One-loop binding corrections to the electron $g$ factor

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We calculate the one-loop electron self-energy correction of order $\alpha (Z \alpha)^5$ to the bound electron $g$ factor. Our result is in agreement with the extrapolated numerical value and paves the way for the calculation of the analogous, but as yet unknown two-loop correction.

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I. INTRODUCTION

The $g$ factor of a bound electron is the coupling constant of the spin to an external, homogeneous magnetic field. In natural units $\hbar = c = \varepsilon_0 = 1$, it is defined by the relation

$$\delta E = - \frac{e}{2 m} (\vec{\sigma} \vec{B}) \frac{g}{2},$$

where $\delta E$ is the energy shift of the electron due to the interaction with the magnetic field $\vec{B}$, $m$ is the mass of the electron, and $e$ is the electron charge ($e < 0$). It was found long ago [1] that in a relativistic (Dirac) theory, the $g$ factor of a bound electron differs from the value $g = 2$ due to the so-called binding corrections. For an $nS$ state, they are given by

$$g = \frac{2}{3} \left( 1 + 2 \frac{E}{m} \right)$$

$$= 2 - \frac{2}{3} (Z \alpha)^2 \frac{n^2}{n^3} + \left( \frac{1}{2n} - \frac{2}{3} \right) \frac{(Z \alpha)^4}{n^3} + \ldots,$$

where $E$ is the Dirac energy. In addition, there are many QED corrections, and the dominant one comes from the so-called electron self-energy. When expanded in powers of $Z \alpha$, the one-loop electron self-energy correction reads (for the $nS$ state)

$$g_{SE} = \alpha \pi \left[ 1 + \frac{(Z \alpha)^2}{6 n^2} + \frac{(Z \alpha)^4}{n^3} \left( \frac{32}{9} \ln[(Z \alpha)^{-2}] + b_{60}(n) \right) + \frac{(Z \alpha)^5}{n^3} b_{50} + \frac{(Z \alpha)^6}{n^3} \left( b_{62} \ln^2[(Z \alpha)^{-2}] + b_{61}(n) \ln[(Z \alpha)^{-2}] + b_{60}(n) \right) + \ldots \right],$$

where $b_{60}(1S) = -10.23652432$ [2, 3], $b_{50} = 23.65(5)$ [4], and higher order coefficients remains unknown. What is approximately known, however, is the sum of $b_{50}$ and higher-orders terms for individual nuclear charges from all-order numerical calculations [4–7]. The subject of this work is the one-loop electron self-energy correction of the order of $\alpha (Z \alpha)^5$, namely the coefficient $b_{50}$. Although it has been obtained by extrapolation of numerical results, we aim to calculate it directly, in order to find out the best approach for the analogous two-loop correction, which currently is the main source of the uncertainty of theoretical predictions. Due to extremely accurate measurements in hydrogen-like carbon [8], the bound electron $g$ factor is presently used for the most accurate determination of the electron mass [9], and in the future it can be used for determination of the fine structure constant [10] and for precision tests of the Standard Model.

II. $\alpha (Z \alpha)^5$ CORRECTION TO THE LAMB SHIFT

Before turning to the $g$ factor we present a simple derivation of the analogous correction to the Lamb shift as proof of concept because the computational approach for the $g$ factor will be very similar. The one-loop electron self-energy contribution to the Lamb shift is

$$E_{SE} = e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \langle \psi | \gamma^\mu \frac{1}{\vec{p} + \vec{k}} - \gamma^0 V - m | \psi \rangle,$$

where $V = -Z \alpha/r$. The $(Z \alpha)^5$ contribution is obtained from the hard two-Coulomb exchange

$$E_{SE}^{(5)} = e^2 \phi^2(0) (Z \alpha)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{f(q^2)}{q^2},$$

$$f(q^2) = \int \frac{d^4 k}{i \pi^2} \frac{1}{k^2} \text{Tr} \left[ (T_1 + 2T_2 + T_3) \left( \frac{\gamma^0 + i}{4} \right) \right],$$

where

$$T_1 = (\gamma^\mu \frac{1}{\vec{p} + \vec{k} - m} \gamma^0 \frac{1}{\vec{p} + \vec{k} + \vec{q} - m} \frac{1}{\vec{p} + \vec{k} - m} \gamma_\mu),$$

$$T_2 = (\gamma^0 \frac{1}{\vec{p} + \vec{k} - m} \gamma^\mu \frac{1}{\vec{p} + \vec{k} + \vec{q} - m} \frac{1}{\vec{p} + \vec{k} - m} \gamma_\mu),$$

$$T_3 = (\gamma^0 \frac{1}{\vec{p} + \vec{k} - m} \gamma^\mu \frac{1}{\vec{p} + \vec{k} + \vec{q} - m} \gamma_\mu \frac{1}{\vec{p} + \vec{k} - m} \gamma^0),$$

and where $t = (m, 0, 0, 0)$, $t q = 0$, $q^2 = -q^2$. Equation (5) as it stands is divergent at small $q^2$. One subtracts leading terms in small $q^2$, which correspond to lower order contributions to the Lamb shift, so $f(q^2) \sim q^2$, and

$$f(q^2) = q^2 \int \frac{d(p^2)}{p^2} \frac{1}{(q^2 + p^2)^2} fA(p^2)$$

function $f$ can be expressed in terms of its imaginary part $fA$

$$fA(p^2) = \frac{f(-p^2 + i \epsilon) - f(-p^2 - i \epsilon)}{2 \pi i}.$$
The correction to energy in terms of $f^A$ becomes

$$E_{SE}^{(5)} = e^2 \phi^2(0) (Z \alpha)^2 \int \frac{dp}{2\pi} f^A(p^2). \tag{10}$$

The imaginary part $f^A$ is much easier to evaluate because it does not involve any infrared or ultraviolet divergences in $k$ and has much simpler analytic form than the $f$ itself. The calculations go as follows. Traces are performed with FeynCalc package \[11\]. The resulting expression is a linear combination of fractions with the numerator containing powers of $k^2, q^2, k t,$ and $k q,$ while $q t$ vanishes. Any $k$ in the numerator can be reduced with the denominator with the help of

$$k q = \frac{1}{2} [ (k + q + t)^2 - (k + t)^2 - q^2], \quad k t = \frac{1}{2} [ (k + t)^2 - k^2 - q^2]. \tag{11}$$

The resulting expression is a linear combination of

$$\frac{1}{i \pi^2} \int \frac{dk}{k^2 \ln([k^2 (k + t)^2 - 1]^{n} [(k + t + q)^2 - 1])} \tag{12}$$

with integer $n, m, l \geq 0$. Next, the powers $n, m, l$ are reduced to 1 or 0 using integration by parts identities

$$\int \frac{d^4 k}{\partial k^\mu} \frac{\rho^\mu}{[k^2 (k + t)^2 - 1]^{n} [(k + t + q)^2 - 1]} = 0 \tag{13}$$

with $p = k, q, t$. The resulting expression contains the integral

$$J = \frac{1}{i \pi^2} \int d^4 k \frac{1}{k^2 \ln([k^2 (k + t)^2 - 1]^{n} [(k + t + q)^2 - 1])} \tag{14}$$

and simpler integrals without any of these denominators. Analytic expressions for all such integrals can be taken from \[12\], but it is much easier to calculate the imaginary part using Feynman parameters. For example, the imaginary part of the $J$-integral is

$$J^A(p^2) = \frac{1}{p} \left[ \arctan(p) - \Theta(p - 2) \arccos \left( \frac{2}{p} \right) \right]. \tag{15}$$

Using $J^A$ and simpler formula for other integrals the result for $f^A$ is

$$f^A(p^2) = \frac{7}{3} - \frac{16}{p^2} - \frac{1}{1 + p^2} + \left( \frac{16}{p^2} \right) \arctan(p) + 4 \left( \frac{1}{1 + p^2} - \frac{12}{p^4} \right) \Theta(p - 2) / \sqrt{1 - 4/p^2} - \left( \frac{16}{p^2} \right) \Theta(p - 2) \arccos \left( \frac{2}{p} \right). \tag{16}$$

The one dimensional integration in Eq. (10) leads to

$$\int \frac{dp}{2\pi} f^A(p^2) = \frac{139}{128} - \frac{\ln 2}{2} = C. \tag{17}$$

Finally, the result for the $\alpha (Z \alpha)^5$ electron self-energy contribution to the Lamb shift

$$E_{SE}^{(5)} = m \frac{\alpha (Z \alpha)^5}{\gamma^3} 4C, \tag{18}$$

is in agreement with the well-known value \[9, 13\]. The same integration technique is used in the next paragraph for the evaluation of the analogous correction to the $g$ factor.

### III. $\alpha (Z \alpha)^5$ CORRECTION TO THE $g$ FACTOR

The one-loop correction to the $g$ factor is similar to Eq. (4)

$$\delta E = e^2 C \int \frac{d^4 k}{(2\pi)^4 i k^2} \langle \bar{\psi} | \gamma^\mu p^+ k - e A - m \gamma^0 V p^+ k - e A - m \gamma^0 V \rangle \tag{19}$$

where $\psi$ is the electron wave function which includes perturbation due to external magnetic field $A$, and $p^0$ includes the corresponding energy shift

$$p^0 = E + \langle \bar{\psi} | e A | \psi \rangle. \tag{20}$$

The $(Z \alpha)^5$ contribution is given in analogy to the Lamb shift, by the hard two-Coulomb exchange

$$\delta E^{(5)} = e^2 A^2 \int \frac{d^4 k}{(2\pi)^4 i k^2} \langle \bar{\psi} | \gamma^\mu \frac{1}{p^+ k - e A - m} \gamma^0 V \frac{1}{p^+ k - e A - m} \gamma^0 V \frac{1}{p^+ k - e A - m} \gamma^\mu \rangle + 2 \gamma^0 V \frac{1}{p^+ k - e A - m} \gamma^\mu \frac{1}{p^+ k - e A - m} \gamma^0 V \frac{1}{p^+ k - e A - m} \gamma^\mu + \gamma^0 V \frac{1}{p^+ k - e A - m} \gamma^\mu \frac{1}{p^+ k - e A - m} \gamma^0 V \langle \bar{\psi} | \psi \rangle, \tag{21}$$

and by the expansion in $A$ and in the momentum carried by $A$. The expansion of $\psi$ in $A$ is not very trivial. Since only the low momenta of the wave function $\psi$ contribute to $(Z \alpha)^5$, we apply the Foldy-Wouthuysen transformation in the presence of the magnetic field

$$S = \frac{1}{2 m} \vec{\gamma} \cdot \vec{\sigma}, \tag{22}$$
and the wave function can be represented as
\[ |\psi\rangle = e^{-iS} \left| \phi \right\rangle = \left( I - \frac{1}{2 m} \gamma^{2} \pi + \frac{e}{8 m^{2}} \sigma B \right) |\phi\rangle, \]
where \( \phi \) is the spinor wave function which corresponds to the transformed Hamiltonian
\[ H' = e^{iS} (H - i \partial_{t}) e^{-iS} = \frac{p^{2}}{2 m} - Z \alpha \frac{\gamma}{r} - \frac{e^{2}}{2 m^{2}} \sigma B \left( 1 - \frac{p^{2}}{2 m^{2}} + \frac{Z \alpha}{6 m r} \right). \]

We are now ready to perform an expansion in \( A \) of Eq. (21), and split \( \delta E^{(5)} \) in four parts
\[ \delta E^{(5)} = E_{1} + E_{2} + E_{3} + E_{4}. \]

\( E_{1} \) comes from the last term in Eq. (23)
\[ E_{1} = \frac{e^{2}}{4 m^{2}} \left| \langle \sigma \cdot B \rangle E^{(5)} \right| = - \frac{e^{2}}{2 m^{2}} \left| \langle \sigma \cdot \bar{B} \rangle \right| g_{1}, \]
where
\[ g_{1} = - \frac{E^{(5)}}{m} = - \frac{\alpha (Z \alpha)^{5}}{n^{3}} 4 C. \]

\( E_{2} \) comes from perturbation of \( \phi \) due to the last term in the transformed Hamiltonian
\[ E_{2} = \frac{e^{2}}{m} \left( \langle \sigma \cdot \bar{B} \rangle C \alpha (Z \alpha)^{5} \right) \left( \frac{5}{6 r} \frac{1}{(E-H)^{2}} + \frac{8}{27} \pi^{2} \right), \]
where \( p^{2}/2 \) is replaced by \( 1/r \). Since
\[ \frac{1}{r} \phi = - \partial_{r} \phi, \]
the above matrix element is
\[ \left( \frac{1}{r} \frac{1}{E-H} \right) 4 \pi \delta^{(3)}(r) \delta^{(5)}(r) = - \frac{6}{r^{3}}, \]
and \( g_{2} \) becomes
\[ g_{2} = \frac{\alpha (Z \alpha)^{5}}{n^{3}} 20 C. \]

\( E_{3} \) comes from expansion of Eq. (21) in \( p_{0} - m = - e (\langle \sigma \cdot \bar{B} \rangle)/(2 m) \),
\[ E_{3} = - \frac{e^{2}}{2 m} \left( \langle \sigma \cdot \bar{B} \rangle \right) e^{2} \phi^{2}(0) (Z \alpha)^{2} C', \]
where
\[ C' = \frac{\partial}{\partial E} \left| \int \frac{d^{4}q}{(2 \pi)^{4}} \frac{1}{q^{2}} \int \frac{d^{4}k}{(2 \pi)^{4}} \frac{1}{k^{2}} \right|^{2} \times \text{Tr} \left( T_{1} + 2 T_{2} + T_{3} \right) \left( \gamma^{0} + i \frac{1}{4} \right) \]
[33]
\[ = - \frac{659}{256} + \ln(2) \]
and where \( T_{i} \) are defined in Eq. (7) with \( t = (E, 0, 0, 0) \). The corresponding correction to the \( g \) factor is
\[ g_{3} = \frac{\alpha (Z \alpha)^{5}}{n^{3}} 8 C'. \]

The last term \( E_{4} \) comes from the expansion of \( \delta E^{(5)} \) in \( \vec{r} \cdot \vec{A} \). A typical contribution is of the form
\[ E_{4} = e^{2} \int \frac{d^{4}k}{(2 \pi)^{4}} \frac{1}{k^{2}} \int \frac{d^{3}p}{(2 \pi)^{3}} \frac{Z \alpha}{-\vec{p} \cdot \vec{q} - (1/2)^{2}} \phi^{2}(0) \left( e^{ijk} \sigma^{k} \right) \]
\[ \text{Tr} \left[ \gamma^{0} \left( \frac{1}{\gamma^{0} + i} \right) \right] \]
where by dots we denote all other diagrams. In addition, we perform an expansion in the momentum \( \vec{q} \) transferred by \( A \) and obtain
\[ E_{4} = e^{2} (Z \alpha)^{2} \phi^{2}(0) C'' \left( A_{1} q^{1} - A_{1} q^{1} \right) e^{ijk} \sigma^{k} \]
\[ = -2 e^{2} (Z \alpha)^{2} \phi^{2}(0) C'' \left( e \tilde{\sigma} \tilde{B} \right), \]
where
\[ C'' = \frac{281}{1024} + \frac{\ln(2)}{12}. \]
The numerical value for the coefficient multiplied by $\pi$ is $b_{50} = 23.282005$, in agreement with Yerokhin’s very recent result of $23.6(5)$ [4]. However, what is not in agreement is the difference for $b_{50}(2S) - b_{50}(1S)$, which according to our calculations vanishes, but Yerokhin et al. [4] give $0.12(5)$. All the assumptions in performing the fit in Ref. [4] were correct, so this small discrepancy needs further investigation.

IV. SUMMARY

We have calculated the one-loop electron self-energy contribution of order $\alpha (Z \alpha)^5$ to the bound electron $g$ factor, and found that it is state independent. The principal result, however, is a presentation of the computational approach, which can be extended to the yet unknown two-loop correction. This correction is presently the main source of theoretical uncertainty. The extension of the direct one-loop numerical calculation to the two-loop case is presently out of reach. In contrast, the analytic approach with an expansion in $Z \alpha$ is technically as difficult as the two-loop self-energy correction to the Lamb shift, which has been known for some time [13].

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