Hilbert spaces built over metrics of fixed signature

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Introduction

♠ Given manifold $\mathcal{M}$, there are two Hilbert spaces with diffeomorphism invariant inner products:

- the Hilbert space $\mathcal{H}^{1/2}$ of complex half-densities on $\mathcal{M}$;
- $L^2(\mathcal{M}, d\mu_c)$, where $d\mu_c$ is the counting measure on $\mathcal{M}$.

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♠ I will present two Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ such that

- each space will be constructed over the set of all metrics of fixed but arbitrary signature, defined on $\mathcal{M}$;
- the inner product on each Hilbert space will be diffeomorphism invariant;
- $\mathfrak{H}$ will be a generalization of $\mathcal{H}^{1/2}$, and $\mathfrak{K}$ a generalization of $L^2(\mathcal{M}, d\mu_c)$.

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♠ Diffeomorphism invariance of the inner products may be helpful in taking into account the vector constraint at the quantum level.

♠ Warning: neither $\mathfrak{H}$ nor $\mathfrak{K}$ is an L^2 space over the set of metrics. The application is not straightforward and further research is needed...

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♠ The measure field $x \mapsto d\mu_x$ should be diffeomorphism invariant.

Diffeomorphism invariant measure field

♠ Let $\theta \in \mathrm{Diff}(\mathcal{M})$. If $x_1 = \theta(x_2)$, then

$$\theta^t : T_{x_2}\mathcal{M} \to T_{x_1}\mathcal{M}, \qquad \theta^{t*} : \Gamma_{x_1} \to \Gamma_{x_2}$$

and $(\theta^{t*})_\star d\mu_{x_1}$ is a measure on Γ_{x_2}.

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♠ If $x \mapsto d\mu_x$ is diff. invariant, then each $d\mu_x$ must be invariant w.r.t. the action of $GL(T_x\mathcal{M})$. Does such a measure exist on Γ_x?

Invariant measures on Γ_x

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♠ $(\gamma^I) \equiv (\gamma_{ij})_{i\leq j}$ are global coordinates on Γ_x and

$$d\mu_Q = \sqrt{|\det Q_{IJ}|}\, d\gamma^1 d\gamma^2 \dots d\gamma^{\dim \Gamma_x}$$

is an invariant measure on Γ_x.

Diffeomorphism invariant measure field

♠ Let $l_{xx'} : T_{x'}\mathcal{M} \to T_x\mathcal{M}$ denotes a linear isomorphism and $d\mu_{x_0}$ be an invariant measure on Γ_{x_0}. If

$$x \mapsto d\mu_x := (l^*_{x_0 x})_\star \, d\mu_{x_0}, \tag{1}$$

then

- for every x, $d\mu_x$ is an inv. measure on Γ_x, independent of $l_{x_0 x}$;
- for every two points $x, x' \in \mathcal{M}$ and for every $l_{xx'}$,

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♠ For any two fields (1) there exists $c > 0$ such that:

$$\forall x \in \mathcal{M} \quad d\check{\mu}_x = c \, d\mu_x.$$

How to merge the spaces $\{H_x\}$?

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How to merge the spaces $\{H_x\}$?

♠ Let $\{H_x\}_{x\in\mathcal{M}}$ be defined by a diff. inv. measure field (1), and let $\langle\cdot|\cdot\rangle_x$ denote the inner product on H_x.

♠ If Ψ, Ψ' are sections of the bundle-like set $\bigcup_{x\in\mathcal{M}} H_x$, then

$$x \mapsto \langle\Psi(x)|\Psi'(x)\rangle_x \qquad (2)$$

is a complex function on $\mathcal{M}$, which, once integrated over $\mathcal{M}$, defines an inner product on a set of such sections.

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♠ To get a diffeomorphism invariant inner product one can

- use the counting measure on $\mathcal{M}$;
- "densitize" the function (2).

Half-densities valued in a Hilbert space

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♠ α-density over V valued in a vector space W is a map

$$\tilde{w} : \{ \text{ all bases of } V \ \} \to W,$$

such that

$$\tilde{w}(\Lambda^i{}_j e_i) = |\det(\Lambda^i{}_j)|^\alpha \, \tilde{w}(e_i).$$

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$$(e_i) \mapsto (\tilde{w}|\tilde{w}')(e_i) := \langle \tilde{w}(e_i) | \tilde{w}'(e_i) \rangle \in \mathbb{C}$$

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♠ $(\tilde{w}, \tilde{w}') \mapsto (\tilde{w}|\tilde{w}')$ is then a *density product*.

Hilbert half-densities on $\mathcal{M}$

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♠ Half-densities of finite norm $||\tilde{\Psi}|| := \sqrt{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}$??? Too complicated to prove that they form a Hilbert space..

Regular Hilbert half-densities

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♠ Suppose that (U, φ) is a local chart on $\mathcal{M}$, (x^i) the corresponding coordinate system and $\gamma \in \Gamma_x$. Given Hilbert half-density $\tilde{\Psi}$,

$$\tilde{\Psi}(x) \in \tilde{H}_x, \qquad \tilde{\Psi}(x, \partial_{x^i}) \in H_x, \qquad \tilde{\Psi}(x, \partial_{x^i}, \gamma) \in \mathbb{C}.$$

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♠ Coordinate representation of $\tilde{\Psi}$:

$$(x^i, \gamma_{ij}) \mapsto \psi(x^i, \gamma_{ij}) := \tilde{\Psi}(\varphi^{-1}(x^i), \partial_{x^i}, \gamma_{ij}\, dx^i \otimes dx^j)$$

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♠ Continuity and compact $\mathcal{M}$-support of $\tilde{\Psi}, \tilde{\Psi}'$ is not sufficient yet...

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♠ Fix a continuous Hilbert half-density $\tilde{\Psi}$. Its coordinate representation ψ given by a chart (U, φ), defines a function

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♠ We say that $\tilde{\Psi}$ is of compact and slowly changing Γ-support, if it is such with respect to charts, which altogether cover the manifold $\mathcal{M}$.

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♠ $\mathcal{H}_1$ is a generalization of the Hilbert space $\mathcal{H}^{1/2}$ of complex half-densities on $\mathcal{M}$, but it is not yet the generalization $\mathfrak{H}$ of $\mathcal{H}^{1/2}$ announced in the introduction.

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♠ But $\mathcal{H}_1$ does not contain any tensor product of $\{H_x\}_{x\in\mathcal{M}}$ and, consequently, $\mathcal{H}_1$ alone cannot be used for quantization.

♠ So there is a need for an extension of the construction.

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♠ Now it is (almost) straightforward to repeat the procedure which led to $\mathcal{H}_1$—this results is a Hilbert space $\mathcal{H}_N$ built of Hilbert half-densities on $\mathcal{M}_N$.

♠ The generalization of $\mathcal{H}^{1/2}$ announced in the introduction:

$$\mathfrak{H} := \bigoplus_{N=1}^{\infty} \mathcal{H}_N.$$

Action of $\mathrm{Diff}(\mathcal{M})$ on $\mathcal{H}_1$

♠ Let $\theta \in \mathrm{Diff}(\mathcal{M})$. If $x' = \theta(x)$, then

$$\theta^t : T_x\mathcal{M} \to T_{x'}\mathcal{M}, \qquad \theta^{t*} : \Gamma_{x'} \to \Gamma_x, \qquad \theta^{t**} : H_x \to H_{x'}.$$

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♠ Owing to diff. invariance of the field $x \mapsto d\mu_x$, the map

$$\tilde{\Psi} \to \theta^*\tilde{\Psi} \equiv U_1(\theta^{-1})\tilde{\Psi}$$

is unitary on $\mathcal{H}_1$ and $\theta \mapsto U_1(\theta)$ is a unitary representation of $\mathrm{Diff}(\mathcal{M})$.

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♠ Consequently,

$$\theta \mapsto \bigoplus_{N=1}^{\infty} U_N(\theta)$$

is a unitary representation of $\mathrm{Diff}(\mathcal{M})$ on $\mathfrak{H}$.

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♠ Thus $\mathfrak{H}$ is unique up to distinguished isomorphisms.

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♠ Let $\mathcal{K}_1$ be a set of sections of the bundle-like set $\bigcup_{x\in\mathcal{M}} H_x$:
$\Psi \in \mathcal{K}_1$ iff

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♠ The generalization of $L^2(\mathcal{M}, d\mu_c)$ announced earlier:

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♠ I managed to define on $\mathfrak{K}$ a representation of the so-called *affine commutation relations* for the ADM formalism. Unfortunately, this representation turned out to be highly reducible.

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