Conformal and CR methods in general relativity

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Norway grants

Project: *Conformal and CR methods in general relativity*; acronym: *ConfCRGR*; registration number: *2020/37/K/ST1/02788*;
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Outline

- The POLS fellowship
- Some old ideas from the Golden Age...
- New from Old

Norway grants

Norway grants

- Financial mechanism funded by **Norway** for the period 2014-2021
- Aims to **strengthen ties and cooperation** between Norway and the EU, and **reduce the disparity** in research performance across Europe
- **Poland** is its largest beneficiary

Basic Research programme (in Poland)

- Operated by the **National Science Centre**
- Norwegian partner: The **Research Council of Norway**

Aims

- To **boost the research potential** of Polish research institutions
- To **increase scientific excellence**
- To **support researchers** consolidating their research careers

The POLS fellowship

POLS fellowship

Small grant scheme for incoming mobility of researchers **from abroad to Poland**

Awarded on the basis of the scientific excellence, relevance, quality of implementation and potential impact of the proposal

My proposal

We shall investigate the **conformal** and **complex** properties of spacetimes in dimensions four and higher with a focus on congruences of null geodesics. In particular, we shall

- examine the relation between Lorentzian geometry and almost CR geometry, and
- apply conformal methods to the study of horizons and related geometries.

Outcome

- **Conceptual understanding** of Einstein spacetimes and horizon geometries in arbitrary dimensions
- **New solutions** to Einstein field equations and horizon geometries

TEAM

FACULTY OF PHYSICS, UNIVERSITY OF WARSAW

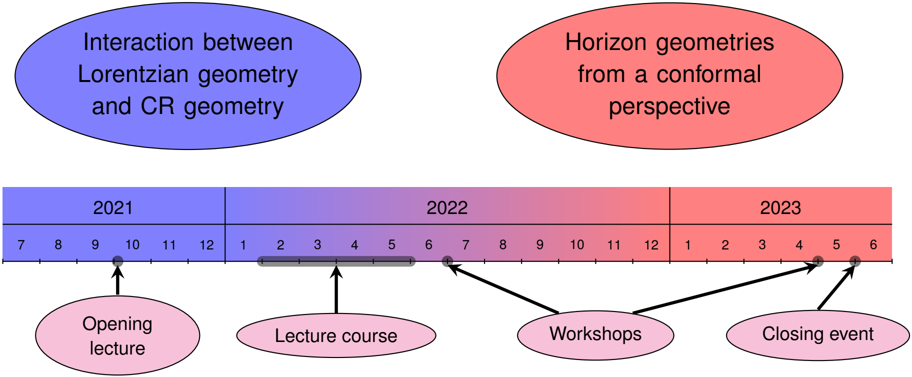
- Principal Investigator: **Arman Taghavi-Chabert**
- Co-investigators: **Jerzy Lewandowski** and two doctoral students

INTERNATIONAL COLLABORATION

- Arctic University of Norway, Tromsø
  Boris Kruglikov and **Dennis The**
  Cartan geometries, CR geometry, invariants, symmetries
- University of Auckland, New Zealand
  Rod Gover
  Tractor calculus, Cartan geometries and applications

Other research groups could also be involved in Warsaw (eg CFT PAN, IM PAN), Poland and beyond

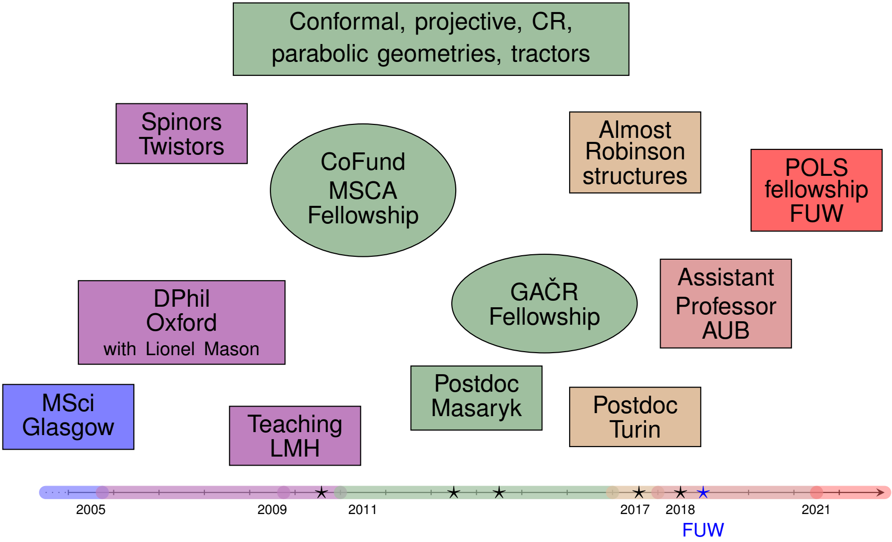
Structure and schedule (subject to COVID fluctuations)



Also on the agenda:

- Research trips to Tromsø, Auckland, etc.
- Dedicated **website**

My timeline



⋆: Visits to Cracow, GR20, CFT PAN, IM PAN, Simons Semester...

Non-shearing congruences of null geodesics

- A **non-shearing congruence of null geodesics (NSCNG)** on a Lorentzian 4-fold $(\mathcal{M}, g)$ is the set of the integrable curves of a non-vanishing null vector field k (ie $g(k,k) = 0$) that satisfies

$$\pounds_k g = \epsilon g + \kappa\alpha \,, \qquad \text{for some function } \epsilon, \text{ 1-form } \alpha,$$

where $\kappa = g(k, \cdot)$.

- Invariant under

$$g \mapsto \mathrm{e}^{\Omega} g \,, \qquad \textbf{Conformal invariance},$$
$$k \mapsto ak \,, \quad (a \neq 0) \,, \qquad \textbf{Null distribution } \langle k \rangle.$$

The Robinson congruence Nurowski–Trautman (2002)

Cast the Minkowski metric as

$$g = 2\,(\mathrm{d}u - \mathrm{i}\bar{z}\mathrm{d}z + \mathrm{i}z\mathrm{d}\bar{z})\,\mathrm{d}v + 2(v^2+1)\mathrm{d}z\mathrm{d}\bar{z} \,,$$

Then $k = \frac{\partial}{\partial v}$, $\kappa = g(k, \cdot) = \mathrm{d}u - \mathrm{i}\bar{z}\mathrm{d}z + \mathrm{i}z\mathrm{d}\bar{z}$ satisfy

$$\pounds_k g = 2vg - 2v\,\kappa\mathrm{d}v \,, \qquad \textbf{NSCNG} \,,$$
$$\kappa \wedge \mathrm{d}\kappa = 2\mathrm{i}\mathrm{d}u \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z} \neq 0 \,, \qquad \textbf{twisting} \,.$$

The Robinson theorem (1961)

- Set $\phi = \mathrm{d}z \wedge \kappa$. Then ϕ is a **closed totally null self-dual complex** 2-form, i.e.

$$\mathrm{d}\phi = 0\,, \qquad \phi \wedge \phi = 0\,, \qquad \star\phi = \mathrm{i}\phi\,.$$

- Then $F := \phi + \overline{\phi}$ is a **real null** 2-form F, ie $F \wedge F = F \wedge \star F = 0$, which satisfies the **vacuum Maxwell equations**:

$$\mathrm{d}F = \mathrm{d} \star F = 0\,,$$

Also conformally invariant! More generally:

Robinson theorem (1961)

Locally, any **analytic** non-shearing congruence of null geodesics gives rise to a null solution to the vacuum Maxwell equations. Conversely (Mariot (1954)), any such field locally arises in this way.

The Goldberg–Sachs theorem (1962)

- Obstruction to the existence of NSCNG given by the **Weyl tensor**, ie the **conformally invariant** part of the Riemann tensor.

  ### Integrability condition Sachs (1961)

  If k generates a NSCNG then k must be a **principal null direction (PND)** of the Weyl tensor, ie

  $$W(k, v, k, v) = 0 \,, \qquad \text{for any vector field } v \text{ st } g(k, v) = 0.$$

- Very weak condition: always satisfied for some k (Cartan (1922))

  ### Goldberg–Sachs (1962)

  For any **Einstein** spacetime, k generates a NSCNG if and only if k is a **repeated PND** of the Weyl tensor, ie

  $$W(k, v, k, \cdot) = 0 \,, \qquad \text{for any vector field } v \text{ st } g(k, v) = 0.$$

  ie W is **algebraically special** (Petrov (1954)).

- **Conformally invariant** version: Kundt–Thompson (1962), Robinson – Schild (1962)

The Kerr metric (1963) and the Kerr theorem

- Many important Einstein metrics are algebraically special: Schwarzschild, Robinson–Trautman, Kerr, Taub–NUT, etc.
- The **Kerr metric** (1963) is a Petrov type D Einstein metric describing a rotating black hole, and admits two NSCNGs. It can be cast into **Kerr–Schild** form:

$$g = \eta + H\kappa\kappa\,,$$

where η is the Minkowski metric, H a function and $\kappa = g(k, \cdot)$ for some null vector k.

Fact

The congruence generated by k is geodesic and non-shearing for g if and only if it is for η.

- Seek NSCNG in Minkowski space...

Kerr theorem

Any **analytic** NSCNG in Minkowski space can be locally obtained from an analytic function of three **complex** variables.

α-PLANE DISTRIBUTIONS

- Null coframe $(\kappa, \mu, \overline{\mu}, \lambda)$ adapted to null distribution $\langle k \rangle$ so that

$$g = 2\kappa\lambda + 2\mu\overline{\mu}\,, \qquad \kappa = g(k, \cdot)\,.$$

Any other adapted coframe $(\widehat{\kappa}, \hat{\mu}, \overline{\hat{\mu}}, \hat{\lambda})$ is given by

$$\widehat{\kappa} = a\kappa\,, \qquad \widehat{\mu} = \mathrm{e}^{\mathrm{i}\phi}\mu + b\kappa\,,$$

$$\widehat{\lambda} = a^{-1}\left(\lambda - b\mathrm{e}^{\mathrm{i}\phi}\mu - \overline{b}\mathrm{e}^{-\mathrm{i}\phi}\overline{\mu} + \frac{|b|^2}{2}\kappa\right)\,, \qquad 0 \neq a, \phi \in \mathbf{R}, b \in \mathbf{C}$$

- (Self-dual) α**-plane** and (anti-self-dual) β**-plane distributions**

$$N_{\langle k \rangle} := \left\{ v \in {}^{\mathbf{C}}T\mathcal{M} : \kappa(v) = \mu(v) = 0 \right\} \qquad \text{and} \qquad \overline{N}_{\langle k \rangle}$$

are complex and totally null.

KEY FACT
Null line distributions are equivalent to α-plane distributions.

- **Spinorial approach to GR**: Witten (1958), Penrose (1959), Newman–Penrose (1962)
α-plane distributions are spinor fields up to scale!

FOLIATIONS BY α-SURFACES

KEY FACT

k generates a **NSCNG** if and only if $N_{\langle k \rangle}$ is **involutive**, i.e. $[N_{\langle k \rangle}, N_{\langle k \rangle}] \subset N_{\langle k \rangle}$

- Issue: Involutivity is **not** equivalent to integrability in general
- Solution: assume **analyticity** and analytically extend $(\mathcal{M}, g)$ to a **complex Riemannian manifold** $({}^{\mathbf{C}}\mathcal{M}, {}^{\mathbf{C}}g)$
- View $(\mathcal{M}, g)$ as a 'real' slice of $({}^{\mathbf{C}}\mathcal{M}, {}^{\mathbf{C}}g)$.
- By the Frobenius theorem, an integrable $N_{\langle k \rangle}$ gives rise to a foliation by α**-surfaces** in ${}^{\mathbf{C}}\mathcal{M}$.
- Similarly, one has a foliation by β**-surfaces** in ${}^{\mathbf{C}}\mathcal{M}$ associated to $\overline{N}_{\langle k \rangle}$.
- The intersection of a α-surface with a β-surface is a complex null curve.

FACT

A NSCNG arises from the intersection of an α-surface foliation and a β-surface foliation.

The Robinson theorem II

Robinson theorem (1961)

Locally, any **analytic** non-shearing congruence of null geodesics gives rise to a null solution to the vacuum Maxwell equations. Conversely (Mariot (1954)), any such field locally arises in this way.

Proof of the Robinson theorem (Eastwood (1984))

- analytic NSCNG $\langle k \rangle \Longleftrightarrow \alpha$-surface foliation in ${}^{\mathbf{C}}\mathcal{M}$
- Local submersion ${}^{\mathbf{C}}\mathcal{M} \xrightarrow{\varpi} \underline{\mathcal{M}}_{\mathcal{N}}$ where $\underline{\mathcal{M}}_{\mathcal{N}}$ 2-dim leaf space
- Take any 2-form $\underline{\phi}$ on $\underline{\mathcal{M}}_{\mathcal{N}}$

$\Longrightarrow$ $\mathrm{d}\underline{\phi} = 0$

$\Longrightarrow$ $\phi := \varpi^* \underline{\phi}$ is a closed null (self-dual) 2-form.

$\Longrightarrow$ $F := \phi + \overline{\phi}$ satisfies the vacuum Maxwell equation.

The converse is immediate.

The Kerr theorem II and twistor space

Kerr theorem - version 1

Any **analytic** NSCNG in Minkowski space can be locally obtained from an analytic function of three **complex** variables.

Kerr theorem - version 2

Any foliation by **α-surfaces** in **complexified** Minkowski space can be locally obtained from an analytic function of three (complex) variables.

Kerr theorem - version 3

Any foliation by **α-planes** in **complexified** Minkowski space can be locally obtained from an analytic function of three (complex) variables.

Penrose (1967):

- View $^{\mathbf{C}}\mathbb{M}$ as a dense open set of a smooth projective quadric $\mathcal{Q}$
- Define the **twistor space** $\mathbb{PT}$ as the space of all α-planes of $\mathcal{Q}$
- Twistor space is complex projective space $\mathbf{C}\mathbb{P}^3$
- The leaf space of an α-plane foliation in $^{\mathbf{C}}\mathbb{M} \subset \mathcal{Q}$ is thus a complex **hypersurface** in $\mathbb{PT}$, ie it is prescribed by an analytic function of three complex variables.

Kerr theorem à la Penrose

Any local foliation of $\mathcal{Q}$ by α-planes gives rise to a ‘certain’ complex hypersurface in $\mathbb{PT}$. Conversely, any such foliation arises in this way.

The real picture: CR manifolds

Example

Consider the metric

$$g = 2\,(\mathrm{d}u - \mathrm{i}\bar{z}\mathrm{d}z + \mathrm{i}z\mathrm{d}\bar{z})\,\mathrm{d}v + 2(v^2+1)\mathrm{d}z\mathrm{d}\bar{z}\,,$$

with NSCNG generated by $k = \frac{\partial}{\partial v}$. Then the 1-forms

$$\kappa = \mathrm{d}u - \mathrm{i}\bar{z}\mathrm{d}z + \mathrm{i}z\mathrm{d}\bar{z}\,, \qquad \mu = \mathrm{d}z\,, \qquad \overline{\mu} = \mathrm{d}\bar{z}\,,$$

descend to the leaf space $\underline{\mathcal{M}}$ of the congruence. The spans $\langle\kappa\rangle$ and $\langle\kappa,\mu\rangle$ define a **non-degenerate almost Cauchy-Riemman (CR) structure** $(\underline{H},\underline{J})$ on $\underline{\mathcal{M}}$, where $\underline{H} = \mathrm{Ann}(\kappa)$ and $\underline{J}(\mu) = \mathrm{i}\mu \pmod{\kappa}$.

More generally,

Key fact Penrose, Robinson, Trautman, Tafel, etc.

The leaf space of a NSCNG is a CR 3-fold. Conversely, to any CR 3-fold, one can associate a spacetime equipped with a NSCNG.

- We can also recover this description from the Kerr theorem according to the tortuous 'historical' narrative...

SPIN-OFF...

- **CR methods to seek Einstein metrics**: Lewandowski–Nurowski (1990), Lewandowski–Nurowski–Tafel (1991)
- **Embeddability of CR manifolds**: Penrose (1983), Tafel (1985), Lewandowski–Nurowski–Tafel (1990), Hill–Lewandowski–Nurowski (2008)
- **Fefferman spaces**: Fefferman (1976), Sparling, Graham (1987), Lewandowski (1988)
- **Analogies between Lorentzian and Riemannian geometries**:
  - Riemannian Goldberg–Sachs Theorem: Przanowski–Broda (1983)
  - Riemannian Kerr Theorem: Eels–Salamon (1985)
  - NSCNG $\longleftrightarrow$ Hermitian structures: Nurowski (1990,1996,1997)
- **Twistor theory** $\longrightarrow$ Penrose tranform, Tractor calculus, parabolic geometries...

HIGHER DIMENSIONS

- For a pseudo-Riemannian manifold $(\mathcal{M}, g)$ of dimension n and **any** signature, we define an **almost null structure** to be a field N of totally null complex $\lfloor \frac{n}{2} \rfloor$-planes.

  ### EXAMPLE

  For a Riemannian manifold $(\mathcal{M}, g)$ of even dimension, an almost null structure N is equivalent to an almost Hermitian structure J:

  $$^{\mathbb{C}}T\mathcal{M} = N \oplus \overline{N}, \qquad \text{and} \qquad J(v) = \mathrm{i}v, \quad v \in N.$$

- Intrinsically connected to Cartan's notion of **pure** or **simple spinors**
  Cartan (1967), Budinich–Trautman (1988,1989), Kopczyński–Trautman (1992), Kopczyński (1997)
- Geometric properties: Hughston (1990, 1995), Jeffryes (1995), TC (2016, 2017b)
- Twistors, Kerr–Robinson theorems: Hughston–Mason (1988), TC (2017a)
- Goldberg–Sachs theorems: TC (2011, 2012)

Almost Robinson geometry

Robinson manifolds Nurowski–Trautman (2002)

Lorentzian analogues of Hermitian manifolds

Almost Robinson manifold Fino–Leistner–TC (2021)

Quadruple $(\mathcal{M}, g, N, K)$ where $(\mathcal{M}, g)$ is Lorentzian $(2m+2)$-fold, N totally null complex $(m+1)$-plane distribution, and $K = T\mathcal{M} \cap N$

- **Nearly Robinson manifold** when $[K, N] \subset N$
- **Robinson manifold** when $[N, N] \subset N$

Lifts of (almost) CR manifolds

(Almost) CR manifold $\longrightarrow$ (nearly) Robinson manifold!

$$\begin{pmatrix} \underline{\mathcal{M}} \\ \underline{H} \\ \underline{J} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{R} \times \underline{\mathcal{M}}, g, K \\ H_K = K^{\perp}/K \\ J \end{pmatrix} \sim (\mathcal{M}, g, N, K)$$

Conversely, any nearly Robinson manifold locally arises in this way.

Examples

Fefferman spaces

- Fefferman (1976): Canonical conformal structure with a conformal Killing field on a circle bundle over any contact CR manifold.
- Leitner (2010): Generalised to **almost** CR structures

Multi-Robinson structures: Mason–TC (2010)

- **Kerr–NUT–(A)dS metrics** (Chen–Lü–Pope (2008), Plebański–Demiański (1976)):
  discrete set of Robinson structures
  shearing congruences (unlike in dim 4)
- Related to **conformal Killing–Yano** 2**-forms**

Twisting non-shearing congruences of null geodesics in even dimensions: TC (2021)

- **Twist-induced** nearly Robinson structure
- Einstein metrics $\longleftrightarrow$ **almost CR–Einstein structures**
- Generalised Fefferman–Einstein and Taub-NUT-(A)dS metrics

OBJECTIVES OF THE POLS FELLOWSHIP

INTERACTION BETWEEN LORENTZIAN AND CR GEOMETRIES: (NEARLY) ROBINSON MANIFOLDS

- **Reduction** of the Einstein field equations to **CR data**:
  - dim 4: recent progress, almost there!
  - dim >4: for NSCNG, see TC (2021) ✓
    Now focus on Robinson geometries with shearing congruences...
- Goldberg–Sachs and Kerr theorems in higher dimensions
- Differential equations on (almost) CR manifolds
- Global properties
- Homogeneous spaces

Not treated in this talk:

CONFORMAL APPROACH TO HORIZON GEOMETRIES

For another time!

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Thank you for your attention!

Dziękuję za uwagę!

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