d^4x\,\sqrt{|\theta^{-1}_{\mu\nu}|}\,\theta^{\mu'\mu}\partial_{\mu'}x^a\partial_\mu\varphi\,\theta^{\nu'\nu}\partial_{\nu'}x^b\partial_\nu\varphi\,g_{ab} \\
&= \int d^4x\,\sqrt{|G_{\mu\nu}|}\,G^{\mu\nu}(x)\,\partial_\mu\varphi\partial_\nu\varphi
\end{aligned}
$$

$$
\begin{aligned}
G^{\mu\nu}(x) &= e^{-\sigma}\theta^{\mu\mu'}(x)\theta^{\nu\nu'}(x)\;g_{\mu'\nu'}(x) \quad \text{effective metric} \\
g_{\mu\nu}(x) &= \partial_\mu x^a\partial_\nu x^b g_{ab} \quad \text{induced metric on } \mathcal{M}^4_\theta
\end{aligned}
$$

$$
e^{-2\sigma} = \frac{|\theta^{-1}_{\mu\nu}|}{|g_{\mu\nu}|}, \qquad |G_{\mu\nu}| = |g_{\mu\nu}| \qquad \text{for } \dim(\mathcal{M}) = 4
$$

φ couples to metric $G^{\mu\nu}(x)$, determined by $\theta^{\mu\nu}(x)$ & embedding $\phi^i(x)$

... quantized Poisson manifold with metric $(\mathcal{M},\theta^{\mu\nu}(x),G_{\mu\nu}(x))$

same metric $G_{\mu\nu}$ for gauge fields, fermions

$\rightarrow$ all matter couples to dynamical metric $G_{\mu\nu}$ $\Rightarrow$ effective gravity

however: metric is not fundamental d.o.f.
rather: matrices X^a resp. $(\phi^i, \theta^{\mu\nu})$ resp. $(\phi^i, F_{\mu\nu})$
$\Rightarrow$ *dynamics* of gravity NOT given by Einstein equations

not GR (long distances!),
may be close enough to observation (?)

note: $D = 10$ just enough to describe most general $g_{\mu\nu}(x)$ in $d = 4$ (locally)

A. Friedman (1961)

class of embedded NC spaces

$$X^a: \quad \mathcal{M} \to \mathbb{R}^D$$

is stable under small deformations

consider small deformation

$$\tilde{X}^a = X^a + A^a$$

by assumption locally $\quad X^a = (X^\mu, \phi^i(X^\mu)) \ \sim \ (x^\mu, \phi^i(x^\mu))$

X^μ generate $\mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C})$

$\Rightarrow \ A^a = A^a(x^\mu)$, smooth

$$\tilde{X}^a = (X^\mu + A^\mu, \phi + A^i) \quad \sim \quad (\tilde{x}^\mu, \tilde{\phi}^i(\tilde{x}^\mu)): \quad \tilde{\mathcal{M}} \to \mathbb{R}^D$$

$$[\tilde{X}^\mu, \tilde{X}^\nu] \quad \sim \quad i\{\tilde{x}^\mu, \tilde{x}^\nu\}$$ ...new Poisson bracket

... deformed embedded NC space $\tilde{\mathcal{M}}$

dynamics of geometry

def. $\Box := [X^a,[X^b,.]]g_{ab}$ ... matrix Laplacian on $\mathcal{A}$

<u>result:</u>
$(\mathcal{M},\omega)$ symplectic manifold, $\omega = \frac{1}{2}\theta^{-1}_{\mu\nu} dx^\mu \wedge dx^\nu$
$x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D$ ... embedding in $\mathbb{R}^D$
induced metric $g_{\mu\nu}$ and $G^{\mu\nu}$ as above. Then:

$$\begin{aligned} \{x^a,\{x^b,\varphi\}\}g_{ab} &= e^\sigma \Delta_G \varphi \\ \nabla^\mu_G(e^\sigma\theta^{-1}_{\mu\nu}) &= G_{\nu\rho}\,\theta^{\rho\mu}\left(e^{-\sigma}\partial_\mu\eta + \partial_\mu x^a\,\Delta_G x^b g_{ab}\right) \end{aligned}$$

for $\varphi \in \mathcal{C}^\infty(\mathcal{M})$, ∇_G ... Levi-Civita, Δ_G ... Laplace- Op. w.r.t. $G_{\mu\nu}$, and

$$\eta(x) := \frac{1}{4} e^\sigma\, G^{\mu\nu} g_{\mu\nu}.$$

cf. fuzzy sphere, torus etc! (H.S., 2008)

Hence: $$\boxed{\Box\phi \sim -e^\sigma \Delta_G \phi(x)}$$

proof: either

$$
\begin{aligned}
-\mathrm{Tr}\,\varphi'[X^a,[X_a,\varphi]] &= \mathrm{Tr}\,[X^a,\varphi'][X_a,\varphi] \\
\int d^4x\,\sqrt{|\theta^{-1}_{\mu\nu}|}\,\varphi'\{X^a,\{X_a,\varphi\}\} &= -\int d^4x\sqrt{|G_{\mu\nu}|}\,G^{\mu\nu}(x)\,\partial_\mu\varphi'\partial_\nu\varphi \\
\int d^4x\,\sqrt{|G_{\mu\nu}|}\,e^{-\sigma}\,\varphi'\{X^a,\{X_a,\varphi\}\} &= \int d^4x\,\sqrt{|G_{\mu\nu}|}\,\varphi'\Delta_G\varphi
\end{aligned}
$$

or

$$
\begin{aligned}
\{X^a,\{X^a,\varphi\}\} &= \theta^{\mu\rho}\partial_\mu x^a\partial_\rho(\theta^{\nu\eta}\partial_\nu x_a\partial_\eta\varphi) \\
&= \theta^{\mu\rho}\partial_\rho(\partial_\mu x^a\theta^{\nu\eta}\partial_\nu x_a\partial_\eta\varphi) \\
&= \theta^{\mu\rho}\partial_\rho(\theta^{\nu\eta}g_{\mu\nu}\partial_\eta\varphi) \\
&= \theta^{\mu\rho}\theta^{\nu\eta}g_{\mu\nu}\partial_\rho\partial_\eta\varphi + \theta^{\mu\rho}\partial_\rho(\theta^{\nu\eta}g_{\mu\nu})\partial_\eta\varphi \\
&= e^\sigma(G^{\rho\eta}\partial_\rho\partial_\eta\varphi - \Gamma^\eta\partial_\eta\varphi) = e^\sigma\Delta_G\varphi,
\end{aligned}
$$

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$$
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&= \theta^{\mu\rho}\theta^{\nu\eta}g_{\mu\nu}\partial_\rho\partial_\eta\varphi+\theta^{\mu\rho}\partial_\rho(\theta^{\nu\eta}g_{\mu\nu})\partial_\eta\varphi \\
&= e^\sigma(G^{\rho\eta}\partial_\rho\partial_\eta\varphi-\Gamma^\eta\partial_\eta\varphi)=e^\sigma\Delta_G\varphi,
\end{aligned}
$$

in particular:

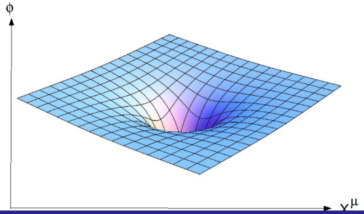
matrix e.o.m: $[X^a,[X^b,X^{a'}]]g_{aa'} = 0 \quad \Longleftrightarrow$

$$\begin{aligned} \Delta_G \Phi^i &= 0, \quad \Delta_G x^\mu = 0 \\ \nabla^\mu(e^\sigma \theta^{-1}_{\mu\nu}) &= e^{-\sigma}\, G_{\rho\nu}\theta^{\rho\mu}\partial_\mu\eta \\ \eta &= \tfrac{1}{4} e^\sigma\, G^{\mu\nu} g_{\mu\nu} \end{aligned}$$

... covariant formulation in semi-classical limit

in particular:

$\mathcal{M}^4 \hookrightarrow \mathbb{R}^D$ is harmonic embedding (w.r.t. $G_{\mu\nu}$)
minimal surface



dynamics of NC structure $\theta^{\mu\nu}$:

$$S_{YM} = -Tr[X^a, X^b][X_a, X_b] \sim \int d^4x \sqrt{g} e^{-\sigma} \eta$$

Euclidean case: at $p \in \mathcal{M}$, diagonalize $g_{\mu\nu} = \mathrm{diag}(1,1,1,1)$
using $SO(4)$ $\rightarrow$ standard form

$$\theta^{\mu\nu} = \theta \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & & 0 & \pm\alpha^{-1} \\ 0 & 0 & \mp\alpha^{-1} & 0 \end{pmatrix}.$$

effective metric $G^{\mu\nu} = \mathrm{diag}(\alpha^2, \alpha^2, \alpha^{-2}, \alpha^{-2})$.
Note

$$\begin{aligned} \tfrac{1}{4} G^{\mu\nu} g_{\mu\nu} &= e^{-\sigma}\eta = \tfrac{1}{2}(\alpha^2 + \alpha^{-2}) \geq 1 \\ \star\omega &= \pm\omega \Leftrightarrow e^{-\sigma}\eta = 1 \Leftrightarrow G_{\mu\nu} = g_{\mu\nu} \Leftrightarrow S_{YM} \text{ minimal} \end{aligned}$$

minimum of S_{YM} $\Leftrightarrow$ $\theta^{\mu\nu}$ (A)SD $\Leftrightarrow$ $G_{\mu\nu} = g_{\mu\nu}$.

more structure:

define

$$\mathcal{J}^{\eta}_{\gamma} = e^{-\sigma/2}\,\theta^{\eta\gamma'}\,g_{\gamma'\gamma} = -e^{\sigma/2}\,G^{\eta\gamma'}\,\theta^{-1}_{\gamma'\gamma}.$$

Then

$$G^{\mu\nu} = \mathcal{J}^{\mu}_{\rho}\,\mathcal{J}^{\nu}_{\rho'}\,g^{\rho\rho'} = -(\mathcal{J}^2)^{\mu}_{\rho}\,g^{\rho\nu},$$

hence

$$G^{\mu\nu}g_{\nu\rho} = -(\mathcal{J}^2)^{\mu}_{\rho}, \quad \mathcal{J}^2 = -\delta \quad \Leftrightarrow \quad g = G$$

... "almost-complex" structure

$\rightarrow \quad (\mathcal{M}, \mathcal{J}, e^{-\sigma/2}g_{\mu\nu})$ "almost-Kähler" $\Leftrightarrow$ $g = G$

---

note: $g = G \Rightarrow$ e.o.m. for $\theta^{\mu\nu}$ reduces to

$$\nabla^{\mu}\theta^{-1}_{\mu\nu} = 0$$

follows from $\star\omega = \pm\omega$

more structure:

define

$$\mathcal{J}^\eta_\gamma = e^{-\sigma/2}\,\theta^{\eta\gamma'}\,g_{\gamma'\gamma} = -e^{\sigma/2}\,G^{\eta\gamma'}\,\theta^{-1}_{\gamma'\gamma}.$$

Then

$$G^{\mu\nu} = \mathcal{J}^\mu_\rho\,\mathcal{J}^\nu_{\rho'}\,g^{\rho\rho'} = -(\mathcal{J}^2)^\mu_\rho\,g^{\rho\nu},$$

hence

$$G^{\mu\nu}g_{\nu\rho} = -(\mathcal{J}^2)^\mu_\rho, \quad \mathcal{J}^2 = -\delta \quad \Leftrightarrow \quad g = G$$

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$\rightarrow \quad (\mathcal{M}, \mathcal{J}, e^{-\sigma/2}g_{\mu\nu})$ “almost-Kähler” $\Leftrightarrow$ $g = G$

---

note: $g = G \Rightarrow$ e.o.m. for $\theta^{\mu\nu}$ reduces to

$$\nabla^\mu\theta^{-1}_{\mu\nu} = 0$$

follows from $\star\omega = \pm\omega$

special class of solutions:

$$\begin{aligned} g_{\mu\nu} &= G_{\mu\nu}, \\ \Delta_G \phi^i &= 0 \\ \nabla^\mu \theta^{-1}_{\mu\nu} &= 0 \end{aligned}$$

holds for (anti)self-dual symplectic structure $\theta^{-1}_{\mu\nu}$,

$$\begin{aligned} \star(\theta^{-1}) &= \pm\theta^{-1} && \text{Euclidean} \\ \star(\theta^{-1}) &= \pm i\theta^{-1} && \text{Minkowski (Wick rotation } X^0 \to it \text{)} \end{aligned}$$

then

$$S_{MM} \sim Tr[X^a, X^b][X_a, X_b] = \int d^4x \sqrt{|g_{\mu\nu}|}$$

... same structure as vacuum energy / cosm. const.

semi-classical derivation of e.o.m.:

$$S = -\mathrm{Tr}[X^a, X^b][X_a, X_b] \sim \frac{1}{(2\pi)^n} \int d^{2n}x \sqrt{|G|}\, e^{-\sigma}\eta.$$

geometrical d.o.f:

$$\begin{aligned} \delta\theta^{-1}_{\mu\nu} &= \nabla_\mu \delta A_\nu - \nabla_\nu \delta A_\mu \\ \delta\phi^i & \end{aligned}$$

$$\begin{aligned} \delta S &= \tfrac{1}{2} \int d^{2n}x \sqrt{|\theta^{-1}_{\mu\nu}|}\Big(g_{\mu\nu}\theta^{\mu\mu'}\delta\theta^{\nu\nu'} g_{\mu'\nu'} + g_{\mu\nu}\theta^{\mu\mu'}\theta^{\nu\nu'}\delta g_{\mu'\nu'} + \eta(x)\,\theta^{\mu\nu}\delta\theta^{-1}_{\nu\mu}\Big) \\ &= \tfrac{1}{2} \int d^{2n}x \sqrt{|\theta^{-1}_{\mu\nu}|}\Big(e^{2\sigma} G^{\eta\mu}\theta^{-1}_{\mu\nu} G^{\nu\rho}\delta\theta^{-1}_{\rho\eta} + e^{\sigma} G^{\mu\nu}\delta g_{\mu\nu} + \eta(x)\,\theta^{\mu\nu}\delta\theta^{-1}_{\nu\mu}\Big) \\ &= \int d^{2n}x \sqrt{G}\Big(G^{\eta\mu} G^{\nu\rho} e^{\sigma}\theta^{-1}_{\mu\nu}\nabla_\rho\delta A_\eta - e^{-\sigma}\eta\,\theta^{\rho\eta}\nabla_\rho\delta A_\eta + G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\delta\phi_i\Big) \\ &= -\int d^{2n}x\sqrt{G}\delta A_\eta\Big(G^{\eta\mu}G^{\nu\rho}\nabla_\rho(e^{\sigma}\theta^{-1}_{\mu\nu}) - \nabla_\rho(e^{-\sigma}\eta\,\theta^{\rho\eta})\Big) + \delta\phi^i\partial_\nu\Big(\sqrt{G}\,G^{\mu\nu}\partial_\mu\phi_i\Big) \\ &= -\int d^{2n}x\sqrt{G}\,\Big(\delta A_\eta\Big(G^{\eta\mu}G^{\nu\rho}\nabla_\rho(e^{\sigma}\theta^{-1}_{\mu\nu}) - \tfrac{1}{\sqrt{G}}\partial_\rho(\sqrt{G}e^{-\sigma}\eta\theta^{\rho\eta})\Big) + \delta\phi^i\,\Delta_G\phi_i\Big) \\ &= -\int d^{2n}x\,\sqrt{G}\,\Big(\delta A_\eta\Big(G^{\eta\mu}G^{\nu\rho}\nabla_\rho(e^{\sigma}\theta^{-1}_{\mu\nu}) - e^{-\sigma}\theta^{\rho\eta}\,\partial_\rho\eta\Big) + \delta\phi^i\Delta_G\phi_i\Big) \end{aligned}$$

(sufficiently) generic 4D geometry in M.M.:

1. take some nice $(\mathcal{M}^4, g_{\mu\nu})$ (e.g. asympt. flat, glob. hyperbolic, ...)
2. choose embedding $x^a : \mathcal{M} \hookrightarrow \mathbb{R}^{10}$ (Friedman etal)
3. equip $\mathcal{M}$ with (anti)selfdual symplectic form $\omega = \theta^{-1}_{\mu\nu} dx^\mu \wedge dx^\nu$,
   $\star_g(\omega) = \pm\omega$ (almost-Kähler)
   $\rightarrow$ construct quantization of $(\mathcal{M}, \omega)$:

   $$\mathcal{I}: \ \mathcal{C}(\mathcal{M}) \to \mathcal{A} \cong Mat(\infty, \mathbb{C})$$

   in particular: $X^a \sim x^a$
4. $\rightarrow$ effective metric $G^{\mu\nu} \sim g^{\mu\nu}$, encoded in $\Box$ in M.M.

(examples: fuzzy spaces = quantized coadjoint orbits, e.g. $S^2_N \subset \mathbb{R}^3$)

$su(n)$ gauge fields: same model, new vacuum

$$Y^a = \begin{pmatrix} Y^\mu \\ Y^i \end{pmatrix} = \begin{pmatrix} X^\mu \otimes \mathbf{1}_n \\ \phi^i \otimes \mathbf{1}_n \end{pmatrix}$$

(n coinciding branes)

include fluctuations:

$$Y^a = (1 + \mathcal{A}^\rho \partial_\rho) \begin{pmatrix} X^\mu \otimes \mathbf{1}_n \\ \phi^i \otimes \mathbf{1}_n + \Phi^i \end{pmatrix}$$

where

$$\begin{aligned} \mathcal{A}^\mu &= -\theta^{\mu\nu} A_{\nu,\alpha} \otimes \lambda^\alpha, \qquad \lambda^\alpha \in \mathfrak{su}(n) \\ \Phi^i &= \Phi^i_\alpha \otimes \lambda^\alpha \end{aligned}$$

$\Rightarrow$ effective action:

$$S_{YM} = \int d^4x \sqrt{G}\, e^\sigma\, G^{\mu\mu'} G^{\nu\nu'}\, tr\, F_{\mu\nu} F_{\mu'\nu'} + 2 \int \eta(x)\, tr\, F \wedge F$$

(H.S., JHEP 0712:049 (2007), JHEP 0902:044,(2009))

... $\mathfrak{su}(n)$ Yang-Mills coupled to metric $G^{\mu\nu}(x)$

fermions

$$S[\Psi] = \mathrm{Tr}\,\overline{\Psi}\not{D}\Psi = \mathrm{Tr}\,\overline{\Psi}\Gamma_a[X^a,\Psi]$$
$$\sim \int d^4x\,\rho(x)\,\overline{\Psi}i\gamma^\mu(x)\partial_\mu\Psi,$$

$$\gamma^\mu(x) = \Gamma_a\theta^{\nu\mu}\partial_\nu x^a$$

note

$$\begin{aligned}\{\gamma^\mu,\gamma^\nu\} &= \{\Gamma_a,\Gamma_b\}\theta^{\mu'\mu}\partial_{\mu'}x^a\theta^{\nu'\nu}\partial_{\nu'}x^b \\ &= 2\theta^{\mu'\mu}\theta^{\nu'\nu}g_{\mu'\nu'} \\ &= 2e^\sigma\,G^{\mu\nu}(x)\end{aligned}$$

naturally SUSY (IKKT model with $D = 10$)

couple to $G_{\mu\nu}$, but non-standard spin connection (submanifold!)

global $SO(9,1)$ symmetry:

- can use to fix $\phi^i|_p = 0 = \partial\phi^i|_p$
  ... analogous to Riemannian normal coordinates

---

bottom line: $U(1)$ sector is geometry

- scalar fields describe embedding $\mathcal{M}^4 \subset \mathbb{R}^{10}$,
  $\theta^{\mu\nu}$ & ϕ^i completely absorbed in $g_{\mu\nu}, G_{\mu\nu}$ (semi-classically)
- dynamics, propagators due to $[X^a, .][X_a, .]$
- fluctuations of branes $\to$ dyn. geometry, nonabelian gauge fields
- couples naturally to matter

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- expect good quantum theory (including gravity):
  action $\equiv$ NC $\mathcal{N} = 4$ $U(1)$ SYM

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$U(1)$ gauge fields as gravitons $\quad \delta\theta^{-1}_{\mu\nu} = F_{\mu\nu}$ on $\mathbb{R}^4_\theta$

$$G^{\mu\nu}(x) = \overline{\eta}^{\mu\nu} - h^{\mu\nu} \qquad (+O(F^2))$$

$F_{\mu\nu}(x)$ ... $\mathfrak{u}(1)$ field strength

therefore

$$\boxed{h_{\mu\nu} = \bar\eta_{\nu\nu'}\bar\theta^{\nu'\rho} F_{\rho\mu} + \bar\eta_{\mu\mu'}\bar\theta^{\mu'\eta} F_{\eta\nu} - \tfrac{1}{2}\bar\eta_{\mu\nu}\,(\bar\theta^{\rho\eta}F_{\rho\eta})}$$

... linearized metric fluctuation

e.o.m:

$$\begin{aligned}
[X^\mu,[X^\nu,X^{\mu'}]]\eta_{\mu\mu'} &= 0\\
\Rightarrow\ \partial^\mu F_{\mu\nu} &= 0\\
\Rightarrow\ R_{\mu\nu}[G] &= 0 \quad (\partial^\mu h_{\mu\nu} = 0 \ldots \text{harm. gauge})
\end{aligned}$$

cf. Rivelles [hep-th/0212262]

while $R_{\mu\nu\rho\eta} \neq 0$

$\Rightarrow$ on-shell d.o.f. of gravitons on Minkowski space

i.e.: NC $U(1)$ on $\mathbb{R}^4_\theta$ as gravitons cf. Kitazawa [hep-th/0512204]

higher-order terms, curvature

$$H^{ab} := \frac{1}{2}[[X^a, X^c], [X^b, X_c]]_+$$
$$T^{ab} := H^{ab} - \frac{1}{4}g^{ab}H, \qquad H := H^{ab}g_{ab} = [X^c, X^d][X_c, X_d]\,,$$
$$\Box X := [X^b, [X_b, X]]$$

result:

for 4-dim. $\mathcal{M} \subset \mathbb{R}^D$ with $g_{\mu\nu} = G_{\mu\nu}$:

$$Tr\left(2T^{ab}\Box X_a \Box X_b - T^{ab}\Box H_{ab}\right) \sim \frac{2}{(2\pi)^2}\int d^4x\sqrt{g}\, e^{2\sigma} R$$
$$Tr([[X^a, X^c], [X_c, X^b]][X_a, X_b] - 2\Box X^a \Box X^a)$$
$$\sim \frac{1}{(2\pi)^2}\int d^4x\sqrt{g}\, e^{\sigma}\left(\frac{1}{2}e^{-\sigma}\theta^{\mu\eta}\theta^{\rho\alpha}R_{\mu\eta\rho\alpha} - 2R + \partial^\mu\sigma\partial_\mu\sigma\right)$$

(Blaschke, H.S. arXiv:1003.4132)

(cf. Arnlind, Hoppe, Huisken arXiv:1001.2223)

$\Rightarrow$ Einstein-Hilbert- type action for gravity as matrix model
pre-geometric version of (quantum?) gravity, background indep.!

derivation: (assume $g = G$)

$$
\begin{aligned}
H^{ab} &= \tfrac{1}{2}[[X^a, X^c], [X^b, X_c]]_+ \sim -e^\sigma G^{\mu\nu}\partial_\mu x^a \partial_\nu x^b \overset{g=G}{=} e^\sigma \mathcal{P}_T^{ab}, \\
T^{ab} &= H^{ab} - \tfrac{1}{4}\eta^{ab} H \sim e^\sigma \mathcal{P}_N^{ab}
\end{aligned}
$$

$\mathcal{P}_N, \mathcal{P}_T$ ... projector on normal / tangential bundle of $\mathcal{M} \subset \mathbb{R}^D$. note

$$
\begin{aligned}
R_{\nu\mu\lambda\kappa} &= \mathcal{P}_N^{ab}(-\partial_\kappa\partial_\nu x_a \partial_\lambda\partial_\mu x_b + \partial_\kappa\partial_\mu x_a \partial_\nu\partial_\lambda x_b) \\
&= -\nabla_\kappa\nabla_\nu x^a \nabla_\lambda\nabla_\mu x_a + \nabla_\kappa\nabla_\mu x^a \nabla_\nu\nabla_\lambda x_a
\end{aligned}
$$

(i.e. Gauss-Codazzi theorem) and

$$
\begin{aligned}
T^{bc}[X^a, [X_a, T_{bc}]] &\sim e^{2\sigma}\mathcal{P}_N^{bc}\nabla_\mu\nabla^\mu(e^\sigma g_{bc} - e^\sigma \partial^\nu x_b \partial_\nu x_c)) \\
&= e^{2\sigma}\Big((D-4)\Box e^\sigma - 2P_N^{bc}(e^\sigma \nabla^\mu\partial^\nu x_b \nabla_\mu \partial_\nu x_c)\Big) \\
&= e^{2\sigma}\Big((D-4)\Box e^\sigma - 2e^\sigma \nabla^\mu\partial^\nu x^a \nabla_\mu\partial_\nu x_a\Big)
\end{aligned}
$$

hence

$$
2T^{ab}\Box X^a \Box X^b - T^{bc}\Box T_{bc} \sim e^{2\sigma}\Big((D-4)\Box e^\sigma - 2e^\sigma R\Big)
$$

noting that $H \sim -e^\sigma = \eta$, result follows.

vacuum energy / cosm.const in matrix model:

recall $|g| = |G|$ for general $G \neq g$

can show

$$\mathrm{Tr}\frac{L^4}{\sqrt{\frac{1}{2}H^2 - H^{ab}H_{ab}}} \sim \frac{1}{2}\frac{1}{(2\pi)^2}\int d^4x\, \Lambda^4(x)\sqrt{g}\,.$$

where

$$\Lambda(x) = L\Lambda_{\rm NC}^2 = Le^{-\sigma/2}$$

L ... cutoff "length" in matrix model

(recall $\Lambda_{\rm NC}^{-4} = \frac{|g_{\mu\nu}|}{|\theta^{-1}_{\mu\nu}|} = e^{\sigma}$)

note: is different from action

$$-\mathrm{Tr}[X^a, X^b][X_a, X_b] \sim \frac{1}{(2\pi)^2}\int d^4x\,\sqrt{|g|}\,e^{-\sigma}\eta \overset{g=G}{=} \frac{1}{(2\pi)^n}\int d^{2n}x\,\sqrt{|g|}.$$

proof:

$$H^{ab} = \frac{1}{2}[[X^a, X^c], [X^b, X_c]]_+ \sim -e^\sigma G^{\mu\nu}\partial_\mu x^a \partial_\nu x^b = -e^\sigma (\mathcal{J}^2 \circ \mathcal{P}_T)^{ab},$$

in normal coords, $\mathcal{J}^2 = -\mathrm{diag}(\alpha^2, \alpha^2, \alpha^{-2}, \alpha^{-2})$
2 EV $\rightarrow$ char. equation

$$\mathcal{J}^4 = \frac{1}{2}(tr\mathcal{J}^2)\,\mathcal{J}^2 - \delta$$

implies (note $H = -e^\sigma\, tr\mathcal{J}^2$)

$$H^{ab}H_{ab} - \frac{1}{2}H^2 \sim -e^{2\sigma}\,\mathcal{P}_T^{ab}(\mathcal{P}_T)_{ab} = -4\Lambda_{\rm NC}^{-8}$$

hence

$$\mathrm{Tr}\frac{L^4}{\sqrt{\frac{1}{2}H^2 - H^{ab}H_{ab}}} \sim \frac{L^4}{2}\frac{1}{(2\pi)^2}\int d^4x\,\sqrt{g}e^{-\sigma}e^{-\sigma} = \frac{1}{2(2\pi)^2}\int d^4x\,\Lambda^4(x)\sqrt{g}$$

Quantization of M.M.

$$Z = \int dX^a d\Psi \, e^{-S[X]-S[\Psi]}$$

...non-perturbative!

- includes integration over geometries !!
- probably ill-defined in general (UV/IR mixing = ∞ ind. gravity)
- $\exists$ ONE model with well-defined (finite !?) quantization:
  $\mathcal{N} = 4$ NC SYM on $\mathbb{R}^4_\theta$ $\Leftrightarrow$ (IKKT) model, $D = 10$

  Ishibashi, Kawai, Kitazawa and Tsuchiya 1996, ff

  fully $SO(9,1)$ and $U(\mathcal{H})$ invariant

$$Z = \int dX^a d\Psi\, e^{-S[X]-S[\Psi]} \;=\; e^{-S_{eff}}$$

2 interpretations:

1. on $\mathbb{R}^4_\theta$: $X^\mu = \bar{X}^\mu + \bar{\theta}^{\mu\nu} A_\nu,$ $\bar{X}^\mu$...Moyal-Weyl

   $\to$ NC SYM on $\mathbb{R}^4_\theta$, UV/IR mixing in $U(1)$ sector

   IKKT model, $D = 10$: $\mathcal{N} = 4$ SYM, perturb. finite !(?)

2. on $\mathcal{M}^4 \subset \mathbb{R}^{10}$: $U(1)$ absorbed in $\theta^{\mu\nu}(x)$, $g_{\mu\nu}$

   $\to$ gravity, induced E-H. action

$$S_{eff} \sim \int d^4x \sqrt{|G|}\, (\Lambda^4 + c\Lambda_4^2\, R[G] + ...)$$

($R[G]$ due to UV/IR mixing in NC gauge theory)

- explanation for UV/IR mixing & $U(1)$ entanglement
- $D = 10$ $\Rightarrow$ good quantization !! (maximal SUSY)

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- explanation for UV/IR mixing & $U(1)$ entanglement
- $D = 10$ $\Rightarrow$ good quantization !! (maximal SUSY)

induced action: fermionic loop

$$S[\Psi] = \mathrm{Tr}\, \overline{\Psi}\gamma_a[X^a, \Psi]$$

induced effective action:

$$\Gamma_{\text{eff}} := \frac{1}{2}\mathrm{Tr}\Big(\log \not{D}^2\Big) \;\to\; -\frac{1}{2}\mathrm{Tr}\int\limits_0^\infty \frac{d\alpha}{\alpha}\, e^{-\alpha\not{D}^2} e^{-\frac{1}{\alpha L^2}} \;=:\; \Gamma_L[X]\,.$$

L ... cutoff length

heat kernel expansion:

$$\mathrm{Tr} e^{-\alpha\not{D}^2} = \sum_n \alpha^{\frac{n-4}{2}}\, \Gamma^{(n)}$$

<u>commutative case:</u> $\Gamma^{(n)}$ ... Seeley-de Witt coeff.,

$$\Gamma_{\text{eff}} = \int d^4x\, (\Lambda^4\sqrt{G} + \Lambda^2 R[G] + \ldots)$$

... induced gravity (Sakharov 1967)

<u>NC case:</u> coupling to gravity $\Rightarrow$ compute induced gravity

NC heat kernel expansion

perturbation of flat background $\not{D}^2 = \not{D}_0^2 + V$

$$-\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\alpha}{\alpha}\left(e^{-\alpha\not{D}^2} - e^{-\alpha\not{D}_0^2}\right)e^{-\frac{1}{\alpha L^2}} = \sum_{k>0} O(V^k) =: Tr\mathcal{L}(X)$$

(cf. Grosse Wohlgenannt 2008)

expect: effective M.M.

$\Gamma_{\rm eff} = \mathrm{Tr}\mathcal{L}(X)$, invar. under $SO(D)$ and $U(\infty)$

complication: UV/IR mixing, additional divergences $\sim \Lambda^n, \; n \in \mathbb{Z}$,

$$\Lambda = L\Lambda_{\rm NC}^2, \qquad \Lambda_{\rm NC} = |\theta^{-1}_{\mu\nu}|^{1/8} \text{ ... NC scale}$$

mild UV/IR mixing: finite Λ, such that $\frac{p^2\Lambda^2}{\Lambda_{\rm NC}^4} \ll 1$,
then semi-class. approx. ok even in loops

or: $\mathcal{N} = 4$ model: finite (?!)

for computation: use NC gauge theory point of view

perturbation of $\mathbb{R}^4_\theta$: $X^a = \begin{pmatrix} \bar{X}^\mu \\ 0 \end{pmatrix} + \begin{pmatrix} -\bar{\theta}^{\mu\nu} A_\nu \\ \phi^i \end{pmatrix}$

$$\not{D}^2 = \bar{\Box} + V, \qquad \bar{G}^{\mu\nu} = \Lambda_{\rm NC}^4 \bar{\theta}^{\mu\mu'} \bar{\theta}^{\nu\nu'} \delta_{\mu'\nu'}$$

$$\begin{aligned} V\Psi &= -i\bar{G}^{\mu\nu}\Big(2[A_\mu, \partial_\nu \Psi] + [\partial_\mu A_\nu, \Psi] + i[A_\mu, [A_\nu, \Psi]]\Big) + \delta_{ij}[\varphi^i, [\varphi^j, \Psi]] \\ &\quad + \Lambda_{\rm NC}^4 \big(\Sigma_{\mu\nu}[\mathcal{F}^{\mu\nu}, \Psi] + 2\Sigma_{\mu i} \bar{\theta}^{\mu\nu}[\partial_\nu \phi^i + i[A_\nu, \phi^i], \Psi] - i\Sigma_{ij}[[\phi^i, \phi^j], \Psi]\big) \end{aligned}$$

compute all dimension 6 operators in effective gauge theory

... long computation (one-loop NC YM, generic external fields A_μ, φ^i)

effective gauge theory

induced gauge theory up to dimension 6:

$$\Gamma_{\rm eff} = \frac{tr\mathbf{1}}{16}\frac{\Lambda^4}{\Lambda_{\rm NC}^4}\int\frac{d^4x}{(2\pi)^2}\sqrt{g}\Big(g^{\alpha\beta}D_\alpha\varphi^i D_\beta\varphi_i$$

$$-\tfrac{1}{2}\Lambda_{\rm NC}^4\big(\bar\theta^{\mu\nu}F_{\nu\mu}\bar\theta^{\rho\sigma}F_{\sigma\rho} + (\bar\theta^{\sigma\sigma'}F_{\sigma\sigma'})(F\bar\theta F\bar\theta)\big)$$

$$-2\bar\theta^{\nu\mu}F_{\mu\alpha}g^{\alpha\beta}\partial_\nu\varphi^i\partial_\beta\varphi_i + \tfrac{1}{2}(\bar\theta^{\mu\nu}F_{\mu\nu})g^{\alpha\beta}\partial_\beta\varphi^i\partial_\alpha\varphi_i + \text{h.o.}\Big)$$

$$+\frac{tr\mathbf{1}}{4}\frac{\Lambda^2}{24}\int\frac{d^4x}{(2\pi)^2}\sqrt{g}\Big(-\frac{11}{2}\Lambda_{\rm NC}^4 F_{\rho\eta}\bar\Box_g F_{\sigma\tau}\bar G^{\rho\sigma}\bar G^{\eta\tau} - 12\Lambda_{\rm NC}^8\bar\Box_g\varphi^i\bar\Box\varphi_i$$

$$+\tfrac{1}{2}\Lambda_{\rm NC}^4(\bar\theta^{\mu\nu}F_{\mu\nu})\bar\Box_G(\bar\theta^{\rho\sigma}F_{\rho\sigma}) \quad + \ldots\Big)$$

$$+\frac{\Lambda^6}{\Lambda_{\rm NC}^8}\int\frac{d^4x}{(2\pi)^2}\sqrt{g}(\ldots)$$

$$+\ldots$$

all of this is due to UV/IR mixing !

(D. Blaschke, H.S. M. Wohlgenannt arXiv:1012.4344)

effective generalized matrix model:

re-assemble effective action: $X^a = \begin{pmatrix} \bar{X}^\mu \\ 0 \end{pmatrix} + \begin{pmatrix} -\bar{\theta}^{\mu\nu} A_\nu \\ \phi^i \end{pmatrix}$

$$\Gamma_L[X] = \mathrm{Tr} \frac{L^4}{\sqrt{\frac{1}{2} H^2 - H^{ab} H_{ab} + \frac{1}{L^2} \mathcal{L}_{10,\mathrm{curv}}[X] + \ldots}} \sim \int d^4x\, \Lambda^4(x) \sqrt{g(x)}$$

$$\mathcal{L}_{10,\mathrm{curv}}[X] = c_1 [X^c, H^{ab}][X_c, H_{ab}] + c_2 H^{cd} [X_c, [X^a, X^b]][X_d, [X_a, X_b]] + \ldots$$

$$H^{ab} = [X^a, X^c][X^b, X_c] + (a \leftrightarrow b), \qquad H = H^{ab} \eta_{ab}$$

(D. Blaschke, H.S. M. Wohlgenannt arXiv:1012.4344)

$SO(D)$ manifest, broken by background (e.g. $\mathbb{R}^4_\theta$)
$\Rightarrow$ highly non-trivial predictions for (NC) gauge theory
expect generalization to nonabelian $\mathcal{N} = 4$ SYM: full $SO(9,1)$!

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= powerful new tool for (NC) gauge theory and (emergent) gravity

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$SO(9,1)$ resp. $SO(10)$ symmetry:

- e.g. $[X^a, X^b] = \begin{pmatrix} \bar\theta^{\mu\nu} + \bar\theta^{\mu\mu'}\bar\theta^{\nu\nu'}F_{\mu'\nu'} & \bar\theta^{\mu\nu}D_\nu\phi^i \\ \bar\theta^{\mu\nu}D_\nu\phi^j & [\phi^i,\phi^j] \end{pmatrix}$ is $SO(9,1)$ multiplet

  only possible due to NC !

  $SO(9,1)$ acts on $X^a = \begin{pmatrix} \bar X^\mu - \bar\theta^{\mu\nu}A_\nu \\ \phi^i \end{pmatrix}$,

  non-linearly realized (cf. SSB) on $\begin{pmatrix} -\bar\theta^{\mu\nu}A_\nu \\ \phi^i \end{pmatrix}$

- can use to fix $\phi^i|_p = 0 = \partial\phi^i|_p$

  ... analogous to Riemannian normal coordinates full IKKT model around $\mathbb{R}^4_\theta$: ($\equiv \mathcal{N}=4$ SYM on $\mathbb{R}^4_\theta$!)

background field method $X^a \to X^a + Y^a$,
fully $SO(9,1)$ covariant, e.g.

$$\begin{aligned}
\Gamma_{1-\mathrm{loop}} &= \tfrac{1}{2}\mathrm{Tr}\left(\log(\mathbb{1} + \Sigma^{(Y)}_{ab}\Box^{-1}[\Theta^{ab},.]) - \tfrac{1}{2}\left(\log(\mathbb{1} + \Sigma^{(\psi)}_{ab}\Box^{-1}[\Theta^{ab},.]\right)\right) \\
\Box &= [X^a,[X^a,.]] \\
\Theta^{rs} &= [X^r, X^s] \\
\Sigma_{rs} &= SO(9,1) \text{ generator}
\end{aligned}$$

$\to$ effective generalized M.M.

(work in progress, D. Blaschke, H.S.)

$SO(9,1)$ invariant formalism, broken spontaneously through $\mathbb{R}^4_\theta$
NC essential.

dynamics of emergent NC gravity

assume effective action

$$S \sim \int d^4x \sqrt{|g|}\,(-2\Lambda^4 + \Lambda_4^2 R) + S_{\text{matter}}$$

leads to

$$\begin{aligned} \delta S &= \int d^4x \sqrt{|g|}\,\delta g_{\mu\nu}(-\Lambda^4 g^{\mu\nu} + 8\pi T^{\mu\nu} - \Lambda_4^2 \mathcal{G}^{\mu\nu}) \\ &= -2\int \delta\phi^i \partial_\mu(\sqrt{|g|}\,(-\Lambda^4 g^{\mu\nu} + 8\pi T^{\mu\nu} - \Lambda_4^2 \mathcal{G}^{\mu\nu}))\partial_\nu\phi^i \end{aligned}$$

since $g_{\mu\nu} = g_{\mu\nu} + \partial_\mu\phi^i\partial_\nu\phi^i$

1. "Einstein branch"

$$\Lambda^4 g^{\mu\nu} + \Lambda_4^2 \mathcal{G}^{\mu\nu} = 8\pi T^{\mu\nu}$$

2. "harmonic branch"

$$\Lambda^4 \Box_g \phi = (8\pi T^{\mu\nu} - \Lambda_4^2 \mathcal{G}^{\mu\nu})\nabla_\mu\partial_\nu\phi$$

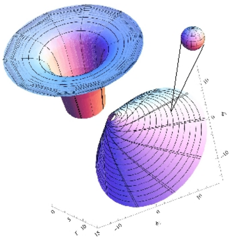
prototype: flat space $\mathbb{R}^4_\theta \subset \mathbb{R}^{10}$, even for $\Lambda \gg 0$!

illustration of Einstein branch

example: Schwarzschild geometry (Blaschke, H.S. arXiv:1005:0499)

embedding $\mathcal{M} \subset \mathbb{R}^7$, asymptotically flat (harmonic), $e^{\sigma} \to const$

$$x^a = \begin{pmatrix} t \\ r\cos\varphi\sin\vartheta \\ r\sin\varphi\sin\vartheta \\ r\cos\vartheta \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\cos(\omega(t+r)) \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\sin(\omega(t+r)) \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}} \end{pmatrix},$$



with $g_{ab} = \mathrm{diag}(-,+,+,+,+,+,-)$.

central singularity: embedding $\hookrightarrow \infty$
with complexified SD symplectic form

$$\star\theta^{-1} = i\theta^{-1}, \qquad \theta^{-1} \to const \quad \text{for} \quad r \to \infty$$

issues remain:

- $\theta^{\mu\nu}$ degenerate on a circle on horizon
  $\rightarrow$ gauge coupling depends on $e^\sigma \sim |\theta^{\mu\nu}|$, not good
- extrinsic term such as $\mathrm{Tr}\Box X^a \Box X^a \sim \int \Delta_G x^a \Delta_G x^a$ may arise

  $\rightarrow$ need to understand (quantum) effective action
  show: predominantly intrinsic geometry $\approx$ GR
- Lorentz violating effects due to $\theta^{\mu\nu}$ must be very small
  (maybe average out $\theta^{\mu\nu}$?)
- probably need something like $\mathcal{M}^4 \times K$, intersecting branes, ...
- singularities ? $\rightarrow$ presumably resolved by fuzzyness
- ...

Fuzzy extra dimensions in field theory

e.g. S^2_N may arise in ordinary 4D gauge theory through Higgs effect:

consider $SU(\mathcal{N})$ Yang-Mills theory on 4-D Minkowski space M^4

$$\mathcal{S}_{YM} = \int d^4y\, Tr\left(\frac{1}{4g^2} F^\dagger_{\mu\nu} F_{\mu\nu} + (D_\mu\phi_a)^\dagger D_\mu\phi_a\right) - V(\phi)$$

A_μ ... $su(\mathcal{N})$ - valued gauge fields, $D_\mu = \partial_\mu + [A_\mu, .]$, and
$\phi_a = \phi^\dagger_a,\ \ a = 1,2,3$ ... scalar fields in adjoint of $SU(\mathcal{N})$

global $SO(3)$ symmetry, gauge symmetry

$$\phi_a \quad \rightarrow \quad U^\dagger \phi_a U, \qquad\qquad U = U(y) \in U(\mathcal{N})$$

$V(\phi)$ ... renormalizable potential respecting the symmetries

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$V(\phi)$ ... renormalizable potential respecting the symmetries (almost) most general potential respecting the symmetries:

$$V(\phi) = Tr\left(a^2(\phi_a\phi_a - b\,\mathbf{1})^2 + c + \frac{1}{\tilde{g}^2}F^\dagger_{ab}F_{ab}\right)$$

for suitable constants $a, b, c, \tilde{g}$, where

$$F_{ab} = [\phi_a, \phi_b] - i\varepsilon_{abc}\phi_c$$

vacuum = minimum of $V(\phi)$, achieved if

$$F_{ab} = [\phi_a, \phi_b] - i\varepsilon_{abc}\phi_c = 0, \qquad a(\phi_a\phi_a - \tilde{b}) = 0$$

$\Rightarrow$ ϕ_a ... representation of $SU(2)$
with Casimir $b = C_2(N)$ for some $N \in \mathbb{N}$

$$\boxed{\phi_a = J_a^{(N)} \otimes \mathbf{1}_n}$$

$J_a^{(N)}$ ... generator of the N-dimensional irrep of $SU(2)$

(assume $\mathcal{N} = Nn$)

note: $\mathrm{Mat}(\mathcal{N},\mathbb{C}) \cong \mathrm{Mat}(\mathrm{N},\mathbb{C}) \otimes \mathrm{Mat}(\mathrm{n},\mathbb{C}) \cong \mathcal{C}(\mathrm{S}^2_\mathrm{N}) \otimes \mathrm{Mat}(\mathrm{n},\mathbb{C})$

interpretation:
$\phi^a(y)$ ... generate $u(n)$-valued functions on $S^2_N \times M^4$

therefore: $\begin{pmatrix} y^\mu \\ \phi^a \end{pmatrix}$ ... functions on $M^4 \times S^2_N \hookrightarrow \mathbb{R}^7$

Higgs effect: $U(\mathcal{N})$ gauge symmetry broken to $U(n)$

(= commutant of $\phi_a = J_a^{(N)}$)

spontaneously generated extra dimensions

model describes 6-dimensional $U(n)$ gauge theory on $M^4 \times S^2_N$
finite tower of massive Kaluza-Klein modes due to Higgs effect

(also true if add fermions to model)
P. Aschieri, T. Grammatikopoulos, H.S., G. Zoupanos 2006; Madore, Manousselis; etc.

... same mechanisms as in string theory, within renormalizable QFT!

full matrix model $\rightarrow$ same applies to space-time itself

Y-M. Matrix Models + fermions:
contain all ingredients for theory of fund. interactions.

a priori only $SU(n)$ gauge groups

symmetry breaking, contact with particle physics:
possible mechanisms:

- extra-dimensional fuzzy spaces $\mathcal{M}^4 \times K \subset \mathbb{R}^{10}$

  add cubic terms to matrix model $\Rightarrow$ extra-dim. fuzzy S^2, interesting low-energy gauge groups, including $SU(3) \times SU(2) \times U(1)\ (\times U(1)_{\text{anomalous}})$

  (P. Aschieri, T. Grammatikopoulos, H.S., G. Zoupanos 2006; Madore, Manousselis; Aoki, Azuma, Iso, ...; H. Grosse, F. Lizzi, H.S. arXiv:1001.2703)

  however non-chiral

- difference to string theory: $\mathbb{R}^D$ "bulk" unphysical, nothing propagates in bulk

  predictive framework

Summary, outlook

- ”Matrix (fuzzy) geometry“: quantized symplectic spaces $\mathcal{M} \subset \mathbb{R}^D$
  generic class, many examples
- matrix-model $Tr[X^a, X^b][X^{a'}, X^{b'}]\, g_{aa'} g_{bb'}$

  dynamical matrix geometries

  $\rightarrow$ emergent gravity & gauge thy
- *not* same as G.R., but might be close enough
  (extrinsic geometry, physics of vacuum energy, ...)
- can address curvature, etc.
- suitable for quantizing gravity !
  (IKKT model $D = 10$, maximal SUSY)
- new powerful techniques: effective generalized matrix models
  ... to be developed
- ... ”new”, more work is needed

Cosmological solution

D. Klammer, H. S., PRL 102 (2009)

<u>assume:</u> vacuum energy $\Lambda^4 \gg$ energy density ρ

$\Rightarrow$ look for harmonic embedding $\Delta x^a = 0$ of FRW metric

$$ds^2 = -dt^2 + a(t)^2(d\chi^2 + \sinh^2(\chi)d\Omega^2),$$

Ansatz

$$x^a(t,\chi,\theta,\varphi) = \begin{pmatrix} a(t)\begin{pmatrix}\cos\psi(t)\\ \sin\psi(t)\end{pmatrix} \otimes \begin{pmatrix}\sinh(\chi)\sin\theta\cos\varphi\\ \sinh(\chi)\sin\theta\sin\varphi\\ \sinh(\chi)\cos\theta\\ \cosh(\chi)\end{pmatrix} \\ 0 \\ x_c(t)\end{pmatrix} \in \mathbb{R}^{10}$$

(cf. B. Nielsen, JGP **4**, (1987))

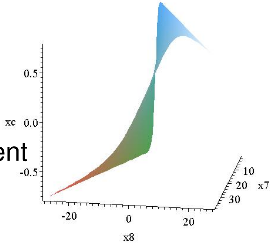
Evolution $a(t), \Psi(t), x_c(t)$ determined by $\Delta x^a = 0$

solution of M.M + leading term $\int d^4x\sqrt{G}\Lambda^4$ in Γ_{1-loop}

harmonic embedding $\Delta_g x^a = 0$ leads to

analog of Friedmann equations

$$\begin{aligned} H^2 = \frac{\dot{a}^2}{a^2} &= -b^2 a^{-10} + d^2 a^{-8} - \frac{k}{a^2}. \\ \frac{\ddot{a}}{a} &= -3d^2 a^{-8} + 4b^2 a^{-10}. \end{aligned}$$



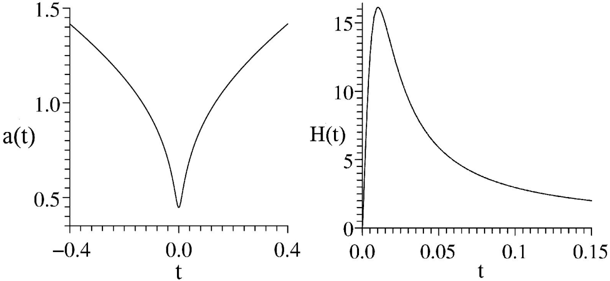
largely independent of detailed matter/energy content
as long as $\Lambda^4 \gg \rho$

$k = -1$ (negative spatial curvature) most interesting

Implications:

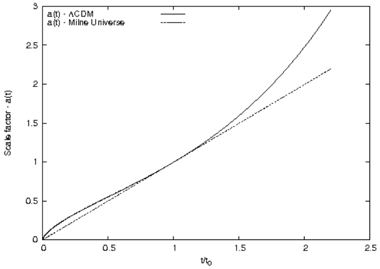
1) early universe:

- big bounce: $\dot{a} = 0$ for $a = a_{min} \sim b^{1/4}$
  ($\exists$ bound for energy density ρ vs. vacuum energy Λ^4)
- inflation-like phase $a(t) \sim t^2$, ends at $a(t_{\rm exit}) = \sqrt{\frac{4}{3}\frac{b}{d}}$
  geometric mechanism (no scalar field required),
  no fine-tuning



2) late evolution (now): $\dot{a} \to 1$

approaches Milne-like universe ($k = -1$, spatial curvature),



in remarkably good agreement with observation
(age $13.8 \cdot 10^9\ yr$, type Ia supernovae)
different physics for early universe (recombination etc.)

A. Benoit-Levy and G. Chardin, [arXiv:0903.2446]

CMB acoustic peak argued to be at correct scale (?)

no fine-tuning of cosm. const., no need for dark energy !