Rigidity of the extremal Kerr-Newman horizon

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- AC, David Katona, James Lucietti (2024). Rigidity of the extremal Kerr-Newman horizon. `arXiv:2406.07128`
- AC, Maciej Dunajski (2024). Quasi-Einstein structures, Hitchin's equations and isometric embeddings. `arXiv:24**.*****`

Motivation: classifying black hole spacetimes

- BH uniqueness [Israel, Hawking, Carter, Robinson '70s]: all (analytic) stationary, asymptotically flat solutions of the 4D vacuum Einstein equations with a connected, non-degenerate event horizon are Kerr.

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- In higher dimensions uniqueness is violated, e.g. 5D Myers-Perry (horizon S^3) and black rings [Emparan-Reall '01] (horizon $S^2 \times S^1$). Classification still largely open.

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- For extremal black holes intrinsic geometry of the horizon decouples from extrinsic geometry. Intrinsic near-horizon geometries can be studied and classified independent of exterior BH spacetime.

Extremal horizons [Kunduri-Lucietti '13]

- Let $(N, \mathbf{g})$ be an $(n+2)$-dimensional spacetime containing an extremal Killing horizon $\mathcal{H}$ with normal $\mathcal{K}$ (so d$(|\mathcal{K}|^2) = 0$ on $\mathcal{H}$). Suppose M is a compact n-dimensional cross-section of $\mathcal{H}$.

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- Introduce Gaussian null coordinates s.t. $\mathcal{H} = \{r = 0\}$, $\mathcal{K} = \frac{\partial}{\partial v}$

$$\mathbf{g} = 2\mathrm{d}v\left(\mathrm{d}r + rX_a(r,x)\mathrm{d}x^a + \tfrac{1}{2}r^2 F(r,x)\mathrm{d}v\right) + g_{ab}(r,x)\mathrm{d}x^a\mathrm{d}x^b.$$

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- Near-horizon limit: $r \mapsto \epsilon r, v \mapsto \frac{v}{\epsilon}$, take $\epsilon \to 0$.

$$\mathbf{g}_{\mathsf{NH}} = 2\mathrm{d}v \left(\mathrm{d}r + rX_a(x)\mathrm{d}x^a + \tfrac{1}{2}r^2 F(x)\mathrm{d}v\right) + g_{ab}(x)\mathrm{d}x^a \mathrm{d}x^b.$$

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- $\mathbf{g}_{\mathsf{NH}}$ determined by near-horizon data (g, F, X) on M.

Near-horizon equations

- Energy-momentum tensor T also has near-horizon limit

$$T_{\mathsf{NH}} = 2\mathrm{d}v\left(T_{vr}(x)\mathrm{d}r + r\beta_a(x)\mathrm{d}x^a + \tfrac{1}{2}r^2\alpha(x)\mathrm{d}v\right) + T_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b.$$

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- In NH limit this determines $F = F(X, g, T_{ab}, T_{vr})$ and imposes

$$R_{ab} = \frac{1}{2}X_aX_b - \nabla_{(a}X_{b)} + \lambda g_{ab} + P_{ab},$$
$$P_{ab} = T_{ab} - \frac{1}{n}(g^{cd}T_{cd} + 2T_{vr})g_{ab}.$$

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- Together with matter eqns: near-horizon equations (NHE) on M.

Outline

1. Vacuum extremal horizons
2. Rigidity of the extremal Kerr-Newman horizon
3. Rigidity of quasi-Einstein metrics
4. Topology of generalized extremal horizons

The NHE in vacuum

- Vacuum NHE: compact Riemannian manifold (M, g) with a vector field $X \in \mathfrak{X}(M)$ satisfying

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- Example: extremal Kerr horizon. $M = S^2, \lambda = 0$.

$$g = \frac{a^2(1+x^2)\mathrm{d}x^2}{1-x^2} + \frac{4a^2(1-x^2)\mathrm{d}\phi^2}{1+x^2},$$
$$X = \frac{K - \nabla\Gamma}{\Gamma}, \text{ where } \Gamma = \frac{1}{2}(1+x^2), \ K = \frac{1}{2a^2}\frac{\partial}{\partial\phi}.$$

a rotation parameter, $x \in [-1, 1], \phi \in [0, 2\pi)$.

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- Q: Are there other (global) solutions to the $n = 2$ vacuum NHE?

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Theorem [Dunajski-Lucietti '23, Colling-Dunajski-Kunduri-Lucietti '24]

Let (M, g) be a compact Riemannian manifold without boundary admitting a non-gradient vector field X such that the vacuum NHE hold. Then (M, g) admits a Killing vector field K. Moreover, $[K, X] = 0$.

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Let (M, g) be a compact Riemannian manifold without boundary admitting a non-gradient vector field X such that the vacuum NHE hold. Then (M, g) admits a Killing vector field K. Moreover, $[K, X] = 0$.

- Corollary: The general non-trivial solution to the $n = 2$ vacuum NHE is given by the extremal Kerr-(A)dS horizon.

Killing vector Ansatz

- K is constructed using an Ansatz inspired by extremal Kerr horizon.

Lemma

Given a vector field X on a compact Riemannian manifold (M, g) there exists a (unique up to scale) smooth function $\Gamma > 0$ such that $\nabla_a K^a = 0$, where K is defined by

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- L has formal adjoint

$$L^*\psi = -\Delta\psi + X^a\nabla_a\psi.$$

Now use Fredholm alternative + maximum principle.

Tensor identity

- NHE in terms of $K = \Gamma X + \nabla\Gamma$:

$$R_{ab} = \frac{K_a K_b}{2\Gamma^2} - \frac{(\nabla_a\Gamma)(\nabla_b\Gamma)}{2\Gamma^2} - \frac{1}{\Gamma}\nabla_{(a}K_{b)} + \frac{1}{\Gamma}\nabla_a\nabla_b\Gamma + \lambda g_{ab}.$$

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Proposition

For any solution to the NHE the following identity holds

$$\frac{1}{4}|\mathcal{L}_K g|^2 = \nabla^a\left(K^b\nabla_{(a}K_{b)} - \tfrac{1}{2}K_a\Delta\Gamma - \tfrac{1}{2}K_a\nabla_b K^b - \lambda\Gamma K_a\right)$$
$$+ \nabla_b K^b\left(-\frac{1}{2\Gamma}|K|^2 + \frac{1}{2}\Delta\Gamma + \frac{1}{2}\nabla_b K^b + \frac{1}{2\Gamma}K^b\nabla_b\Gamma + \lambda\Gamma\right).$$

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- Proof: apply $K^b\nabla^a(R_{ab} - \frac{1}{2}Rg_{ab}) = 0$ to the NHE and calculate. The result relies on many mysterious cancellations.

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- L has formal adjoint of the form

$$L^*\psi = -\Delta\psi + B^a\nabla_a\psi + C\psi,$$

with $C = \Gamma^{-2}|K|^2 \geq 0$. Maximum principle + Fredholm alternative imply ker $L = \{0\}$. □

The NHE in Einstein-Maxwell theory

- Energy-momentum tensor for Einstein-Maxwell

$$T_{\mu\nu} = 2\left(\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho} - \frac{1}{4}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}\mathbf{g}_{\mu\nu}\right).$$

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- Einstein-Maxwell NHE: compact Riemannian manifold (M, g) with $X \in \mathfrak{X}(M)$, $\psi \in C^\infty(M)$, $B \in \Omega^2(M)$ satisfying $\mathsf{d}B = 0$ and

$$R_{ab} = \frac{1}{2}X_a X_b - \nabla_{(a}X_{b)} + \lambda g_{ab} + P_{ab},$$
$$(\nabla^a - X^a)B_{ab} = -(\nabla_b - X_b)\psi,$$

where

$$P_{ab} = 2B_{ac}B_b{}^c + \frac{1}{n}g_{ab}(2\psi^2 - B_{cd}B^{cd}).$$

Extremal Kerr-Newman horizon

- Example: extremal Kerr-Newman horizon. $M = S^2, \lambda = 0$.

$$g = \frac{\rho_+^2}{1-x^2}\mathrm{d}x^2 + \frac{(1-x^2)(a^2+r_+^2)^2}{\rho_+^2}\mathrm{d}\phi^2,$$

$$X = \frac{K - \nabla\Gamma}{\Gamma}, \quad \text{where} \quad \Gamma = \frac{\rho_+^2}{2ar_+}, \quad K = \frac{1}{a^2+r_+^2}\frac{\partial}{\partial\phi},$$

$$\psi = \frac{a^2Q^2x^2 - 2aPr_+x - Qr_+^2}{\rho_+^4},$$

$$B = -\frac{(a^2+r_+^2)(a^2Px^2 + 2aQr_+x - Pr_+^2)}{\rho_+^4}\mathrm{d}x \wedge \mathrm{d}\phi.$$

Here $\rho_+^2 = r_+^2 + a^2x^2, r_+^2 = a^2 + P^2 + Q^2$. a rotation parameter, P, Q magnetic resp. electric charge. $x \in [-1, 1], \phi \in [0, 2\pi)$.

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Theorem [Colling-Katona-Lucietti '24]

Let (M, g) be a compact, oriented Riemannian surface (without boundary) admitting a non-gradient vector field X such that the Einstein-Maxwell NHE hold. Then (M, g) admits a Killing vector field K. Moreover, $[K, X] = 0$, $\mathcal{L}_K \psi = 0$ and $\mathcal{L}_K B = 0$.

The matter equation for $n = 2$

- Define the function $\beta = \star B$. The matter equation becomes

$$\star(\mathsf{d}\beta - \beta X^\flat) = \mathsf{d}\psi - \psi X^\flat.$$

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- Other proof: [Dobkowski-Ryłko, Kamiński, Lewandowski, Szereszewski '18].

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$$\frac{1}{4}|\mathcal{L}_K g|^2 + 2|\nabla(\Gamma\rho)|^2 =$$

$$\nabla^a \left(K^b \nabla_{(a}K_{b)} - \tfrac{1}{2}K_a \Delta\Gamma - \tfrac{1}{2}K_a \nabla_b K^b - \lambda\Gamma K_a + \Gamma\rho\nabla_a(\Gamma\rho) \right)$$

$$+ \nabla_b K^b \left(-\frac{1}{2\Gamma}|K|^2 + \frac{1}{2}\Delta\Gamma + \frac{1}{2}\nabla_b K^b + \frac{1}{2\Gamma}K^b \nabla_b \Gamma + \lambda\Gamma - \Gamma\rho^2 \right).$$

Proof of theorem

- The vacuum calculation gives

$$\frac{1}{4}|\mathcal{L}_K g|^2 = \nabla_a(\dots^a) + \nabla_a K^a(\dots) - \rho^2 K^a \nabla_a \Gamma.$$

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- Corollary: The general non-trivial solution to the $n = 2$ Einstein-Maxwell NHE is given by extremal KN-(A)dS horizon.

Quasi-Einstein equations

- Quasi-Einstein equations (QEE): Riemannian manifold (M, g) of dimension n together with $X \in \mathfrak{X}(M)$ satisfying

$$R_{ab} = \frac{1}{m} X_a X_b - \nabla_{(a} X_{b)} + \lambda g_{ab}.$$

$m \neq 0$ and λ constants.

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- $m = \infty$: Ricci solitons [Hamilton '98].

Rigidity of quasi-Einstein metrics

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$$\int_M |\mathcal{L}_K g|^2 \,\mathrm{vol}_g = \frac{4}{m}(2-m)\int_M R_{ab}K^a\nabla^b\Gamma \,\mathrm{vol}_g.$$

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Theorem

Let (M, g) be a compact Riemannian n-manifold without boundary admitting a non-gradient vector field X such that the QEE hold with either (i) $m > 2$ or (ii) $m \leq 2 - n$. Then (M, g) admits a Killing vector field K. Moreover, $[K, X] = 0$.

QEE Tensor identity

- QEE in terms of $K = \frac{2}{m}\Gamma X + \nabla\Gamma$:

$$R_{ab} = \frac{m}{4\Gamma^2} K_a K_b - \frac{m}{2\Gamma}\nabla_{(a}K_{b)} + \frac{m}{2\Gamma}\nabla_a\nabla_b\Gamma - \frac{m}{4\Gamma^2}(\nabla_a\Gamma)(\nabla_b\Gamma) + \lambda g_{ab}.$$

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Proposition

For any solution to the QEE with $m \neq 2$ the following identity holds

$$\frac{1}{4}\Gamma^{\frac{m-2}{2}}|\mathcal{L}_K g|^2 + \frac{1}{m-2}\Gamma^{\frac{m-2}{2}}(\nabla_a K^a)^2 = \nabla_a\left(\Gamma^{\frac{m-2}{2}}K^a\right)H + \nabla_a V^a.$$

Here

$$H = -\frac{|K|^2}{2\Gamma} + \tfrac{1}{2}\Delta\Gamma + \tfrac{1}{4}(m-2)\frac{|\nabla\Gamma|^2}{\Gamma} + \tfrac{m}{2(m-2)}\nabla_a K^a + \lambda\Gamma,$$

$$V^a = \Gamma^{\frac{m-2}{2}}K_b\nabla^{(a}K^{b)} - \tfrac{m-2}{4}|\nabla\Gamma|^2\Gamma^{\frac{m-4}{2}}K^a - \tfrac{1}{2}\Gamma^{\frac{m-2}{2}}(\nabla_b K^b)K^a - \tfrac{1}{2}\Gamma^{\frac{m-2}{2}}(\Delta\Gamma)K^a - \lambda\Gamma^{\frac{m}{2}}K^a.$$

QEE Killing vector

- Let Ψ be a smooth positive function satisfying

$$\Delta\Psi + \nabla_a(\Psi X^a) = 0.$$

Choose Γ to be $\Psi^{\frac{2}{m}}$, so that $\Gamma^{\frac{m-2}{2}}K$ is divergence-free.

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For $m < 2 - n$ same result holds after using $|\mathcal{L}_K g|^2 \geq \frac{4}{n}(\nabla_a K^a)^2$.
For $m = 2 - n$ find that K is CKVF. Use $\nabla_a V^a = 0 \implies \mathcal{L}_K g = 0$.

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  For $m = 2 - n$ find that K is CKVF. Use $\nabla_a V^a = 0 \implies \mathcal{L}_K g = 0$.

- $\nabla_a K^a = \nabla_a (\Gamma^{\frac{m-2}{2}} K^a) = 0$ implies $\mathcal{L}_K \Gamma = 0$ and so $[K, X] = 0$. □

Generalized extremal horizon equations

Definition [Kamiński-Lewandowski '24]

A metric g and vector field X on a surface M satisfy the generalized extremal horizon equation (GEHE) for some $f \in C^\infty(M)$ and $c \neq 0$ if

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$$\nabla_{(a}X_{b)} + cX_aX_b + fg_{ab} = 0.$$

- $c = -\frac{1}{2}, f = \frac{1}{2}R - \lambda$: vacuum NHE.
- $c = -\frac{1}{2}, f = \frac{1}{2}R - \lambda - \rho^2$: Einstein-Maxwell NHE.
- $c = -\frac{1}{m}, f = \frac{1}{2}R - \lambda$: QEE.

Topological rigidity

Theorem

Let (g, X) be a solution to the GEHE on a closed, connected and oriented surface M with X not identically zero. Then M is diffeomorphic to S^2.

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- Poincaré-Hopf theorem: Let M be a closed manifold and X a vector field on M having only isolated zeros. The sum of the indices of the zeros of X equals the Euler characteristic $\chi(M)$.

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- [Kamiński-Lewandowski '24]: proof based on holomorphic vector fields.
- Poincaré-Hopf theorem: Let M be a closed manifold and X a vector field on M having only isolated zeros. The sum of the indices of the zeros of X equals the Euler characteristic $\chi(M)$.
- Recall the index of X at an isolated zero $p \in M$ is defined as the degree of the map $X/|X| : \partial D \to S^{n-1}$, where D is a coordinate disk around p s.t. p is the only zero of X in D.

Proof of theorem: step 1

- Outline of proof: show that

1. X has at least one zero.
2. Any zero of X is isolated.
3. The index of X at any zero is positive.

This implies $\chi(M) > 0$ and hence $M \cong S^2$.

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- Step 1: Use the trace of GEHE to express f in terms of X. Then contract the GEHE twice with X to find [Jezierski '09]

$$\nabla_a \left(\frac{X^a}{|X|^2} \right) = c.$$

  On a closed manifold M this shows X must have zero.

Proof of theorem: step 2

- Step 2: Introduce complex coords $(z, \bar{z})$ around a zero $p \in U$ and functions $H : U \to \mathbb{R}$, $P : U \to \mathbb{C}$ s.t.

$$g = 2e^H \mathrm{d}z\mathrm{d}\bar{z}, \qquad X^\flat = P\mathrm{d}z + \bar{P}\mathrm{d}\bar{z}.$$

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- Define a complex function F locally by $\partial_{\bar{z}} F = \bar{P}$. The $(\bar{z}\bar{z})$-component of the GEHE gives

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- Note: if $M = S^2$ we can define F globally by $\bar{\partial} F = (X^\flat)^{(0,1)}$. The computation above then shows that $V = e^{cF} X^{(1,0)}$ is a holomorphic vector field.

Proof of theorem: step 3

- Step 3: motivated by [Chruściel-Szybka-Tod '17]. Key ingredient
- Lemma [Milnor '65]: Let p be a zero of a vector field X. If (in some coordinates) $\det(\partial_\mu X^\nu) > 0$ at p, then then the zero is isolated and of index 1.

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  Hence $\det(\partial_\mu X^\nu) > 0$ at p unless X, f, Ω vanish simultaneously.
- In this degenerate case it can be shown that $\det(\partial_\mu X^\nu)$ has a strict minimum at p, which still implies that the index is positive.

Summary

- Main results
  - 4D Einstein-Maxwell Theory: every non-trivial extremal horizon cross-section admits a Killing vector and hence is given by the extremal KN family.
  - Quasi-Einstein equation: every compact non-gradient solution to the QEE with $m > 2$ or $m \leq 2 - n$ admits a Killing vector preserving X.
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  - Generalized extremal horizon equation: every non-trivial solution is (up to a double cover) on the two-sphere S^2.
- Open problems
  - Killing vector for the QEE with $m \in (2 - n, 2)$?
  - Other theories, e.g. 5D Einstein-Maxwell Chern-Simons.

Thank you