Theory of Relativity Seminar, Warsaw

Newman-Janis algorithms application to regular black hole models
Regular rotating black hole: to Kerr or not to Kerr?

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Newman-Janis algorithm's application to regular black hole models,
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and

Leonardo Chataignier, Alexander Kamenshchik, Alessandro Tronconi and Giovanni Venturi,
Regular black holes, universes without singularities, and phantom-scalar field transitions,
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Introduction and Motivations

- Appearance of singularities is one of the most important phenomena in General Relativity and in its generalizations and modifications.
- The singularities were first discovered in such simple geometries as those of Friedmann and Schwarzschild and later their general character was established (Penrose, Hawking).
- The investigation of the oscillatory approach to the cosmological singularity (Belinsky, Khalatnikov, Lifshitz) known also as Mixmaster universe (Misner) has opened the way to the birth of a new branch of the mathematical physics chaotic cosmology and hyperbolic Kac-Moody algebras (Damour, Henneaux, Nicolai).

Introduction and Motivations

- The investigation of the non-singular (or so-called regular) black holes and their regular rotating counterparts is extremely popular nowadays.
- There are also attempts to construct non-singular cosmological models.
- It is easy to construct a regular black-hole type solution, it is enough to put some regularizing parameters into the metric.
- It is difficult to do it in such a way to have a plausible matter content.
- The Newman-Janis algorithm permits to obtain the metric of the Kerr rotating black hole starting from the Schwarzschild metric.

- We have obtained Kerr-like regular rotating black hole starting from the Schwarzschild-like black hole, sustained by the phantom scalar field.
- There is no (simple) matter, which could sustain our construction.

Seed (spherically symmetric static) regular geometry

The study of the regular black holes has rather a long history. Recently, Simpson and Visser, 2018 suggested a very simple modification of the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2m}{\sqrt{u^2+b^2}}\right) dt^2 - \left(1 - \frac{2m}{\sqrt{u^2+b^2}}\right)^{-1} du^2$$
$$-(u^2+b^2)(d\theta^2 + \sin^2\theta d\varphi^2),$$

where the singularity at $r = 0$ is replaced by a regular minimum of $r(u)$ at $u = 0$, a sphere of radius b. If $b > 2m$, this metric describes a wormhole with a throat at $u = 0$; if $b < 2m$, one has a black hole with two horizons at $u = \pm\sqrt{4m^2 - b^2}$; and if $b = 2m$, we see an extremal black hole with a single horizon at $u = 0$.

What kind of matter can sustain this black hole? It is difficult to invent it.

We would like to have a regular black hole where the role of matter is played by a scalar field.

Consider the following spherically symmetric metric

$$ds^2 = A(u)\left(dx^0\right)^2 - \frac{du^2}{A(u)} - r^2(u)\, d\Omega_2^2,$$

where $d\Omega_2^2 = \left(dx^2\right)^2 + \sin^2 x^2 \left(dx^3\right)^2$ is the line element on a unit sphere; the area function $r(u)$ is regular and positive everywhere and has at least one minimum at some $u = u_{\min}$, at which $r(u_{\min}) > 0$, $r'(u_{\min}) = 0$, and $r''(u_{\min}) > 0$, providing the existence of two asymptotic regions with $r(u) \sim |u|$ at $u \to \pm\infty$.

The Einstein tensor:

$$G_0^0 = -A'\frac{r'}{r} - 2A\frac{r''}{r} - A\frac{r'^2}{r^2} + \frac{1}{r^2},$$
$$G_u^u = -A'\frac{r'}{r} - A\frac{r'^2}{r^2} + \frac{1}{r^2},$$
$$G_2^2 = G_3^3 = -\frac{1}{2}A'' - A'\frac{r'}{r} - A\frac{r''}{r}.$$

For a minimally coupled scalar field $\phi(r)$:

$$T_0^0[\phi] = T_2^2[\phi] = T_3^3[\phi].$$

Hence,

$$\frac{1}{2}A'' - A\frac{r''}{r} - A\frac{r'^2}{r^2} + \frac{1}{r^2} = 0.$$

To find a globally regular geometry, we choose the simplest possible area function

$$r(u) = \sqrt{u^2 + b^2}, \quad b = \text{const} > 0.$$

The exact solution of the equation for A is

$$A(u) = 1 + c_1\left(u^2 + b^2\right) + c_2\left(\left(u^2 + b^2\right)\tan^{-1}\frac{u}{b} + ub\right).$$

By setting $c_1 = -\pi c_2/2$ and $c_2 b^3 \equiv u_0$ to ensure the regularity of $A(u)$ at $b \to 0$ and the Schwarzschild form, i.e., $A(u) \simeq 1 - 2u_0/3u$ at $u \to +\infty$ correspondingly, we get

$$A(u) = 1 - \frac{u_0}{b^3}\left(\left(u^2 + b^2\right)\cot^{-1}\frac{u}{b} - ub\right).$$

We have a traversable wormhole if $2b > \pi u_0$; a regular black hole if $0 < 2b < \pi u_0$ with a single horizon at u_h, which is a regular zero of $A(u_h) = 0$; or a regular black hole with a single extremal horizon (black throat) at $u = 0$ if $2b = \pi u_0$. Beyond the event horizon (if it exists), there is a bounce to anisotropic Kantowski–Sachs cosmology with two scale factors, $A(u)$ and $r(u)$.

This solution was first obtained in
K.A. Bronnikov and J.C. Fabris,
Regular Phantom Black Holes,
Phys. Rev. Lett. 96 (2006) 251101.

From the regular Schwarzschild-like black hole to the regular Kerr-like black hole

The Newman–Janis algorithm provides the derivation of the rotating solutions from the static ones.
One introduces a vierbein of null vectors:

$$\mathbf{e}_\alpha = \left(\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\right)$$

and a series of complex conjugation transformations.
We switch to a null coordinate system, replacing in the time coordinate x^0 with the null time coordinate τ via $dx^0 \to d\tau = dx^0 - du\,/A(u)$.
In the Eddington–Finkelstein type coordinates:

$$ds^2 = A(u)d\tau^2 + 2d\tau du - r^2(u)d\Omega_2^2.$$

In the Newman–Penrose tetrad formalism:

$$ds^2 = \left(l_\mu n_\nu - m_\mu \bar{m}_\nu\right) dx^\mu dx^\nu,$$

$$l^\mu = \delta^\mu_u, \qquad n^\mu = \delta^\mu_\tau - \frac{A(u)}{2}\delta^\mu_u,$$
$$m^\mu = \frac{1}{\sqrt{2}r(u)}\left(\delta^\mu_2 + \frac{i}{\sin x^2}\,\delta^\mu_3\right),$$

where l^μ and n^μ being real null vectors, and m^μ and its complex conjugate $\bar{m}^\mu$ being complex null vectors.

$$l_\mu m^\mu = l_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0, l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1.$$

Complexification of the seed metric:

$$A(u,\bar{u}) = 1 + \frac{u_0}{2b^2}(u+\bar{u}) - \\ - \frac{u_0}{2b^3}\left(\left(u^2+b^2\right)\cot^{-1}\frac{u}{b} + \left(\bar{u}^2+b^2\right)\cot^{-1}\frac{\bar{u}}{b}\right),$$

$$r(u) = \sqrt{u^2+b^2}, \quad \bar{r}(u) = \sqrt{\bar{u}^2+b^2},$$

requiring at $u = \bar{u}$ the recovery of initial vierbein.
We apply complex transformation:

$$x^\mu \to x'^\mu = x^\mu - ia\cos x^2\left(\delta^\mu_\tau - \delta^\mu_u\right),$$

treating the primed coordinates as real.

Through the null complex tetrad transform,
$e^{\mu}_{\alpha} \to e'^{\mu}_{\alpha} = e^{\nu}_{\alpha}\, \partial x'^{\mu}/\partial x^{\nu}$, and the use of the new light-like vectors

$$l'^{\mu} = \delta^{\mu}_{u'}, \qquad n'^{\mu} = \delta^{\mu}_{\tau'} - \frac{A\left(u', x^{2'}\right)}{2}\delta^{\mu}_{u'},$$

$$m'^{\mu} = \frac{ia \sin x^{2'}\left(\delta^{\mu}_{\tau'} - \delta^{\mu}_{u'}\right) + \delta^{\mu}_{2'} + \dfrac{i}{\sin x^{2'}}\,\delta^{\mu}_{3'}}{\sqrt{2}\,\bar{r}\left(u', x^{2'}\right)},$$

$$\bar{m}'^{\mu} = \frac{-ia \sin x^{2'}\left(\delta^{\mu}_{\tau'} - \delta^{\mu}_{u'}\right) + \delta^{\mu}_{2'} - \dfrac{i}{\sin x^{2'}}\,\delta^{\mu}_{3'}}{\sqrt{2}\,r\left(u', x^{2'}\right)},$$

it yields the new $g'^{\mu\nu} = 2l'^{(\mu}n'^{\nu)} - 2m'^{(\mu}\bar{m}'^{\nu)}$ expression

$$g'^{\mu\nu} = \begin{pmatrix} -\frac{a^2\sin^2 x^{2\prime}}{r\bar{r}(u',x^{2\prime})} & 1+\frac{a^2\sin^2 x^{2\prime}}{r\bar{r}(u',x^{2\prime})} & 0 & -\frac{a}{r\bar{r}(u',x^{2\prime})} \\ 1+\frac{a^2\sin^2 x^{2\prime}}{r\bar{r}(u',x^{2\prime})} & -A(u',x^{2\prime})-\frac{a^2\sin^2 x^{2\prime}}{r\bar{r}(u',x^{2\prime})} & 0 & \frac{a}{r\bar{r}(u',x^{2\prime})} \\ 0 & 0 & -\frac{1}{r\bar{r}(u',x^{2\prime})} & 0 \\ -\frac{a}{r\bar{r}(u',x^{2\prime})} & \frac{a}{r\bar{r}(u',x^{2\prime})} & 0 & -\frac{1}{r\bar{r}(u',x^{2\prime})\sin^2 x^{2\prime}} \end{pmatrix}.$$

The covariant metric in the ingoing Eddington–Finkelstein coordinates is

$$\begin{aligned} ds'^2 &= A(u',x^{2'})\left(d\tau' - a\sin^2 x^{2'}dx^{3'}\right)^2 \\ &+2\left(d\tau' - a\sin^2 x^{2'}dx^{3'}\right)\left(du' + a\sin^2 x^{2'}dx^{3'}\right) \\ &-r\bar{r}(u',x^{2'})\left(\left(dx^{2'}\right)^2 + \sin^2 x^{2'}\left(dx^{3'}\right)^2\right). \end{aligned}$$

$$
\begin{aligned}
A\left(u', x^{2'}\right) &= 1 + \frac{u_0 u'}{b^2} \\
&+ \frac{u_0}{2b^3}\left(u'^2 - a^2\cos^2 x^{2'} + b^2\right) \\
&\times \left(\tan^{-1}\frac{u'}{b + a\cos x^{2'}} + \tan^{-1}\frac{u'}{b - a\cos x^{2'}} - \pi\right) \\
&+ \frac{a u_0 u'}{2b^3}\cos x^{2'} \ln \frac{{u'}^2 + \left(b - a\cos x^{2'}\right)^2}{{u'}^2 + \left(b + a\cos x^{2'}\right)^2},
\end{aligned}
$$

$$
r\bar{r}\left(u', x^{2'}\right) = \sqrt{\left(u'^2 - a^2\cos^2 x^{2'} + b^2\right)^2 + 4a^2 u'^2 \cos^2 x^{2'}}.
$$

The obtained geometry does not contain Kerr's usual ring coordinate singularity at $u' = 0$ and $x^{2'} = \pi/2$, and turns into the Kerr original one at the $b \to 0$ limit:

$$g'_{\mu\nu}\Big|_{b=0} =$$

$$\begin{pmatrix} 1 - \frac{2u_0 u'}{3\left(u'^2 + a^2\cos^2 x^{2'}\right)} & 1 & 0 & \frac{2au_0 u' \sin^2 x^{2'}}{3\left(u'^2 + a^2\cos^2 x^{2'}\right)} \\ 1 & 0 & 0 & -a\sin^2 x^{2'} \\ 0 & 0 & -u'^2 - a^2\cos^2 x^{2'} & 0 \\ \frac{2au_0 u' \sin^2 x^{2'}}{3\left(u'^2 + a^2\cos^2 x^{2'}\right)} & -a\sin^2 x^{2'} & 0 & -\left(u'^2 + a^2 + \frac{2a^2 u_0 u' \sin^2 x^{2'}}{3\left(u'^2 + a^2\cos^2 x^{2'}\right)}\right)\sin^2 x^{2'} \end{pmatrix},$$

and the Schwarzschild one in the Eddington–Finkelstein null coordinates for $a = 0$.

The curvature invariants for the obtained rotated solution are finite in the entire range of the u' coordinate.

The Ricci scalar, the Ricci tensor squared, and the Kretschmann scalar

$$R \sim \left(r\bar{r}\right)^{-3}, \quad R_{\alpha\beta}R^{\alpha\beta} \sim \left(r\bar{r}\right)^{-6},$$
$$\text{and} \quad \mathcal{K} \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \sim \left(r\bar{r}\right)^{-6}$$

are globally regular if $a \neq b$, and at the $b \rightarrow 0$ limit the standard features of the Kerr spacetime are observed.

As the final step of the NJA, one reverts the metric to the Boyer–Lindquist coordinates, which have only a single off-diagonal component, $g_{\tau x^3}$. However, it is not always possible to find such integrable coordinate transformation, $d\tau' \rightarrow d\tau = d\tau' - \alpha(u')du'$ and $dx'^3 \rightarrow dx^3 = dx'^3 - \beta(u')du'$, that leave $\alpha(u')$ and $\beta(u')$ independent of $x^{2'}$ as in our non-empty case here.

For a small regularization parameter b, these functions

$$\alpha(u', x^{2'}) = \frac{g'^{\tau u}}{g'^{uu}} \simeq \frac{u'^2 + a^2}{u'^2 + a^2 - 2u_0 u'/3} + O(b^2),$$

$$\beta(u', x^{2'}) = \frac{g'^{ux^3}}{g'^{uu}} \simeq \frac{a}{u'^2 + a^2 - 2u_0 u'/3} + O(b^2)$$

provide the well-known Boyer–Lindquist transform, and spacetime being algebraically general, degenerates to an algebraically special and of Petrov type D up to $O(b^2)$; and in the slow rotation approximation:

$$\alpha(u', x^{2'}) \simeq \frac{1}{A(u')} - \frac{a^2(1 - A(u'))}{A^2(u')r^2(u')} + O(a^4),$$

$$\beta(u', x^{2'}) \simeq \frac{a}{A(u')r^2(u')} + O(a^3).$$

The metric can also be reduced to the Boyer–Lindquist representation:

$$
\begin{aligned}
ds^2_{slow} &\simeq \Big(A(u') + O\big(a^2\big)\Big)d\tau^2 \\
&+\Big(2a\sin^2 x^{2'}(1 - A(u')) + O\big(a^3\big)\Big)d\tau du' \\
&-\Big(A^{-1}(u') + O\big(a^2\big)\Big)du'^2 \\
&-\Big(r^2(u') + O\big(a^2\big)\Big)\Big(\big(dx^{2'}\big)^2 + \sin^2 x^{2'}\big(dx^3\big)^2\Big),
\end{aligned}
$$

coinciding at $u' \to +\infty$ with the slow rotation limit of the standard Kerr solution in these coordinates.

Scalar field

We apply a series of complex conjugation transformations to the scalar field in the Newman–Janis spirit.

The scalar field's stress-energy tensor is

$$T^{\mu}_{\nu}[\phi] = \epsilon \phi^{,\mu}\, \phi_{,\nu} - \frac{\delta^{\mu}_{\nu}}{2}\, \epsilon\, \phi^{,\alpha}\, \phi_{,\alpha} + \delta^{\mu}_{\nu} V(\phi),$$

where $\epsilon = +1$ corresponds to a canonical scalar field ϕ_{c} and $\epsilon = -1$ to a phantom one ϕ_{ph}.
Assuming $\phi = \phi(u)$, we obtain

$$-2\frac{r''}{r} = \epsilon \phi'^2 \quad \rightarrow \quad \phi_{\mathrm{ph}}(u) = \pm\sqrt{2}\tan^{-1}\frac{u}{b} + \phi_0.$$

We chose the minus sign and $\phi_0 = \pi/\sqrt{2}$, resulting in $\phi_{\mathrm{ph}}(u) = \sqrt{2}\cot^{-1}(u/b)$.

The sum of G_τ^τ and G_u^u components leads to an expression for potential in terms of the radial coordinate u:

$$V(u) = \frac{u_0\left(\left(3u^2+b^2\right)\cot^{-1}\frac{u}{b} - 3ub\right)}{b^3\left(u^2+b^2\right)}.$$

We can reconstruct the exact expression for the potential via the phantom scalar field:

$$V(\phi_{\mathrm{ph}}) = \frac{u_0\phi_{\mathrm{ph}}}{\sqrt{2}b^3}\left(3 - 2\sin^2\frac{\phi_{\mathrm{ph}}}{\sqrt{2}}\right) - \frac{3u_0}{2b^3}\sin\sqrt{2}\phi_{\mathrm{ph}},$$

where we have used the inversion $u = b\cot\frac{\phi_{\mathrm{ph}}}{\sqrt{2}}$.

We would like to complexify the scalar field, the potential, and the Lagrangian density, introducing $\phi_{\mathrm{ph}}(u,\bar{u})$, $V(u,\bar{u})$, and $L(u,\bar{u})$, and to apply the complex transformation of the coordinates $x^\mu \to x'^\mu$. Then, for the scalar field, dropping coordinate prime indices, we have

$$\phi_{\mathrm{ph}}\left(u, x^2\right) = \frac{\pi}{\sqrt{2}} - \frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{b + a\cos x^2} - \\ -\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{b - a\cos x^2}\,.$$

The energy-momentum tensor is

$$T^{\mu}_{\nu} =$$

$$\begin{pmatrix} -L(u,x^2) & -\left(1+\frac{a^2\sin^2 x^2}{r\bar{r}(u,x^2)}\right)(\phi_{\rm ph})'^{\,2}_u & -\left(1+\frac{a^2\sin^2 x^2}{r\bar{r}(u,x^2)}\right)(\phi_{\rm ph})'_u(\phi_{\rm ph})'_{x^2} & 0 \\ 0 & \left(A(u,x^2)+\frac{a^2\sin^2 x^2}{r\bar{r}(u,x^2)}\right)(\phi_{\rm ph})'^{\,2}_u - L(u,x^2) & \left(A(u,x^2)+\frac{a^2\sin^2 x^2}{r\bar{r}(u,x^2)}\right)(\phi_{\rm ph})'_u(\phi_{\rm ph})'_{x^2} & 0 \\ 0 & \frac{(\phi_{\rm ph})'_u(\phi_{\rm ph})'_{x^2}}{r\bar{r}(u,x^2)} & \frac{(\phi_{\rm ph})'^{\,2}_{x^2}}{r\bar{r}(u,x^2)} - L(u,x^2) & 0 \\ 0 & -\frac{a(\phi_{\rm ph})'^{\,2}_u}{r\bar{r}(u,x^2)} & -\frac{a(\phi_{\rm ph})'_u(\phi_{\rm ph})'_{x^2}}{r\bar{r}(u,x^2)} & -L(u,x^2) \end{pmatrix},$$

where

$$
\begin{aligned}
L(u,x^2) &= \frac{1}{b^2\left(r\bar{r}(u,x^2)\right)^4}\Bigg(\left(u^2+b^2\right)^2\left(3u_0u^3+4u_0b^2u+b^4\right)\\
&+\Big(u_0u\left(9u^4+4b^2u^2-b^4\right)-2b^4\left(3u^2+b^2\right)\Big)a^2\cos^2x^2\\
&+\left(9u_0u^3-6u_0b^2u+b^4\right)a^4\cos^4x^2+3u_0u\,a^6\cos^6x^2\Bigg)\\
&+\frac{u_0u\,a\cos x^2}{2b\left(r\bar{r}(u,x^2)\right)^2}\ln\frac{u^2+\left(b-a\cos x^2\right)^2}{u^2+\left(b+a\cos x^2\right)^2}\\
&+\frac{u_0}{2b^3\left(r\bar{r}(u,x^2)\right)^2}\Bigg(\left(3u^2+2b^2\right)\left(u^2+b^2\right)\\
&+\left(6u^2-5b^2\right)a^2\cos^2x^2+3a^4\cos^4x^2\Bigg)\\
&\times\left(\tan^{-1}\frac{u}{b+a\cos x^2}+\tan^{-1}\frac{u}{b-a\cos x^2}-\pi\right).
\end{aligned}
$$

The non-trivial components of the obtained stress-energy tensor are asymptotically trivial, see at $u \to +\infty$, and $T^{\mu}_{\nu}[\phi_{\rm ph}] \sim O(b^2)$ at the $b \to 0$ limit, and turn out coinciding with the exact non-rotation ones if $a = 0$.

The mixed Einstein tensor components G^{μ}_{ν} are all non-trivial. The Einstein equations, $G^{\mu}_{\nu} = T^{\mu}_{\nu}[\phi_{\rm ph}]$, are satisfied asymptotically, being noticeably violated only at distances of the order of the regularization parameter b.

Regular cosmological models

We can construct non-singular cosmological models, using Simpson-Visser-like method.

Flat Friedmann model with a scalar field

Let us consider a flat Friedmann universe filled with a massless scalar field.

$$ds^2 = dt^2 - t^{\frac{2}{3}}(dx_1^2 + dx_2^2 + dx_3^2),$$

$$\dot{\phi} = \sqrt{\frac{2}{3}}\frac{1}{t}.$$

Let us now construct the regularized metric:

$$ds^2 = dt^2 - (t^2 + b^2)^{\frac{1}{3}}(dx_1^2 + dx_2^2 + dx_3^2).$$

$$R_0^0 = \frac{2t^2 - 3b^2}{3(t^2 + b^2)^4},$$

$$R_1^1 = R_2^2 = R_3^3 = -\frac{b^2}{3(t^2 + b^2)^2}.$$

$$R = \frac{2t^2 - 6b^2}{3(t^2 + b^2)^2}.$$

The Friedmann equations give the expressions for the energy density and for the isotropic pressure of matter

$$\rho = \frac{t^2}{3(t^2 + b^2)^2},$$

$$p = \frac{t^2 - 2b^2}{3(t^2 + b^2)^2}.$$

Let us suppose that the universe is filled with a spatially homogeneous scalar field with a potential $V(\phi)$.

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

Then

$$\dot{\phi} = \pm\sqrt{\frac{2}{3}}\frac{\sqrt{t^2 - b^2}}{t^2 + b^2},$$

$$V = \frac{b^2}{3(t^2 + b^2)^2}.$$

What happens at $|t| < b$?

The kinetic energy of ϕ changes sign and the standard scalar field transforms into a phantom scalar field.

We can study the behaviour of the potential V in the vicinity of $t = b$.

$$t = b + \tau,$$

where τ is small.

$$\frac{d\phi}{d\tau} = \frac{\sqrt{\tau}}{\sqrt{3b^3}},$$

$$\phi(\tau) = \phi_0 + \frac{2\tau^{3/2}}{3\sqrt{3b^3}}.$$

$$\tau = 3b\left(\frac{\phi - \phi_0}{2}\right)^{\frac{2}{3}}.$$

In the vicinity of the critical point:

$$V(\phi) = \frac{1}{3b^2\left[\left(1+3\left(\frac{\phi-\phi_0}{2}\right)^{\frac{2}{3}}\right)^2+1\right]^2}.$$

By keeping only the leading terms:

$$V(\phi) = \frac{1}{12b^2}\left[1-6\left(\frac{\phi-\phi_0}{2}\right)^{\frac{2}{3}}\right].$$

The distinguishing feature of this expressions is the presence of a non-analyticity of the cusp type, which is responsible for the transition from the standard scalar field to its phantom counterpart and vice versa.

It is interesting that a similar phenomenon of the transition from the phantom and non-phantom phases of the scalar field was found in another context in Andrianov, Cannata and Kamenshchik, 2005, Smooth dynamical crossing of the phantom divide line of a scalar field in simple cosmological models, Phys. Rev. D 72, 043531.

The potential of the scalar field had also a cusp with the same type of non-analyticity $(\phi - \phi_0)^{2/3}$.

A slightly more general model

$$ds^2 = dt^2 - t^{2\alpha}(dx_1^2 + dx_2^2 + dx_3^2).$$

Such an evolution arises in a universe filled with a perfect fluid with the equation of state parameter

$$w = \frac{2 - 3\alpha}{3\alpha}.$$

The results obtained in this model are similar to those described above.

An analogous study was undertaken also for a Bianchi-I model.

Conclusions

- The appearance of the singularities in the cosmological and other gravitational systems is not drawback of models or theories
- It is their distinguishing feature.
- Rather than avoid singularities, it is better to study how their presence influences the non-singular quantities (just like in quantum field theory).

- For simple systems (like Friedmann universes or Schwarzschild black holes) it is possible to construct their regular analogs sustained by more or less reasonable energy-momentum tensors.
- It is not easy to find the regular version of the rotating black hole sustained by a plausible matter source.

G.F.R. Ellis and D. Garfinkle,
The Synge G-Method: Cosmology, Wormholes, Firewalls, Geometry,
arXiv: 2311.06881 [gr-qc].

"J.L. Synge many years ago showed how a simple process (his "G-Method") could lead purely by differentiation to exact solutions of the Einstein Field Equations.
However, this often leads to a negative inertial mass density, hence they are unphysical."

J.L. Synge, Relativity: The General Theory,
North-Hollad Publishing Company, Amsterdam, 1960.

Chapter IV. The Material Continuum.
Paragraph 6. Survey of field equations and coordinate conditions.

Any set of ten functions $g_{ij}(x)$, sufficiently smooth, define a Riemannian spacetime. If then we choose such functions arbitrarily, we have a universe in which the energy tensor is

$$T_{ij} = -k^{-1} G_{ij},$$

the Einstein tensor having been calculated from g_{ij} - this involves no more than finding g^{ij} algebraically and carrying out the required differentiations. There are no partial differential equations to solve. Since the procedure is based on chosen values of g_{ij}, we shall call it g-method.

Reversing the roles, we now regards T_{ij} as given (T-method), so that

$$G_{ij} = -kT_{ij}$$

is a set of ten non-linear second-order partial differential equations to be satisfied by g_{ij}.

C. W. Misner, Absolute Zero of Time,
Physical Review 186 (1969) 1328.

I prefer a more optimistic viewpoint ("Nature and Einstein are subtle but tolerant") which views the initial singularity in cosmological theory not as a proof of our ignorance, but as a source from which we can much valuable understanding of cosmology.

Thus, while I presume that relativity, like other physical theories, will be improved from time to time, I do not see that these changes need bear directly on the problem of cosmological singularity.

We should stretch our minds, find some more acceptable set of words to describe the mathematical situation, now identified as "singular", and then proceed to incorporate this singularity into our physical thinking until observational difficulties force revision on us.

The concept of a true initial singularity (as distinct from an indescribable early era at extravagant but finite high densities and temperatures) can be a positive and useful element in cosmological theory.

The Universe is meaningfully infinitely old because infinitely many things have happened since the beginning.