Towards (post)quantum information relativity

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Plan

1. **Nonlinear generalisations of quantum dynamics:**
    - Geometric structures on quantum state spaces $\rightarrow$ relative entropies & Poisson brackets
    - Lüders' rules $\rightarrow$ constrained relative entropy maximisations
    - Unitary evolution $\rightarrow$ nonlinear hamiltonian flows
2. **Geometric (post)quantum information foundations:**
    - Mathematical and physical principles
    - Global and local dynamics
    - Global and local reconstruction of QM
3. **Category-theoretic operational semantics:**
    - Adjointness in foundations, functorial localisation
    - Resource theories a la LdR–LK–RR
    - Beyond adjointness: local monad–comonad systems
4. **Towards [(post)quantum] local information relativity:**
    - From equlibrium to nonequilibrium space-time thermodynamics
    - Two-dimensional surfaces (θ, σ) and geometry in Klauder–Daubechies quantisation
    - Quantum dynamics of (θ, σ)-spaces

1. Nonlinear generalisation of quantum dynamics

- Geometric structures on spaces of quantum states: relative entropies & Poisson brackets
- Lüders' rules $\rightarrow$ constrained relative entropy maximisations
- Unitary evolution $\rightarrow$ nonlinear hamiltonian flows

trace class operators: $\mathcal{T}(\mathcal{H}) := \{\rho \in \mathfrak{B}(\mathcal{H}) \mid \rho \geq 0,\ \mathrm{tr}_{\mathcal{H}}|\rho| < \infty\}$
we will consider arbitrary sets of denormalised quantum states: $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^{+}$

Quantum information distances $D : \mathcal{M}(\mathcal{H}) \times \mathcal{M}(\mathcal{H}) \to [0,\infty]$ s.t. $D(\rho,\sigma) = 0 \iff \rho = \sigma$.

- E.g.
  - $D_1(\rho,\sigma) := \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$ [Umegaki'62]
  - $D_{1/2}(\rho,\sigma) := 2\|\sqrt{\rho} - \sqrt{\sigma}\|^2_{\mathfrak{G}_2(\mathcal{H})} = 4\mathrm{tr}_{\mathcal{H}}(\frac{1}{2}\rho + \frac{1}{2}\sigma - \sqrt{\rho}\sqrt{\sigma})$ (Hilbert–Schmidt norm^2)
  - $D_{L_1(\mathcal{N})}(\rho,\sigma) := \frac{1}{2}\|\rho - \sigma\|_{\mathcal{T}(\mathcal{H})} = \frac{1}{2}\mathrm{tr}_{\mathcal{H}}|\rho - \sigma|$ (L_1/predual norm)
  - $D_\gamma(\rho,\sigma) := \frac{1}{\gamma(1-\gamma)}\mathrm{tr}_{\mathcal{H}}(\gamma\rho + (1-\gamma)\sigma - \rho^\gamma\sigma^{1-\gamma})$; $\gamma \in \mathbb{R} \setminus \{0,1\}$ [Hasegawa'93]
  - $D_{\alpha,z}(\rho,\sigma) := \frac{1}{1-\alpha}\log \mathrm{tr}_{\mathcal{H}}(\rho^{\alpha/z}\sigma^{(1-\alpha)/z})^z$; $\alpha, z \in \mathbb{R}$ [Audenauert–Datta'14]
  - $D_{\mathfrak{f}}(\rho,\sigma) := \mathrm{tr}_{\mathcal{H}}(\sqrt{\rho}\,\mathfrak{f}(\mathfrak{L}_\rho\mathfrak{R}_\sigma^{-1})\sqrt{\rho})$; $\mathfrak{f}$ operator convex, $\mathfrak{f}(1) = 0$ [Kosaki'82,Petz'85]

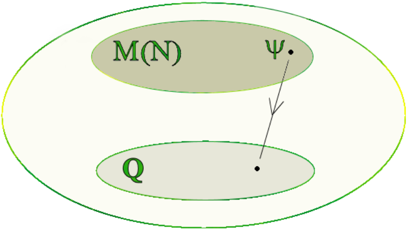
  for $\mathrm{ran}(\rho) \subseteq \mathrm{ran}(\sigma)$, and with all $D(\rho,\sigma) := +\infty$ otherwise.

Quantum entropic projections

Let $\mathcal{Q} \subseteq \mathcal{T}(\mathcal{H})^+$ be such that
for each $\psi \in \mathcal{M}(\mathcal{H})$
there exists a unique solution

$$\mathfrak{P}^D_{\mathcal{Q}}(\psi) := \arg\inf_{\rho\in\mathcal{Q}} \{D(\rho,\psi)\}\,.$$

It will be called an entropic projection.



E.g.

- for $D_{1/2}(\rho,\sigma) = 2\left\|\sqrt{\rho}-\sqrt{\sigma}\right\|^2_{\mathcal{H}}$,
consider the entropic projections $\mathfrak{P}^{D_{1/2}}_{\mathcal{Q}}$
where $\mathcal{Q}$ are images of closed convex subspaces $\tilde{\mathcal{Q}} \subseteq \mathcal{K}^+ := \mathfrak{G}_2(\mathcal{H})^+$
under the mapping $\tilde{\mathcal{Q}} \ni \sqrt{\rho} \mapsto \rho \in \mathcal{Q}$.
They coincide with the ordinary projection operators in $\mathfrak{B}(\mathcal{K}) \cong \mathfrak{B}(\mathcal{H}\otimes\mathcal{H}^\star)$.

Quantum measurement, bayesianity, and maximum relative entropy

- Lüders' rules:

$$\rho \mapsto \rho_{\mathrm{new}} := \sum_i P_i \rho P_i \quad \text{('weak')}$$

$$\rho \mapsto \rho_{\mathrm{new}} := \frac{P\rho P}{\mathrm{tr}_{\mathcal{H}}(P\rho)} \quad \text{('strong')}$$

- Bub'77'79, Caves–Fuchs–Schack'01, Fuchs'02, Jacobs'02: Lüders' rules should be considered as rules of inference (conditioning) that are quantum analogues of

the Bayes–Laplace rule: $$p(x) \mapsto p_{\mathrm{new}}(x) := \frac{p(x)p(b|x)}{p(b)}.$$

- Williams'80, Warmuth'05, Caticha&Giffin'06: the Bayes–Laplace rule is a special case of

$$p(\chi) \mapsto p_{\mathrm{new}}(\chi) := \arg\inf_{q \in \mathcal{Q}} \{D_1(q,p)\}; \quad D_1(q,p) := \int_{\mathcal{X}} \mu(\chi) q(\chi) \log\left(\frac{q(\chi)}{p(\chi)}\right).$$

- Douven&Romeijn'12: the Bayes–Laplace rule is also a special case of

$$p \mapsto \arg\inf_{q \in \mathcal{Q}} \{D_1(p,q)\} = \mathfrak{P}^{D_0}_{\mathcal{Q}}(p),$$

where $D_0(p,q) = D_1(q,p)$.

Quantum bayesian inference from quantum entropic projections

- RPK'13'14, F.Hellmann–W.Kamiński–RPK'14:

1. weak Lüders' rule is a special case of
$$\rho \mapsto \arg\inf_{\sigma\in\mathcal{Q}} \{D_1(\rho,\sigma)\}$$
with
$$\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i,\sigma] = 0\ \forall i\}$$
2. strong Lüders' rule derived from
$$\rho \mapsto \arg\inf_{\sigma\in\mathcal{Q}} \{D_1(\rho,\sigma)\}$$
with
$$\mathcal{Q} = \{\sigma \in \mathcal{T}(\mathcal{H})^+ \mid [P_i,\sigma] = 0,\ \mathrm{tr}_{\mathcal{H}}(\sigma P_i) = p_i\ \forall i\}$$
under the limit $p_2,\ldots,p_n \to 0$.
3. hence, weak and strong Lüders' rules are special cases of quantum entropic projection $\mathfrak{P}^{D_0}_{\mathcal{Q}}$ based on relative entropy $D_0(\sigma,\rho) = D_1(\rho,\sigma)$.

Bayes–Laplace and Lüders' conditionings are special cases of entropic projections
$\Rightarrow$ "quantum bayesianism $\subseteq$ quantum relative entropism".

Meaning: the rule of maximisation of relative entropy (entropic projection on the subspace of constraints) can be considered as a nonlinear generalisation of the dynamics describing "quantum measurement". [RPK'10'11]

Quantum Poisson structure

- Consider the space of self-adjoint trace-class operators: $\mathcal{T}(\mathcal{H})^{\mathrm{sa}} := \mathcal{T}(\mathcal{H}) \cap \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$.

- It can be equipped with a following real Banach smooth manifold structure:
  - tangent spaces: $\mathbf{T}_\phi(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}) \cong \mathcal{T}(\mathcal{H})^{\mathrm{sa}}$
  - cotangent spaces: $\mathbf{T}^{\circledast}_\phi(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}) \cong (\mathcal{T}(\mathcal{H})^{\mathrm{sa}})^\star \cong \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$

- Bóna'91,'00: a Poisson manifold structure on $\mathcal{T}(\mathcal{H})^{\mathrm{sa}}$ is defined by a commutator of an algebra. Given any $f, h \in \mathrm{C}^\infty(\mathcal{T}(\mathcal{H})^{\mathrm{sa}}; \mathbb{R})$, $\rho \in \mathcal{T}(\mathcal{H})^{\mathrm{sa}}$,

$$\{h, f\}(\rho) := \mathrm{tr}_{\mathcal{H}}\left(\rho\, \mathrm{i}[\mathbf{d}h(\rho), \mathbf{d}f(\rho)]\right).$$

- So, if $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$ is a smooth submanifold of $\mathcal{T}(\mathcal{H})^{\mathrm{sa}}$, then every $f \in \mathrm{C}^\infty(\mathcal{M}(\mathcal{H}); \mathbb{R})$ determines a hamiltonian vector field:

$$\mathfrak{X}_f(\rho) = -\{\cdot, f\}(\rho) = \mathrm{tr}_{\mathcal{H}}(\rho\, \mathrm{i}[\mathbf{d}(\cdot), \mathbf{d}f(\rho)]).$$

- More generally, we can choose arbitrary real Banach Lie subalgebra $\mathcal{A}$ of $\mathfrak{B}(\mathcal{H})$ such that: (i) it has a unique Banach predual $\mathcal{A}_\star$ in $\mathcal{T}(\mathcal{H})$; (ii) there exists at least one $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{T}(\mathcal{H})^+$ which is a smooth submanifold of $\mathcal{A}_\star$.

Nonlinear quantum hamiltonian dynamics

For each hamiltonian vector field, the corresponding Hamilton equation is equivalent to the Bóna equation ['91'00]

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [\mathbf{d}h(\rho(t)), \rho(t)].$$

Hence,

The Poisson structure $\{\cdot,\cdot\}$ induced by a commutator of $\mathfrak{B}(\mathcal{H})$ allows to introduce various nonlinear hamiltonian evolutions on spaces $\mathcal{M}(\mathcal{H})$ of quantum states, generated by arbitrary real-valued smooth functions on $\mathcal{M}(\mathcal{H})$.

The solutions of Bóna equation are state-dependent unitary operators $U(\rho, t)$.
They do not form a group, but satisfy a cocycle relationship:

$$U(\rho, t+s) = U((\mathrm{Ad}(U(\rho,t)))(\rho), s)U(\rho, t) \;\; \forall t, s \in \mathbb{R}.$$

In a special case, when $h(\rho) = \mathrm{tr}_{\mathcal{H}}(\rho H)$ for $H \in \mathfrak{B}(\mathcal{H})^{\mathrm{sa}}$,
the Bóna equation turns to the von Neumann equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [H, \rho(t)].$$

Quantum causal inferences by entropic-hamiltonian dynamics

- **Two elementary geometric structures:**
  - $D(\cdot,\cdot)$ represents the convention of "best estimation/inference"
  - $\{h,\cdot\}$ represents a convention of causality ("internal dynamics")
- **Two elementary forms of quantum dynamics:**
  - entropic projections $\mathfrak{P}^D_Q$ generated by quantum distances $D(\cdot,\cdot)$
  - hamiltonian flows w^h_t generated by nonlinear hamiltonian vector fields $\{h,\cdot\}$

A general form of quantum dynamics is defined as a causal inference $\mathfrak{P}^D_\mathcal{Q} \circ w^h_t$.

- It generalises unitary evolution followed by a "projective measurement".
- Postulate: consider the setting of causal inferences $\mathfrak{P}^D_\mathcal{Q} \circ w^h_t$ as an alternative to the paradigm of semigroups of CPTP maps.
- Basic idea: every CPTP instrument [Davies–Lewis'70] can be decomposed into:

(1) tensor product of initial state with uncorrelated environment,
(2) unitary evolution,
(3) projective measurement,
(4) partial trace.

It remains to prove that (4) and (3+4) are entropic projections.
M.Munk-Nielsen'15: (4) is entropic projection at least for strictly positive states. *Ongoing work RPK+MMN'16:* prove (3+4) for all states and (4) for nonfaithful ones.

2. Geometric framework for quantum information theories beyond quantum mechanics

- Principles of geometric (post)quantum kinematics
- Global/sequential and local/parallel dynamics
- Global and local reconstruction of QM

Towards new foundations

Key mathematical and conceptual change:
A shift from ontology of eigenvalues (implemented by operators on Hilbert spaces and probabilistic statistics) to epistemology of expectations (implemented by geometry of state spaces of W^*-algebras and quantum statistics).

Idea:

- consider spaces $\mathcal{M}(\mathcal{H})$ as fundamental
- allow any nonlinear functions $\mathcal{M}(\mathcal{H}) \to \mathbb{R}$ as observables (smooth determine hamiltonian functions, affine determine self-adjoint operators)
- define geometry of $\mathcal{M}(\mathcal{H})$ by means of $D(\cdot, \cdot)$ and $\{\cdot, \cdot\}$
- define dynamics of $\mathcal{M}(\mathcal{H})$ by means of $\mathfrak{P}^D_{\mathcal{Q}}(\cdot, \cdot)$ and $w_t^{\{h, \cdot\}}$

Questions:

- what's up with Hilbert spaces?
- what's up with spectral theory, probability, Born rule, etc?

Answers:

- replace Hilbert spaces $\mathcal{H}$ by W^*-algebras $\mathcal{N}$
- replace sets $\mathcal{M}(\mathcal{H})$ of density matrices on $\mathcal{H}$ by sets $\mathcal{M}(\mathcal{N})$ of positive integrals on W^*-algebras $\mathcal{N}$
- this setting is an exact generalisation of Kolmogorov's measure theoretic setting for probability theory

W^*-algebras and integration

- A W^*-algebra $\mathcal{N}$:
  - an algebra over $\mathbb{C}$ with unit $\mathbb{I}$,
  - with * operation s.t. $(xy)^* = y^*x^*$, $(x+y)^* = x^* + y^*$, $(x^*)^* = x$, $(\lambda x)^* = \lambda^* x^*$,
  - that is also a Banach space,
  - with $\cdot$, $+$, * continuous in the norm topology (implied by the condition $\|x^*x\| = \|x\|^2$),
  - such that there exists a Banach space $\mathcal{N}_\star$ satisfying Banach duality: $(\mathcal{N}_\star)^\star \cong \mathcal{N}$,
- Special cases:
  - if $\mathcal{N}$ is commutative then $\exists$ a measure space $(\mathcal{X}, \mu)$ s.t. $\mathcal{N}^+ \cong L_\infty(\mathcal{X}, \mu)^+$ and $\mathcal{N}_\star \cong L_1(\mathcal{X}, \mu)$
  - if $\mathcal{N}$ is "type I factor" then $\exists$ a Hilbert space $\mathcal{H}$ s.t. $\mathcal{N} \cong \mathfrak{B}(\mathcal{H})$ and $\mathcal{N}_\star^+ \cong \mathcal{T}(\mathcal{H})^+$.
- Hence, the element $\phi \in (\mathcal{N}_\star)^+$ provides a joint generalisation of probability density and of density operator. By means of embedding of $\mathcal{N}_\star$ into $\mathcal{N}^\star$, it is also an integral on $\mathcal{N}$.
- We chose $\mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_\star^+$ as our generic quantum state spaces.

Noncommutative integration on W^*-algebras

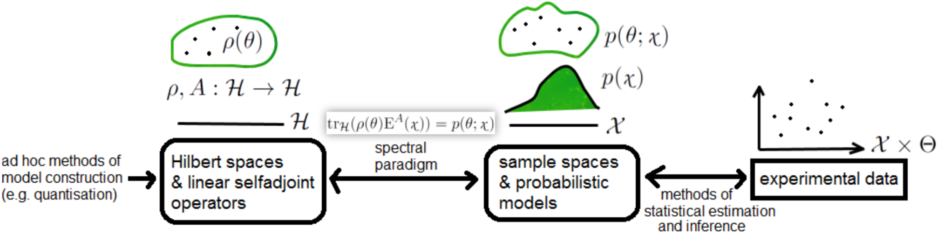
- Commutative integration:

	spatial representation	algebraic formulation
underlying object	localisable measure space: $(\mathcal{X}, \mho(\mathcal{X}), \mu)$	localisable boolean algebra: $\mathcal{A}$
L_p-spaces	$L_p(\mathcal{X}, \mho(\mathcal{X}), \mu)$	$Lp(\mathcal{A})$
states	$q \in \mathcal{M}(\mathcal{X}, \mho(\mathcal{X}), \mu) \subseteq L_1(\mathcal{X}, \mho(\mathcal{X}), \mu)^+$	$\phi \in \mathcal{M}(\mathcal{A}) \subseteq L_1(\mathcal{A})^+$
expectations of observables	$L_\infty(\mathcal{X}, \mho(\mathcal{X}), \mu) \ni f \mapsto \int_{\mathcal{X}} \mu q f \in \mathbb{R}$	$\phi \in L_\infty(\mathcal{A}) \ni f \mapsto \phi(f) \in \mathbb{R}$

- Noncommutative integration:

	spatial representation	algebraic formulation
underlying object	Hilbert space with std. trace: $(\mathcal{H}, \mathrm{tr}_{\mathcal{H}})$	W^*-algebra: $\mathcal{N}$
L_p-spaces	$\mathfrak{G}_p(\mathcal{H}) = L_p(\mathfrak{B}(\mathcal{H}), \mathrm{tr})$	$L_p(\mathcal{N})$
states	$\rho \in \mathcal{M}(\mathcal{H}) \subseteq \mathfrak{G}_1(\mathcal{H})^+ \cong \mathfrak{B}(\mathcal{H})^+_\star$	$\phi \in \mathcal{M}(\mathcal{N}) \subseteq L_1(\mathcal{N})^+ \cong \mathcal{N}^+_\star$
expectations of observables	$\mathfrak{B}(\mathcal{H}) = \mathfrak{G}_\infty(\mathcal{H}) \ni x \mapsto \mathrm{tr}(\rho x) \in \mathbb{C}$	$\mathcal{N} = L_\infty(\mathcal{N}) \ni x \mapsto \phi(x) \in \mathbb{C}$

Orthodox quantum mechanical paradigm (von Neumann, 1926-1932):

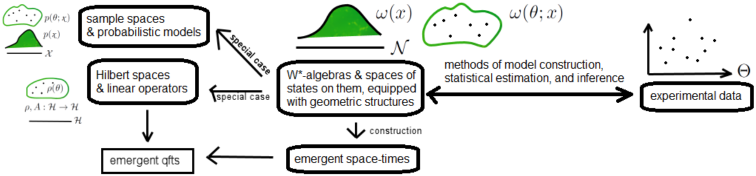


- a solution of a particular problem (solid mathematical framework providing unifying foundations for 'wave mechanics' and 'matrix mechanics')
- von Neumann'1935: "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space anymore."

Some key observations:

- Probability theory is just a special case of integration theory on W^*-algebras.
- From the perspective of this theory, quantum states are **just** integrals, so there is no a priori reason why "general" quantum theory (beyond QM) should depend on probabilities.
- Quantum states (and structures over them) can be associated directly with the epistemic data by generalising the methods of associating epistemic data with probabilities (and with structures over them).

New paradigm:



Basic object of interest: spaces $\mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_\star^+$ of states over W^*-algebras $\mathcal{N}$.

- Quantum theoretic kinematics generalises and replaces probability theory.
- Quantum theoretic dynamics generalises and replaces causal statistical inference.
- Nonlinear information geometry of spaces of quantum states replaces the role of (linear) spectral theory of quantum mechanics.
- Replace the use of eigenvalues and expectations of self-adjoint operators on $\mathcal{H}$ (or in $\mathcal{N}$) by observables $f : \mathcal{M}(\mathcal{N}) \to \mathbb{R}$. Given any model construction rule $\mathbb{R}^n \supset \Theta \ni \theta \mapsto \rho(\theta) \in \mathcal{M}(\mathcal{N})$, and the set of experimental functions $f_\Theta : \Theta \to \mathbb{R}$ the set of observables relevant to the problem is given by $\{f : \mathcal{M}(\mathcal{N}) \to \mathbb{R} \mid f_\Theta = f \circ \theta\}$

New dynamics: information geometric causal inference

- Main change: Consider nonlinear state changes defined by geometric structures on quantum models as more important than linear CP maps (= "no-initial-correlations + unitary evolution + projective measurement + partial trace")
- **Two fundamental dynamic structures on $\mathcal{M}(\mathcal{N})$:**
  - a) **Inference:** Entropic projections $\phi \mapsto \arg\inf_{\omega \in \mathcal{Q}(\eta)} \{D(\omega, \phi)\}$ [RPK'10]
    - ★ nonlinear and nonlocal
    - ★ requires convexity
    - ★ represents ("active/external") information dynamics due to learning/measuring
    - ★ allows to encode experimental constraints
    - ★ reduces in special cases to Lüders', Jeffrey's, Bayes' rules
  - b) **Causality:** Hamiltonian flows $\phi \mapsto w^h_t(\phi)$, $\frac{\mathrm{d}}{\mathrm{d}t} f(w^h_t(\phi)) = \{h, f(w^h_t)\}(\phi)\ \forall f$ [Bóna'00]
    - ★ nonlinear and local
    - ★ requires smoothness
    - ★ represents ("passive/internal") changes of information states when no inference is made
    - ★ allows to encode theoretical symmetries
    - ★ reduces in a special case to the von Neumann equation.
- **They allow for two main descriptions of total information dynamics**:
  - a) **Sequential processing:** entropic projections composed with hamiltonian flows: $\phi \mapsto \mathfrak{P}^D_{\mathcal{Q}}(\eta) \circ w^h_t(\phi)$
    - ★ nonlinear and nonmarkovian
    - ★ allows for arbitrary correlations between subsystems
    - ★ from the bayesian perspective, $w^h_t(\phi)$ is a prior for $\mathfrak{P}^D_{\mathcal{Q}}(\eta)$-updating
  - b) **Parallel processing:** infinitesimal hamiltonian flows perturbed by dissipative dynamics given by free falls along geodesics determined by entropic projections

Smooth quantum information geometries

Under some conditions, D induces a generalisation of smooth riemannian geometry on $\mathcal{M}(\mathcal{N})$.

- $\mathcal{M}(\mathcal{H}) := \{\rho(\theta) \in \mathcal{T}(\mathcal{H}) \mid \rho(\theta) > 0, \theta \in \Theta \subseteq \mathbb{R}^n \text{ open}, \theta \mapsto \rho(\theta) \text{ smooth}\}$ is a C-manifold

- Jenčová'05: a general construction of smooth manifold structure on the space of all strictly positive states over arbitrary W^*-algebra, with tangent spaces given by noncommutative Orlicz spaces.

- Eguchi'83/Ingarden et al'82/Lesniewski–Ruskai'99/Jenčová'04:
Every smooth distance D with positive definite hessian determines
a riemannian metric $\mathbf{g}^D$ and a pair $(\nabla^D, \nabla^{D^\dagger})$ of torsion-free affine connections:

$$\mathbf{g}_\phi(u,v) := -\partial_{u|\phi}\partial_{v|\omega} D(\phi,\omega)|_{\omega=\phi},$$

$$\mathbf{g}_\phi((\nabla_u)_\phi v, w) := -\partial_{u|\phi}\partial_{v|\phi}\partial_{w|\omega} D(\phi,\omega)|_{\omega=\phi},$$

$$\mathbf{g}_\phi(v, (\nabla^\dagger_u)_\phi w) := -\partial_{u|\omega}\partial_{w|\omega}\partial_{v|\phi} D(\phi,\omega)|_{\omega=\phi},$$

which satisfy the characteristic equation of the Norden['37]–Sen['44] geometry,

$$\mathbf{g}^D(u,v) = \mathbf{g}^D(\mathbf{t}_c^{\nabla^D}(u), \mathbf{t}_c^{\nabla^{D^\dagger}}(v)) \;\; \forall u,v \in \mathbf{T}\mathcal{M}(\mathcal{N}).$$

- A riemannian geometry $(\mathcal{M}(\mathcal{N}), \mathbf{g}^D)$ has Levi-Civita connection $\bar{\nabla} = (\nabla^D + \nabla^{D^\dagger})/2$.

Example

- Example 1: $\mathcal{M}(\mathcal{N}) = \mathcal{T}(\mathcal{H}) \cap \{\rho > 0, \mathrm{tr}_{\mathcal{H}}(\rho) = 1\}$
$D_1(\rho, \sigma) = \mathrm{tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \sigma)$
give Mori['55]–Kubo['56]–Bogolyubov['62] $\mathbf{g}^{D_1}$ and Nagaoka['94]–Hasegawa['95] $(\nabla^{D_1}, \nabla^{D_1 \dagger})$:
$$\mathbf{g}^{D_1}_{\rho}(x, y) = \mathrm{tr}_{\mathcal{H}} \left(\int_0^{\infty} \mathrm{d}\lambda x \frac{1}{\lambda \mathbb{I} + \rho} y \frac{1}{\lambda \mathbb{I} + \rho} \right),$$
$$\mathbf{t}^{\nabla^{D_1}}_{\rho,\omega}(x) = x - \mathrm{tr}_{\mathcal{H}}(\omega x), \quad \mathbf{t}^{\nabla^{D_1 \dagger}}_{\rho,\omega}(x) = x.$$

Hessian geometries = dually flat Norden–Sen geometries

If $(\mathcal{M}, \mathbf{g}, \nabla, \nabla^\dagger)$ is a Norden–Sen geometry with flat ∇ and $\nabla^\dagger$, then:

1. there exists a unique pair of functions $\Phi : \mathcal{M} \to \mathbb{R}$, $\Phi^{\mathbf{L}} : \mathcal{M} \to \mathbb{R}$ such that $\mathbf{g}$ is their hessian metric,

$$\mathbf{g}_{ij}(\rho) = \frac{\partial^2 \Phi(\rho(\theta))}{\partial\theta^i \partial\theta^j} \mathrm{d}\theta^i \otimes \mathrm{d}\theta^j, \quad \mathbf{g}_{ij}(\rho) = \frac{\partial^2 \Phi^{\mathbf{L}}(\rho(\eta))}{\partial\eta^i \partial\eta^j} \mathrm{d}\eta^i \otimes \mathrm{d}\eta^j,$$

where: $\{\theta^i\}$ is a coordinate system s.t. $\Gamma^{\nabla}_{ijk}(\rho(\theta)) = 0 \; \forall\rho \in \mathcal{M}$,
$\{\eta^i\}$ is a coordinate system s.t. $\Gamma^{\nabla^\dagger} ijk(\rho(\eta)) = 0 \; \forall\rho \in \mathcal{M}$.

2. the Eguchi equations applied to the Brègman distance

$$D_\Phi(\rho, \sigma) := \Phi(\rho) + \Phi^{\mathbf{L}}(\sigma) - \sum_i \theta^i(\rho)\eta^i(\sigma)$$

yield $(\mathbf{g}, \nabla, \nabla^\dagger)$ above.

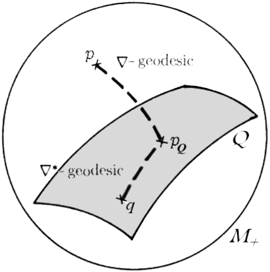
Smooth generalised pythagorean theorem

Let $(\mathcal{M}, \mathbf{g}, \nabla, \nabla^\dagger)$ be a hessian geometry. Then for any $\mathcal{Q} \subseteq \mathcal{M}$ which is:

- $\nabla^\dagger$-autoparallel $:= \nabla^\dagger_u v \in \mathbf{T}\mathcal{Q}\ \forall u, v \in \mathbf{T}\mathcal{Q}$;
- $\nabla^\dagger$-convex $:= \forall \rho_1, \rho_2 \in \mathcal{Q}\ \exists!\ \nabla^\dagger$-geodesics in $\mathcal{Q}$ connecting ρ_1 and ρ_2;

there exists a unique projection

$$\mathcal{M} \ni \rho \mapsto \mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho) := \arg\inf_{\sigma \in \mathcal{Q}} \{D_\Phi(\sigma, \rho)\} \in \mathcal{Q}.$$



- it is equal to a unique projection of ρ onto $\mathcal{Q}$ along a ∇-geodesic that is $\mathbf{g}$-orthogonal at $\mathcal{Q}$.
- it satisfies a generalised pythagorean equation

$$D_\Phi(\omega, \mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho)) + D_\Phi(\mathfrak{P}^{D_\Phi}_{\mathcal{Q}}(\rho), \rho) = D_\Phi(\omega, \rho) \quad \forall(\omega, \rho) \in \mathcal{Q} \times \mathcal{M}.$$

Hence, for Brègman distances D_Φ the local entropic projections are equivalent with geodesic projections.

Local effective dynamics

- One can combine locally the entropic projections with hamiltonian flows, by passing to the derived geodesic projections, and combining both in a single formula for effective dynamics.
- Given a hamiltonian observable h and a relative entropy D, the 1-form $dh(\phi) - d_{\nabla^D}(\phi)$ represents a local perturbation of causal dynamics by the information flow along entropic geodesics.
- In particular, $D_{1/2} = 2\|\sqrt{\rho} - \sqrt{\sigma}\|^2_{\mathcal{H}}$ gives Wigner–Yanase metric $\mathbf{g}^{1/2}$, with $d_{\mathbf{g}^{1/2}}(\rho,\sigma) = 2\arccos(\mathrm{tr}_{\mathcal{H}}(\sqrt{\rho}\sqrt{\sigma}))$. The free fall along the geodesics of Levi-Civita connection $\nabla^{1/2}$ encodes the continuous process of projective measurement.
- The resulting effective dynamics can be given mathematically exact form in terms of a continuous-time regularised path-integral

$$\lim_{\varepsilon\to+0}\int\mathcal{D}\phi(\cdot)\mathrm{e}^{\mathrm{i}\int_\gamma \mathrm{d}t\left\langle\Omega_{\phi(t)},\mathbf{d}_{\nabla^{1/2}}(\phi(t)\Omega_{\phi(t)}\right\rangle_{\mathcal{H}_{\phi(t)}}}\cdot \tag{1}$$

$$\cdot\mathrm{e}^{-\mathrm{i}\int_\gamma \mathrm{d}t\left\langle\Omega_{\phi(t)},\pi_{\phi(t)}(\mathbf{d}h(\phi(t)))\Omega_{\phi(t)}\right\rangle}\mathrm{e}^{-\frac{\varepsilon}{2}\int_\gamma \mathrm{d}t\mathbf{g}^{1/2}_{ab}(\phi(t))\dot{\phi}^a\dot{\phi}^b}, \tag{2}$$

- If evaluated only on boundary pure states, and for $h(\phi) = \phi(\mathcal{H})$, it is known (Daubechies–Klauder'85, Anastopoulos–Savvidou'03) to be equal to $\left\langle\Omega(t=s), e^{-\mathrm{i}Hs}\Omega(t=0)\right\rangle_{\mathcal{H}}$.

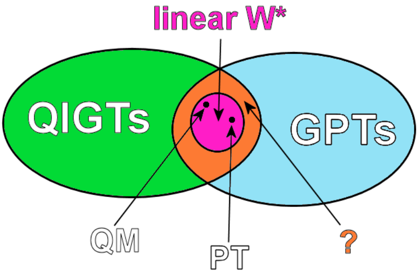
Backwards compatibility: Global reconstructions

1a. **(Global) Reconstruction of quantum mechanics:**

- $\mathcal{N}$: type I W^*-algebras
- $\mathcal{M}(\mathcal{N})$: normalised states
- D: $D_{1/2}$ or D_0
- $\{\cdot,\cdot\}$: generated by Banach Lie algebra $\mathcal{N}^{\mathrm{sa}}$
- observables: linear functions on $\mathcal{M}(\mathcal{N})$

2. **Reconstruction of probability theory:**

- $\mathcal{N}$: commutative algebras
- $\mathcal{M}(\mathcal{N})$: normalised states
- D: arbitrary (but for D_1 or D_0, and specific types of constraints, Bayes' and Jeffrey's rules are recovered)
- $\{\cdot,\cdot\}$: trivialises for commutative algebras
- observables: arbitrary or affine functions on $\mathcal{M}(\mathcal{N})$



Backwards compatibility: Local reconstruction of [W^*-]QM

1. **localc kinematics (only in tangent space)**:
   - states: vectors of $\mathbf{T}_\phi\mathcal{M}(\mathcal{N})$ (configurations: $\phi(\theta) \to \theta \to \frac{\partial}{\partial\theta}$)
   - effects: vectors of $\mathbf{T}^{\circledast}_\phi\mathcal{M}(\mathcal{N})$ (observables: $f \to \mathbf{d}f(\phi)$)
2. **local dynamics (only in tangent space)**:
   - causality: hamiltonian causality is local
   - inference: arbitrary entropic projections are nonlocal, but the Norden–Sen geometries derived from relative entropies allow to localise entropic projections
   - causality+inference: as presented few slides ago
3. **reconstruction of W^*-algebras**: Can we start from *arbitrary* sets $\mathcal{M}$, equipped with geometric structures $\{\cdot,\cdot\}$ and $D(\cdot,\cdot)$, without knowing that they are over W^*-algebras, and reconstruct $\mathcal{M} = \mathcal{M}(\mathcal{N})$ from some conditions? $\to$ **work in progress!**
4. Basic idea of a proof: W^*-algebras = LJBW*-algebras = BLP submanifolds extendible to convex hull, with observables having Jordan structure = BLP submanifolds (=Poisson spaces) $\mathcal{M}$ with riemannian structure induced from relative entropy and Kähler compatibility condition on the convex hull of $\mathcal{M}$ $\leftarrow$ **main conjecture**

Plan

1. **Nonlinear generalisations of quantum dynamics:**
    - Geometric structures on quantum state spaces → relative entropies & Poisson brackets
    - Lüders' rules → constrained relative entropy maximisations
    - Unitary evolution → nonlinear hamiltonian flows
2. **Geometric (post)quantum information foundations:**
    - Mathematical and physical principles
    - Global and local dynamics
    - Global and local reconstruction of QM
3. **Category-theoretic operational semantics:**
    - Adjointness in foundations, functorial localisation
    - Resource theories a la LdR–LK–RR
    - Beyond adjointness: local monad–comonad systems
4. **Towards [(post)quantum] local information relativity:**
    - From equlibrium to nonequilibrium space-time thermodynamics
    - Two-dimensional surfaces (θ, σ) and geometry in Klauder–Daubechies quantisation
    - Quantum dynamics of (θ, σ)-spaces

What is the predictive content of semantics of probability theory?

Consider:

- Θ: a space of possible configurations
- $\Theta \ni \theta = (\theta_1, \ldots, \theta_n)$: average values of n types of experimental configuration variables
- Ξ: a space of registrations

Example: **MaxEnt + entropic projections (or Bayes' rule) + prediction:**

$$\textit{configurations \& registrations} \underset{\text{predictive verification}}{\overset{\text{model construction}}{\rightleftarrows}} \textit{beliefs \& updatings}$$

$$\Theta \underset{}{\overset{\theta_i = \mathrm{tr}_{\mathcal{H}}(\rho h_i)}{\rightleftarrows}} \{\rho(\theta) = \mathrm{e}^{-\sum_i \beta(\theta_i) h_i} \mid \theta \in \Theta\}$$

$$\Theta \xrightarrow{\Xi} \tilde{\Theta}, \qquad \{\rho(\theta)\} \xrightarrow{\mathfrak{P}^{D_1}_{\mathcal{Q}(\Xi)}} \{\tilde{\rho}\}$$

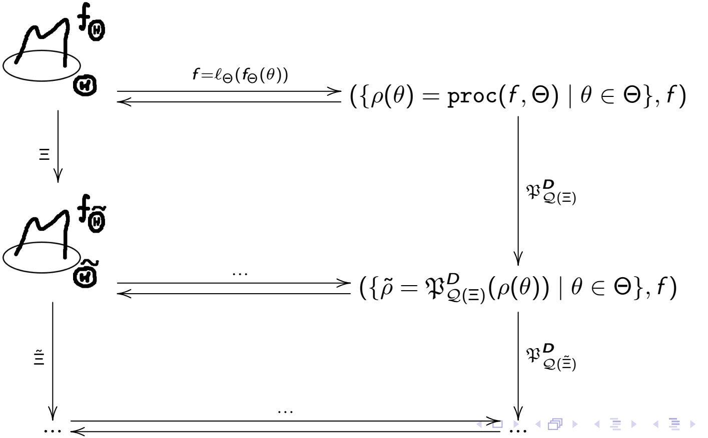
$$\tilde{\Theta} \underset{}{\overset{\tilde{\theta}_i = \mathrm{tr}_{\mathcal{H}}(\tilde{\rho} h_i)}{\rightleftarrows}} \{\tilde{\rho} = \mathfrak{P}^{D_1}_{\mathcal{Q}(\Xi)}(\rho(\theta)) \mid \theta \in \Theta\}$$

$$\tilde{\Theta} \xrightarrow{\tilde{\Xi}} \hat{\Theta}, \qquad \{\tilde{\rho}\} \xrightarrow{\mathfrak{P}^{D_1}_{\mathcal{Q}(\tilde{\Xi})}} \{\hat{\rho}\}$$

$$\hat{\Theta} \underset{}{\overset{\hat{\theta}_i = \mathrm{tr}_{\mathcal{H}}(\hat{\rho} h_i)}{\rightleftarrows}} \{\hat{\rho} \mid \ldots\}$$

Towards new semantics

- One can use other model construction principles
- There is no need to use linear expectation type constraints
- This what we should care about is the relationship between model construction (information encoding), inference (information processing), and predictive verifiability (information decoding).



Adjointness in the foundations (of inductive inference)

- We can relax the condition of bijectivity of arrows between models and configurations to one that makes the relationship between encoding and decoding to be optimal in the following sense:

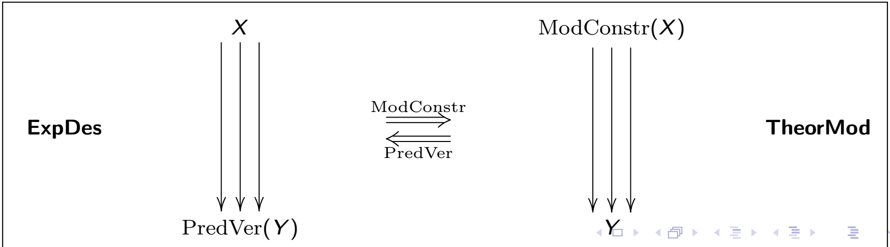
> The method of encoding (model construction) should be the most effective solution of the problem provided by the given decoding (prediction).

- Let **TheorMod** be a category of theoretical models as objects and inferences as arrows.
Let **ExpDes** be a category of experimental designs as objects and registrations as arrows.
Model construction is defined as a functor $\mathrm{ModConstr} : \mathbf{ExpDes} \to \mathbf{TheorMod}$.
Predictive verification is defined as a functor $\mathrm{PredVer} : \mathbf{TheorMod} \to \mathbf{ExpDes}$.

- Mutual consistency condition: $\mathrm{ModConstr} \dashv \mathrm{PredVer}$

- This means: there is a natural bijection

$$\hom_{\mathbf{ExpDes}}(X, \mathrm{PredVer}(Y)) \cong \hom_{\mathbf{TheorMod}}(\mathrm{ModConstr}(X), Y)$$



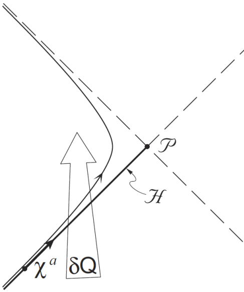
(Some blackboard, dependently on time)

- F&f functors
- Localisation by groupoids
- A resource theory of knowledge is $(\mathcal{C}, \mathcal{T})$, where $\mathcal{T}$ is a submonoid of a functor category $\mathcal{C}^{\mathcal{C}}$
- LdR–LK–RR setting as an example of this framework:
  - Posets as categories
  - Galois insertions as adjoint functors
  - Localisation by division by equivalence as a coarse-grained groupoidal localisation
  - Compatibility of local descriptions as isomorphism to a terminal object
  - ...
- Monad–comonad systems as more pluralistic multi-agency.

Jacobson'95: Einstein equations *from* space-time thermodynamics

Consider:

- a space-time $(\mathcal{M}, g_{ab})$
- a point $p \in \mathcal{M}$
- a small 2-dimensional surface element $\mathcal{P}$
- a Killing vector field χ^a generating local boost orthogonal to $\mathcal{P}$



Define:

- a local causal horizon $\mathcal{H}$ as a boundary of the past of $\mathcal{P}$, generated by χ^a
- a heat flow δQ as an energy flux across a local causal horizon: $\delta Q := \int_{\mathcal{H}} \mathrm{d}\Sigma^a T_{ab}\chi^b$
- a temperature T as an Unruh temperature associated with a uniformly accelerated observer.

Assume:

- that entropy S is proportional to the area of $\mathcal{H}$: $S = \lambda A$
- that Clausius' law holds: $\delta Q = T\mathrm{d}S$.

Then:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{2\pi}{\lambda}T_{ab}.$$

Beyond quantum gravity

Jacobson'95:

- "This perspective suggests that it may be no more appropriate to canonically quantize the Einstein equation than it would be to quantize the wave equation for sound in air."
- "one might expect that sufficiently high frequency or large amplitude disturbances of the gravitational field would no longer be described by the Einstein equation, not because some quantum operator nature of the metric would become relevant, but because the local equilibrium condition would fail. It is my hope that, by following this line of inquiry, we shall eventually reach an understanding of the nature of "non-equilibrium spacetime". "

Our conclusions:

- Instead of quantising gravity lets try to derive kinematics and dynamics of space-time from suitably generalized kinematics and dynamics of quantum theory [recall Mielnik'76: "instead of modifying general relativity to fit quantum mechanics one should rather modify quantum mechanics to fit general relativity"]
- Equilibrium and nonequilibrium thermodynamics can be described using information geometry. It seems plausible that quantum information geometry may play a key role in the space-time emergence.

(Some blackboard, dependently on time)

- Nonequilibrium setting: Chirco–Liberati observation
- Daubechies–Klauder Wiener measure
- Idea: emergence of local space-time from local quantum geometry of 2-surfaces.

Summary of open problems and ongoing work

- Kinematics and global dynamics of (post)quantum geometric information theory: quite ok!
- Open problem: characterise CPTP maps as a special case of composition of $\otimes$, hamiltonian flow, and entropic projections (ongoing project with Morten Munk-Nielsen)
- Local dynamics: quite ok & new! to be submitted today on arXiv :).
- Semantics for operational consistency: quite ok, based on adjointness, LdR-LK-RR scheme as its special case (soon on arXiv).
- Open problem: make kinematics and dynamics canonical in ∞-dimensional case, using quantum Brègman entropies and noncommutative Orlicz spaces as tangent/cotangent spaces (project with Anna Jenčová)
- Open problem: local reconstruction of QM (projects with Berna Lessel [Jordan/riemannian part] and Wojtek Kamiński [Kähler part])
- Open problem: emergent space-times via noneq thermodynamics and D–K approach (project with Daniel Guariento)
- & ...
- Open problem: more explicit models of nonstandard composition of submodels and subdynamics (planned work with Karol Horodecki)
- Open problem: generalisation of adjointness semantics to monad-comonad localisation (planned work with Tobias Fritz)