The variational principles in the general relativity theory

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Action:

$$
\begin{aligned}
S &= \int_\Omega \mathcal{L}\,\mathrm{d}x^4\,, \\
\delta S &= \int_\Omega \delta\mathcal{L}\,\mathrm{d}x^4 = 0\,,
\end{aligned}
$$

Lagrangian:

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}\left(\phi,\, \phi_{,\nu}\right)\,, \\
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi_{,\nu}}\,\delta\phi_{,\nu}\,,
\end{aligned}
$$

Momenta canonically conjugated:

$$
p^\nu := \frac{\partial\mathcal{L}}{\partial\phi_{,\nu}}\,,
$$

A short retrospection

Variation principle:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi + p^{\nu}\,\delta\phi_{,\nu} = \left(\frac{\partial \mathcal{L}}{\partial \phi} - p^{\nu}{}_{,\nu}\right)\delta\phi + \partial_{\nu}\left(p^{\nu}\,\delta\phi\right)\;,$$

Special configuration – respecting of the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} - p^{\nu}{}_{,\nu} = 0\,,$$

Final result – *on shell* philosophy:

$$\delta\mathcal{L} = \partial_{\nu}\left(p^{\nu}\,\delta\phi\right) = p^{\nu}{}_{,\nu}\,\delta\phi + p^{\nu}\,\delta\phi_{,\nu}\,,$$

The equivalence between variational principles

Metric picture:

$$\begin{aligned}\mathcal{L}_g &= \mathcal{L}_g\left(g_{\mu\nu},\, g_{\mu\nu,\alpha},\, g_{\mu\nu,\alpha\beta},\, \phi^I_J,\, \phi^I_{J,\nu}\right) = \\ &= \frac{\sqrt{|\det g|}}{16\pi}\, \overset{\circ}{R}_{\mu\nu}\, g^{\mu\nu} + \mathcal{L}_{matt}\left(\phi^I_J,\, \overset{\circ}{\nabla}_\nu\, \phi^I_J,\, g_{\mu\nu}\right)\,,\end{aligned}$$

Palati picture

$$\begin{aligned}\mathcal{L}_P &= \mathcal{L}_P\left(g_{\mu\nu},\, \Gamma^\kappa{}_{\lambda\mu},\, \Gamma^\kappa{}_{\lambda\mu,\nu},\, \phi^I_J,\, \phi^I_{J,\nu}\ \right) = \\ &= \frac{\sqrt{|\det g|}}{16\pi}\, R_{\mu\nu}\, g^{\mu\nu} + \widetilde{\mathcal{L}}_{matt}\left(g_{\alpha\beta},\, \Gamma^\kappa{}_{\lambda\mu},\, \phi^I_J\,, \phi^I_{J,\nu}\right)\,,\end{aligned}$$

Affine picture

$$\mathcal{L}_A = \mathcal{L}_A\left(\Gamma^\kappa{}_{\lambda\mu},\, \Gamma^\kappa{}_{\lambda\mu,\nu},\, \phi^I_J\,, \phi^I_{J,\nu}\right)\,.$$

The equivalence between variational principles

Concepts/Conclusions

- It is possible to analyse the general relativity theory in all those pictures equivalently,
- It is a unique way to rewrite the theory from the one way of description to others,
- Main assumption: The connection has to be **symmetric**!!!

Scheme of proof

The metric picture

$$\mathcal{L}_g = \mathcal{L}_H + \mathcal{L}_{matt} = \frac{\sqrt{|\det g|}}{16\pi}\ \overset{\circ}{R}_{\mu\nu}\, g^{\mu\nu} + \mathcal{L}_{matt}\left(\phi^I_J,\ \overset{\circ}{\nabla}_\nu\, \phi^I_J,\ g_{\mu\nu}\right),$$

Variation of the Hilbert Lagrangian

$$\begin{aligned}
\delta\mathcal{L}_H &= -\frac{1}{16\pi}\, \overset{\circ}{\mathcal{G}}{}^{\mu\nu}\ \delta g_{\mu\nu} + \partial_\nu\left(\pi_\kappa{}^{\lambda\mu\nu}\, \delta\, \overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu}\right) = \\
&= \overset{\circ}{R}_{\mu\nu}\ \delta\pi^{\mu\nu} + \partial_\nu\left(\pi_\kappa{}^{\lambda\mu\nu}\, \delta\, \overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu}\right), \\
\pi^{\mu\nu} &:= \frac{\sqrt{|\det g|}}{16\pi}\, g^{\mu\nu}, \\
\pi_\kappa{}^{\lambda\mu\nu} &:= \delta^\nu_\kappa\, \pi^{\lambda\mu} - \delta^{(\lambda}_\kappa\, \pi^{\mu)\nu}, \\
\mathcal{L}_H &= \pi^{\mu\nu}\ \overset{\circ}{R}_{\mu\nu}\ .
\end{aligned}$$

Scheme of proof

The metric picture

$$\mathcal{L}_g \quad = \quad \mathcal{L}_H + \mathcal{L}_{matt} = \pi^{\mu\nu} \overset{\circ}{R}_{\mu\nu} + \mathcal{L}_{matt}\left(\phi^I_J, \overset{\circ}{\nabla}_\nu \phi^I_J, g_{\mu\nu}\right),$$

Variation of the matter Lagrangian

$$\delta\mathcal{L}_{matt} \quad = \quad \left[\frac{\partial \mathcal{L}_{matt}}{\partial \phi^I_J} - \overset{\circ}{\nabla}_\mu p_I^{J\mu}\right] \delta\phi^I_J + \partial_\mu\left(p_I^{J\mu}\, \delta\phi^I_J\right) + \\ + \mathcal{P}^{\lambda\mu}{}_\kappa\, \delta\, \overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu} + \frac{\partial \mathcal{L}_{matt}}{\partial g_{\mu\nu}}\, \delta g_{\mu\nu}\,.$$

Scheme of proof

The metric picture

$$\begin{aligned}
\mathcal{L}_g &= \mathcal{L}_H + \mathcal{L}_{matt} = \pi^{\mu\nu}\,\mathring{R}_{\mu\nu} + \mathcal{L}_{matt}\left(\phi^I_J,\,\mathring{\nabla}_\nu\,\phi^I_J,\,g_{\mu\nu}\right), \\
\delta\mathcal{L}_g &= \left[\frac{\partial\mathcal{L}_{matt}}{\partial\phi^I_J} - \mathring{\nabla}_\mu\,p_I^{J\mu}\right]\delta\phi^I_J + \left[\frac{\partial\mathcal{L}_{matt}}{\partial g_{\mu\nu}} - \frac{1}{16\pi}\,\mathring{\mathcal{G}}^{\mu\nu}\right]\delta g_{\mu\nu} + \\
&\quad + \mathcal{P}^{\lambda\mu}{}_{\kappa}\,\delta\,\mathring{\Gamma}^{\kappa}{}_{\lambda\mu} + \partial_\nu\left(p_I^{J\nu}\,\delta\phi^I_J + \pi_{\kappa}{}^{\lambda\mu\nu}\,\delta\,\mathring{\Gamma}^{\kappa}{}_{\lambda\mu}\right),
\end{aligned}$$

Crucial part

$$\begin{aligned}
\mathcal{P}^{\mu\nu}{}_{\kappa}\,\delta\,\mathring{\Gamma}^{\kappa}{}_{\mu\nu} &= \partial_\kappa\left(\mathcal{R}^{\mu\nu\kappa}\,\delta g_{\mu\nu}\right) - \left(\mathring{\nabla}_\kappa\,\mathcal{R}^{\mu\nu\kappa}\right)\delta g_{\mu\nu}, \\
\mathcal{R}^{\mu\nu\kappa} &:= \frac{1}{2}\left(\mathcal{P}^{\kappa\mu\nu} + \mathcal{P}^{\kappa\nu\mu} - \mathcal{P}^{\mu\nu\kappa}\right),
\end{aligned}$$

Scheme of proof

The metric picture

$$\mathcal{L}_g = \mathcal{L}_H + \mathcal{L}_{matt} = \pi^{\mu\nu} \overset{\circ}{R}_{\mu\nu} + \mathcal{L}_{matt}\left(\phi^I_J, \overset{\circ}{\nabla}_\nu \phi^I_J, g_{\mu\nu}\right),$$

$$\begin{aligned} \delta\mathcal{L}_g &= \left[\frac{\partial\mathcal{L}_{mat}}{\partial\phi^I_J} - \overset{\circ}{\nabla}_\mu p_I^{J\mu}\right]\delta\phi^I_J + \\ &+ \left[\frac{\partial\mathcal{L}_{matt}}{\partial g_{\mu\nu}} - \overset{\circ}{\nabla}_\kappa \mathcal{R}^{\mu\nu\kappa} - \frac{1}{16\pi}\overset{\circ}{\mathcal{G}}^{\mu\nu}\right]\delta g_{\mu\nu} + \\ &+ \partial_\nu\left(p_I^{J\nu}\,\delta\phi^I_J + \pi_\kappa{}^{\lambda\mu\nu}\,\delta\,\overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu} + \mathcal{R}^{\alpha\beta\nu}\,\delta g_{\alpha\beta}\right), \end{aligned}$$

On shell:

$$\delta\mathcal{L}_g = \partial_\nu\left(p_I^{J\nu}\,\delta\phi^I_J + \pi_\kappa{}^{\lambda\mu\nu}\,\delta\,\overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu} + \mathcal{R}^{\alpha\beta\nu}\,\delta g_{\alpha\beta}\right),$$

Scheme of proof

Variation of the metric Lagrangian *on shell*

$$\delta \mathcal{L}_g = \partial_\nu \left(p_I^{J\nu}\, \delta\phi^I_J + \pi_\kappa{}^{\lambda\mu\nu}\, \delta\, \mathring{\Gamma}^\kappa{}_{\lambda\mu} + \mathcal{R}^{\alpha\beta\nu}\, \delta g_{\alpha\beta} \right) ,$$

Appearance of the non-metricity tensor

$$\begin{aligned}
\partial_\nu \left(\mathcal{R}^{\alpha\beta\nu}\, \delta g_{\alpha\beta} \right) &= \partial_\nu \left[\pi_\kappa{}^{\lambda\mu\nu}\, \delta N^\kappa{}_{\lambda\mu} \right] + \delta \left[\mathring{\nabla}_\kappa\, \mathcal{R}_\sigma{}^{\sigma\kappa} \right] , \\
N^\kappa{}_{\lambda\mu} &= 16\pi \Big[R_{\lambda\mu}{}^\kappa - \frac{1}{2} R_\sigma{}^{\sigma\kappa}\, g_{\lambda\mu} + \\
&\quad - \frac{2}{3} \left(\delta^\kappa_{(\lambda}\, R_{\mu)\sigma}{}^\sigma - \frac{1}{2} \delta^\kappa_{(\lambda}\, R^\sigma{}_{|\sigma|\mu)} \right) \Big] , \\
R^{\alpha\beta\gamma} &= \frac{1}{\sqrt{|\det g|}} \mathcal{R}^{\alpha\beta\gamma} ,
\end{aligned}$$

Scheme of proof

Variation of the metric Lagrangian *on shell*

$$\delta \mathcal{L}_g = \partial_\nu \left[p_I^{J\nu}\, \delta\phi^I_J + \pi_\kappa{}^{\lambda\mu\nu}\, \delta\left(\overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu} + N^\kappa{}_{\lambda\mu} \right)\right] + \delta\left[\overset{\circ}{\nabla}_\kappa\, \mathcal{R}_\sigma{}^{\sigma\kappa} \right],$$

$$\Gamma^\kappa{}_{\lambda\mu} = \overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu} + N^\kappa{}_{\lambda\mu},$$

$$\delta \mathcal{L}_g = \partial_\nu \left[p_I^{J\nu}\, \delta\phi^I_J + \pi_\kappa{}^{\lambda\mu\nu}\, \delta\Gamma^\kappa{}_{\lambda\mu} \right] + \delta\left[\overset{\circ}{\nabla}_\kappa\, \mathcal{R}_\sigma{}^{\sigma\kappa} \right],$$

Conclusion:

- if the Lagrangian of matter does not depend on the covariant derivative of the matter field, then the connection stays metric,
- for the matter Lagrangian which depends on the first covariant derivative of the matter field exist an **unique** way to define a general symmetric connection,

Scheme of proof

Metric Lagrangian – variables

$$
\begin{aligned}
\mathcal{L}_g &= \mathcal{L}_g\left(g_{\mu\nu},\, g_{\mu\nu,\alpha},\, g_{\mu\nu,\alpha\beta},\, \phi^I{}_J,\, \phi^I{}_{J,\nu}\right) = \\
&= \mathcal{L}_g\left(g_{\mu\nu},\, \mathring{\Gamma}^\kappa{}_{\lambda\mu},\, \mathring{\Gamma}^\kappa{}_{\lambda\mu,\nu},\, \phi^I{}_J,\, \phi^I{}_{J,\nu}\right) = \\
&= \mathcal{L}_g\left(g_{\mu\nu},\, \mathring{R}_{\mu\nu},\, \phi^I{}_J,\, \mathring{\nabla}_\nu\, \phi^I{}_J\right), \\
\delta\mathcal{L}_g &= \partial_\nu\left[p_I^{J\nu}\, \delta\phi^I{}_J + \pi_\kappa{}^{\lambda\mu\nu}\, \delta\Gamma^\kappa{}_{\lambda\mu}\right] + \delta\left[\mathring{\nabla}_\kappa\, \mathcal{R}_\sigma{}^{\sigma\kappa}\right],
\end{aligned}
$$

Scheme of proof

Palatini picture

$$\mathcal{L}_P := \mathcal{L}_g - \overset{\circ}{\nabla}_\kappa \mathcal{R}_\sigma{}^{\sigma\kappa} = \pi^{\mu\nu} \overset{\circ}{R}_{\mu\nu} + \mathcal{L}_{matt} - \overset{\circ}{\nabla}_\kappa \mathcal{R}_\sigma{}^{\sigma\kappa} ,$$
$$\delta\mathcal{L}_P = \partial_\nu \left[p_I^{J\nu} \, \delta\phi^I_J + \pi_\kappa{}^{\lambda\mu\nu} \, \delta\Gamma^\kappa{}_{\lambda\mu} \right] ,$$

Palatini Lagrangian – variables

$$\mathcal{L}_P = \mathcal{L}_P \left(g_{\mu\nu}, \Gamma^\kappa{}_{\lambda\mu}, \Gamma^\kappa{}_{\lambda\mu,\nu}, \phi^I_J, \phi^I_{J,\nu} \right) ,$$

Scheme of proof

Transformation of the Palatini Lagrangian

$$
\begin{aligned}
\Gamma^{\kappa}{}_{\lambda\mu} &= \mathring{\Gamma}^{\kappa}{}_{\lambda\mu} + N^{\kappa}{}_{\lambda\mu}\,,\\
R_{\mu\nu} &= \mathring{R}_{\mu\nu} + \mathring{\nabla}_{\kappa}\, N^{\kappa}{}_{\mu\nu} - \mathring{\nabla}_{(\mu}\, N^{\kappa}{}_{\nu)\kappa} + N^{\sigma}{}_{\mu\nu}\, N^{\kappa}{}_{\kappa\sigma} - N^{\sigma}{}_{\kappa\mu}\, N^{\kappa}{}_{\nu\sigma}\,,\\
\pi^{\mu\nu} R_{\mu\nu} &= \pi^{\mu\nu}\, \mathring{R}_{\mu\nu} - \mathring{\nabla}_{\kappa}\, \mathcal{R}_{\sigma}{}^{\sigma\kappa} + \pi^{\mu\nu} \left(N^{\sigma}{}_{\mu\nu}\, N^{\kappa}{}_{\kappa\sigma} - N^{\sigma}{}_{\kappa\mu}\, N^{\kappa}{}_{\nu\sigma} \right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_P &= \pi^{\mu\nu}\, \mathring{R}_{\mu\nu} + \mathcal{L}_{matt} - \mathring{\nabla}_{\kappa}\, \mathcal{R}_{\sigma}{}^{\sigma\kappa} = \\
&= \pi^{\mu\nu}\, R_{\mu\nu} + \tilde{\mathcal{L}}_{matt}\,,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{L}}_{matt} &= \mathcal{L}_{matt} - \pi^{\mu\nu} \left(N^{\sigma}{}_{\mu\nu}\, N^{\kappa}{}_{\kappa\sigma} - N^{\sigma}{}_{\kappa\mu}\, N^{\kappa}{}_{\nu\sigma} \right)\\
&= \mathcal{L}_P - \pi^{\mu\nu}\, R_{\mu\nu}\,,
\end{aligned}
$$

Scheme of proof

Transformation of the variation $\delta\mathcal{L}_P$

$$
\begin{aligned}
\mathcal{L}_P &= \pi^{\mu\nu} R_{\mu\nu} + \tilde{\mathcal{L}}_{matt} , \\
\delta\mathcal{L}_P &= \partial_\nu \left[p_I^{J\nu}\, \delta\phi^I_J + \pi_\kappa{}^{\lambda\mu\nu}\, \delta\Gamma^\kappa{}_{\lambda\mu} \right] = \\
&= \partial_\nu \left[p_I^{J\nu}\, \delta\phi^I_J \right] + \pi_\kappa{}^{\lambda\mu\nu}{}_{,\nu}\, \delta\Gamma^\kappa{}_{\lambda\mu} + \pi_\kappa{}^{\lambda\mu\nu}\, \delta\Gamma^\kappa{}_{\lambda\mu,\nu} = \\
&= p_I^{J\nu}{}_{,\nu}\, \delta\phi^I_J + p_I^{J\nu}\, \delta\phi^I_{J,\nu} + \pi^{\mu\nu}\, \delta R_{\mu\nu} + \nabla_\nu\, \pi_\kappa{}^{\lambda\mu\nu}\, \delta\Gamma^\kappa{}_{\lambda\mu} ,
\end{aligned}
$$

Connection character equation

$$
\nabla_\nu\, \pi_\kappa{}^{\lambda\mu\nu} = \frac{\partial\mathcal{L}_P}{\partial\Gamma^\kappa{}_{\lambda\mu}} = \frac{\partial\tilde{\mathcal{L}}_{matt}}{\partial\Gamma^\kappa{}_{\lambda\mu}} = 0 ,
$$

$$
\nabla_\nu\, g_{\alpha\beta} = 0 \quad \Rightarrow \quad \Gamma^\kappa{}_{\lambda\mu} = \overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu} ,
$$

Scheme of proof

Transformation of the variation $\delta\mathcal{L}_P$ *on shell*

$$\begin{aligned}
\mathcal{L}_P &= \pi^{\mu\nu}\,R_{\mu\nu} + \tilde{\mathcal{L}}_{matt}\,, \\
\delta\mathcal{L}_P &= p_I^{J\nu}{}_{,\nu}\,\delta\phi^I_J + p_I^{J\nu}\,\delta\phi^I_{J,\nu} + \pi^{\mu\nu}\,\delta R_{\mu\nu} + \nabla_\nu\,\pi_\kappa{}^{\lambda\mu\nu}\,\delta\Gamma^\kappa{}_{\lambda\mu}\,,
\end{aligned}$$

Connection character equation

$$\begin{aligned}
\nabla_\nu\,\pi_\kappa{}^{\lambda\mu\nu} &= \frac{\partial\mathcal{L}_P}{\partial\Gamma^\kappa{}_{\lambda\mu}} = \frac{\partial\tilde{\mathcal{L}}_{matt}}{\partial\Gamma^\kappa{}_{\lambda\mu}} = \mathcal{P}^{\lambda\mu}{}_\kappa\,, \\
N^\kappa{}_{\lambda\mu} &= 16\pi\left[R_{\lambda\mu}{}^\kappa - \frac{1}{2}\,R_\sigma{}^{\sigma\kappa}\,g_{\lambda\mu} + \right. \\
&\quad \left. -\frac{2}{3}\left(\delta^\kappa_{(\lambda}\,R_{\mu)\sigma}{}^\sigma - \frac{1}{2}\,\delta^\kappa_{(\lambda}\,R^\sigma{}_{|\sigma|\mu)}\right)\right], \\
R^{\mu\nu\kappa} &= \frac{1}{2\sqrt{|\det g|}}\left(\mathcal{P}^{\kappa\mu\nu} + \mathcal{P}^{\kappa\nu\mu} - \mathcal{P}^{\mu\nu\kappa}\right).
\end{aligned}$$

Scheme of proof

Comments/Conclusions

- there is a very strength relation between matter fields and non-metricity of the connection,
- we **cannot** randomly assume the non-metric character of the connection and the structure of matter Lagrangian simultaneously,
- the non-metricity tensor N is **uniquely** defined by the matter Lagrangian.

Scheme of proof

Palatini picture

$$\mathcal{L}_P = \pi^{\mu\nu} R_{\mu\nu} + \tilde{\mathcal{L}}_{matt} ,$$
$$\delta\mathcal{L}_P = \partial_\nu \left(p_I^{J\nu} \, \delta\phi^I_J \right) + \pi^{\mu\nu} \, \delta R_{\mu\nu} + \nabla_\nu \, \pi_\kappa{}^{\lambda\mu\nu} \, \delta\Gamma^\kappa{}_{\lambda\mu}$$

Metric as a pathological degree of freedom in the Palatini picture

If the metric had been an independent degree of freedom it has had a non-vanishing momenta:

$$\partial_\kappa \left(\Lambda^{\mu\nu\kappa} \, \delta g_{\mu\nu} \right) = \Lambda^{\mu\nu\kappa}{}_{,\kappa} \, \delta g_{\mu\nu} + \Lambda^{\mu\nu\kappa} \, \delta g_{\mu\nu,\kappa}$$

Scheme of proof

Analogue from classical mechanic

$$L(q,\,\dot q,\,p) = p\dot q - H(q,\,p)\,,$$

$$\frac{\delta L}{\delta p} = \frac{\partial L}{\partial p} = \dot q - \frac{\partial H}{\partial p} = 0 \Rightarrow p(q,\,\dot q)\,,$$

Elimination of the metric tensor

Euler-Lagrange equation (Einstein equation):

$$\frac{\delta \mathcal{L}_P}{\delta g_{\mu\nu}} = \frac{\partial \mathcal{L}_P}{\partial g_{\mu\nu}} = -\frac{1}{16\pi}\,\mathcal{G}^{\mu\nu} + \frac{\partial \tilde{\mathcal{L}}_{matt}}{\partial g_{\mu\nu}} = 0\,,$$

$$\frac{\delta \mathcal{L}_g}{\delta g_{\mu\nu}} = -\frac{1}{16\pi}\,\overset{\circ}{\mathcal{G}}{}^{\mu\nu} + \frac{\partial \mathcal{L}_{matt}}{\partial g_{\mu\nu}} - \overset{\circ}{\nabla}_\kappa\,\mathcal{R}^{\mu\nu\kappa} = -\frac{1}{16\pi}\,\overset{\circ}{\mathcal{G}}{}^{\mu\nu} + \frac{\delta \mathcal{L}_{matt}}{\delta g_{\mu\nu}} = 0\,,$$

Scheme of proof

Elimination of the metric tensor

$$
\begin{aligned}
\pi^{\mu\nu} &= \frac{\sqrt{|\det g|}}{16\pi}\, g^{\mu\nu}\,, \\
\mathcal{L}_P &= \pi^{\mu\nu}\, R_{\mu\nu} + \tilde{\mathcal{L}}_{matt}\,, \\
\delta\mathcal{L}_P &= \partial_\nu \left(p_I^{J\nu}\, \delta\phi^I_{\ J} \right) + \pi^{\mu\nu}\, \delta R_{\mu\nu} + \nabla_\nu\, \pi_\kappa^{\ \lambda\mu\nu}\, \delta\Gamma^\kappa_{\ \lambda\mu}
\end{aligned}
$$

Euler-Lagrange equation (Einstein equation):

$$
\begin{aligned}
\frac{\delta\mathcal{L}_P}{\delta\pi^{\mu\nu}} &= \frac{\partial\mathcal{L}_P}{\partial\pi^{\mu\nu}} = R_{\mu\nu} + \frac{\partial\tilde{\mathcal{L}}_{matt}}{\partial\pi^{\mu\nu}} = 0\,, \\
\frac{\delta\mathcal{L}_g}{\delta\pi^{\mu\nu}} &= \overset{\circ}{R}_{\mu\nu} + \frac{\delta\mathcal{L}_{matt}}{\delta\pi^{\mu\nu}} = 0\,,
\end{aligned}
$$

Affine picture

$$\begin{aligned}
\mathcal{L}_A &= \mathcal{L}_P\left(\Gamma^{\kappa}{}_{\lambda\mu}\,, \Gamma^{\kappa}{}_{\lambda\mu,\nu}\,, \phi^I_J\,, \phi^I_{J,\nu}\right)\,, \\
\mathcal{L}_A &= \mathcal{L}_A\left(R_{\mu\nu}\,, \phi^I_J\,, \nabla_\nu\,\phi^I_J\right)\,, \\
\delta\mathcal{L}_A &= \partial_\nu\left(p_I^{J\nu}\,\delta\phi^I_J\right) + \pi^{\mu\nu}\,\delta R_{\mu\nu} + \nabla_\nu\,\pi_\kappa{}^{\lambda\mu\nu}\,\delta\Gamma^{\kappa}{}_{\lambda\mu}\,,
\end{aligned}$$

Einstein equation:

$$\frac{\partial\mathcal{L}_A}{\partial R_{\mu\nu}} = \pi^{\mu\nu}\,,$$

Whole idea in a nutshell

$$\mathcal{L}_g \Rightarrow \mathcal{L}_P \Rightarrow \mathcal{L}_A$$

$$\begin{aligned}
\overset{\circ}{R}_{\mu\nu} + \frac{\delta \mathcal{L}_{matt}}{\delta \pi^{\mu\nu}} &= 0\,, \\
\overset{\circ}{\Gamma}{}^{\kappa}{}_{\lambda\mu} &\Rightarrow \overset{\circ}{\Gamma}{}^{\kappa}{}_{\lambda\mu} + N^{\kappa}{}_{\lambda\mu} = \Gamma^{\kappa}{}_{\lambda\mu}\,, \\
R_{\mu\nu} + \frac{\partial \tilde{\mathcal{L}}_{matt}}{\partial \pi^{\mu\nu}} &= 0\,, \\
g_{\mu\nu} &= g_{\mu\nu}\left(\Gamma^{\kappa}{}_{\lambda\mu},\, R_{\mu\nu},\, \phi^{I}_{J}\,, \phi^{I}_{J,\nu}\right)\,, \\
\frac{\partial \mathcal{L}_A}{\partial R_{\mu\nu}} &= \pi^{\mu\nu} = \frac{\sqrt{|\det g|}}{16\pi}\, g^{\mu\nu}\,.
\end{aligned}$$

Why is it useful?

- possibility to rewrite and analyze theories in other pictures,
- new options of generalisation well-known theories,
- verification of theories which have some "extra" assumptions,

Example – gravitation with cosmological constant

Metric picture

$$\mathcal{L}_g = \frac{\sqrt{|\det g|}}{16\pi} \overset{\circ}{R}_{\mu\nu} \; g^{\mu\nu} - \frac{\sqrt{|\det g|}\Lambda}{8\pi} \,,$$

Einstein Equation:

$$\begin{aligned} \frac{\delta \mathcal{L}_g}{\delta g_{\mu\nu}} &= 0 \,, \\ \overset{\circ}{R}_{\mu\nu} &= \Lambda \, g_{\mu\nu} \,, \end{aligned}$$

Derivative over the metric connection:

$$\frac{\partial \mathcal{L}_{matt}}{\partial \; \overset{\circ}{\Gamma}{}^{\kappa}{}_{\lambda\mu}} = 0 \Rightarrow \Gamma^{\kappa}{}_{\lambda\mu} = \overset{\circ}{\Gamma}{}^{\kappa}{}_{\lambda\mu} \,,$$

Example – gravitation with cosmological constant

Palatini picture

$$\mathcal{L}_P = \mathcal{L}_g \left(\Gamma^{\kappa}{}_{\lambda\mu} = \mathring{\Gamma}^{\kappa}{}_{\lambda\mu} \right) = \frac{\sqrt{|\det g|}}{16\pi} R_{\mu\nu}\, g^{\mu\nu} - \frac{\sqrt{|\det g|}\,\Lambda}{8\pi} \, ,$$

Einstein equation

$$\begin{aligned} \frac{\partial \mathcal{L}_P}{\partial g_{\mu\nu}} &= -\frac{1}{16\pi} \mathcal{G}^{\mu\nu} - \frac{\sqrt{|\det g|}\,\Lambda}{16\pi} g^{\mu\nu} = 0 \, , \\ R_{\mu\nu} &= \Lambda\, g_{\mu\nu} \, , \end{aligned}$$

Connection character equation:

$$\begin{aligned} \frac{\partial \mathcal{L}_P}{\partial \Gamma^{\kappa}{}_{\lambda\mu}} &= \nabla_\nu \, \pi_\kappa{}^{\lambda\mu\nu} \, , \\ \nabla_\nu \, \pi_\kappa{}^{\lambda\mu\nu} &= 0 \Rightarrow \Gamma^{\kappa}{}_{\lambda\mu} = \mathring{\Gamma}^{\kappa}{}_{\lambda\mu} \, , \end{aligned}$$

Example – gravitation with cosmological constant

Affine picture

$$\mathcal{L}_A = \mathcal{L}_P\left(R_{\mu\nu} = \Lambda\, g_{\mu\nu}\right) = \frac{\sqrt{|\det R|}}{8\pi\Lambda}\,,$$

Einstein equation:

$$\begin{aligned} \frac{\partial \mathcal{L}_A}{\partial R_{\mu\nu}} &= \pi^{\mu\nu}\,, \\ R_{\mu\nu} &= \Lambda\, g_{\mu\nu}\,. \end{aligned}$$

Connection character equation:

$$\begin{aligned} \frac{\partial \mathcal{L}_A}{\partial \Gamma^{\kappa}{}_{\lambda\mu}} &= \nabla_\nu\, \pi_\kappa{}^{\lambda\mu\nu}\,, \\ \nabla_\nu\, \pi_\kappa{}^{\lambda\mu\nu} &= 0 \Rightarrow \Gamma^{\kappa}{}_{\lambda\mu} = \mathring{\Gamma}^{\kappa}{}_{\lambda\mu}\,, \end{aligned}$$

Decomposition:

$$
\begin{aligned}
\Gamma^{\kappa}{}_{\lambda\mu} &= \overset{\circ}{\Gamma}{}^{\kappa}{}_{\lambda\mu} + N^{\kappa}{}_{\lambda\mu} \,, \\
R_{\mu\nu} &= K_{\mu\nu} + F_{\mu\nu} \,, \\
K_{\mu\nu} &= \overset{\circ}{K}_{\mu\nu} + \overset{\circ}{\nabla}_{\kappa} N^{\kappa}{}_{\mu\nu} - \overset{\circ}{\nabla}_{(\mu} N^{\kappa}{}_{\nu)\kappa} + N^{\sigma}{}_{\mu\nu} N^{\kappa}{}_{\kappa\sigma} - N^{\sigma}{}_{\kappa\mu} N^{\kappa}{}_{\nu\sigma} \,, \\
F_{\mu\nu} &= \overset{\circ}{\nabla}_{[\mu} N^{\kappa}{}_{\nu]\kappa} = \partial_{[\mu} N^{\kappa}{}_{\nu]\kappa} \,,
\end{aligned}
$$

Proposition

$$
\mathcal{L}_A = \frac{\sqrt{|\det K|}}{8\pi\Lambda} \mapsto \mathcal{L}_A = \frac{\sqrt{|\det(K+F)|}}{8\pi\Lambda} = \frac{\sqrt{|\det R|}}{8\pi\Lambda} \,,
$$

Properties of generalised theory

- skew-symmetric part of Ricci tensor could be interpret as an electromagnetism: $F_{\mu\nu}$ has the same properties as a Faraday 2-form $f_{\mu\nu}$, moreover:

$$F_{\mu\nu} = \pm\sqrt{8\pi|\Lambda|}\, f_{\mu\nu}$$

- Non-metric connection:

$$\Gamma^{\kappa}{}_{\lambda\mu} = \overset{\circ}{\Gamma}{}^{\kappa}{}_{\lambda\mu} \pm\sqrt{8\pi|\Lambda|}\left(\delta^{\kappa}_{\lambda}\, a_{\mu} + \delta^{\kappa}_{\mu}\, a_{\lambda} - 3\, a^{\kappa}\, g_{\lambda\mu}\right),$$

- Equation for electromagnetic potential a_{μ} in unified theory:

$$\overset{\circ}{\Box}\, a_{\mu} = 5|\Lambda|\, a_{\mu}\,,$$

- Einstein equation:

$$K_{\mu\nu} = \Lambda\, g_{\mu\nu} + 8\pi\left[f_{\mu\sigma}\, f^{\sigma}_{\ \nu} - \frac{1}{4}\, f_{\alpha\beta}\, f^{\alpha\beta}\, g_{\mu\nu}\right],$$

Properties of generalised theory – approximation

- Covariant derivative of a metric:

$$\nabla_\alpha g_{\mu\nu} = \pm 2\sqrt{8\pi|\Lambda|}\,\left(g_{\alpha\mu}\,a_\nu + g_{\alpha\nu}\,a_\mu - g_{\mu\nu}\,a_\alpha\right)$$

- Cosmological constant $\Lambda < 0$,
- such theory generate (as an approximation for weak electromagnetic field) the generalised (and also pure) Born-Infeld theory:

$$\begin{aligned}\mathcal{L}_{BI} &= \frac{\sqrt{|\det g|}}{16\pi}\,g^{\mu\nu}\,K_{\mu\nu} - b^2\sqrt{\left|\det\left(g + \frac{1}{b}\,f\right)\right|} + 2\,b^2\,\sqrt{|\det g|}\,,\\ b &= \mp\sqrt{\frac{|\Lambda|}{8\pi}}\,,\end{aligned}$$

- Finally we have (as an approximation) the well-known Einstein-Maxwell theory with cosmological constant Λ:

$$\mathcal{L}_{EM\Lambda} = \frac{\sqrt{|\det g|}}{16\pi}\,g^{\mu\nu}\,\overset{\circ}{R}_{\mu\nu} - \frac{\sqrt{|\det g|}}{4}\,f_{\mu\nu}\,f^{\mu\nu} - \frac{\sqrt{|\det g|}\,\Lambda}{8\pi}\,,$$

Weyl conformal gravity

Covariant derivative

$$\nabla_\alpha \, g_{\mu\nu} \quad = \quad -q \, \omega_\alpha \, g_{\mu\nu} \, ,$$

Connection

$$\Gamma^\kappa{}_{\lambda\mu} \quad = \quad \mathring{\Gamma}^\kappa{}_{\lambda\mu} + N^\kappa{}_{\lambda\mu} = \mathring{\Gamma}^\kappa{}_{\lambda\mu} + \frac{q}{2}\left(\delta^\kappa_\lambda \, \omega_\mu + \delta^\kappa_\mu \, \omega_\lambda - \omega^\kappa \, g_{\lambda\mu}\right) \, ,$$

Palatini picture

Connection character equation:

$$\frac{\partial \mathcal{L}_P}{\partial \Gamma^\kappa{}_{\lambda\mu}} \quad = \quad \frac{\partial \tilde{\mathcal{L}}_{matt}}{\partial \Gamma^\kappa{}_{\lambda\mu}} = \nabla_\nu \, \pi_\kappa{}^{\lambda\mu\nu} = -q \, \omega_\nu \, \pi_\kappa{}^{\lambda\mu\nu} \, ,$$

$$\pi_\kappa{}^{\lambda\mu\nu} \quad = \quad \frac{\sqrt{|\det g|}}{16\pi}\left(\delta^\nu_\kappa \, g^{\lambda\mu} - \delta^{(\lambda}_\kappa \, g^{\mu)\nu}\right) \, ,$$

Palatini picture

Matter Lagrangian

$$\begin{aligned}
\tilde{\mathcal{L}}_{matt} &\simeq -q\,\omega_\nu\,\pi_\kappa{}^{\lambda\mu\nu}\,\Gamma^\kappa{}_{\lambda\mu} = \\
&= \frac{\sqrt{|\det g|}}{16\pi}\,q\,\left(\Gamma^\kappa{}_{\kappa\mu}\,g^{\mu\nu}\,\omega_\nu - g^{\lambda\mu}\,\Gamma^\kappa{}_{\lambda\mu}\,\omega_\kappa\right)\,, \\
\tilde{\mathcal{L}}_{matt} &= \frac{\sqrt{|\det g|}}{16\pi}\left(q\,\nabla_\mu j^\mu + q\,g^{\lambda\mu}\,\nabla_\lambda\,\omega_\mu + (j_\mu - \omega_\mu)(j^\mu - \omega^\mu)\right)\,,
\end{aligned}$$

It is really strange.....

The nontrivial example

Proca theory with a little perturbation

Lagrangian:

$$\mathcal{L}_{matt} = -\sqrt{|\det g|}\left(\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + q \overset{\circ}{\nabla}_\nu b^\nu + \frac{1}{2} m^2 b_\nu b^\nu\right) ,$$

$$B_{\mu\nu} = b_{\nu,\mu} - b_{\mu,\nu} ,$$

Proca equation:

$$\overset{\circ}{\Box} b_\mu = m^2 b_\mu + b_\sigma \overset{\circ}{K}{}^\sigma{}_\mu - \overset{\circ}{\nabla}_\mu \left(\overset{\circ}{\nabla}_\nu b^\nu\right) ,$$

Non-metricity tensor:

$$N^\kappa{}_{\lambda\mu} = -\frac{2 \cdot 16\pi q}{3} \delta^\kappa_{(\lambda} b_{\mu)} ,$$

The nontrivial example

Proca theory with a little perturbation

Einstein equation:

$$\overset{\circ}{R}_{\mu\nu} = 8\pi \left[B_{\alpha\mu}\, B^{\alpha}_{\ \nu} + m^2\, b^{\mu}\, b^{\nu} - \frac{1}{4} B_{\alpha\beta}\, B^{\alpha\beta}\, g_{\mu\nu} \right]$$

Covariant derivative:

$$\begin{aligned} \overset{\circ}{\nabla}_{\alpha}\, b^{\alpha} &= g^{\alpha\beta}\, \overset{\circ}{\nabla}_{\alpha}\, b_{\beta} = g^{\alpha\beta}\, \nabla_{\alpha}\, b_{\beta} + g^{\alpha\beta}\, N^{\sigma}{}_{\alpha\beta}\, b_{\sigma} = \\ &= g^{\alpha\beta}\, \nabla_{\alpha}\, b_{\beta} - \frac{2 \cdot 16\pi\, q}{3}\, b_{\alpha}\, b^{\alpha} \end{aligned}$$

The nontrivial example

Proca theory with a little perturbation

Matter Lagrangian in Palatini picture:

$$\tilde{\mathcal{L}}_{matt} = \mathcal{L}_{matt} - \frac{16\pi\, q^2 \sqrt{|\det g|}}{3}\, b_\alpha\, b^\alpha =$$
$$= -\sqrt{|\det g|}\left[\frac{1}{4}\, B_{\mu\nu}\, B^{\mu\nu} + q\, g^{\alpha\beta}\, \nabla_\alpha\, b_\beta + \frac{1}{2}\left(m^2 - \frac{32\pi\, q^2}{3}\right) b_\alpha\, b^\alpha\right],$$

Palatini Lagrangian *on shell*:

$$\mathcal{L}_P = -\frac{\sqrt{|\det g|}}{4}\, B_{\mu\nu}\, B^{\mu\nu}\,.$$