Quantum information geometric foundations: Beyond the spectral paradigm

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Information Theoretic Foundations for Physics

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$$\frac{\textbf{SR}}{\textbf{GR}} = \frac{\textbf{QM}}{\textbf{General quantum theory}} \ ?$$

$$\frac{\text{Ptolemaic epicycles}}{\text{Newtonian theory}} = \frac{\text{Feynman diagrams}}{\text{General quantum theory}} \ ?$$

Plan

1. **Nonlinear generalisation of quantum dynamics**
   - Geometric structures on quantum states: relative entropies & Poisson brackets
   - Lüders' rules $\rightarrow$ constrained relative entropy maximisations
   - Unitary evolution $\rightarrow$ nonlinear hamiltonian flows
2. **Geometric framework for quantum information theories beyond quantum mechanics**
   - Quantum states $=$ integrals on W^*-algebras
   - Quantum theoretic kinematics $=$ a generalisation of probability theory
   - Quantum theoretic dynamics $=$ a generalisation of causal statistical inference
   - Reconstruction of QM and probability theory
   - Quantum theoretic semantics beyond spectral theory, probabilities, and Born rule
   - Intersubjective bayesian coherence
3. **Emergence of space-time theories**
   - Space-time geometry $=$ geometry of local correlations and causality
   - Emergent QFTs?

Quantum information models and quantum information distances

trace class operators: $\mathcal{T}(\mathcal{H}) := \{\rho \in \mathfrak{B}(\mathcal{H}) \mid \rho \geq 0,\ \mathrm{tr}_{\mathcal{H}}|\rho| < \infty\}$
we will consider arbitrary sets of denormalised quantum states: $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$

Quantum information distances $D : \mathcal{M}(\mathcal{H}) \times \mathcal{M}(\mathcal{H}) \to [0, \infty]$s.t. $D(\rho, \sigma) = 0 \iff \rho = \sigma$.

- E.g.
  - $D_1(\rho, \sigma) := \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$ [Umegaki'62]
  - $D_{1/2}(\rho, \sigma) := 2\left\|\sqrt{\rho} - \sqrt{\sigma}\right\|^2_{\mathfrak{G}_2(\mathcal{H})} = 4\mathrm{tr}_{\mathcal{H}}(\frac{1}{2}\rho + \frac{1}{2}\sigma - \sqrt{\rho}\sqrt{\sigma})$ (Hilbert–Schmidt norm^2)
  - $D_{L_1(\mathcal{N})}(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_{\mathcal{T}(\mathcal{H})} = \frac{1}{2}\mathrm{tr}_{\mathcal{H}}|\rho - \sigma|$ (L_1/predual norm)
  - $D_\gamma(\rho, \sigma) := \frac{1}{\gamma(1-\gamma)}\mathrm{tr}_{\mathcal{H}}(\gamma\rho + (1-\gamma)\sigma - \rho^\gamma\sigma^{1-\gamma})$; $\gamma \in \mathbb{R} \setminus \{0, 1\}$ [Hasegawa'93]
  - $D_{\alpha,z}(\rho, \sigma) := \frac{1}{1-\alpha}\log \mathrm{tr}_{\mathcal{H}}(\rho^{\alpha/z}\sigma^{(1-\alpha)/z})^z$; $\alpha, z \in \mathbb{R}$ [Audenauert–Datta'14]

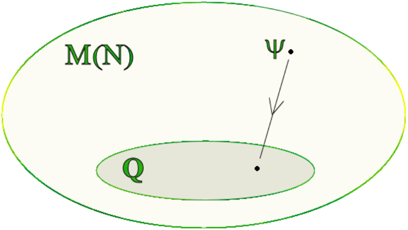
  for $\mathrm{ran}(\rho) \subseteq \mathrm{ran}(\sigma)$, and with all $D(\rho, \sigma) := +\infty$ otherwise.
- Various "quantum geometries" will arise from different additional conditions imposed on pairs $(\mathcal{M}(\mathcal{H}), D)$:
  - Different choices of $\mathcal{M}(\mathcal{H})$ reflect different assumptions on the available possible knowledge (description of experimental situation).
  - Different choices of D reflect different assumptions regarding the convention of "best/optimal" estimation/inference.
  - Both choices are case-to-case-dependent and should be operationally justified.

Quantum entropic projections

Let $\mathcal{Q} \subseteq \mathcal{T}(\mathcal{H})^+$ be such that
for each $\psi \in \mathcal{M}(\mathcal{H})$
there exists a unique solution

$$\boxed{\mathfrak{P}^D_{\mathcal{Q}}(\psi) := \arg\inf_{\rho\in\mathcal{Q}} \{D(\rho,\psi)\}\,.}$$

It will be called an entropic projection.



E.g.

- for $D_{1/2}(\rho,\sigma) = 4\mathrm{tr}_{\mathcal{H}}(\frac{1}{2}\rho + \frac{1}{2}\sigma - \sqrt{\rho}\sqrt{\sigma})$,
  consider the entropic projections $\mathfrak{P}^{D_{1/2}}_{\mathcal{Q}}$
  where $\mathcal{Q}$ are images of closed convex subspaces $\tilde{\mathcal{Q}} \subseteq \mathcal{K}^+ := \mathfrak{G}_2(\mathcal{H})^+$
  under the mapping $\tilde{\mathcal{Q}} \ni \sqrt{\rho} \mapsto \rho \in \mathcal{Q}$.
  They coincide with the ordinary projection operators in $\mathfrak{B}(\mathcal{K}) \cong \mathfrak{B}(\mathcal{H}\otimes\mathcal{H}^\star)$.
- for $D_1(\rho,\sigma) = \mathrm{tr}_{\mathcal{H}}(\rho\log\rho - \rho\log\sigma)$
  and $\mathcal{M}(\mathcal{H}) = \mathcal{T}(\mathcal{H})^+_1$, $\psi \in \mathcal{T}(\mathcal{H})^+_1$, $h \in \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$, then [Araki'77, Donald'90]

$$\exists!\ \psi^h := \arg\inf_{\rho\in\mathcal{T}(\mathcal{H})^+_1} \{D_1(\rho,\psi) + \mathrm{tr}_{\mathcal{H}}(\rho h)\}\,.$$

Quantum measurement, bayesianity, and maximum relative entropy

- Lüders' rules:

$$\rho \mapsto \rho_{\mathrm{new}} := \sum_i P_i \rho P_i \quad \text{('weak')}$$

$$\rho \mapsto \rho_{\mathrm{new}} := \frac{P\rho P}{\mathrm{tr}_{\mathcal{H}}(P\rho)} \quad \text{('strong')}$$

- Bub'77'79, Caves–Fuchs–Schack'01, Fuchs'02, Jacobs'02: Lüders' rules should be considered as rules of inference (conditioning) that are quantum analogues of

the Bayes–Laplace rule: $\quad p(x) \mapsto p_{\mathrm{new}}(x) := \dfrac{p(x)p(b|x)}{p(b)}.$

- Williams'80, Warmuth'05, Caticha&Giffin'06: the Bayes–Laplace rule is a special case of

$$p(\chi) \mapsto p_{\mathrm{new}}(\chi) := \arg\inf_{q\in\mathcal{Q}} \{D_1(q,p)\}\,; \quad D_1(q,p) := \int_{\mathcal{X}} \mu(\chi)q(\chi)\log\left(\frac{q(\chi)}{p(\chi)}\right).$$

- Douven&Romeijn'12: the Bayes–Laplace rule is also a special case of

$$p \mapsto \arg\inf_{q\in\mathcal{Q}} \{D_1(p,q)\} = \mathfrak{P}^{D_0}_{\mathcal{Q}}(p),$$

where $D_0(p,q) = D_0(q,p)$.

Quantum bayesian inference from quantum entropic projections

- RPK'13'14, F.Hellmann–W.Kamiński–RPK'14:
  1. weak Lüders' rule is a special case of
  $$\rho \mapsto \arg\inf_{\sigma\in\mathcal{Q}} \{D_1(\rho,\sigma)\}$$
  with
  $$\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i,\sigma] = 0 \ \forall i\}$$
  2. strong Lüders' rule derived from
  $$\rho \mapsto \arg\inf_{\sigma\in\mathcal{Q}} \{D_1(\rho,\sigma)\}$$
  with
  $$\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i,\sigma] = 0,\ \mathrm{tr}_{\mathcal{H}}(\sigma P_i) = p_i \ \forall i\}$$
  under the limit $p_2,\ldots,p_n \to 0$.
  3. hence, weak and strong Lüders' rules are special cases of quantum entropic projection $\mathfrak{P}^{D_0}_{\mathcal{Q}}$ based on relative entropy $D_0(\sigma,\rho) = D_1(\rho,\sigma)$.

Bayes–Laplace and Lüders' conditionings are special cases of entropic projections $\Rightarrow$ "quantum bayesianism $\subseteq$ quantum relative entropism".

Quantum Jeffrey's rule

- Caticha&Giffin'06: under more general constraints, one can derive also Jeffrey's rule (generalising the Bayes–Laplace rule):

$$p(x|\eta) \mapsto p_{\mathrm{new}}(x|\eta) := \sum_{i=1}^{n} p(x|b_i)\lambda_i = \sum_{i=1}^{n} \frac{p(x \wedge b_i|\eta)}{p(b_i|\eta)}\lambda_i,$$

  where $n \in \mathbb{N}$,
  - $\{b_1, \ldots, b_n\}$ is a set of exhaustive and mutually exclusive elements of boolean algebra,
  - $\lambda_i = p_{\mathrm{new}}(b_i|\eta)\ \forall i \in \{1, \ldots, n\}$,
  - $p(b_i|\eta) \neq 0$.
- RPK'14: derivation of a quantum analogue of Jeffrey's rule:

$$\mathcal{T}(\mathcal{H})_1^+ \ni \rho \mapsto \rho_{\mathrm{new}} := \underset{\sigma \in \mathcal{Q}}{\arg\inf} \{D_1(\rho, \sigma)\} = \sum_{i=1}^{n} \frac{P_i \rho P_i}{\mathrm{tr}_{\mathcal{H}}(\rho P_i)}\lambda_i \in \mathcal{T}(\mathcal{H})_1^+,$$

  where $n \in \mathbb{N}$,
  - $\{P_1, \ldots, P_n\} \subseteq \mathrm{Proj}(\mathfrak{B}(\mathcal{H}))$, $\sum_{i=1}^{n} P_i = \mathbb{I}$, $P_i P_j = \boldsymbol{\delta}_{ij} P_i$,
  - $\lambda_i = \mathrm{tr}_{\mathcal{H}}(\rho_{\mathrm{new}} P_i)\ \forall i \in \{1, \ldots, n\}$,
  - $\mathrm{tr}_{\mathcal{H}}(\rho P_i) \neq 0$.

  It generalises Lüders' rule.

Quantum Poisson structure

- Consider the space of self-adjoint trace-class operators: $\mathcal{T}(\mathcal{H})^{\mathrm{sa}} := \mathcal{T}(\mathcal{H}) \cap \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$.
- It can be equipped with a following real Banach smooth manifold structure:
  - tangent spaces: $\mathbf{T}_\phi(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}) \cong \mathcal{T}(\mathcal{H})^{\mathrm{sa}}$
  - cotangent spaces: $\mathbf{T}^{\circledast}_\phi(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}) \cong (\mathcal{T}(\mathcal{H})^{\mathrm{sa}})^\star \cong \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$
- Bóna'91,'00: a Poisson manifold structure on $\mathcal{T}(\mathcal{H})^{\mathrm{sa}}$ is defined by a commutator of an algebra:

$$\{h, f\}(\rho) := \mathrm{tr}_{\mathcal{H}}\left(\rho\, \mathrm{i}[\mathbf{d}h(\rho), \mathbf{d}f(\rho)]\right) \; \forall f, h \in \mathrm{C}^\infty(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}; \mathbb{R}) \; \forall \rho \in \mathcal{T}(\mathcal{H})^{\mathrm{sa}}.$$

- So, if $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$ is a smooth submanifold of $\mathcal{T}(\mathcal{H})^{\mathrm{sa}}$, then every $f \in \mathrm{C}^\infty(\mathcal{M}(\mathcal{H}); \mathbb{R})$ determines a hamiltonian vector field:

$$\mathfrak{X}_f(\rho) = -\{\cdot, f\}(\rho) = \mathrm{tr}_{\mathcal{H}}(\rho\, \mathrm{i}[\mathbf{d}(\cdot), \mathbf{d}f(\rho)]).$$

- More generally, we can choose arbitrary real Banach Lie subalgebra $\mathcal{A}$ of $\mathfrak{B}(\mathcal{H})$ such that: (i) it has a unique Banach predual $\mathcal{A}_\star$ in $\mathcal{T}(\mathcal{H})$; (ii) there exists at least one $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$ which is a smooth submanifold of $\mathcal{A}_\star$.

Nonlinear quantum hamiltonian dynamics

For each hamiltonian vector field, the corresponding Hamilton equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\rho(t)) = \{h, f\}(\rho(t)) = \mathrm{i}\,\mathrm{tr}_{\mathcal{H}}\left([\rho(t), \mathbf{d}h(\rho(t))]\mathbf{d}f(\rho(t))\right).$$

The above equation is equivalent to the Bóna equation ['91'00]

$$\boxed{\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [\mathbf{d}h(\rho(t)), \rho(t)].}$$

Hence,

The Poisson structure $\{\cdot,\cdot\}$ induced by a commutator of $\mathfrak{B}(\mathcal{H})$ allows to introduce various nonlinear hamiltonian evolutions on spaces $\mathcal{M}(\mathcal{H})$ of quantum states, generated by arbitrary real-valued smooth functions on $\mathcal{M}(\mathcal{H})$.

The solutions of Bóna equation are state-dependent unitary operators $U(\rho, t)$.
They do not form a group, but satisfy a cocycle relationship:

$$U(\rho, t+s) = U((\mathrm{Ad}(U(\rho, t)))(\rho), s)U(\rho, t) \ \ \forall t, s \in \mathbb{R}.$$

In a special case, when $h(\rho) = \mathrm{tr}_{\mathcal{H}}(\rho H)$ for $H \in \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$,
the Bóna equation turns to the von Neumann equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [H, \rho(t)].$$

Quantum causal inferences by entropic-hamiltonian dynamics

- **Two elementary geometric structures:**
  - $D(\cdot,\cdot)$ represents the convention of "best estimation/inference"
  - $\{h,\cdot\}$ represents a convention of causality ("internal dynamics")
- **Two elementary forms of quantum dynamics:**
  - entropic projections $\mathfrak{P}^D_Q$ generated by quantum distances $D(\cdot,\cdot)$
  - hamiltonian flows w^h_t generated by nonlinear hamiltonian vector fields $\{h,\cdot\}$

A general form of quantum dynamics is defined as a causal inference $\mathfrak{P}^D_{\mathcal{Q}} \circ w^h_t$.

- It generalises unitary evolution followed by a "projective measurement".
- Postulate: consider the setting of causal inferences $\mathfrak{P}^D_{\mathcal{Q}} \circ w^h_t$ as an alternative to the paradigm of semigroups of CPTP maps.
- Basic idea: every CPTP map can be decomposed into:
  1) tensor product of initial state with uncorrelated environment,
  2) unitary evolution,
  3) projective measurement,
  4) partial trace.

  It remains to prove that 4 and 3+4 are entropic projections (ongoing work with M.Munk-Nielsen).

Towards new foundations

Idea:

- consider spaces $\mathcal{M}(\mathcal{H})$ as fundamental
- allow any nonlinear functions $\mathcal{M}(\mathcal{H}) \to \mathbb{R}$ as observables
- define geometry of $\mathcal{M}(\mathcal{H})$ by means of $D(\cdot,\cdot)$ and $\{\cdot,\cdot\}$
- define dynamics of $\mathcal{M}(\mathcal{H})$ by means of $\mathfrak{P}^D_{\mathcal{Q}}(\cdot,\cdot)$ and $w_t^{\{h,\cdot\}}$

Questions:

- what's up with Hilbert spaces? (are they necessary? if not, then what?)
- what's up with spectral theory, probability, Born rule, etc?

Answers:

- replace Hilbert spaces by W^*-algebras
- replace density matrices by positive integrals on W^*-algebras
- this setting is an exact generalisation of Kolmogorov's measure theoretic setting for probability theory
- build up all remaining semantics for quantum theory
  in the analogy to semantics of probability theory and statistical inference
  (hence: no Born rule, no probabilities, no spectral theory)

Probability theory:

- Underlying structure: measure space $(\mathcal{X}, \mu)$
- Main spaces: **Probabilistic models:**

$$\mathcal{M}(\mathcal{X}, \mu) \subseteq L_1(\mathcal{X}, \mu)^+ := \{p : \mathcal{X} \to \mathbb{R} \mid \int_{\mathcal{X}} \mu |p| < \infty, p \geq 0\}$$

- e.g. Gaussian models: $\{p(\chi, (m, s)) = \frac{1}{\sqrt{2\pi}s} \mathrm{e}^{-\frac{(\chi - m)^2}{2s^2}} \mid (m, s) \in \Theta \subseteq \mathbb{R} \times \mathbb{R}^+\}$.
- Observables (estimators): functions $f : \mathcal{X} \to \mathbb{R}$
- The mapping $L_1(\mathcal{X}, \mu) \times L_\infty(\mathcal{X}, \mu) \ni (p, f) \mapsto \int_{\mathcal{X}} \mu p f \in \mathbb{R}$ determines Banach space duality $L_1(\mathcal{X}, \mu)^\star \cong L_\infty(\mathcal{X}, \mu)$.

Quantum mechanics:

- Underlying structure: Hilbert space $\mathcal{H}$
- Main spaces: **Spaces of density matrices:**

$$\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+ := \{\rho \in \mathfrak{B}(\mathcal{H}) \mid \mathrm{tr}_{\mathcal{H}}(|\rho|) < \infty, \rho \geq 0\}$$

- e.g. Gibbs states: $\{\mathrm{e}^{-\beta H} \mid \beta \in]0, \infty[\}$, for a fixed self-adjoint H.
- Observables: self-adjoint operators $x : \mathcal{H} \to \mathcal{H}$
- The mapping $\mathcal{T}(\mathcal{H}) \times \mathfrak{B}(\mathcal{H}) \ni (\rho, x) \mapsto \mathrm{tr}_{\mathcal{H}}(\rho x) \in \mathbb{C}$ determines Banach space duality $\mathcal{T}(\mathcal{H})^\star \cong \mathfrak{B}(\mathcal{H})$.

W^*-algebras and integration

- A W^*-algebra $\mathcal{N}$:
  - an algebra over $\mathbb{R}$ or $\mathbb{C}$ with unit $\mathbb{I}$,
  - with * operation s.t. $(xy)^* = y^*x^*$, $(x+y)^* = x^* + y^*$, $(x^*)^* = x$, $(\lambda x)^* = \lambda^* x^*$,
  - that is also a Banach space,
  - with $\cdot$, $+$, * continuous in the norm topology (implied by the condition $\|x^*x\| = \|x\|^2$),
  - such that there exists a Banach space $\mathcal{N}_\star$ satisfying the Banach space duality: $(\mathcal{N}_\star)^\star \cong \mathcal{N}$,
- Special cases:
  - if $\mathcal{N}$ is commutative
    then $\exists$ a measure space $(\mathcal{X}, \mu)$ s.t. $\mathcal{N} \cong L_\infty(\mathcal{X}, \mu)$ and $\mathcal{N}_\star \cong L_1(\mathcal{X}, \mu)$
  - if $\mathcal{N}$ is "type I factor"
    then $\exists$ a Hilbert space $\mathcal{H}$ s.t. $\mathcal{N} \cong \mathfrak{B}(\mathcal{H})$ and $\mathcal{N}_\star \cong \mathcal{T}(\mathcal{H})$.
- Hence, the element $\phi \in (\mathcal{N}_\star)^+$ provides a joint generalisation of probability density and of density operator. By means of embedding of $\mathcal{N}_\star$ into $\mathcal{N}^\star$, it is also an integral on $\mathcal{N}$.
- Key fact: The above setting allows to develop full-fledged integration theory on noncommutative W^*-algebras, which generalises integration theory on measure spaces (with partial integration, conditional expectations, $L_p(\mathcal{N})$ spaces,...).

New kinematics: quantum models and observables

General quantum information models:

For any W^*-algebra $\mathcal{N}$, $\mathcal{M}(\mathcal{N})$ will be defined as an arbitrary subset of a positive part of a Banach predual space of $\mathcal{N}$, $\mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_\star^+$.

Special cases:

- $\mathcal{N}$ is commutative $\Rightarrow \mathcal{M}(\mathcal{N}) = \mathcal{M}(\mathcal{X}, \mu)$
- $\mathcal{N}$ is type I factor $\Rightarrow \mathcal{M}(\mathcal{N}) = \mathcal{M}(\mathcal{H})$.

We do not assume that:

- $\mathcal{M}(\mathcal{N})$ is convex ($\iff$ probabilistic mixing)
- $\mathcal{M}(\mathcal{N})$ is smooth ($\iff$ asymptotic estimation)
- $\mathcal{M}(\mathcal{N})$ is normalised ($\iff$ frequentist interpretation)

Observables:

Observables are defined as arbitrary functions $f : \mathcal{M}(\mathcal{N}) \to \mathbb{R}$.

Hence: smooth observables define hamiltonian vector fields.

Each "observable in the old sense" $x \in \mathcal{N}^{\mathrm{sa}}$ determines a corresponding "observable in the new sense" by $f_x(\phi) := \phi(x)$.

New kinematics: quantum information geometry

- Main change: Consider expectation values as more fundamental than eigenvalues $\Rightarrow$ foundational role of spectral theory replaced by quantum information geometry
- Kinematic setting:
  (1) **spaces**: Hilbert spaces $\mathcal{H}$ of eigenvectors
  $\rightarrow$ spaces $\mathcal{M}(\mathcal{N})$ of denormalised expectation functionals on W^*-algebras $\mathcal{N}$.
  (2) **observables**: linear functions $\mathcal{H} \rightarrow \mathcal{H}$ that have real eigenvalues
  $\rightarrow$ nonlinear real valued functions $\mathcal{M}(\mathcal{N}) \rightarrow \mathbb{R}$.
  (3) **geometry**: geometry of Hilbert spaces $\mathcal{H}$ defined by scalar product $\langle\cdot,\cdot\rangle$
  $\rightarrow$ geometry of spaces $\mathcal{M}(\mathcal{N})$ defined by quantum relative entropies $D(\cdot,\cdot)$ and quantum Poisson structures $\{\cdot,\cdot\}$.
- **Two fundamental geometric structures on $\mathcal{M}(\mathcal{N})$:**
  a) Quantum distances $D(\cdot,\cdot)$
     - ⋆ large variety of choices
     - ⋆ allows to derive riemannian geometry (via $\partial_i\partial_j D$) and Hilbert space projective geometry (via $\mathfrak{P}^D_{\mathcal{Q}}$ for $D = D_{1/2}$) as special cases
  b) Quantum Poisson structures $\{\cdot,\cdot\}$
     - ⋆ depend on the choice of a real Banach Lie subalgebra of $\mathcal{N}$
     - ⋆ generalises symplectic geometry
- No Hilbert spaces, no probability theory in foundations (derived as special cases)

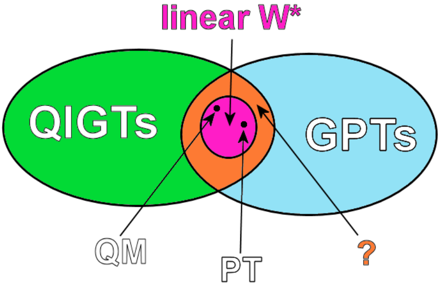
New dynamics: information geometric causal inference

- Main change: Consider nonlinear state changes defined by geometric structures on quantum models as more important than unitary evolution with no-initial-correlations assumption
  $\Rightarrow$ linear CP maps replaced by nonlinear entropic-hamiltonian maps
- **Two fundamental dynamic structures on $\mathcal{M}(\mathcal{N})$:**
  a) Inference: Entropic projections $\phi \mapsto \arg\inf_{\omega\in\mathcal{Q}(\eta)}\{D(\omega,\phi)\}$
     - ★ nonlinear and nonlocal
     - ★ requires convexity
     - ★ allows to encode experimental constraints
     - ★ reduces in special cases to Lüders' rules

  b) Causality: Hamiltonian flows $\phi \mapsto w^h_t(\phi)$, where $\frac{\mathrm{d}}{\mathrm{d}t}f(w^h_t(\phi)) = \{h, f(w^h_t)\}(\phi)\ \forall f$
     - ★ nonlinear and local
     - ★ requires smoothness
     - ★ allows to encode theoretical symmetries
     - ★ reduces in a special case to the von Neumann equation.
- Dynamic setting:
  - ▶ entropic projections composed with hamiltonian flows: $\phi \mapsto \mathfrak{P}^D_{\mathcal{Q}}(\eta)\circ w^h_t(\phi)$

Backwards compatibility

1. **Reconstruction of quantum mechanics:**
   - $\mathcal{N}$: type I W^*-algebras
   - $\mathcal{M}(\mathcal{N})$: normalised states
   - D: $D_{1/2}$ or D_0
   - $\{\cdot,\cdot\}$: generated by Banach Lie algebra $\mathcal{N}^{\mathrm{sa}}$
   - observables: affine functions on $\mathcal{M}(\mathcal{N})$
2. **Reconstruction of probability theory:**
   - $\mathcal{N}$: commutative algebras
   - $\mathcal{M}(\mathcal{N})$: normalised states
   - D: arbitrary
   - $\{\cdot,\cdot\}$: trivialises for commutative algebras
   - observables: arbitrary or affine functions on $\mathcal{M}(\mathcal{N})$



Smooth quantum information geometries

Under some conditions, D induces a generalisation of smooth riemannian geometry on $\mathcal{M}(\mathcal{N})$.

- Jenčová'05: a general construction of smooth manifold structure on the space of all strictly positive states over arbitrary W^*-algebra.
- E.g. $\mathcal{M}(\mathcal{H}) := \{\rho(\theta) \in \mathcal{T}(\mathcal{H}) \mid \rho(\theta) > 0, \theta \in \Theta \subseteq \mathbb{R}^n \ \text{open}, \theta \mapsto \rho(\theta) \ \text{smooth}\}$
- Eguchi'83/Ingarden et al'82/Lesniewski–Ruskai'99/Jenčová'04:
  Every smooth distance D with positive definite hessian determines
  a riemannian metric $\mathbf{g}^D$ and a pair $(\nabla^D, \nabla^{D^\dagger})$ of torsion-free affine connections:

$$\mathbf{g}_\phi(u,v) := -\partial_{u|\phi}\partial_{v|\omega} D(\phi,\omega)|_{\omega=\phi},$$
$$\mathbf{g}_\phi((\nabla_u)_\phi v, w) := -\partial_{u|\phi}\partial_{v|\phi}\partial_{w|\omega} D(\phi,\omega)|_{\omega=\phi},$$
$$\mathbf{g}_\phi(v, (\nabla^\dagger_u)_\phi w) := -\partial_{u|\omega}\partial_{w|\omega}\partial_{v|\phi} D(\phi,\omega)|_{\omega=\phi},$$

  which satisfy the characteristic equation of the Norden['37]–Sen['44] geometry,

$$\mathbf{g}^D(u,v) = \mathbf{g}^D(\mathbf{t}_c^{\nabla^D}(u), \mathbf{t}_c^{\nabla^{D^\dagger}}(v)) \ \ \forall u,v \in \mathbf{T}\mathcal{M}(\mathcal{N}).$$

- A riemannian geometry $(\mathcal{M}(\mathcal{N}), \mathbf{g}^D)$ has Levi-Civita connection $\bar{\nabla} = (\nabla^D + \nabla^{D^\dagger})/2$.
- E.g., $\mathcal{M}(\mathcal{N}) = \mathcal{T}(\mathcal{H}) \cap \{\rho > 0, \mathrm{tr}_{\mathcal{H}}(\rho) = 1\}$ and $D_1(\rho,\sigma) = \mathrm{tr}(\rho\log\rho - \rho\log\sigma)$ give Mori['55]–Kubo['56]–Bogolyubov['62] $\mathbf{g}^{D_1}$ and Nagaoka['94]–Hasegawa['95] $(\nabla^{D_1}, \nabla^{D_1^\dagger})$:

$$\mathbf{g}^{D_1}_\rho(x,y) = \mathrm{tr}\left(\int_0^\infty \mathrm{d}\lambda\, x \frac{1}{\lambda\mathbb{I}+\rho} y \frac{1}{\lambda\mathbb{I}+\rho}\right), \quad \mathbf{t}^{\nabla^{D_1}}_{\rho,\omega}(x) = x - \mathrm{tr}(\omega x), \quad \mathbf{t}^{\nabla^{D_1^\dagger}}_{\rho,\omega}(x) = x.$$

Quantum mechanics as a local theory

- Apart from tangent bundle $\bigcup_\phi \mathbf{T}_\phi\mathcal{M}(\mathcal{N})$, there is also a bundle of complex (GNS) Hilbert spaces $\mathcal{H}\mathcal{M}(\mathcal{N}) \to \mathcal{M}(\mathcal{N})$.
- Vectors in $\mathbf{T}_\phi\mathcal{M}(\mathcal{N})$ are defined by self-adjoint operators, which can be represented uniquely as elements of $(\mathcal{H}_\phi\mathcal{M}(\mathcal{N}))_\mathbb{R}$.
- Under some (mild) conditions: $\mathbf{T}_\phi\mathcal{M}(\mathcal{N}) \sqsubseteq (\mathcal{H}_\phi\mathcal{M}(\mathcal{N}))_\mathbb{R}$.
- Thus, as opposed to C^*-algebraic approach:
  - Spaces of quantum states are equipped with rich geometric structure, allowing for model construction, state estimation, and nonlinear dynamics.
  - Quantum mechanics is reconstructed not only as a global special case of a framework, but also is present locally at each point of a manifold, as an extension of a tangent space.
  - Our framework allows also for a geometric description of renormalisation procedures (see Cedric Bény's talk).

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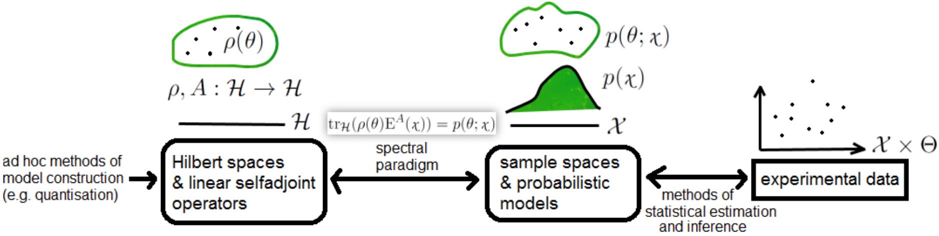
$$\frac{\textbf{SR}}{\textbf{GR}} = \frac{\textbf{QM}}{\textbf{Quantum information geometric QT}} \; ?$$

Semantics I

Orthodox quantum mechanical paradigm:

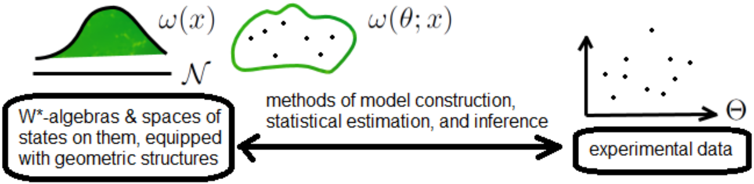


Some key observations:

- Probability theory is just a special case of integration theory on W^*-algebras, and quantum states are **just** integrals, so there is no a priori reason why "general" quantum theory (beyond QM) should depend on probabilities.
- Quantum states (and structures over them) can be associated directly with the epistemic data by generalising the methods of associating epistemic data with probabilities (and with structures over them).

Semantics II

New paradigm:



Basic object of interest: spaces $\mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_\star^+$ of states over W^*-algebras $\mathcal{N}$.

1) Quantum theoretic kinematics generalises and replaces probability theory.
2) Quantum theoretic dynamics generalises and replaces causal statistical inference.
3) Nonlinear information geometry of spaces of quantum states replaces (linear) spectral theory of quantum mechanics.
4) But if probability theory disappears from the framework, then...

Main question:

- What is the measurement theory of nonlinear observables and nonnormalised states?

What is the predictive content of semantics of probability theory?

Consider:

- Θ: a space of possible configurations
- $\Theta \ni \theta = (\theta_1, \ldots, \theta_n)$: average values of n types of experimental configuration variables
- Ξ: a space of registrations

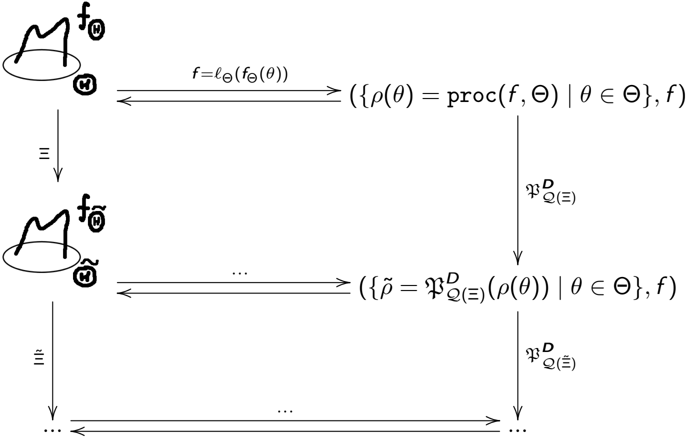
MaxEnt + entropic projections (or Bayes' rule) + prediction:

$$\textit{configurations \& registrations} \underset{\text{predictive verification}}{\overset{\text{model construction}}{\rightleftarrows}} \textit{beliefs \& updatings}$$

$$\begin{tikzcd}
\Theta \arrow[r, shift left, "\theta_i = \mathrm{tr}_{\mathcal{H}}(\rho h_i)"] \arrow[d, "\Xi"'] & \{\rho(\theta) = \mathrm{e}^{-\sum_i \beta(\theta_i) h_i} \mid \theta \in \Theta\} \arrow[l, shift left] \arrow[d, "\mathfrak{P}^{D_1}_{\mathcal{Q}(\Xi)}"] \\
\tilde{\Theta} \arrow[r, shift left, "\tilde{\theta}_i = \mathrm{tr}_{\mathcal{H}}(\tilde{\rho} h_i)"] \arrow[d, "\tilde{\Xi}"'] & \{\tilde{\rho} = \mathfrak{P}^{D_1}_{\mathcal{Q}(\Xi)}(\rho(\theta)) \mid \theta \in \Theta\} \arrow[l, shift left] \arrow[d, "\mathfrak{P}^{D_1}_{\mathcal{Q}(\tilde{\Xi})}"] \\
\hat{\Theta} \arrow[r, shift left, "\hat{\theta}_i = \mathrm{tr}_{\mathcal{H}}(\hat{\rho} h_i)"] & \{\hat{\rho} \mid \ldots\} \arrow[l, shift left]
\end{tikzcd}$$

Towards new semantics

- One can use other model construction principles
- There is no need to use linear expectation type constraints
- This what we should care about is the relationship between model construction (information encoding), inference (information processing), and predictive verifiability (information decoding).



Adjointness in the foundations (of inductive inference)

- We can relax the condition of bijectivity of arrows between models to one that makes the relationship between encoding and decoding to be optimal in the following sense:

> The method of encoding (model construction) should be the most effective solution of the problem provided by the given decoding (prediction).

- Let **TheorMod** be a category of theoretical models as objects and inferences as arrows.
- Let **ExpDes** be a category of experimental designs as objects and registrations as arrows.
- Model construction is defined as a functor $\mathrm{ModConstr} : \textbf{ExpDes} \to \textbf{TheorMod}$.
- Predictive verification is defined as a functor $\mathrm{PredVer} : \textbf{TheorMod} \to \textbf{ExpDes}$.
- $\boxed{\text{Mutual consistency condition: } \mathrm{ModConstr} \dashv \mathrm{PredVer}}$
- This means: there is a natural bijection

$$\hom_{\textbf{ExpDes}}(X, \mathrm{PredVer}(Y)) \cong \hom_{\textbf{TheorMod}}(\mathrm{ModConstr}(X), Y)$$

$$\begin{tikzcd}
 & X \arrow[dd] \arrow[dd, shift left=2] \arrow[dd, shift right=2] & & \mathrm{ModConstr}(X) \arrow[dd] \arrow[dd, shift left=2] \arrow[dd, shift right=2] & \\
\textbf{ExpDes} & & {} \arrow[r, Rightarrow, shift left, "\mathrm{ModConstr}"] & {} \arrow[l, Rightarrow, shift left, "\mathrm{PredVer}"] & \textbf{TheorMod} \\
 & \mathrm{PredVer}(Y) & & Y &
\end{tikzcd}$$

Intersubjective bayesian coherence

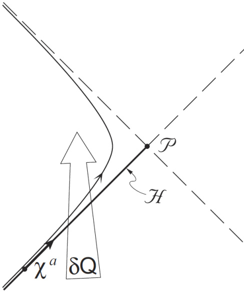
- Personalistic bayesianism does not address intersubjective scientific practice of construction and verification of experiments and theories:
  1. they depend on each other,
  2. they are based not on personal betting preferences, but on the socially shared beliefs and thought styles
     (see Ludwik Fleck, 1935, *Genesis and development of a scientific fact* for more).
- The above semantics provides an account for this, while remaining ontically noncommital, agent based, and compatible with the idea that the values taken by variables in individual registrations are just observed (not determined by a theory).

What if space-time and its dynamics are emergent from quantum theory?

Jacobson'95: Einstein equations *from* space-time thermodynamics

Consider:

- a space-time $(\mathcal{M}, g_{ab})$
- a point $p \in \mathcal{M}$
- a small 2-dimensional surface element $\mathcal{P}$
- a Killing vector field χ^a generating local boost orthogonal to $\mathcal{P}$



Define:

- a local causal horizon $\mathcal{H}$ as a boundary of the past of $\mathcal{P}$, generated by χ^a
- a heat flow δQ as an energy flux across a local causal horizon: $\delta Q := \int_{\mathcal{H}} \mathrm{d}\Sigma^a T_{ab}\chi^b$
- a temperature T as an Unruh temperature associated with a uniformly accelerated observer.

Assume:

- that entropy S is proportional to the area of $\mathcal{H}$: $S = \lambda A$
- that Clausius' law holds: $\delta Q = T\mathrm{d}S$.

Then:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{2\pi}{\lambda}T_{ab}.$$

Emergent space-times

- Basic idea: Consider a principle of equivalence of euclidean QSM with lorentzian QFT via Wick rotation as a fundamental principle, analogous to $m_{\mathrm{grav}} = m_{\mathrm{inert}}$.
- Basic mathematical data:
  - $\mathbf{g}^D_\rho(\cdot,\cdot)$ is a correlation functional, representing a convention of a local (asymptotic) estimation/inference at ρ.
  - $\{h(\rho),\cdot\}$ is a dynamical evolution, representing a convention of a local temporal causality at ρ.
- Required assumptions:
  - choice of a manifold Σ that is determined by operational parameters of measurement of "space" and "time"
  - split $\mathcal{M}(\mathcal{N}) \cong \Sigma \times \widetilde{\mathcal{M}}(\mathcal{N})$
  - $\{h(\rho),\cdot\}$ is well defined on Σ
- Implementation:
  - consider a riemannian metric $\mathbf{g}^D_\Sigma$ induced by $\mathbf{g}^D$ on Σ
  - "Poincaré–Wick rotation" of $\mathbf{g}^D_\Sigma$ to a lorentzian $\hat{\mathbf{g}}^{D,h}_\Sigma$ along a vector field $\{h,\cdot\}$:

$$\mathbf{g}^D_\Sigma = \mathbf{g}^D_\perp + \mathbf{e}_h \otimes \mathbf{e}_h \mapsto \mathbf{g}^D_\perp - \mathbf{e}_h \otimes \mathbf{e}_h =: \hat{\mathbf{g}}^{D,h}_\Sigma,$$

where $\mathbf{g}^D_\perp$ is a riemannian metric induced by $\mathbf{g}^D_\Sigma$ on the submanifolds orthogonal to $\mathbf{e}_h$, while $\mathbf{e}_h := \frac{\mathbf{g}^D_\Sigma(\{h,\cdot\},\cdot)}{\sqrt{\mathbf{g}^D(\{h,\cdot\},\{h,\cdot\})}}$ is a normalised 1-form of $\{h,\cdot\}$.

$$\boxed{\text{An emergent space-time is a triple } (\Sigma, \hat{\mathbf{g}}^{D,h}_\Sigma, \mathbf{e}_h).}$$

Emergent space-times: comments

- Operational assumptions that may lead to derivation of 4-dimensionality of Σ? $\rightarrow$ see the talk of Markus Müller for very interesting ideas.
- Instead of a split $\mathcal{M}(\mathcal{N}) \cong \Sigma \times \widetilde{\mathcal{M}}(\mathcal{N})$, one can consider also a nontrivial fibre bundle with locally (but not globally) defined operational space-times $\pi : \mathcal{M}(\mathcal{N}) \rightarrow \Sigma$.
- Every section of a bundle $\widetilde{\mathcal{M}}(\mathcal{N})$ over Σ defines a global quantum state $\phi(\xi)$ over space-time, and this determines a bundle $\mathcal{H}\Sigma \rightarrow \Sigma$ of GNS Hilbert spaces $\mathcal{H}_{\phi(\xi)}\Sigma$, $\xi \in \Sigma$.
- This allows to use Prugovečki's approach to defining quantum propagators over a curved space-time. $\Rightarrow$ construction of emergent QFTs over curved space-time.

Overview

1. **Nonlinear generalisation of quantum dynamics**
   - Geometric structures on quantum states: relative entropies & Poisson brackets
   - Lüders' rules $\rightarrow$ constrained relative entropy maximisations
   - Unitary evolution $\rightarrow$ nonlinear hamiltonian flows
2. **Geometric framework for quantum information theories beyond quantum mechanics**
   - Quantum states = integrals on W^*-algebras
   - Quantum theoretic kinematics = a generalisation of probability theory
   - Quantum theoretic dynamics = a generalisation of causal statistical inference
   - Reconstruction of QM and probability theory
   - Quantum theoretic semantics beyond spectral theory, probabilities, and Born rule
   - Intersubjective bayesian coherence
3. **Emergence of space-time theories**
   - Space-time geometry = geometry of local correlations and causality
   - Emergent QFTs?

References

Main paper:

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