Geometric properties of noncommutative Orlicz spaces with p-Amemiya norms in the context of metric and Vaĭnberg–Brègman projections

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Three main ideas underlying the results presented in this talk

1. Deep analogy between the "internal" characterisation of the properties of an Orlicz space $(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,s})$ by the properties of an Orlicz/Young function Φ and the "external" characterisation of the properties of a Banach space $(X,\|\cdot\|_X)$ by the properties of an Asplund Ψ_φ function.

2. The latter properties provide sufficient conditions to determine the properties of nonlinear projections generated by the Vaĭnberg–Brègman information D_{Ψ_φ}.

Orlicz space	Banach space
$\Phi(u)=\int_0^u \mathrm{d}t\,\phi(t)$	$\Psi_\varphi(x)=\int_0^{\|x\|_X}\mathrm{d}t\,\varphi(t)$
$\Phi^{\mathbf{Y}}(v):=\sup\{u\lvert v\rvert-\Phi(u)\,:\,u\geq 0\}$	$\Psi_\varphi^{\mathbf{F}}(y):=\sup\{[\![x,y]\!]_{X\times X^\star}-\Psi_\varphi(x)\,:\,x\in X\}$
$\underbrace{\Phi\in\ldots\iff\|\cdot\|_{\Phi,s}\in\ldots}_{\text{characterisation of norm geometry}}$	$\underbrace{\Psi_\varphi\in\ldots\iff\|\cdot\|_X\in\ldots}_{\text{characterisation of norm geometry}}$
$\vdots$	$\vdots$
$\underbrace{\Phi(u)+\Phi^{\mathbf{Y}}(v)-uv\geq 0}_{\Rightarrow\text{ Rogers–Hölder inequality}}$	$\underbrace{\Psi_\varphi(x)-\Psi_\varphi^{\mathbf{F}}(y)-[\![x,y]\!]_{X\times X^\star}\geq 0}_{\iff:\ \underbrace{D_{\Psi_\varphi}(x,y)\geq 0}_{\text{VKB information}}}$

3. The bijective relationship between the geometric properties of norms of commutative and noncommutative rearrangement invariant spaces, $(E(\mathcal{X},\mu),\|\cdot\|_{E(\mathcal{X},\mu)})\xleftrightarrow{(\cdot)^\tau}(E(\mathcal{N},\tau),\|\cdot\|_{E(\mathcal{N},\tau)})$, allows to extend the above analogy (and results that follow) to the noncommutative setting.

I. Spaces

Orlicz spaces with p-Amemiya norms:

- $(\mathcal{X},\mu)$:= a countably finite, complete measure space.
- we will denote: $\mathrm{type}(\mathcal{X},\mu) = \mathrm{I}_\infty^{\text{s.f.}}$ (resp., II_1; II_∞) := $(\mathcal{X},\mu)$ is atomic and infinite (resp., nonatomic and finite; nonatomic and infinite).
- $\Phi : \mathbb{R} \to [0,\infty]$ is Young := $\Phi(0)=0$, $\Phi \not\equiv 0$, $\exists u > 0$ $\Phi(u) < \infty$, Φ is convex on the domain of its finiteness, and left continuous on $]0,\infty[$. Zaanen'49
- $\Phi \in \mathrm{N}$:= Φ is Young, finite, $\Phi(x)=0$ iff $x=0$, $\lim_{x\to^+0} \frac{\Phi(x)}{x} = 0$, $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$.
- $\mathbb{R} \ni y \mapsto \Phi^{\mathsf{Y}}(y) := \sup\{x|y| - \Phi(x) \,:\, x \geq 0\} \in [0,\infty]$. Young'1912'26, Birnbaum–Orlicz'31
- a modular := $I_\Phi(f) := \int_{\mathcal{X}} \mu(\chi)\Phi(f(\chi))$.
- an Orlicz space := $L_\Phi(\mathcal{X},\mu) := \{x \in L_0(\mathcal{X},\mu) \,:\, \exists\lambda > 0\ I_\Phi(\lambda x) < \infty\}$.
- $\|x\|_\Phi^{\mathrm{O}} := \sup\{\int \mu|xy| \,:\, y \in L_{\Phi^{\mathsf{Y}}}(\mathcal{X},\mu),\ I_{\Phi^{\mathsf{Y}}}(y) \leq 1\}$. Orlicz'32
- $\|x\|_\Phi := \inf\{\lambda > 0 \,:\, I_\Phi(\frac{x}{\lambda}) \leq 1\}$. Morse–Transue'50, Nakano'50
- $\|x\|_{\Phi,p} := \inf_{k>0}\{\frac{1}{k} s_p(I_\Phi(kx))\}$, $s_\infty(t) := \lim_{p\to\infty}(1+t^p)^{1/p} = \max\{1,t\}$, $p=\infty$: Orlicz'61
  $s_{p\in[1,\infty[}(t) := (1+t^p)^{1/p}$. $p=1$: Amemiya'54; $p \in]1,\infty[$: Hudzik–Maligranda'00
- $\|\cdot\|_\Phi = \|\cdot\|_{\Phi,\infty}$. Orlicz'61
- $\|\cdot\|_\Phi^{\mathrm{O}} = \|\cdot\|_{\Phi,1}$. Shragin'67, Hudzik–Maligranda'00

Orlicz–Lorentz spaces and rearrangement invariant Banach spaces:

- chr_Z := the characteristic function of the set Z.
- a rearrangement :=
$L_0(\mathcal{X},\mu) \ni f \mapsto f^\mu(t) := \inf\{s \geq 0 \,:\, \int \mu\, \mathrm{chr}_{\{x\in\mathcal{X}\,:\,|f(x)|>s\}} \leq t\} \in [0,\infty]^{[0,\infty[}$.
Hardy–Littlewood'30.
- a Banach space $(E(\mathcal{X},\mu), \|\cdot\|_{E(\mathcal{X},\mu)})$ with $E(\mathcal{X},\mu) \subseteq L_0(\mathcal{X},\mu)$ is rearrangement invariant := $(f \in E(\mathcal{X},\mu), g \in L_0(\mathcal{X},\mu), |g|^\mu(t) \leq |f|^\mu(t)\ \forall t \in [0,\infty[) \ \Rightarrow$ $(g \in E(\mathcal{X},\mu), \|g\|_{E(\mathcal{X},\mu)} \leq \|f\|_{E(\mathcal{X},\mu)})$. (Lorentz'51,...) Semënov'64, Kreĭn–Petunin–Semënov'78

- $\mathrm{type}(\mathcal{X},\mu) = \mathrm{I}^{\text{s.f.}}_\infty \ \Rightarrow \ \begin{cases} \text{let } w : \mathbb{N} \to \mathbb{R}^+ \text{ nonincreasing}, w \not\equiv 0 \\ I_{\Phi,w}(x) := \sum_{i=1}^\infty w_i \Phi(x^\mu(i)). \end{cases}$
- $\mathrm{type}(\mathcal{X},\mu) \in \{\mathrm{II}_1, \mathrm{II}_\infty\} \ \Rightarrow$
$\begin{cases} \text{let } w : [0,\mu(\mathcal{X})[\to \mathbb{R}^+ \text{ nonincreasing, locally integrable w.r.t. } (\mathbb{R}^+, \mathrm{d}t), w \not\equiv 0 \\ I_{\Phi,w}(x) := \int_{[0,\mu(\mathcal{X})[} \mathrm{d}t\, w(t)\Phi(x^\mu(t)). \end{cases}$
- $\|x\|_{\Phi,w} := \inf\{\lambda > 0 \,:\, I_{\Phi,w}(\frac{x}{\lambda}) \leq 1\}$.
- $(L_{\Phi,w}(\mathcal{X},\mu), \|\cdot\|_{\Phi,w})$: Nakamura'70: $\Phi \in \mathrm{N}$, $\mathrm{type}(\mathcal{X},\mu) = \mathrm{II}_1$; Maligranda'85: Φ finite nondegenerate, $\mathrm{type}(\mathcal{X},\mu) \in \{\mathrm{II}_1, \mathrm{II}_\infty\}$; Yáo–Chéng–Sòng'92: Young $\Phi \in \triangle_2^0$, $\mathrm{type}(\mathcal{X},\mu) = \mathrm{I}^{\text{s.f.}}_\infty$; for $\Phi = (\cdot)^p$: Lorentz'50: $L_{p,q}$; Lorentz'51, Halperin'53: $L_{p,w}$
- $\|x\|^{\mathrm{A}}_{\Phi,w} := \inf_{k>0}\{1 + I_{\Phi,w}(kx)\}$. Wú–Rèn'99

- examples of r.i.B.s.: $(L_\Phi(\mathcal{X},\mu), \|\cdot\|_{\Phi,p})$, $(L_{\Phi,w}(\mathcal{X},\mu), \|\cdot\|_{\Phi,w})$, $(L_{\Phi,w}(\mathcal{X},\mu), \|\cdot\|^{\mathrm{A}}_{\Phi,w})$.

Noncommutative integration:

- $\mathcal{N}$:= a von Neumann algebra on a Hilbert space $\mathcal{H}$:= von Neumann'29, Sakai'56
  - a (noncommutative) algebra over $\mathbb{C}$ with unit $\mathbb{I}$,
  - with * operation s.t. $(xy)^* = y^*x^*$, $(x+y)^* = x^* + y^*$, $(x^*)^* = x$, $(\lambda x)^* = \lambda^* x^*$,
  - that is also a Banach space,
  - with $\cdot$, $+$, * continuous in the norm topology (implied by the condition $\|x^*x\| = \|x\|^2$),
  - s.t. there exists a Banach space $\mathcal{N}_\star$ satisfying the Banach space duality: $(\mathcal{N}_\star)^\star \cong \mathcal{N}$.
- $\tau : \mathcal{N}^+ \to [0,\infty]$:= a faithful, normal, semifinite trace on $\mathcal{N}$: von Neumann'29
  - $\tau(0) = 0$, $\tau(x+y) = \tau(x) + \tau(y)$, $\lambda > 0 \Rightarrow \tau(\lambda x) = \lambda\tau(x)$,
  - $\forall x \in \mathcal{N}^+ \exists \mathcal{N}^+ \setminus \{0\} \ni y \leq x \quad \tau(y) < +\infty$,
  - $\tau(\sup \mathcal{F}) = \sup_{x\in\mathcal{F}} \tau(x)$ $\forall$ directed filters $\mathcal{F} \subseteq \mathcal{N}^+$,
  - $\tau(x^*x) = 0 \Rightarrow x = 0 \forall x \in \mathcal{N}$,
  - $\tau(u^*xu) = \tau(x)$ $\forall$ unitary $u \in \mathcal{N}$.
- $\mathrm{type}(\mathcal{N},\tau) = \mathrm{I}_\infty^{\text{s.f.}} :\iff \mathcal{N} = \mathfrak{B}(\mathcal{H})$, $\mathcal{H}$ is separable, and $\dim(\mathcal{H}) = \infty$;
  $\mathrm{type}(\mathcal{N},\tau) = \mathrm{II}_1 :\iff \tau(\mathbb{I}) < \infty$ and $\mathcal{N} \subsetneq \mathfrak{B}(\mathcal{H})$;
  $\mathrm{type}(\mathcal{N},\tau) = \mathrm{II}_\infty :\iff \tau(\mathbb{I}) = \infty$ and $\mathcal{N} \subsetneq \mathfrak{B}(\mathcal{H})$. Murray–von Neumann'36
- Let $\mathcal{Y} \subseteq \mathcal{H}$. $x : \mathcal{Y} \to \mathcal{H}$ is called $(\mathcal{N},\tau)$-measurable iff
  0) x is closed, densely defined, linear;
  1) $u_x \in \mathcal{N}$, $\mathrm{E}^{|x|}(\lambda) \in \mathcal{N}$ $\forall \lambda \in \mathbb{R}^+$, where: $x = u_x|x|$, $|x| = \int_{\mathbb{R}^+} \mathrm{E}^{|x|}(\lambda)|\lambda|$ (i.e. a unique polar decomposition & a unique spectral decomposition);
  2) $\exists \lambda \geq 0$ $\tau(\mathbb{I} - \mathrm{E}^{|x|}(\lambda)) < \infty$. Nelson'74, Yeadon'75
- $\mathscr{M}(\mathcal{N},\tau)$:= a set of all $(\mathcal{N},\tau)$-measurable operators on $\mathcal{H}$.

Noncommutative rearrangement invariant Banach spaces:

- a noncommutative rearrangement :=
$\mathscr{M}(\mathcal{N},\tau) \ni x \mapsto x^{\tau}(t) := \inf\{s \geq 0 : \tau(\mathbb{I} - \mathrm{E}^{|x|}(s)) \leq t\} \in [0,\infty]^{[0,\infty[}$.
Grothendieck'55, Ovchinnikov'70, Yeadon'75
- a noncommutative r.i.B.s. := a Banach space $(E(\mathcal{N},\tau), \|\cdot\|_{E(\mathcal{N},\tau)})$ with
$E(\mathcal{N},\tau) \subseteq \mathscr{M}(\mathcal{N},\tau)$ s.t. $(f \in E(\mathcal{N},\tau),\ g \in \mathscr{M}(\mathcal{N},\tau),$
$|g|^{\tau}(t) \leq |f|^{\tau}(t)\ \forall t \in [0,\infty[) \Rightarrow (g \in E(\mathcal{N},\tau),\ \|g\|_{E(\mathcal{N},\tau)} \leq \|f\|_{E(\mathcal{N},\tau)})$.
Ovchinnikov'70, Yeadon'80, Medzhitov'87
- Theorem: Yeadon'80, Sukochev'86, Dodds–Dodds–de Pagter'89, Chilin–Sukochev'90, Kalton–Sukuchev'08
If $\mathrm{type}(\mathcal{N},\tau) = \mathrm{type}(\mathcal{X},\mu)$,
$(E(\mathcal{X},\mu), \|\cdot\|_{E(\mathcal{X},\mu)})$ is r.i.B.s.,
$E(\mathcal{N},\tau) := \{x \in \mathscr{M}(\mathcal{N},\tau) : x^{\tau} \in E(\mathcal{X},\mu)\}$,
$\|\cdot\|_{E(\mathcal{N},\tau)} := \|(\cdot)^{\tau}\|_{E(\mathcal{X},\mu)}$,
then: $(E(\mathcal{N},\tau), \|\cdot\|_{E(\mathcal{N},\tau)})$ is n.r.i.B.s.

Noncommutative Orlicz and Orlicz–Lorentz spaces (I):

- $I_\Phi(x) := \tau(\Phi(x)) = \begin{cases} \sum_{i=1}^{\infty} \Phi(x^\tau(i)) & : \mathrm{type}(\mathcal{N},\tau) = \mathrm{I}_\infty^{\mathrm{s.f.}} \\ \int_{[0,\tau(\mathbb{I})[} \mathrm{d}t\, \Phi(x^\tau(t)) & : \mathrm{type}(\mathcal{N},\tau) \in \{\mathrm{II}_1, \mathrm{II}_\infty\} \end{cases}$ $\forall x \in \mathscr{M}(\mathcal{N},\tau)$.
- $L_\Phi(\mathcal{N},\tau) := \{x \in \mathscr{M}(\mathcal{N},\tau) : \exists \lambda > 0\ I_\Phi(\lambda x) < \infty\}$.
- $\|x\|_\Phi := \inf\{\lambda > 0 : I_\Phi(\frac{x}{\lambda}) \leq 1\}$. $\mathrm{I}_\infty^{\mathrm{s.f.}}$: Rao'71, II_1: Tadzhibaev'86
- $\|x\|_\Phi^{\widetilde{\mathrm{O}}} := \sup\{|\tau(xy)| : y \in \mathscr{M}(\mathcal{N},\tau),\ \tau(\Phi^{\mathsf{Y}}(|y|)) \leq 1\}$. II_1: Muratov'78
- $\|x\|_\Phi^{\sim} := \sup\{|\tau(xy)| : L_{\Phi^{\mathsf{Y}}}(\mathcal{N},\tau),\ \|y\|_{\Phi^{\mathsf{Y}}}^{\widetilde{\mathrm{O}}} \leq 1\}$. II_1: Muratov'78
- $\|x\|_\Phi^{\mathrm{O}} := \sup\{\tau(|xy|) : y \in \mathscr{M}(\mathcal{N},\tau),\ \tau(\Phi^{\mathsf{Y}}(|y|)) \leq 1\}$. II_1, II_∞: Kunze'90
- $\|x\|_{\Phi,p} := \inf_{k>0}\{\frac{1}{k} s_p(I_\Phi(kx))\}$. RPK'23
- $(L_{\Phi,w}(\mathcal{N},\tau), \|\cdot\|_{\Phi,w})$: Hán'13
  ($L_{p,q}$; $\mathrm{I}_\infty^{\mathrm{s.f.}}$): Triebel'67, ($L_{p,q}$; $\{\mathrm{II}_1, \mathrm{II}_\infty\}$): Peetre–Sparr'75, Kosaki'81, $L_{1,w}$: Ovchinnikov'71, $L_{p,w}$: Ciach'83
- $(L_{\Phi,w}(\mathcal{N},\tau), \|\cdot\|_{\Phi,w}^{\mathrm{A}})$: RPK'23

Noncommutative Orlicz and Orlicz–Lorentz spaces (II):

- Let:
  - $\Phi \in \triangle_2^0 := \lim_{u \to {}^+0} \frac{\Phi(2u)}{\Phi(u)} < \infty$; Birnbaum–Orlicz'31
  - $\Phi \in \triangle_2^\infty := \limsup_{u \to \infty} \frac{\Phi(2u)}{\Phi(u)} < \infty$; Birnbaum–Orlicz'30
  - $\Phi \in \triangle_2 := \sup_{u > 0} \frac{\Phi(2u)}{\Phi(u)} < \infty$ ($\iff \Phi \in \triangle_2^0 \cap \triangle_2^\infty$); Burkill'28
  - $\Phi \in \triangle_2^{\mathrm{type}(\mathcal{N},\tau)} :=$
    $\Phi \in \triangle_2^0$ (resp., $\triangle_2^\infty$; $\triangle_2$) if $\mathrm{type}(\mathcal{N},\tau) = \mathrm{I}_\infty^{\text{s.f.}}$ (resp., II_1; II_∞).
- Theorem: RPK'23
  1) ($\Phi, \Phi^\mathsf{Y} \in \triangle_2^\infty$, $\mathrm{type}(\mathcal{N},\tau) = \mathrm{II}_1$, $\mathcal{N}$ has a commutative von Neumann subalgebra) $\Rightarrow$ ($\|\cdot\|_\Phi^{\widetilde{\mathrm{O}}} = \|\cdot\|_\Phi^{\mathrm{O}}$, $(L_\Phi(\mathcal{N},\tau), \|\cdot\|_\Phi) \cong (L_\Phi(\mathcal{N},\tau), \|\cdot\|_\Phi^\sim)$).
  2) $\Phi \in \triangle_2^{\mathrm{type}(\mathcal{N},\tau)} \Rightarrow (L_\Phi(\mathcal{N},\tau), \|\cdot\|_\Phi)^\star \cong (L_{\Phi^\mathsf{Y}}(\mathcal{N},\tau), \|\cdot\|_{\Phi^\mathsf{Y}}^{\mathrm{O}})$.
  3) $\Phi, \Phi^\mathsf{Y} \in \triangle_2^{\mathrm{type}(\mathcal{N},\tau)} \Rightarrow (L_\Phi(\mathcal{N},\tau), \|\cdot\|_\Phi^{\mathrm{O}})^\star \cong (L_{\Phi^\mathsf{Y}}(\mathcal{N},\tau), \|\cdot\|_{\Phi^\mathsf{Y}})$.
  4) $\Phi^\mathsf{Y} \in \triangle_2^{\mathrm{type}(\mathcal{N},\tau)} \Rightarrow \|\cdot\|_\Phi^{\mathrm{O}} = \|\cdot\|_{\Phi,1}$.
  5) ($\Phi \in \triangle_2^{\mathrm{type}(\mathcal{N},\tau)}$, Φ finite, Φ not linear on $[0,\infty[$, $\gamma \in]0,1[$) $\Rightarrow$
     $(L_\Phi(\mathcal{N},\tau), \|\cdot\|_{\Phi,1/\gamma})^\star \cong (L_{\Phi^\mathsf{Y}}(\mathcal{N},\tau), \|\cdot\|_{\Phi^\mathsf{Y},1/(1-\gamma)})$.

II. Geometry of spaces

Geometric properties of Banach spaces $(X, \|\cdot\|_X)$:

- $(X, \|\cdot\|_X) \in \mathbf{R} := X \ni x \mapsto [\![x, \cdot]\!]_{X \times X^\star} \in X^{\star\star}$ is an isometric isomorphism. Hahn'27
- $(X, \|\cdot\|_X) \in \mathbf{SC} := \forall x, y \in S(X, \|\cdot\|_X)$
  $x \neq y \Rightarrow \|x + y\|_X < 2$. Kreĭn'38; equiv.: Fréchet'25, Clarkson'36
- $(X, \|\cdot\|_X) \in \mathbf{LUC} := \forall \varepsilon_1 > 0\ \forall x \in S(X, \|\cdot\|_X)\ \exists \varepsilon_2 > 0\ \forall y \in S(X, \|\cdot\|_X)$
  $\|x - y\|_X \geq \varepsilon_1 \Rightarrow \|x + y\|_X \leq 2(1 - \varepsilon_2)$. Lovaglia'55
- $(X, \|\cdot\|_X) \in \mathbf{UC} := \forall \varepsilon_1 > 0\ \forall \varepsilon_2 > 0\ \forall x, y \in S(X, \|\cdot\|_X)$
  $\|x - y\|_X \geq \varepsilon_1 \Rightarrow \|x + y\|_X \leq 2(1 - \varepsilon_2)$. Clarkson'36
- $(X, \|\cdot\|_X) \in \mathbf{RRS} := \forall \{x_n \in X : n \in \mathbb{N}\}$ (x_n is convergent to $x \in X$ in weak topology, $\lim_{n\to\infty} \|x_n\|_X = \|x\|_X$) $\Rightarrow \lim_{n\to\infty} \|x_n - x\|_X = 0$.
  l_p: Radon'1913, $L_p(\mathcal{X}, \mu)$: Riesz'29, $(X, \|\cdot\|_X)$: Shmul'yan'39
- $(X, \|\cdot\|_X) \in \mathbf{G} := \|\cdot\|_X$ is Gateaux differentiable on $S(X, \|\cdot\|_X)$. Mazur'33
- $(X, \|\cdot\|_X) \in \mathbf{F} := \|\cdot\|_X$ is Fréchet differentiable on $S(X, \|\cdot\|_X)$. Mazur'33
- $(X, \|\cdot\|_X) \in \mathbf{UF} := \forall \varepsilon_1 > 0\ \exists \varepsilon_2\ \forall x, y \in S(X, \|\cdot\|_X)$
  $\|x - y\|_X \leq \varepsilon_1 \Rightarrow \|x + y\|_X \geq 2(1 - \varepsilon_2 \|x - y\|_X)$. Day'44; equiv.: Shmul'yan'40, Nakano'51
- $(X, \|\cdot\|_X) \in \mathbf{UC}_{r \in [2,\infty[} := \exists \lambda > 0\ \forall x, y \in X$
  $\|x + y\|_X^r + \|x - y\|_X^r \geq 2(\|x\|_X^r + \left\|\lambda^{-1} y\right\|_X^r)$.
  Assouad'75; equiv.: Woyczyński'75, r=2: Lindenstrauss'63
- $(X, \|\cdot\|_X) \in \mathbf{UF}_{r \in]1,2]} := \exists \lambda > 0\ \forall x, y \in X\ \ \|x + y\|_X^r + \|x - y\|_X^r \leq 2(\|x\|_X^r + \|\lambda y\|_X^r)$.
  Assouad'75; equiv.: Hoffmann-Jørgensen'74, $p = 2$: Lindenstrauss'63

Characterisation of geometry of $\|\cdot\|_{\Phi,p}$ by the properties of Φ:

Each property has a priori $3\times 3=9$ different cases of $(\mathrm{type}(\mathcal{X},\mu),p)$:

- $\|\cdot\|_{\Phi,p}\in$ **SC**: $(\mathrm{I}^{\text{s.f.}}_\infty,1)$: Cuī–Hudzik–Nowak–Płuciennik'99; $(\mathrm{I}^{\text{s.f.}}_\infty,]1,\infty[)$: Lǐ–Cuī'22; $(\mathrm{I}^{\text{s.f.}}_\infty,\infty)$: Kamińska'81; $(\mathrm{II}_{1,\infty},[1,\infty])$: Cuī–Duàn–Hudzik–Wisła'08.
- $\|\cdot\|_{\Phi,p}\in$ **LUC**: $(\mathrm{I}^{\text{s.f.}}_\infty,1)$: Cuī–Hudzik–Nowak–Płuciennik'99; $(\mathrm{I}^{\text{s.f.}}_\infty,]1,\infty[)$: Chéng–Duàn–Zuǒ'15 $\boxed{\Phi\in\mathrm{N}}$; $(\{\mathrm{I}^{\text{s.f.}}_\infty,\mathrm{II}_1,\mathrm{II}_\infty\},\infty)$: Kamińska'84; $(\mathrm{II}_1,1)$: Chén'86 $\boxed{\Phi\in\mathrm{N}}$; $(\mathrm{II}_1,]1,\infty[)$: Duàn–Zuǒ–Cuī'14 $\boxed{\Phi\in\mathrm{N}}$; $(\mathrm{II}_\infty,1)$: Níng'10 $\boxed{\Phi\in\mathrm{N}}$; $(\mathrm{II}_\infty,]1,\infty[)$: $\boxed{??!}$.
- $\|\cdot\|_{\Phi,p}\in$ **RRS**: $(\mathrm{I}^{\text{s.f.}}_\infty,1)$: Cuī–Hudzik–Nowak–Płuciennik'99; $(\mathrm{I}^{\text{s.f.}}_\infty,]1,\infty[)$: Zhào–Cuī'21; $(\mathrm{I}^{\text{s.f.}}_\infty,\infty)$: Hudzik–Pallaschke'97; $(\mathrm{II}_1,1)$: Cuī–Zhào'22; $(\mathrm{II}_1,]1,\infty[)$: $\boxed{??!}$; (II_1,∞): Medzhitov–Sukochev'92, Wáng–Cuī'98; $(\mathrm{II}_\infty,[1,\infty[)$: $\boxed{??!}$; $(\mathrm{II}_\infty,\infty)$: Wáng–Cuī–Zhāng'98.
- $\|\cdot\|_{\Phi,p}\in$ **UC**: $(\mathrm{I}^{\text{s.f.}}_\infty,1)$: Táo'86 $\boxed{\Phi\in\mathrm{N}}$; $(\mathrm{I}^{\text{s.f.}}_\infty,]1,\infty[)$: $\boxed{??!}$; $(\{\mathrm{I}^{\text{s.f.}}_\infty,\mathrm{II}_1,\mathrm{II}_\infty\},\infty)$: Kamińska'82; $(\mathrm{II}_1,\mathrm{II}_\infty\},1)$: Milnes'57; $(\{\mathrm{II}_1,\mathrm{II}_\infty\},]1,\infty[)$: Kaczmarek'18.
- $\|\cdot\|_{\Phi,p}\in$ **G**: $(\mathrm{I}^{\text{s.f.}}_\infty,1)$: Cuī–Hudzik–Nowak–Płuciennik'99 ∪ Lǐ–Cuī–Wisła'23; $(\mathrm{I}^{\text{s.f.}}_\infty,]1,\infty[)$: Lǐ–Cuī–Wisła'23; $(\{\mathrm{I}^{\text{s.f.}}_\infty,\mathrm{II}_1,\mathrm{II}_\infty\},\infty)$: Grząślewicz–Hudzik'92; $(\{\mathrm{II}_1,\mathrm{II}_\infty\},1)$: Vigelis'11; $(\mathrm{II}_1,]1,\infty[)$: Lǐ–Cuī–Wisła'21; $(\mathrm{II}_\infty,]1,\infty[)$: $\boxed{??!}$.
- $\|\cdot\|_{\Phi,p}\in$ **R** & $\|\cdot\|_{\Phi,p}\in$ **OC**: $(\{\mathrm{I}^{\text{s.f.}}_\infty,\mathrm{II}_1,\mathrm{II}_\infty\},\{1,\infty\})$: Luxemburg'55; $(\{\mathrm{I}^{\text{s.f.}}_\infty,\mathrm{II}_1,\mathrm{II}_\infty\},]1,\infty[)$: follows from equivalence of $\|\cdot\|_{\Phi,p}$ and $\|\cdot\|_\Phi$.

Known results on UC_s and UF_r of $\|\cdot\|_\Phi$:

- Φ finite, nondegenerate, $\Phi(1)=1$, $u \mapsto \Phi(u^{1/2})$ convex, $\Phi \in \triangle_2$ (resp., $\triangle_2^0$) and $\frac{1}{\gamma} := \sup_{t>0} \frac{t\Phi'_+(t)}{\Phi(t)}$ (resp., $\sup_{t\in\,]0,1]} \frac{t\Phi'_+(t)}{\Phi(t)}$) for $\mathrm{type}(\mathcal{X},\mu) \in \{\mathrm{II}_1, \mathrm{II}_\infty\}$ (resp., $\mathrm{I}_\infty^{\text{s.f.}}$)
  $\Rightarrow$ $(L_\Phi(\mathcal{X},\mu), \|\cdot\|_\Phi) \in \mathbf{UC}_{1/\gamma}$. Hudzik'91
- Φ is an inverse of $u \mapsto (\Phi_0^{-1}(u))^{1-2\gamma} u^\gamma$, $\gamma \in\,]0, \frac{1}{2}]$, $\Phi_0 \in \mathrm{N}$, $\|\cdot\| \in \{\|\cdot\|_\Phi, \|\cdot\|_\Phi^{\mathrm{O}}\}$
  $\Rightarrow$ $(L_\Phi(\mathcal{X},\mu), \|\cdot\|) \in \mathbf{UC}_{1/\gamma} \cap \mathbf{UF}_{1/(1-\gamma)}$. Rèn'97
- Φ finite, nondegenerate, $\Phi(1)=1$, $1 < r \leq 2 \leq s < \infty$, $u \mapsto \Phi(u^{1/r})$ convex, $u \mapsto \Phi(u^{1/s})$ concave, $\mathrm{type}(\mathcal{X},\mu) \in \{\mathrm{II}_1, \mathrm{II}_\infty\}$
  $\Rightarrow$ $(L_\Phi(\mathcal{X},\mu), \|\cdot\|_\Phi) \in \mathbf{UC}_s \cap \mathbf{UF}_r$. Hăo–Kamińska–Tomczak-Jaegermann'06

- no characterisation results (so far).
- no results for $\|\cdot\|_{\Phi,p\in\,]1,\infty[}$ (so far).

Charact. of geometry of $\{\|\cdot\|_{\Phi,w}, \|\cdot\|^{\mathrm{A}}_{\Phi,w}\}$ by the properties of $\{\Phi, w\}$:

- $\|\cdot\|_{\Phi,w} \in$ **SC**: $\mathrm{I}^{\text{s.f.}}_{\infty}$: Cerdà–Hudzik–Kamińska–Mastyło'98; $\{\mathrm{II}_1, \mathrm{II}_\infty\}$: Kamińska'90.
- $\|\cdot\|^{\mathrm{A}}_{\Phi,w} \in$ **SC**: $\mathrm{I}^{\text{s.f.}}_{\infty}$: Níng'10 $\boxed{\Phi \in \mathrm{N}}$; II_1: Wú–Rèn'99 $\boxed{\Phi \in \mathrm{N}}$; II_∞: Chén'09 $\boxed{\Phi \in \mathrm{N}}$.
- $\|\cdot\|_{\Phi,w} \in$ **RRS**: $\mathrm{I}^{\text{s.f.}}_{\infty}$: Hudzik–Kowalewski–Lewicki'06; $\{\mathrm{II}_1, \mathrm{II}_\infty\}$: $\boxed{??!}$.
- $\|\cdot\|^{\mathrm{A}}_{\Phi,w} \in$ **RRS**: $\mathrm{I}^{\text{s.f.}}_{\infty}$: Cuī–Foralewski–Hudzik–Kaczmarek'21; $\{\mathrm{II}_1, \mathrm{II}_\infty\}$: $\boxed{??!}$.
- $\|\cdot\|_{\Phi,w} \in$ **LUC**: $\mathrm{I}^{\text{s.f.}}_{\infty}$: Cerdà–Hudzik–Kamińska–Mastyło'98; II_1: Hudzik–Kamińska–Mastyło'97, Wú–Rèn'97; II_∞: Hudzik–Kamińska–Mastyło'97.
- $\|\cdot\|^{\mathrm{A}}_{\Phi,w} \in$ **LUC**: $\mathrm{I}^{\text{s.f.}}_{\infty}$: Mǎ'11 $\boxed{\Phi \in \mathrm{N}}$; II_1: $\boxed{??!}$; II_∞: Níng'10 $\boxed{\Phi \in \mathrm{N}}$.
- $\|\cdot\|_{\Phi,w} \in$ **UC**: $\mathrm{I}^{\text{s.f.}}_{\infty}$: Cerdà–Hudzik–Kamińska–Mastyło'98; $\{\mathrm{II}_1, \mathrm{II}_\infty\}$: Kamińska'91.
- $\|\cdot\|^{\mathrm{A}}_{\Phi,w} \in$ **UC**: $\mathrm{I}^{\text{s.f.}}_{\infty}$: Níng'10 $\boxed{\Phi \in \mathrm{N}}$; II_1: Chén'09 $\boxed{\Phi \in \mathrm{N}}$; II_∞: Wáng–Chén'12 $\boxed{\Phi \in \mathrm{N}}$.
- $\|\cdot\|_{\Phi,w} \in$ **R**: $\mathrm{I}^{\text{s.f.}}_{\infty}$: Yáo–Chéng–Sòng'92; $\{\mathrm{II}_1, \mathrm{II}_\infty\}$: Lín–Sūn'95.
- $\|\cdot\|_{\Phi,w} \in$ **OC**: $\mathrm{I}^{\text{s.f.}}_{\infty}$: Yáo–Chéng–Sòng'92.
- $\|\cdot\|^{\mathrm{A}}_{\Phi,w} \in$ **OC**: $\{\mathrm{II}_1, \mathrm{II}_\infty\}$: Foralewski–Kończak'23.
- the rest of **R** and **OC** cases follows from the equivalence of $\|\cdot\|_{\Phi,w}$ with $\|\cdot\|^{\mathrm{A}}_{\Phi,w}$.

Transfer of geometric properties between r.i.B.s. and n.r.i.B.s.:

- Let $\mathrm{type}(\mathcal{N},\tau)=\mathrm{type}(\mathcal{X},\mu)$, $(E,\|\cdot\|)_{(\mathcal{X},\mu)}:=(E(\mathcal{X},\mu),\|\cdot\|_{E(\mathcal{X},\mu)})$ be r.i.B.s., $(E,\|\cdot\|)_{(\mathcal{N},\tau)}:=(\{x\in\mathscr{M}(\mathcal{N},\tau)\,:\,x^{\tau}\in E(\mathcal{X},\mu)\},\|(\cdot)^{\tau}\|_{E(\mathcal{X},\mu)})$.
- $(E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{SC}\iff(E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{SC}$. Arazy'81, Chilin–Krygin–Sukochev'92, Czerwińska–Kamińska'17
- $(E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{RRS}\iff(E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{RRS}$. Arazy'81, Chilin–Dodds–Sukochev'97
- $(E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{LUC}\iff(E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{LUC}$. Krygin–Sukochev–Chilin'91, Chilin–Krygin–Sukochev'92
- $(E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{UC}\iff(E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{UC}$. Krygin–Sukochev–Chilin'91, Chilin–Krygin–Sukochev'92
- $(\mathrm{type}(\mathcal{N},\tau)=\mathrm{I}_{\infty}^{\mathrm{s.f.}},\ (E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{OC})$
  $\Rightarrow((E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{G}\iff(E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{G})$; Arazy'81
  $(\mathrm{type}(\mathcal{N},\tau)\in\{\mathrm{II}_1,\mathrm{II}_\infty\},\ (E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{OC},\ \lim_{t\to\infty}x^{\mu}(t)=0\ \forall x\in(E(\mathcal{X},\mu))^{\times})$
  $\Rightarrow((E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{G}\iff(E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{G})$. Czerwińska–Kamińska–Kubiak'12
- $(\mathrm{type}(\mathcal{N},\tau)=\mathrm{I}_{\infty}^{\mathrm{s.f.}},\ (E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{R})\ \boxed{\Rightarrow}\ (E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{R}$; Arazy'81
  $\mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_1\Rightarrow((E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{R}\iff(E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{R})$; Sukochev'87
  $(\mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_\infty,\ (E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{OC})\Rightarrow$
  $((E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{R}\iff(E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{R})$. Dodds–Dodds'95
- $(E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{UC}_q\ \boxed{\Rightarrow}\ (E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{UC}_q$. Tomczak-Jaegermann'84, Xŭ'89
- $(E,\|\cdot\|)_{(\mathcal{X},\mu)}\in\mathbf{UF}_q\ \boxed{\Rightarrow}\ (E,\|\cdot\|)_{(\mathcal{N},\tau)}\in\mathbf{UF}_q$. Tomczak-Jaegermann'84, Xŭ'89

Ψ_φ:

- $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a gauge $:=$ φ is strictly increasing, continuous, $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$. Matuszewska'60 (without strictly increasing), Beurling–Livingston'62
- For any Banach space $(X, \|\cdot\|_X)$, a duality map $:=$ $j_\varphi : X \ni x \mapsto \{y \in X^\star : [\![x, y]\!]_{X\times X^\star} = \|x\|_X \|y\|_{X^\star}, \|y\|_{X^\star} = \varphi(\|x\|_X)\} \subseteq X^\star$. Beurling–Livingston'62
- a subdifferential $:=$ $\partial\Psi(x) := \{y \in X^\star : \Psi(z) - \Psi(x) \geq [\![z - x, y]\!]_{X\times X^\star} \ \forall z \in X\}$. Rockafellar'63, Moreau'63, Minty'64
- if Ψ is Gateaux differentiable at x, then $\partial\Psi(x) = \{\mathfrak{D}^{\mathrm{G}}\Psi(x)\}$, where $\mathfrak{D}^{\mathrm{G}}\Psi(x) :=$ a Gateaux derivative of Ψ at x.
- $X \ni x \mapsto \Psi_\varphi(x) := \int_0^{\|x\|_X} \mathrm{d}t\, \varphi(t) \in \mathbb{R}^+$. Asplund'67
- Theorem: Asplund'67
  $j_\varphi(x) = \partial\Psi_\varphi(x) \ \forall x \in X \setminus \{0\}$,
  $j_\varphi(0) = 0$.

Charact. of geometry of Banach spaces by the geometry of Ψ_φ:

- $(X, \|\cdot\|_X) \in$ **SC** $\iff$ Ψ_φ is strictly convex on X. Zălinescu'83
- $(X, \|\cdot\|_X) \in$ **LUC** $\iff$ Ψ_φ is uniformly convex at each $x \in X$. Zălinescu'83
- $(X, \|\cdot\|_X) \in$ **UC** $\iff$ Ψ_φ is uniformly convex on bounded subsets of X. Zălinescu'83
- $(X, \|\cdot\|_X) \in$ **G** $\iff$ Ψ_φ is Gateaux differentiable on X. Zălinescu'83
- $(X, \|\cdot\|_X) \in$ **F** $\iff$ Ψ_φ is Fréchet differentiable on X. Zălinescu'02
- $(X, \|\cdot\|_X) \in$ **UF** $\iff$ Ψ_φ is uniformly Fréchet diff. on bounded subsets of X. Zălinescu'02
- $(X, \|\cdot\|_X) \in$ **UC**$_r$ $\iff$ Ψ_φ with $\varphi(t) = rt^{r-1}$ is uniformly convex on X. Xú'91
- $(X, \|\cdot\|_X) \in$ **UF**$_r$ $\iff$ Ψ_φ with $\varphi(t) = rt^{r-1}$ is uniformly Fréchet diff. on X. Shioji'94
- $(X, \|\cdot\|_X) \in$ **R** $\iff$ $\partial\Psi_\varphi$ is is surjective on X. de Figueiredo'67

Characterisation of the geometry of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$ (I):

Theorem: RPK'21–'23

If

$\mathrm{type}(\mathcal{X},\mu)=\mathrm{type}(\mathcal{N},\tau)\in\{\mathrm{I}_\infty^{\text{s.f.}},\mathrm{II}_1,\mathrm{II}_\infty\}$,

Φ is semi-Young (:= Young without left-semicontinuity on $]0,\infty[$),

φ is a gauge,

$\Psi_\varphi : L_\Phi(\mathcal{N},\tau)\to\mathbb{R}$,

$a_\Phi := \sup\{u\geq 0 \,:\, \Phi(u)=0\}$,

$b_\Phi := \sup\{u\geq 0 \,:\, \Phi(u)<\infty\}$,

$\Phi^\vee(t) := \sup\{u\geq 0 \,:\, \Phi(u)\leq t\}$,

Φ'_+ (resp., Φ'_-) := right (resp., left) derivative of Φ,

$\Phi\in\mathrm{C}^1(I) := \Phi'_+$ is continuous on $I\subseteq\mathbb{R}$,

$\Phi\in\mathrm{SC}(I) := (u\neq v\Rightarrow\Phi(\frac{u+v}{2})<\frac{1}{2}(\Phi(u)+\Phi(v)))\ \forall u,v\in I\subseteq\mathbb{R}$,

$\Phi\in\mathrm{UC}(I) := \forall a\in]0,1[\ \exists\delta(a)\in]0,1[\ \forall u\in I\subseteq\mathbb{R}$

$\quad\Phi\left(\frac{u+au}{2}\right)\leq\frac{1}{2}(1-\delta(a))(\Phi(u)+\Phi(au))$,

$\mathrm{UC} := \mathrm{UC}(\mathbb{R})$,

$\mathrm{UC}^\infty := \mathrm{UC}([u_0,\infty[)$ for some $u_0>0$,

$\varpi_\Phi(\lambda) := \sup\{t>0 \,:\, \Phi^{\mathsf{Y}}(\Phi'_+(t))\leq\lambda\}$,

$\varpi_{\Phi,p}(\lambda) := \sup\{t>0 \,:\, 2^p(\Phi(t))^{p-1}\Phi^{\mathsf{Y}}(\Phi'_+(t))\leq\lambda\}$,

$p=1^{\mathrm{O}} := \{$either $\|\cdot\|^{\mathrm{O}}_\Phi$ is used instead of $\|\cdot\|_{\Phi,1}$, or $\Phi^{\mathsf{Y}}\in\triangle_2^{\mathrm{type}(\mathcal{N},\tau)}\}$,

then:

1) if $p\in\{1^{\mathrm{O}}\}\cup]1,\infty]$, and the following conditions are jointly satisfied:
   a) Φ is Young if not $((\mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}}$ and $p=1^{\mathrm{O}})$ or $(\mathrm{type}(\mathcal{N},\tau)\in\{\mathsf{II}_1,\mathsf{II}_\infty\}$ and $p=\infty))$;
   b) (Φ is Young or $\lim_{x\to\infty}\frac{\Phi(x)}{x}=\infty$) if $(\mathrm{type}(\mathcal{N},\tau)\in\{\mathsf{II}_1,\mathsf{II}_\infty\}$ and $p=\infty)$;

   then:

   Ψ_φ is strictly convex $\iff$
   $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})\in\mathbf{SC}$ $\iff$
   $(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,p})\in\mathbf{SC}$ $\iff$

$$
\begin{cases}
\Phi\in\mathrm{SC}([0,\varpi_\Phi(1)]),\ a_\Phi=0,\ \exists u>0\ \Phi^{\mathbf{Y}}(\Phi'_+(u))\geq\frac{1}{2} & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p=1^{\mathrm{O}}\\
\Phi\in\triangle_2^0\cap\mathrm{SC}([0,\Phi^\vee(\frac{1}{2})]),\ \exists u>0\ \Phi(u)=1 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p=\infty\\
\Phi\in\mathrm{SC}([0,\varpi_{\Phi,p}(1)]),\ a_\Phi=0,\ \exists u>0\ \Phi^{\mathbf{Y}}(\Phi'_+(u))>\frac{1}{2} & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p\in]1,\infty[\\
\Phi\in\mathrm{SC}([0,b_\Phi[),\ \lim_{u\to\infty}\Phi^{\mathbf{Y}}(\Phi'_+(u))=\infty, & \\
\quad\sup\{u\geq0:\ \Phi^{\mathbf{Y}}(\Phi'_+(u))\mu(\mathcal{X})<1\}\leq b_\Phi & :\ \mathrm{type}(\mathcal{N},\tau)\in\{\mathsf{II}_1,\mathsf{II}_\infty\},\ p=1^{\mathrm{O}}\\
\Phi\in\triangle_2^\infty\cap\mathrm{SC}([0,b_\Phi[) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_1,\ p=\infty\\
\Phi\in\mathrm{SC}([0,b_\Phi[) & :\ \mathrm{type}(\mathcal{N},\tau)\in\{\mathsf{II}_1,\mathsf{II}_\infty\},\ p\in]1,\infty[\\
\Phi\in\triangle_2\cap\mathrm{SC}(\mathbb{R}) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_\infty,\ p=\infty;
\end{cases}
$$

Characterisation of the geometry of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$ (III):

2) if $p\in[1,\infty]$ and the following conditions are jointly satisfied:
   a) $\Phi\in\mathrm{N}$ if ($p=1$ and $\mathrm{type}(\mathcal{N},\tau)\in\{\mathsf{II}_1,\ \mathsf{II}_\infty\}$) or ($p\in]1,\infty[$ and $\mathrm{type}(\mathcal{N},\tau)\in\{\mathsf{I}_\infty^{\text{s.f.}},\ \mathsf{II}_1\}$);
   b) $b_\Phi=\infty$ if $p=\infty$;
   c) ($\mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_\infty$ and $p\in]1,\infty[$) does not hold;

then:

Ψ_φ is uniformly convex at each $x\in L_\Phi(\mathcal{N},\tau)$ $\iff$
$(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})\in\mathbf{LUC}$ $\iff$
$(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,p})\in\mathbf{LUC}$ $\iff$

$$\begin{cases}
\Phi\in\triangle_2^0\cap\mathrm{SC}([0,\varpi_\Phi(1)]),\ \Phi^{\mathsf{Y}}\in\triangle_2^0, & \\
\quad\exists u>0\ \Phi^{\mathsf{Y}}(\Phi'_+(u))\geq\frac{1}{2} & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p=1\\
\Phi\in\triangle_2^0 \text{ and either } \Phi\in\mathrm{SC}([0,\Phi^\vee(1)]) & \\
\quad\text{or } (\Phi\in\mathrm{SC}([0,\Phi^\vee(\frac{1}{2})]) \text{ and } \Phi^{\mathsf{Y}}\in\triangle_2^0) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p=\infty\\
\Phi\in\triangle_2^0\cap\mathrm{SC}([0,\varpi_{\Phi,p}(1)]),\ \Phi^{\mathsf{Y}}\in\triangle_2^0 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p\in]1,\infty[\\
\Phi\in\triangle_2^\infty\cap\mathrm{SC}(\mathbb{R}),\ \Phi^{\mathsf{Y}}\in\triangle_2^\infty & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_1,\ p\in[1,\infty[\\
\Phi\in\triangle_2^\infty\cap\mathrm{SC}(\mathbb{R}) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_1,\ p=\infty\\
\Phi\in\triangle_2\cap\mathrm{SC}(\mathbb{R}),\ \Phi^{\mathsf{Y}}\in\triangle_2 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_\infty,\ p=1\\
\Phi\in\triangle_2\cap\mathrm{SC}(\mathbb{R}) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_\infty,\ p=\infty;
\end{cases}$$

Characterisation of the geometry of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$ (IV):

3) if $p\in[1,\infty]$ and the following conditions are jointly satisfied:
   a) $\Phi\in\mathrm{N}$ if $(\mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}}$ and $p=1)$;
   b) $b_\Phi=\infty$ if $(\mathrm{type}(\mathcal{N},\tau)\in\{\mathsf{I}_\infty^{\text{s.f.}},\mathsf{II}_1\}$ and $p=\infty)$;
   c) Φ is Young if $(\mathrm{type}(\mathcal{N},\tau)\in\{\mathsf{II}_1,\mathsf{II}_\infty\}$ and $p\in[1,\infty[)$;
   d) $(\mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}}$ and $p\in]1,\infty[)$ does not hold;

   then:

   Ψ_φ is uniformly convex on bounded subsets of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$ $\iff$
   $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})\in\mathbf{UC}$ $\iff$
   $(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,p})\in\mathbf{UC}$ $\iff$

$$\begin{cases}\Phi\in\triangle_2^0\cap\mathrm{UC}([0,\varpi_\Phi(1)]) & : \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p=1\\ \Phi\in\triangle_2^0\cap\mathrm{UC}([0,\Phi^\vee(\frac{1}{2})]) & : \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p=\infty\\ \Phi\in\triangle_2^\infty\cap\mathrm{UC}^\infty\cap\mathrm{SC}(\mathbb{R}) & : \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_1,\ p\in[1,\infty]\\ \Phi\in\triangle_2\cap\mathrm{UC} & : \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_\infty,\ p\in[1,\infty];\end{cases}$$

Characterisation of the geometry of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$ (V):

4) if $p\in\{1^{\mathrm{O}}\}\cup]1,\infty]$ and the following conditions are jointly satisfied:

a) ($b_\Phi=\infty$ and $\lim_{u\to\infty}\frac{\Phi(u)}{u}=\infty$) if either ($\mathrm{type}(\mathcal{N},\tau)\in\{\mathrm{I}^{\text{s.f.}}_\infty,\mathrm{II}_1\}$ and $p\in]1,\infty[$) or ($\mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_1$ and $p=1^{\mathrm{O}}$);

b) (Φ is Young, $a_\Phi=0$, $\exists u>0$ $\Phi(u)=1$) if ($\mathrm{type}(\mathcal{N},\tau)=\mathrm{I}^{\text{s.f.}}_\infty$ and $p=\infty$);

c) $\Phi\in\mathrm{N}$ if ($\mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_1$ and $p=\infty$);

d) $b_\Phi=\infty$ if ($\mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_\infty$ and $p\in\{1^{\mathrm{O}},\infty\}$);

e) ($\mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_\infty$ and $p\in]1,\infty[$) does not hold;

then:

$$
\begin{aligned}
&(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})\in\mathbf{RRS}\iff\\
&(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,p})\in\mathbf{RRS}\iff\\
&\begin{cases}
\Phi\in\triangle_2^0 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{I}^{\text{s.f.}}_\infty,\ p\in\{1^{\mathrm{O}}\}\cup]1,\infty]\\
\Phi\in\triangle_2^\infty\cap\mathrm{SC}(\mathbb{R}) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_1,\ p\in\{1^{\mathrm{O}}\}\cup]1,\infty]\\
\Phi\in\triangle_2\cap\mathrm{SC}(\mathbb{R}) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_\infty,\ p\in\{1^{\mathrm{O}},\infty\};
\end{cases}
\end{aligned}
$$

Characterisation of the geometry of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$ (VI):

5) if $p\in\{1^{\mathrm{O}}\}\cup]1,\infty]$ for $\mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}}$, or $p\in[1,\infty]$ for $\mathrm{type}(\mathcal{N},\tau)\in\{\mathsf{II}_1,\ \mathsf{II}_\infty\}$, and the following conditions are jointly satisfied:

- a) $(b_\Phi<\infty,\ \Phi(b_\Phi)<\infty)\Rightarrow\Phi'_-(b_\Phi)<\infty$ if $(\mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p=\infty)$;
- b) $\Phi^{\mathbf{Y}}\in\triangle_2^\infty$ (resp., $\triangle_2$) if $\mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_1$ (resp., II_∞);
- c) Φ is lower semicontinuous if $(\mathrm{type}(\mathcal{N},\tau)\in\{\mathsf{II}_1,\ \mathsf{II}_\infty\}$ and $p=1^{\mathrm{O}})$ and it is Young in all other cases;
- d) $(\mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_\infty$ and $p\in]1,\infty[)$ does not hold;

then:

Ψ_φ is Gateaux differentiable on $L_\Phi(\mathcal{N},\tau)\iff$
$(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})\in\mathbf{G}\iff(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,p})\in\mathbf{G}\iff$

$$\begin{cases}\Phi\in\triangle_2^0\cap\mathrm{C}^1([0,\varpi_\Phi(\tfrac{1}{2})[),\ \lim_{t\to^+0}\frac{\Phi(t)}{t}=0 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p=1^{\mathrm{O}}\\ \Phi\in\triangle_2^0\cap\mathrm{C}^1([0,\varpi_{\Phi,p}(1)[),\ \lim_{t\to^+0}\frac{\Phi(t)}{t}=0 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p\in]1,\infty[\\ \Phi\in\triangle_2^0\cap\mathrm{C}^1([0,\Phi^\vee(1)[),\ \Phi(b_\Phi)\geq1 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\ p=\infty\\ \Phi\in\triangle_2^\infty\cap\mathrm{C}^1(\mathbb{R}) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_1,\ p\in\{1,\infty\}\\ \Phi\in\triangle_2^\infty\cap\mathrm{C}^1(\mathbb{R}),\ \lim_{t\to^+0}\frac{\Phi(t)}{t}=0 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_1,\ p\in]1,\infty[\\ \Phi\in\triangle_2\cap\mathrm{C}^1(\mathbb{R}) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_\infty,\ p\in\{1,\infty\}.\end{cases}$$

We have also the analogous sets of results for $(L_{\Phi,w}(\mathcal{N},\tau),\|\cdot\|_{\Phi,w})$ and $(L_{\Phi,w}(\mathcal{N},\tau),\|\cdot\|^{\mathrm{A}}_{\Phi,w})$.

III. Nonlinear projections

Metric projections in Banach and Orlicz spaces

- Let $(X, \|\cdot\|_X)$ be a Banach space, $\varnothing \neq C \subseteq X$.
- metric projection $:= \mathfrak{P}_C^{d_{\|\cdot\|_X}}(x) := \arg\inf_{y\in C}\{\|x-y\|_X\} \subseteq X$.
- a set C is Chebyshëv $:= \mathfrak{P}_C^{d_{\|\cdot\|_X}}$ is a function on X $\iff$ $\forall x \in X\ \mathfrak{P}_C^{d_{\|\cdot\|_X}}(x) = \{*\}$. Efimov–Stechkin'58
- if C is closed and convex, then: $\mathfrak{P}_C^{d_{\|\cdot\|_X}}$ is a function on X $\iff$ $(X, \|\cdot\|_X) \in \mathbf{R} \cap \mathbf{SC}$. Klee'61
- if C is closed and convex, and $(X, \|\cdot\|_X) \in \mathbf{R} \cap \mathbf{SC} \cap \mathbf{RRS}$, then $\mathfrak{P}_C^{d_{\|\cdot\|_X}}$ is norm-to-norm continuous. Fán–Glicksberg'58

- Theorem: Wáng–Chén'86
  if $(\mathcal{X}, \mu)$ is nonatomic, $\Phi \in \mathrm{N}$, C is closed and convex, then equivalent are:
  1) $(L_\Phi(\mathcal{X}, \mu), \|\cdot\|_\Phi) \in \mathbf{R} \cap \mathbf{SC}$;
  2) $(L_\Phi(\mathcal{X}, \mu), \|\cdot\|_\Phi^{\mathrm{O}}) \in \mathbf{R} \cap \mathbf{SC}$;
  3) $\mathfrak{P}_C^{d_{\|\cdot\|_\Phi}}$ is norm-to-norm continuous;
  4) $\mathfrak{P}_C^{d_{\|\cdot\|_\Phi^{\mathrm{O}}}}$ is norm-to-norm continuous.
- if $(\mathcal{X}, \mu)$ is nonatomic, Φ is finite, then: $(L_\Phi(\mathcal{X}, \mu), \|\cdot\|_\Phi) \in \mathbf{SC}$ $\iff$ $(L_\Phi(\mathcal{X}, \mu), \|\cdot\|_\Phi) \in \mathbf{LUC}$. Kamińska'84
- if $(\mathcal{X}, \mu)$ is nonatomic, Φ is finite and nondegenerate, then: $(L_{\Phi,w}(\mathcal{X}, \mu), \|\cdot\|_{\Phi,w}) \in \mathbf{SC}$ $\iff$ $(L_{\Phi,w}(\mathcal{X}, \mu), \|\cdot\|_{\Phi,w}) \in \mathbf{LUC}$.
  Hudzik–Kamińska–Mastyło'97

Theorem: RPK'23

Let $\mathrm{type}(\mathcal{X},\mu)=\mathrm{type}(\mathcal{N},\tau)\in\{\mathrm{I}_\infty^{\mathrm{s.f.}},\mathrm{II}_1,\mathrm{II}_\infty\}$, $p\in[1,\infty]$, with an exclusion of $(\mathrm{II}_\infty,\,]1,\infty[)$ case.
Let φ be a gauge, $\Psi_\varphi:(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})\to\mathbb{R}$.
Let $\varnothing\neq M\subseteq L_\Phi(\mathcal{X},\mu)$ and $\varnothing\neq K\subseteq L_\Phi(\mathcal{N},\tau)$ be convex and closed in the topology of respective $\|\cdot\|_{\Phi,p}$.

$$
\text{Let}\begin{cases}
a_\Phi=0,\ \exists u>0\ \Phi(u)=1 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{I}_\infty^{\mathrm{s.f.}},\ p=\infty\\
b_\Phi=\infty,\ \lim_{u\to\infty}\frac{\Phi(u)}{u}=\infty & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{I}_\infty^{\mathrm{s.f.}},\ p\in\,]1,\infty[\\
\lim_{t\to\infty}\Phi^{\mathsf{Y}}(\Phi'_+(t))=\infty, & \\
\quad\sup\{u\geq0:\Phi^{\mathsf{Y}}(\Phi'_+(u))\mu(\mathcal{X})<1\}\leq b_\Phi & :\ \mathrm{type}(\mathcal{N},\tau)\in\{\mathrm{II}_1,\mathrm{II}_\infty\},\ p=1\\
\Phi\in\mathrm{N} & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_1,\ p=\infty\\
b_\Phi=\infty,\ \lim_{u\to\infty}\frac{\Phi(u)}{u}=\infty & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_1,\ p\in\,]1,\infty[\\
b_\Phi=\infty & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_\infty,\ p=\infty.
\end{cases}
$$

Charact. of norm continuity of $\mathfrak{P}^{d_{\|\cdot\|}}_{C}$ on Orlicz spaces (II)

Consider the statements:

1) $(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,p})\in\mathbf{R}\cap\mathbf{SC}$;
2) $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})\in\mathbf{R}\cap\mathbf{SC}$;
3) $(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,p})\in\mathbf{R}\cap\mathbf{SC}\cap\mathbf{RRS}$;
4) $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})\in\mathbf{R}\cap\mathbf{SC}\cap\mathbf{RRS}$;
5) M is Chebyshëv;
6) K is Chebyshëv;
7) $\mathfrak{P}^{d_{\|\cdot\|_{\Phi,p}}}_{M}$ is norm-to-norm continuous on $(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,p})$;
8) $\mathfrak{P}^{d_{\|\cdot\|_{\Phi,p}}}_{K}$ is norm-to-norm continuous on $(L_\Phi(\mathcal{N},\mu),\|\cdot\|_{\Phi,p})$;
9) Ψ_φ is strictly convex and $\mathrm{ran}(\mathfrak{D}^{\mathrm{G}}\Psi_\varphi)=L_\Phi(\mathcal{N},\tau)$.

Then:

(i) if $\mathrm{type}(\mathcal{N},\tau)\in\{\mathrm{II}_1,\mathrm{II}_\infty\}$ and ($\Phi\in\triangle_2$ if $\mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_\infty$), then: 1) iff ... iff 9) iff 10) $:=\begin{cases}\Phi\in\triangle_2^\infty\cap\mathrm{SC}(\mathbb{R}),\ \Phi^{\mathsf{Y}}\in\triangle_2^\infty & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_1,\ p\in[1,\infty]\\ \Phi\in\mathrm{SC}(\mathbb{R}),\ \Phi^{\mathsf{Y}}\in\triangle_2 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_\infty,\ p\in\{1,\infty\};\end{cases}$

(ii) if $\widetilde{\mathrm{type}}(\mathcal{N})=\mathrm{I}_\infty^{\mathrm{s.f.}}$ then:

$$\begin{tikzcd}
8) \arrow[r, Leftrightarrow] & 4) \arrow[dr, Leftrightarrow] & 5) \arrow[r, Leftrightarrow] & 1) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & 2) \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & 6) \\
 & 7) \arrow[r, Leftrightarrow] & 2) \arrow[r, Leftrightarrow] & 10^{\circ}) & 9) &
\end{tikzcd}$$

$$10^{\circ})\ \begin{cases}\Phi\in\triangle_2^0\cap\mathrm{SC}([0,\varpi_\Phi(1)]),\ a_\Phi=0,\ \exists u>0\ \Phi^{\mathsf{Y}}(\Phi'_+(u))\geq\frac{1}{2},\ \Phi^{\mathsf{Y}}\in\triangle_2^0 & :\ p=1\\ \Phi\in\triangle_2^0\cap\mathrm{SC}([0,\Phi^{\mathsf{Y}}(\frac{1}{2})]),\ \Phi^{\mathsf{Y}}\in\triangle_2^0 & :\ p=\infty\\ \Phi\in\triangle_2^0\cap\mathrm{SC}([0,\varpi_{\Phi,p}(1)]),\ a_\Phi=0,\ \exists u>0\ \Phi^{\mathsf{Y}}(\Phi'_+(u))>\frac{1}{2},\ \Phi^{\mathsf{Y}}\in\triangle_2^0 & :\ p\in\,]1,\infty[\,.\end{cases}$$

Charact. of norm continuity of $\mathfrak{P}^{d_{\|\cdot\|}}_C$ on Orlicz–Lorentz spaces (I)

Theorem: RPK'23

Let $\mathrm{type}(\mathcal{X},\mu)=\mathrm{type}(\mathcal{N},\tau)\in\{\mathrm{I}^{\mathrm{s.f.}}_\infty,\mathrm{II}_1,\mathrm{II}_\infty\}$, $p\in[1,\infty]$, $\|\cdot\|\in\{\|\cdot\|_{\Phi,w},\|\cdot\|^{\mathrm{A}}_{\Phi,w}\}$.
Let φ be a gauge, $\Psi_\varphi:(L_{\Phi,w}(\mathcal{N},\tau),\|\cdot\|)\to\mathbb{R}$.
Let $\varnothing\neq M\subseteq L_{\Phi,w}(\mathcal{X},\mu)$ and $\varnothing\neq K\subseteq L_{\Phi,w}(\mathcal{N},\tau)$ be convex and closed in the topology of respective $\|\cdot\|$.

Consider the statements:

1) $(L_{\Phi,w}(\mathcal{X},\mu),\|\cdot\|)\in\mathbf{R}\cap\mathbf{SC}$;
2) $(L_{\Phi,w}(\mathcal{N},\tau),\|\cdot\|)\in\mathbf{R}\cap\mathbf{SC}$;
3) $(L_{\Phi,w}(\mathcal{X},\mu),\|\cdot\|)\in\mathbf{R}\cap\mathbf{SC}\cap\mathbf{RRS}$;
4) $(L_{\Phi,w}(\mathcal{N},\tau),\|\cdot\|)\in\mathbf{R}\cap\mathbf{SC}\cap\mathbf{RRS}$;
5) M is Chebyshëv;
6) K is Chebyshëv;
7) $\mathfrak{P}^{d_{\|\cdot\|}}_M$ is norm-to-norm continuous on $(L_{\Phi,w}(\mathcal{X},\mu),\|\cdot\|)$;
8) $\mathfrak{P}^{d_{\|\cdot\|}}_K$ is norm-to-norm continuous on $(L_{\Phi,w}(\mathcal{N},\mu),\|\cdot\|)$;
9) Ψ_φ is strictly convex and $\mathrm{ran}(\mathfrak{D}^{\mathrm{G}}\Psi_\varphi)=L_{\Phi,w}(\mathcal{N},\tau)$.

Then:

(i) if $\mathrm{type}(\mathcal{N},\tau)\in\{\mathrm{II}_1,\mathrm{II}_\infty\}$ and $\|\cdot\|=\|\cdot\|_{\Phi,w}$, Φ is finite and nondegenerate, $\int_0^1\mathrm{d}t\,w(t)=1$, and (($\Phi\in\triangle_2$ and $\int_0^\infty\mathrm{d}t\,w(t)=\infty$) if $\mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_\infty$), then 1) iff … iff 9) iff

$$\begin{cases}\Phi\in\triangle_2^\infty\cap\mathrm{SC}(\mathbb{R}),\ \Phi^{\mathrm{Y}}\in\triangle_2^\infty,\ w(t)>0\ \forall t\in[0,\tau(\mathbb{I})[& :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_1\\ \Phi\in\mathrm{SC}(\mathbb{R}),\ \Phi^{\mathrm{Y}}\in\triangle_2,\ w(t)>0\ \forall t\in\mathbb{R}^+ & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_\infty;\end{cases}$$

Charact. of norm continuity of $\mathfrak{P}_C^{d_{\|\cdot\|}}$ on Orlicz–Lorentz spaces (II)

(ii) if $\mathrm{type}(\mathcal{N},\tau) = \mathrm{I}_\infty^{\text{s.f.}}$, $\|\cdot\| = \|\cdot\|_{\Phi,w}$, Φ is finite and nondegenerate, $w(t) > 0$ $\forall i \in \mathbb{N}$, $\sum_{i=1}^\infty w(i) = \infty$, $w(1) = 1$, $\Phi \in \triangle_2^0$, then:

$$
\begin{tikzcd}
8) \arrow[r, Leftrightarrow] & 4) \arrow[dr, Leftrightarrow] & 5) \arrow[r, Leftrightarrow] & 1) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & 2) \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & 6) \\
 & 7) \arrow[r, Leftrightarrow] & 2) \arrow[r, Leftrightarrow] & 10^o) & 9) &
\end{tikzcd}
$$

where $10^o)$ $:= \Phi \in \mathrm{SC}[0, \Phi^{-1}(\frac{1}{w(1)+w(2)})]$, $\Phi^{\mathsf{Y}} \in \triangle_2^0$;

(iii) if $\mathrm{type}(\mathcal{N},\tau) = \mathrm{II}_\infty$, $\|\cdot\| = \|\cdot\|_{\Phi,w}^{\mathrm{A}}$, $\Phi \in \mathrm{N}$, $\int_0^1 \mathrm{d}t\, w(t) = 1$, $\Phi \in \triangle_2$, and $\int_0^\infty \mathrm{d}t\, w(t) = \infty$, then 1), 2), 5), 6), 9) are equivalent with 10') $:= \Phi \in \mathrm{SC}(\mathbb{R})$, $\Phi^{\mathsf{Y}} \in \triangle_2$.
If, additionally, $\exists k > 1\ \exists x_0 > 0\ \forall x > x_0\ \ \int_0^{2x} \mathrm{d}t\, w(t) \geq k \int_0^x \mathrm{d}t\, w(t)$, then 10') is also equivalent with 3), 4), 7), 8);

(iv) if $\mathrm{type}(\mathcal{N},\tau) = \mathrm{I}_\infty^{\text{s.f.}}$, $\|\cdot\| = \|\cdot\|_{\Phi,w}^{\mathrm{A}}$, $\Phi \in \mathrm{N}$, $w(i) > 0\ \forall i \in \mathbb{N}$, $\sum_{i=1}^\infty w(i) = \infty$, $w(1) = 1$, $\Phi \in \triangle_2^0$, then the above diagram of implications holds, under replacement of $10^o)$ by: $\Phi \in \mathrm{SC}[0, \varpi(\frac{1}{w(1)})]$, $\Phi^{\mathsf{Y}} \in \triangle_2^0$.

Charact. of norm continuity of $\mathfrak{P}^{d_{\|\cdot\|}}_C$ on L_Φ & $L_{\Phi,w}$ spaces: proof

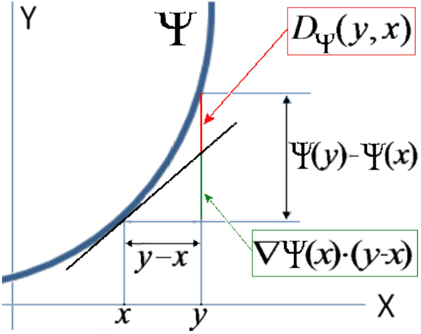
Components of the proof:

1) Characterisations of the geometry of $(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,p\in[1,\infty]})$, $(L_{\Phi,w}(\mathcal{X},\mu),\|\cdot\|_{\Phi,w})$, $(L_{\Phi,w}(\mathcal{X},\mu),\|\cdot\|^{\mathrm{A}}_{\Phi,w})$.
2) Theorems on transfer of geometric properties between r.i.B.s. and n.r.i.B.s.
3) Theorem: Vlasov'81
   For any Banach space $(X,\|\cdot\|_X)$,
   $\mathfrak{P}^{d_{\|\cdot\|_X}}_C$ is norm-to-norm continuous $\iff$ $(X,\|\cdot\|_X)\in\mathbf{R}\cap\mathbf{SC}\cap\mathbf{RRS}$.

Note:
The above characterisation of the norm-to-norm continuity of $\mathfrak{P}^{d_{\|\cdot\|}}_C$ is new also in the commutative case, beyond the original range of the Wáng–Chén theorem.

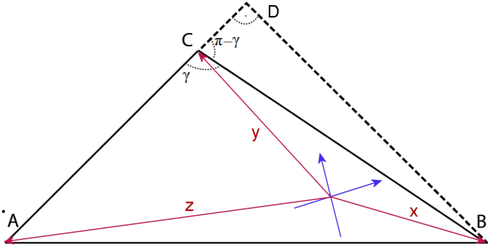
The Vaĭnberg–Brègman information:

- Let $(X, \|\cdot\|_X) \in \mathbf{R}$, let $\Psi : X \to]-\infty, \infty]$ be convex, lower semicontinuous, Gateaux differentiable on $\mathrm{int}(\mathrm{efd}(\Psi)) \neq \varnothing$, $\mathrm{efd}(\Psi) := \{x \in X \,:\, \Psi(x) \neq \infty\}$.
- Vaĭnberg–Brègman information := $D_\Psi(y, x)$:=
$$\begin{cases} \Psi(y) - \Psi(x) - [\![y - x, \mathfrak{D}^{\mathrm{G}}\Psi(x)]\!]_{X \times X^\star} & : (y, x) \in X \times \mathrm{int}(\mathrm{efd}(\Psi)) \\ \infty & : y \in X,\ x \notin \mathrm{int}(\mathrm{efd}(\Psi)). \end{cases}$$
Vaĭnberg'65, Kachurovskiĭ'66, Brègman'66, Al'ber–Butnariu'97
- $D_\Psi(x, y) \geq 0\ \forall x, y \in X$.
- $(D_\Psi(x, y) = 0 \text{ iff } x = y) \iff \Psi$ is strictly convex. Butnariu–Iusem'00



Cosine theorem:

Pythagorean theorem: $c^2 = a^2 + b^2$ is a special case of the cosine theorem:



- Eúkleidos'$^{\sim}$−300

$$\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 + 2(\overline{AC})(\overline{CD}),$$

- Al Kāshī'1427

$$\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 - 2(\overline{AC})(\overline{CB})\cos\gamma.$$

- In cartesian ($\mathbb{R}^n$) and Hilbert ($\mathcal{H}$) spaces:

$$\left\|\overrightarrow{AB}\right\|^2 = \left\|\overrightarrow{AC}\right\|^2 + \left\|\overrightarrow{CB}\right\|^2 - 2\left\langle\overrightarrow{AC}, \overrightarrow{CB}\right\rangle,$$

$$\|z-x\|^2_{\mathcal{H}} = \|z-y\|^2_{\mathcal{H}} + \|y-x\|^2_{\mathcal{H}} - 2\langle z-y, x-y\rangle_{\mathcal{H}}.$$

- $\Psi(x) = \frac{1}{2}\|\cdot\|^2_{\mathcal{H}} \Rightarrow D_\Psi(x,y) = \frac{1}{2}\|x-y\|^2_{\mathcal{H}}$.
- D_Ψ provides a direct generalisation of the cosine theorem: Chen–Teboulle'93

$$D_\Psi(z,x) = D_\Psi(z,y) + D_\Psi(y,x) - [\![z-y, \mathfrak{D}^{\mathrm{G}}\Psi(x) - \mathfrak{D}^{\mathrm{G}}\Psi(y)]\!]_{X\times X^\star}.$$

- What is the corresponding generalisation of the pythagorean theorem?

The Vaĭnberg–Brègman projection (I):

- Let $\varnothing \neq C \subseteq X$.
- left VKB projection $:= \overleftarrow{\mathfrak{P}}^{D_\Psi}_C(x) := \arg\inf_{z\in C}\{D_\Psi(z,x)\}$. Brègman'66
- right VKB projection $:= \overrightarrow{\mathfrak{P}}^{D_\Psi}_C(x) := \arg\inf_{z\in C}\{D_\Psi(x,z)\}$.

  Amari–Nagaoka'93, Bauschke–Noll'02
- if $(X,\|\cdot\|_X)$ is a Hilbert space, then $\overrightarrow{\mathfrak{P}}^{D_\Psi}_C = \mathfrak{P}^{d_{\|\cdot\|_X}}_C = \overleftarrow{\mathfrak{P}}^{D_\Psi}_C$.

- $X^\star \ni y \mapsto \Psi^{\mathbf{F}}(y) := \sup_{x\in X}\{[\![x,y]\!]_{X\times X^\star} - \Psi(x)\}$. Mandelbrojt'39, Fenchel'49, Moreau'62
- $\Psi : X \to]-\infty,\infty]$ is Euler–Legendre $:=$

  Ψ is convex, lower semicontinuous, Gateaux differentiable on $\mathrm{int}(\mathrm{efd}(\Psi)) \neq \varnothing$,

  $\Psi^{\mathbf{F}}$ is Gateaux differentiable on $\mathrm{int}(\mathrm{efd}(\Psi^{\mathbf{F}})) \neq \varnothing$,

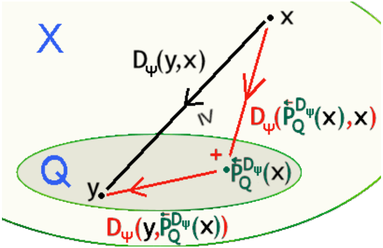
  $\partial\Psi(x) \neq \varnothing$ iff $x \in \mathrm{int}(\mathrm{efd}(\Psi))$ $\forall x \in \mathrm{efd}(\Psi)$,

  $\partial\Psi^{\mathbf{F}}(x) \neq \varnothing$ iff $x \in \mathrm{int}(\mathrm{efd}(\Psi^{\mathbf{F}}))$ $\forall x \in \mathrm{efd}(\Psi^{\mathbf{F}})$.

  Bauschke–Borwein–Combettes'01; $\mathbb{R}^n$: Rockafellar'63

The Vaĭnberg–Brègman projection (II):

- Theorem: Brègman'66, Bauschke–Borwein–Combettes'03
  if Ψ is Euler–Legendre, $\varnothing \neq Q \subseteq \mathrm{int}(\mathrm{efd}(\Psi))$ is convex and closed, then:
  (i) $\overleftarrow{\mathfrak{P}}^{D_\Psi}_Q$ is a function on $\mathrm{int}(\mathrm{efd}(\Psi))$;
  (ii) $D_\Psi(y,x) \geq D_\Psi(y,z) + D_\Psi(z,x)$ $\forall(y,x) \in Q \times \mathrm{int}(\mathrm{efd}(\Psi))$
  $\iff z = \overleftarrow{\mathfrak{P}}^{D_\Psi}_Q(x)$;
  (iii) '$\geq$' turns into '$=$' if Q is affine.
- Theorem: Martín-Marquez–Reich–Sabach'12, RPK'21
  Ψ is Euler–Legendre, $\varnothing \neq K \subseteq \mathrm{int}(\mathrm{efd}(\Psi))$, $\mathfrak{D}^{\mathrm{G}}\Psi(K)$ is convex and closed $\Rightarrow$
  (i) $\overrightarrow{\mathfrak{P}}^{D_\Psi}_K$ is a function on X;
  (ii) $D_\Psi(x,y) \geq D_\Psi(x,z) + D_\Psi(z,y)$ $\forall(x,y) \in X \times K$ $\iff z = \overrightarrow{\mathfrak{P}}^{D_\Psi}_K(x)$;
  (iii) '$\geq$' turns into '$=$' if $\mathfrak{D}^{\mathrm{G}}\Psi(K)$ is affine.



Pythagoreanity, and continuity, of $\overleftarrow{\mathfrak{P}}_K^{\Psi_\varphi}$ and $\overrightarrow{\mathfrak{P}}_K^{\Psi_\varphi}$ (I):

- Theorem: RPK'21
  Let $(X, \|\cdot\|_X)$ be any Banach space.
  Then $\Psi_\varphi : X \to \mathbb{R}$ is Euler–Legendre iff $(X, \|\cdot\|_X) \in \mathbf{SC} \cap \mathbf{G}$.
- Theorem: RPK'21–'23, with an exception of 3).(i) by Schuster–Kaltenbach–Hofmann–Kazimierski'12
  1) For any gauge φ, $\varnothing \neq K \subseteq X$, if $(X, \|\cdot\|_X) \in \mathbf{R} \cap \mathbf{SC} \cap \mathbf{G}$, then:
     (i) if K is convex and closed, then $\overleftarrow{\mathfrak{P}}_K^{D_{\Psi_\varphi}}$ is a function on X, and $\forall x \in X\ \forall z \in K$
     $z = \overleftarrow{\mathfrak{P}}_K^{D_{\Psi_\varphi}}(x) \iff D_{\Psi_\varphi}(y, z) + D_{\Psi_\varphi}(z, x) \leq D_{\Psi_\varphi}(y, x)\ \forall y \in K$;
     (ii) if $\mathfrak{D}^{\mathrm{G}}\Psi_\varphi(K)$ is convex and closed, then $\overrightarrow{\mathfrak{P}}_K^{D_{\Psi_\varphi}}$ is a function on X, and $\forall x \in X\ \forall z \in K$
     $z = \overrightarrow{\mathfrak{P}}_K^{D_{\Psi_\varphi}}(x) \iff D_{\Psi_\varphi}(x, z) + D_{\Psi_\varphi}(z, y) \leq D_{\Psi_\varphi}(x, y)\ \forall y \in K$.
  2) For any gauge φ, $\varnothing \neq K \subseteq X$, if $(X, \|\cdot\|_X) \in \mathbf{R} \cap \mathbf{SC} \cap \mathbf{F} \cap \mathbf{RRS}$, then:
     (i) if K is convex and closed, then $\overleftarrow{\mathfrak{P}}_K^{D_{\Psi_\varphi}}$ is norm-to-norm continuous on X;
     (ii) if $\mathfrak{D}^{\mathrm{G}}\Psi_\varphi(K)$ is convex and closed, then $\overrightarrow{\mathfrak{P}}_K^{D_{\Psi_\varphi}}$ is norm-to-norm continuous on X.

Pythagoreanity, and continuity, of $\overleftarrow{\mathfrak{P}}^{\Psi_\varphi}_K$ and $\overrightarrow{\mathfrak{P}}^{\Psi_\varphi}_K$ (II):

3) If $\beta\in]0,1[$, $r\in]1,2]$, $\varphi_{1,\beta}(t):=t^{1/\beta-1}$, $\varnothing\neq K\subseteq X$, then:
   (i) if $\beta\in]0,\frac{1}{2}]$, $(X,\|\cdot\|_X)\in\mathbf{UC}_{1/\beta}\cap\mathbf{UF}$, K is convex and closed, then $\overleftarrow{\mathfrak{P}}^{D_{\Psi_{\varphi_{1,\beta}}}}_K$ is uniformly continuous on bounded subsets of X;
   (ii) if $\beta\in]0,\frac{1}{2}]$, $(X,\|\cdot\|_X)\in\mathbf{UC}_{1/\beta}\cap\mathbf{UF}_r$, K is convex, closed, and bounded, then $\overleftarrow{\mathfrak{P}}^{D_{\Psi_{\varphi_{1,\beta}}}}_K$ is $\beta(r-1)$-Lipschitz–Hölder continuous on bounded subsets of X;
   (iii) if $\beta\in[\frac{1}{2},1[$, $(X,\|\cdot\|_X)\in\mathbf{UC}\cap\mathbf{UF}_{1/\beta}$, $\mathfrak{D}^{\mathrm{G}}\Psi_{\varphi_{1,\beta}}$ is convex and closed, then $\overrightarrow{\mathfrak{P}}^{D_{\Psi_{\varphi_{1,\beta}}}}_K$ is uniformly continuous on bounded subsets of X;
   (iv) if $\beta\in[\frac{1}{2},1[$, $(X,\|\cdot\|_X)\in\mathbf{UC}_{r/(r-1)}\cap\mathbf{UF}_{1/\beta}$, $\mathfrak{D}^{\mathrm{G}}\Psi_{\varphi_{1,\beta}}(K)$ is convex, closed, and bounded, then $\overrightarrow{\mathfrak{P}}^{D_{\Psi_{\varphi_{1,\beta}}}}_K$ is $\frac{(1-\beta)^2}{\beta}(r-1)^2$-Lipschitz–Hölder continuous on bounded and convex subsets of X.

Theorem: RPK'21–'23

Let $\mathrm{type}(\mathcal{N},\tau)\in\{\mathrm{I}^{\mathrm{s.f.}}_\infty,\mathrm{II}_1,\mathrm{II}_\infty\}$ and $p\in[1,\infty]$, let φ be a gauge, let $\varnothing\neq K\subseteq L_\Phi(\mathcal{N},\tau)$ be convex and closed in the topology of $\|\cdot\|_{\Phi,p}$, let $\varnothing\neq C\subseteq L_\Phi(\mathcal{N},\tau)$, let $\mathfrak{D}^{\mathrm{G}}\Psi_\varphi(C)$ be convex and closed in the topology of $\|\cdot\|_{\Phi^{\mathsf{Y}},p/(p-1)}$. Then:

(i) if

$$\begin{cases}\Phi\in\triangle_2^0\cap\mathrm{SC}([0,\varpi_\Phi(1)]),\,\Phi^{\mathsf{Y}}\in\triangle_2^0\cap\mathrm{SC}([0,(\Phi^{\mathsf{Y}})^\vee(\tfrac12)]), & \\ \quad\exists u>0\ \Phi^{\mathsf{Y}}(u)=1,\exists u>0\ \Phi^{\mathsf{Y}}(\Phi'_+(u))\geq\tfrac12 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{I}^{\mathrm{s.f.}}_\infty,\,p=1\\ \Phi\in\triangle_2^0\cap\mathrm{SC}([0,\Phi^{-1}(\tfrac12)]]),\,\Phi^{\mathsf{Y}}\in\triangle_2^0\cap\mathrm{SC}([0,\varpi_{\Phi^{\mathsf{Y}}}(1)[), & \\ \quad\exists u>0\ \Phi((\Phi^{\mathsf{Y}})'_+(u))\geq\tfrac12 & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{I}^{\mathrm{s.f.}}_\infty,\,p=\infty\\ \Phi\in\triangle_2^0\cap\mathrm{C}^1([0,\varpi_{\Phi,p}(1)[), & \\ \quad\Phi^{\mathsf{Y}}\in\triangle_2^0\cap\mathrm{C}^1([0,\varpi_{\Phi^{\mathsf{Y}},p/(p-1)}(1)[) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{I}^{\mathrm{s.f.}}_\infty,\,p\in]1,\infty[\\ \Phi,\Phi^{\mathsf{Y}}\in\triangle_2^\infty\cap\mathrm{SC}(\mathbb{R}) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_1,\,p\in[1,\infty]\\ \Phi,\Phi^{\mathsf{Y}}\in\triangle_2\cap\mathrm{SC}(\mathbb{R}) & :\ \mathrm{type}(\mathcal{N},\tau)=\mathrm{II}_\infty,\,p\in[1,\infty]\end{cases}$$

then:

1) $D_{\Psi_\varphi}(x,y)=0\iff x=y\ \forall x,y\in L_\Phi(\mathcal{N},\tau)$;
2) $\overleftarrow{\mathfrak{P}}^{D_{\Psi_\varphi}}_K$ and $\overrightarrow{\mathfrak{P}}^{D_{\Psi_\varphi}}_C$ are functions on $L_\Phi(\mathcal{N},\tau)$;
3) D_{Ψ_φ} satisfies left and right generalised pythagorean theorem on $L_\Phi(\mathcal{N},\tau)$;

(ii) under the same conditions as above, with an additional assumption of
$$\begin{cases} \lim_{u\to\infty}\frac{\Phi(u)}{u}=\infty & \textbf{:}\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{I}_\infty^{\text{s.f.}},\, p\in\,]1,\infty[\\ \Phi\in\triangle_2\cap\mathrm{UC},\,\Phi^{\mathsf{Y}}\in\mathrm{UC} & \textbf{:}\ \mathrm{type}(\mathcal{N},\tau)=\mathsf{II}_\infty,\, p\in\,]1,\infty[\,, \end{cases}$$
$\overleftarrow{\mathfrak{P}}_K^{D_{\Psi_\varphi}}$ and $\overrightarrow{\mathfrak{P}}_C^{D_{\Psi_\varphi}}$ are norm-to-norm continuous on $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$;

(iii) if $p=\infty$, $\varphi(t)=\varphi_{1,\gamma}(t):=t^{1/\gamma-1}$ with $\gamma\in\,]0,1[$,
and one of the following (inequivalent) conditions holds:

a) $1<r\leq 2\leq s<\infty$, Φ is finite and nondegenerate, $\Phi(1)=1$, $u\mapsto\Phi(u^{1/r})$ is convex, $u\mapsto\Phi(u^{1/s})$ is concave;

b) Φ is an inverse of $u\mapsto(\Phi_0^{-1}(u))^{1-2\beta}u^\beta$, $\beta\in\,]0,\frac{1}{2}]$, $\Phi_0\in\mathrm{N}$, $s:=\frac{1}{\beta}$, $r:=\frac{1}{1-\beta}$;

then:

Pythagoreanity & continuity of $\overleftarrow{\mathfrak{P}}^{\Psi_\varphi}_K$ & $\overrightarrow{\mathfrak{P}}^{\Psi_\varphi}_K$ on $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$ (III):

1) $\mathfrak{P}^{d_{\|\cdot\|_\Phi}}_K$ is uniformly continuous on bounded subsets of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_\Phi)$, and $\frac{r}{s}$-Lipschitz–Hölder continuous on bounded neighbourhoods of K;
2) $\overleftarrow{\mathfrak{P}}^{D_{\Psi_{\varphi_{1,1/s}}}}_K$ is uniformly continuous on bounded subsets of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_\Phi)$;
3) $\overrightarrow{\mathfrak{P}}^{D_{\Psi_{\varphi_{1,1/r}}}}_C$ is uniformly continuous on bounded subsets of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_\Phi)$;
4) if K bounded, then $\overleftarrow{\mathfrak{P}}^{D_{\Psi_{\varphi_{1,1/s}}}}_K$ is $\frac{1}{s}(r-1)$-Lipschitz–Hölder continuous on bounded subsets of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_\Phi)$;
5) if $\mathfrak{D}^{\mathrm{G}}\Psi_{\varphi_{1,1/r}}(C)$ is bounded, then $\overrightarrow{\mathfrak{P}}^{D_{\Psi_{\varphi_{1,1/r}}}}_C$ is $\frac{(r-1)^2}{r(s-1)}\min\{\frac{1}{r-1},\frac{1}{s-1}\}$-Lipschitz–Hölder continuous on bounded and convex subsets of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_\Phi)$.

Proof:

Follows from the results presented earlier, except of (iii).1), that uses additionally a theorem of Adzhiev'09.

Summary of results

1) Banach space duality for $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_\Phi)$, $(L_\Phi(\mathcal{N},\tau),\|\cdot\|^{\mathrm{O}}_\Phi)$, as well as $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$, and establishing their relationship with Muratov's variants.
2) Characterisation of geometric properties of $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$, $(L_{\Phi,w}(\mathcal{N},\tau),\|\cdot\|_{\Phi,w})$, $(L_{\Phi,w}(\mathcal{N},\tau),\|\cdot\|^{\mathrm{A}}_{\Phi,w})$ using geometry transfer theorems for r.i.B.s.
3) Characterisation of norm-to-norm continuity of metric projection $\mathfrak{P}^{d_{\|\cdot\|}}_C$ on $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$, $(L_{\Phi,w}(\mathcal{N},\tau),\|\cdot\|_{\Phi,w})$, $(L_{\Phi,w}(\mathcal{N},\tau),\|\cdot\|^{\mathrm{A}}_{\Phi,w})$ by $\mathbf{R}\cap\mathbf{SC}$. (This result is new also in the commutative cases, beyond the range of the Wáng–Chén theorem.)
4) Characterisation of Euler–Legendre Ψ_φ on any Banach space $(X,\|\cdot\|_X)$ by $\mathbf{SC}\cap\mathbf{G}$.
5) Sufficient conditions for left and right pythagorean theorems for D_{Ψ_φ}, and for $\{$norm-to-norm, uniform, Lipschitz–Hölder$\}$ continuity of $\overleftarrow{\mathfrak{P}}^{D_{\Psi_\varphi}}_K$ and $\overrightarrow{\mathfrak{P}}^{D_{\Psi_\varphi}}_K$, on $(X,\|\cdot\|_X)$ in terms of geometric properties of $(X,\|\cdot\|_X)$.
6) Sufficient conditions for left and right pythagorean theorems for D_{Ψ_φ}, and for $\{$norm-to-norm, uniform, Lipschitz–Hölder$\}$ continuity of $\overleftarrow{\mathfrak{P}}^{D_{\Psi_\varphi}}_K$ and $\overrightarrow{\mathfrak{P}}^{D_{\Psi_\varphi}}_K$, on $(L_\Phi(\mathcal{N},\tau),\|\cdot\|_{\Phi,p})$ in terms of the properties of Φ. (This result is new also in the commutative case.)

Open problems (which would improve generality of the above results):

- Missing (or limited to $\Phi\in\mathrm{N}$) characterisations of a few geometric properties of $(L_\Phi(\mathcal{X},\mu),\|\cdot\|_{\Phi,p})$ in some particular cases of $(\mathrm{type}(\mathcal{X},\mu),p)$.
- Missing equivalences in n.r.i.B.s. transfer of $\mathbf{R}$ for type $\mathrm{I}^{\text{s.f.}}_\infty$, and of $\mathbf{UC}_r$ for any type.