Revisiting timelike and null geodesics in the Schwarzschild spacetime:

general expressions in terms of Weierstrass elliptic functions

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State of the knowledge

- 1916 K. Schwarzschild - the first vacuum solution of Einstein's equations.
- 1917 J. Droste - independently produced the same solution as Schwarzschild. Additionally solved equations of motion of test particles using Weierstrass elliptic functions.
- 1930 Y. Hagihara - gave a full description of the motion of test particles based on Droste's work.
- 1959-62 C. Darwin, J. Plebański, B. Mielnik - description of the geodesic motion in the language of Legendre elliptic integrals.
- 2011 G. Scharf - description of the geodesic motion using the simplified Biermann Weierstrass formula.
- 2014 U. Kostić - elegant description in a modern language.

Metric

We will work in spherical coordinates (t, r, θ, φ). In its simplest form the Schwarzschild metric is written as

$$g = -N d\bar{t}^2 + \frac{d\bar{r}^2}{N} + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\varphi^2,$$

where

$$N = 1 - \frac{r_s}{\bar{r}},$$

and $r_s = 2M$ is the Schwarzschild radius.

In order to avoid irregularities at the horizon, we choose a new time foliation

$$t = \bar{t} + \int^{\bar{r}} \left[\frac{1}{N(s)} - \eta(s)\right] ds, \quad r = \bar{r},$$

where $\eta = \eta(\bar{r})$ is a function of radius $\bar{r}$, yields the metric in the form

$$g = -N dt^2 + 2(1 - N\eta) dt dr + \eta(2 - N\eta) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2. \quad (1)$$

Conserved quantities

The Hamiltonian H of a single particle moving along the geodesic can be chosen as

$$H = \frac{1}{2} g^{\mu\nu}(x^\alpha) p_\mu p_\nu.$$

Here (x^μ, p_μ) are treated as canonical variables, and H depends on x^α through $g^{\mu\nu}(x^\alpha)$. It is easy to verify that the Hamilton's equations

$$\frac{dx^\mu}{d\tilde{s}} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tilde{s}} = -\frac{\partial H}{\partial x^\mu} \tag{2}$$

lead to the standard geodesic equation of the form

$$\frac{d^2 x^\mu}{d\tilde{s}^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tilde{s}} \frac{dx^\beta}{d\tilde{s}} = 0.$$

We choose the parameter $\tilde{s}$ as a rescaled proper time $\tilde{s} = \tilde{\tau}/m$, where m is the particle rest mass.

The four velocity $u^\mu = dx^\mu/d\tilde{\tau}$ is normalized as $g_{\mu\nu}u^\mu u^\nu = -1$. We require that $p^\mu = dx^\mu/d\tilde{s}$, and that $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = -\frac{1}{2}m^2$.
For metric (1) the Hamiltonian has the form

$$H = \frac{1}{2}\left[g^{tt}(r)p_t^2 + 2g^{tr}(r)p_t p_r + g^{rr}(r)p_r^2 + \frac{1}{r^2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right)\right].$$

Since H depends neither on t nor on φ, the momentum components $E \equiv -p_t$ (the energy) and $l_z \equiv p_\varphi$ are constants of motion. The Hamiltonian H is also independent of $\tilde{s}$, and hence it is also conserved. A simple calculation allows one to check that the total angular momentum

$$l = \sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}} \tag{3}$$

is another constant of motion.

Equations of motion

Thanks to Hamilton's equations (2) we can now write down particle equation of motion

$$\frac{dr}{d\tilde{s}} = \frac{\partial H}{\partial p_r} = \epsilon_r \sqrt{E^2 - \tilde{U}_{l,m}(r)}, \tag{4a}$$

$$\frac{d\varphi}{d\tilde{s}} = \frac{\partial H}{\partial p_\varphi} = \frac{l_z}{r^2 \sin^2\theta}, \tag{4b}$$

$$\frac{d\theta}{d\tilde{s}} = \frac{\partial H}{\partial p_\theta} = \frac{\epsilon_\theta}{r^2} \sqrt{l^2 - \frac{l_z^2}{\sin^2\theta}}, \tag{4c}$$

$$\frac{dt}{d\tilde{s}} = \frac{\partial H}{\partial p_t} = \frac{E}{N} + \epsilon_r \frac{1 - N\eta}{N} \sqrt{E^2 - \tilde{U}_{l,m}(r)}, \tag{4d}$$

where

$$\tilde{U}_{l,m}(r) = \left(1 - \frac{r_s}{r}\right)\left(m^2 + \frac{l^2}{r^2}\right)$$

is the radial effective potential and where we have introduced the signs $\epsilon_\theta = \pm 1$, and $\epsilon_r = \pm 1$, corresponding to the directions of motion.

New coordinate system

It is convenient to work in dimensionless rescaled variables

$$t = M\tau,\ r = M\xi,\ p_r = m\pi_\xi,\ p_\theta = Mm\pi_\theta,\ E = m\varepsilon,\ l = Mm\lambda,\ l_z = Mm\lambda_z.$$

In addition, a new affine parameter s is defined by

$$\tilde{s} = \frac{M}{m}s.$$

In terms of these changes, the equations of motion (4) can be written as

$$\frac{d\xi}{ds} = \epsilon_r\sqrt{\varepsilon^2 - U_\lambda(\xi)}, \tag{5a}$$

$$\frac{d\varphi}{ds} = \frac{\lambda_z}{\xi^2\sin^2\theta}, \tag{5b}$$

$$\frac{d\theta}{ds} = \frac{\epsilon_\theta}{\xi^2}\sqrt{\lambda^2 - \frac{\lambda_z^2}{\sin^2\theta}}, \tag{5c}$$

$$\frac{d\tau}{ds} = \frac{\varepsilon}{N(\xi)} + \epsilon_r\frac{(1 - N(\xi)\eta(\xi))}{N(\xi)}\sqrt{\varepsilon^2 - U_\lambda(\xi)}, \tag{5d}$$

where $N(\xi) = 1 - 2/\xi$, and the dimensionless radial potential reads

$$U_\lambda(\xi) = \left(1 - \frac{2}{\xi}\right)\left(1 + \frac{\lambda^2}{\xi^2}\right) = 1 - \frac{2}{\xi} + \frac{\lambda^2}{\xi^2} - \frac{2\lambda^2}{\xi^3}.$$

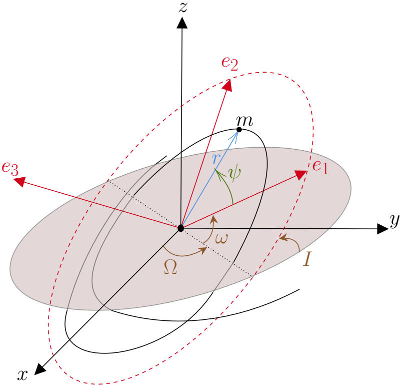
Because the metric in which we work is spherically symmetric, we can always find the plane in which the particle moves, using the so-called true anomaly (a polar angle in the orbital plane). Let us introduce the following orthonormal 3-vectors expressed with respect to the local Cartesian coordinate basis

$$\mathbf{e}_1 = \begin{pmatrix} \cos\omega\cos\Omega - \cos I\sin\omega\sin\Omega \\ \cos\omega\sin\Omega + \cos I\sin\omega\cos\Omega \\ \sin I\sin\omega \end{pmatrix}, \tag{6a}$$

$$\mathbf{e}_2 = \begin{pmatrix} -\sin\omega\cos\Omega - \cos I\cos\omega\sin\Omega \\ -\sin\omega\sin\Omega + \cos I\cos\omega\cos\Omega \\ \sin I\cos\omega \end{pmatrix}, \tag{6b}$$

$$\mathbf{e}_3 = \begin{pmatrix} \sin I\sin\Omega \\ -\sin I\cos\Omega \\ \cos I \end{pmatrix}. \tag{6c}$$

Geometric interpretation of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, as well as parameters ω, Ω, and I (orbital elements) is shown in figure below



Cartesian coordinates of a location on the orbit can be expressed as

$$\begin{pmatrix} r\cos(\Omega+\varphi)\sin\theta \\ r\sin(\Omega+\varphi)\sin\theta \\ r\cos\theta \end{pmatrix} = M\xi(\psi)[\cos(\psi)\mathbf{e}_1 + \sin(\psi)\mathbf{e}_2], \qquad (7)$$

where we have introduced the so-called true anomaly ψ. The sum $\psi+\omega$ is usually refered to as the argument of latitude.

In order to establish a connection with Eqs. (5) we set

$$\cos I = \frac{\lambda_z}{\lambda}. \qquad (8)$$

One can now derive the differential relations between ψ and φ, θ

$$\cos I d\psi = \frac{\lambda_z}{\lambda} d\psi = \sin^2\theta d\varphi.$$

$$\frac{1}{\lambda}\sqrt{\lambda^2 - \frac{\lambda_z^2}{\sin^2\theta}} d\psi = \epsilon_\theta d\theta.$$

In the light of these two differential relations, both equations (5b) and (5c) can be replaced by $d\psi/ds = \lambda/\xi^2$. This means, that in terms of new variables ξ, ψ, and τ, the equations of motion (5) can be rewritten as

$$\frac{d\xi}{ds} = \epsilon_r \sqrt{\varepsilon^2 - U_\lambda(\xi)}, \tag{9a}$$

$$\frac{d\psi}{ds} = \frac{\lambda}{\xi^2}, \tag{9b}$$

$$\frac{d\tau}{ds} = \frac{\varepsilon}{N(\xi)} + \epsilon_r \frac{1 - N(\xi)\eta(\xi)}{N(\xi)} \sqrt{\varepsilon^2 - U_\lambda(\xi)}. \tag{9c}$$

The above system of equations describes the geodesic motion of a massive particle in the four-dimensional Schwarzschild spacetime.

Solution of equations of motion

Given the form of Eqs. (9), it is natural to treat ψ as a parameter and search for the solution of the form $\xi = \xi(\psi)$. From (9a) and (9b) we get immediately

$$\frac{d\xi}{d\psi} = \epsilon_r \frac{\xi^2}{\lambda} \sqrt{\varepsilon^2 - U_\lambda(\xi)} = \epsilon_r \sqrt{\frac{\varepsilon^2 - 1}{\lambda^2}\xi^4 + \frac{2}{\lambda^2}\xi^3 - \xi^2 + 2\xi}. \tag{10}$$

Defining

$$f(\xi) = a_0\xi^4 + 4a_1\xi^3 + 6a_2\xi^2 + 4a_3\xi + a_4, \tag{11}$$

where

$$a_0 = \frac{\varepsilon^2 - 1}{\lambda^2}, \quad 4a_1 = \frac{2}{\lambda^2}, \quad 6a_2 = -1, \quad 4a_3 = 2, \quad a_4 = 0, \tag{12}$$

one can write Eq. (10) as

$$\frac{d\xi}{d\psi} = \epsilon_r \sqrt{f(\xi)}. \tag{13}$$

For a segment of the trajectory for which ϵ_r is constant, we get

$$\psi = \epsilon_r \int_{\xi_0}^{\xi} \frac{d\xi'}{\sqrt{f(\xi')}}, \tag{14}$$

where ξ_0 is an arbitrarily chosen radius corresponding to the angle $\psi = 0$.

Weierstrass elliptic $\wp$ function

Def.1 *Doubly-periodic function*

A function which satisfies the equations

$$f(z+2\omega_1)=f(z),\quad f(z+2\omega_2)=f(z),\quad \forall z\in\mathbb{C},$$

where ω_1, $\omega_2 \in \mathbb{C}$.

Def.2 *Elliptic function*

A doubly-periodic function which is analytic (except at poles), and which has no singularities other than poles in the finite part of the plane.

Def.3 *Weierstrass elliptic function*

$$\wp(z)=\frac{1}{z^2}+\sum_{m,n}{}' \left(\frac{1}{(z-2m\omega_1-2n\omega_2)^2}-\frac{1}{(2m\omega_1+2n\omega_2)^2}\right),$$

where the prime indicated that terms in the sum giving zero denominators are omitted.

Theorem (Biermann-Weierstrass)

Let

$$f(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4, \tag{15}$$

be a quartic polynomial. Denote the invariants of f by g_2 and g_3, i.e.,

$$g_2 \equiv a_0a_4 - 4a_1a_3 + 3a_2^2, \tag{16a}$$

$$g_3 \equiv a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4. \tag{16b}$$

Let

$$z(x) = \int_{x_0}^{x} \frac{dx'}{\sqrt{f(x')}}, \tag{17}$$

where x_0 is any constant, not necessarily a zero of $f(x)$.

Then

$$x = x_0 + \frac{-\sqrt{f(x_0)}\wp'(z) + \frac{1}{2}f'(x_0)\left(\wp(z) - \frac{1}{24}f''(x_0)\right) + \frac{1}{24}f(x_0)f'''(x_0)}{2\left(\wp(z) - \frac{1}{24}f''(x_0)\right)^2 - \frac{1}{48}f(x_0)f^{(4)}(x_0)}, \tag{18}$$

and

$$\wp(z) = \frac{\sqrt{f(x)f(x_0)} + f(x_0)}{2(x-x_0)^2} + \frac{f'(x_0)}{4(x-x_0)} + \frac{f''(x_0)}{24}, \tag{19a}$$

$$\wp'(z) = -\left[\frac{f(x)}{(x-x_0)^3} - \frac{f'(x)}{4(x-x_0)^2}\right]\sqrt{f(x_0)} - \left[\frac{f(x_0)}{(x-x_0)^3} + \frac{f'(x_0)}{4(x-x_0)^2}\right]\sqrt{f(x)}, \tag{19b}$$

where $\wp(z) = \wp(z; g_2, g_3)$ is the Weierstrass function corresponding to invariants (16).

Application of the theorem

Therefore thanks to the Biermann-Weierstrass theorem, we can write the formula for $\xi = \xi(\psi)$ as

$$\xi(\psi) = \xi_0 + \frac{-\epsilon_{r_0}\sqrt{f(\xi_0)}\wp'(\psi) + \frac{1}{2}f'(\xi_0)\left[\wp(\psi) - \frac{1}{24}f''(\xi_0)\right] + \frac{1}{24}f(\xi_0)f'''(\xi_0)}{2\left[\wp(\psi) - \frac{1}{24}f''(\xi_0)\right]^2 - \frac{1}{48}f(\xi_0)f^{(4)}(\xi_0)}. \tag{20}$$

Here $\wp$ is understood to be defined by the invariants

$$g_2 \equiv \frac{1}{12} - \frac{1}{4}\frac{\xi_s^2}{\lambda^2}, \tag{21}$$

$$g_3 \equiv \frac{1}{6^3} - \frac{1}{48}\frac{\xi_s^2}{\lambda^2} - \frac{1}{16}\frac{(\varepsilon^2 - 1)}{\lambda^2}\xi_s^2, \tag{22}$$

where $\xi_s = 2$, and it is the dimensionless Schwarzschild radius.
The above equation is a general solution to equation Eq. (10), and it is valid for all types of allowed trajectories. Sign of ϵ_{r_0} is selected at the initial position ξ_0. **After selecting it, we do not have to worry about whether the particle is in front of its periapsis or not.** It can be checked numerically and demonstrated analytically.

Post-Newtonian approximation

As a first check of the solution (20) we consider the Newtonian limit. However, the most convenient way to do this is for the per/apocenter, i.e. for $f(\xi_1) = 0$. Then Eq. (20) becomes

$$\xi(\psi) = \xi_1 + \frac{f'(\xi_1)}{4\left[\wp(\psi) - \frac{1}{24} f''(\xi_1)\right]}.$$

It is easy to check that for the Newtonian limit invariants $g_2 \to \frac{1}{12}$, $g_3 \to \frac{1}{6^3}$. Then $\wp$ becomes

$$\wp\left(\psi, 12^{-1}, 6^{-3}\right) = -\frac{1}{12} + \frac{1}{2(1-\cos\psi)}.$$

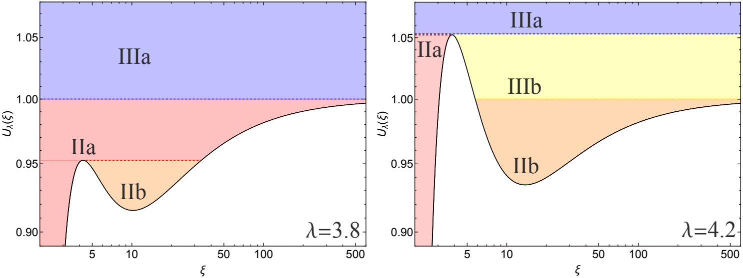
It is convenient to introduce the eccentricity e by

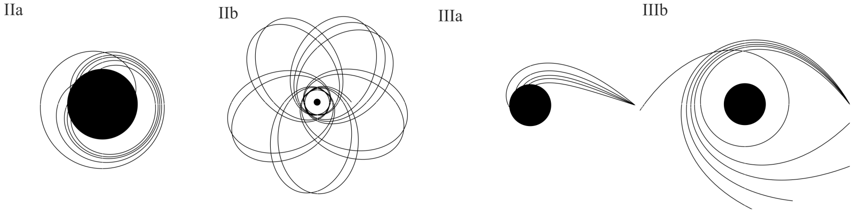
$$\frac{\xi_1}{\xi_2} = \frac{1-e}{1+e},$$

where $f(\xi_2) = 0$. Using all this we get

$$\xi(\psi) = \frac{(1+e)\xi_1}{1+e\cos\psi}.$$

Properties of the effective potential





Numerical tests

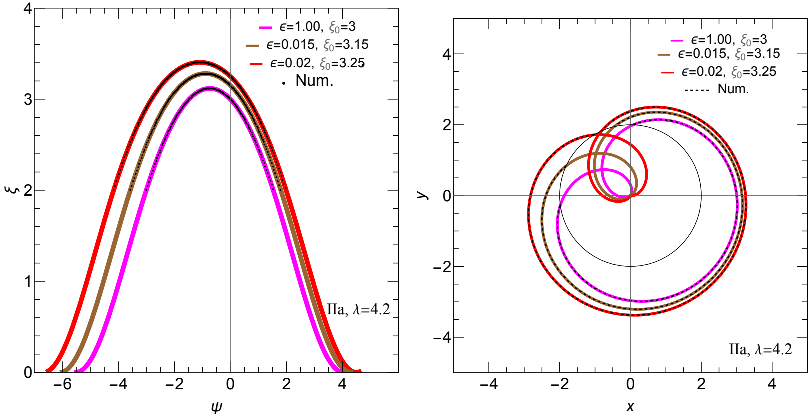


Figure: Sample inner bound orbits (type IIa) for $\lambda = 4.2$. Solid color lines correspond to solutions obtained with Eq. (20). Dotted lines depict corresponding numerical solutions.

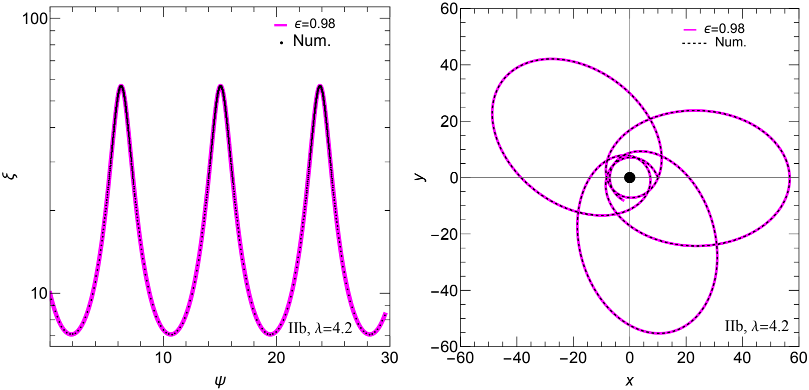


Figure: Sample outer bound orbits (type IIb) for $\lambda = 4.2$. Solid color lines correspond to solutions obtained with Eq. (20). Dotted lines depict corresponding numerical solutions.

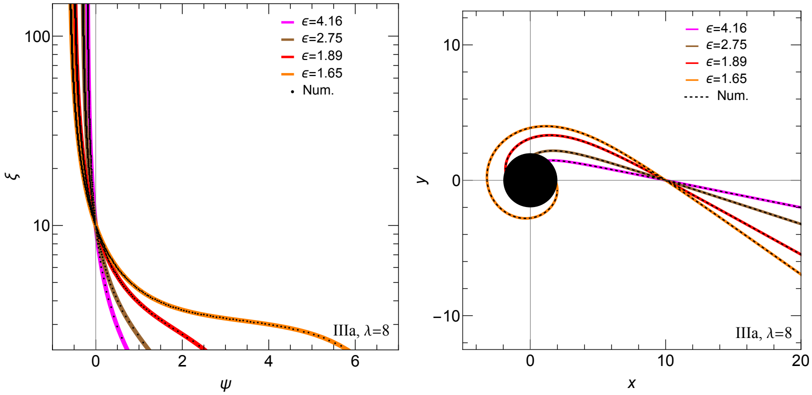


Figure: Sample unbound absorbed orbits (type IIIa) for $\lambda = 8$. Solid color lines correspond to solutions obtained with Eq. (20). Dotted lines depict corresponding numerical solutions.

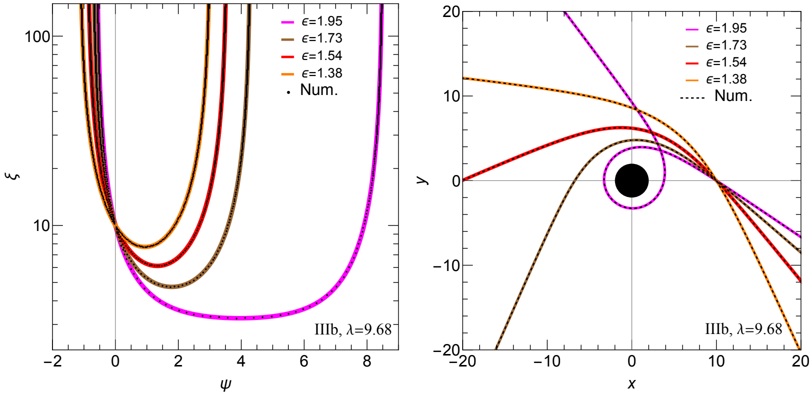


Figure: Sample unobound scattered orbits (type IIIb) for $\lambda = 9.68$. Solid color lines correspond to solutions obtained with Eq. (20). Dotted lines depict corresponding numerical solutions.

Proper time

Given an expression for $\xi = \xi(\psi)$, the corresponding proper time can be computed by integrating Eq. (9b), i.e., as

$$s(\psi) = \frac{1}{\lambda} \int_0^{\psi} \xi^2 \left(\psi'\right) d\psi'. \tag{23}$$

Integrating the square of expression (20) is, in principle, possible, but it is tedious, and the result seems to be too complicated to be useful in practical applications. Much simpler formulas can be obtained by siplified verosion of using Eq. (20)

$$\xi(\psi) = \xi_1 + \frac{f'(\xi_1)}{4\left[\wp(\psi) - \frac{1}{24} f''(\xi_1)\right]},$$

i.e., we describe the motion with respect to periapsis.

The proper time elapsed during the motion from $\psi = 0$ o some $\psi = \psi_2$ can be written as

$$s_*(\psi_2, \xi_1) = \frac{1}{\lambda}\left\{\xi_1^2\psi_2 + \frac{1}{2}f'(\xi_1)\xi_1\left[I_1(\psi_2; y) - I_1(0; y)\right] + \frac{1}{16}\left[f'(\xi_1)\right]^2\left[I_2(\psi_2; y) - I_2(0; y)\right]\right\}, \qquad (24)$$

where $\wp(y) = \frac{1}{24}f''(\xi_1)$ or $y = \wp^{-1}\left(\frac{1}{24}f''(\xi_1)\right)$, and,

$$I_1(x; y) = \frac{1}{\wp'(y)}\left(2\zeta(y)x + \ln\frac{\sigma(x-y)}{\sigma(x+y)}\right), \qquad (25)$$

$$I_2(x; y) = -\frac{1}{\wp'^2(y)}\left(\zeta(x+y) + \zeta(x-y) + \left(2\wp(y) + \frac{2\wp''(y)\zeta(y)}{\wp'(y)}\right)x\right) - \frac{\wp''(y)}{\wp'^3(y)}\ln\frac{\sigma(x-y)}{\sigma(x+y)}. \qquad (26)$$

Consider a motion of a particle starting from an arbitrary location ξ_0, and moving inwards (in the dorection of the BH), up to a periapsis with the radius ξ_1. Next the particle moves outwards, up to a location with a radius ξ. Define the angles ψ_1 and ψ_2 as

$$\psi_1 = -\int_{\xi_0}^{\xi_1} \frac{d\xi'}{\sqrt{f(\xi')}} = \int_{\xi_1}^{\xi_0} \frac{d\xi'}{\sqrt{f(\xi')}},$$

$$\psi_2 = \int_{\xi_1}^{\xi} \frac{d\xi'}{\sqrt{f(\xi')}}.$$

Both angles satisfy $\psi_1 \geq 0$ and $\psi_2 \geq 0$. Let $\psi = \psi_1 + \psi_2$. Because of symmetry, the proper time of the entire motion can be written as

$$s(\psi) = s_*(\psi_1; \xi_1) + s_*(\psi_2; \xi_2) = s_*(\psi_1; \xi_1) + s_*(\psi - \psi_1; \xi_1). \tag{27}$$

Formula Eq. (27) can be understood as a replacement for integral Eq. (23) with $\xi(\psi)$ given by Eq. (20). Note that, since $s_*(\psi_2; \xi_1)$ is an odd function of ψ_2, we get $s(\psi = 0) = 0$, as expected. It can also be checked that the same formula holds for ξ_1 corresponding to an apoapsis.

The coordinate time τ can be obtained in a way similar to the calculation of the proper time s.

Conclusion

- We have a new description of the motion of test particles, which depends only on the constants of motion λ, λ_z, ε and the choice of the initial position (ξ_0, ψ_0).
- Properties of our description:
  - one function $\xi(\psi)$ for the entire trajectory and every type of trajectory,
  - the formula $\xi(\psi)$ does not require the knowledge of turning points,
  - the expression for $\xi(\psi)$ is analytic and it is given in terms of well-known Weierstrass elliptic functions.
- The method was designed to work in simulations of the Vlasov gas on the Schwarzschild background.