Physical interpretation of the Newman-Janis algorithm

Philip Beltracchi

Based on
P. Beltracchi and P. Gondolo, Phys. Rev. D 104, 124066 (2021)
P. Beltracchi and P. Gondolo, Phys. Rev. D 104, 124067 (2021)

Overview

- The Newman-Janis algorithm[1] originally allowed for the derivation of the Kerr-Newman black hole and a rederivation of the Kerr black hole
- Gurses and Gursey[2] showed a version of the algorithm can create rotating Kerr-Schild spacetimes from static spherical Kerr-Schild spacetimes. Popular with nonsingular black hole research
- Drake and Szekeres[3] examined a further extension which could create rotating spacetimes from arbitrary spherically symmetric spacetimes
- While exact solutions to rotating spacetimes are interesting, the physical nature of the algorithm and properties of the resultant spacetime are not immediately apparent.

[1] E. T. Newman and A. I. Janis, J. Math. Phys. 6, 915 (1965).
[2]M. Gurses and F. Gursey, J. Math. Phys. 16, 2385 (1975).
[3]S. P. Drake and P. Szekeres, Gen. Rel. Grav. 32, 445 (2000).

Original Newman Janis 1

-Write Schwarzschild or Reissner-Nordstrom metric in advanced null coordinates,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)du^2 - 2\,du\,dr + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

-express $g^{\mu\nu}$ from null tetrads with complex coordinate r

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + \bar{m}^\nu m^\mu + \bar{m}^\mu m^\nu,$$

$$l^\mu = (0, 1, 0, 0),$$

$$n^\mu = \left(1, -\frac{1}{2}\left(1 - M\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) + \frac{Q^2}{r\bar{r}}\right), 0, 0\right)$$

$$m^\mu = \frac{1}{\sqrt{2}\bar{r}}\left(0, 0, 1, \frac{i}{\sin\theta}\right).$$

Original Newman-Janis 2

-Transform r and u coordinates to new r^*, u^* coordinates,

$$u^* = u - ia\cos\theta, \quad r^* = r + ia\cos\theta$$

so that the new tetrads are

$$\begin{aligned}
l^{*\mu} &= (0,1,0,0),\\
n^{*\mu} &= \left(1, -\frac{1}{2}\left(1 - \frac{2r^*M}{\Sigma} + \frac{Q^2}{\Sigma}\right), 0, 0\right),\\
m^{*\mu} &= \frac{1}{\sqrt{2}(r^* - ia\cos\theta)}\left(ia\sin\theta, -ia\sin\theta, 1, \frac{i}{\sin\theta}\right),
\end{aligned}$$

where we assume r^*, a, θ are all real and $\Sigma = r^{*2} + a^2\cos^2\theta$.

-Construct a metric such that

$$g^{*\mu\nu} = -l^{*\mu}n^{*\nu} - l^{*\nu}n^{*\mu} + \bar{m}^{*\nu}m^{*\mu} + \bar{m}^{*\mu}m^{*\nu},$$

which is the Kerr-Newman metric in u, r^*, θ, ϕ coordinates.

Open questions

- One finds Schwarzschild $\rightarrow$ Kerr and Reissner-Nordstrom $\rightarrow$ Kerr-Newman. What else can we do?
- The complexification of the M term and the Q term look rather different at this level. Is there a more systematic way of doing the complexifications?
- Why did this work?

The Gurses-Gursey Algorithm for Kerr-Schild systems

Start with a spherically symmetric Kerr-Schild metric in Schwarzschild coordinates

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right) dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

convert to advanced null coordinates $du = dt - (1 - 2m(r)/r)^{-1} dr$, obtaining

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right) du^2 - 2\, du\, dr + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Write in terms of l, m, n as before, but with the appropriately modified n

$$n^\mu = \left(1, -\frac{1}{2}\left(1 - \frac{2m(r)}{r}\right), 0, 0\right)$$

Complexify the r coordinate such that

$$n^\mu = \left(1, -\frac{1}{2}\left(1 - \frac{(r+\bar{r})m(\frac{r+\bar{r}}{2})}{r\bar{r}}\right), 0, 0\right),$$

Kerr-Schild systems have metrics that can be written
$g_{\mu\nu} = \eta_{\mu\nu} - S\, k_\mu k_\nu$
where S is a scalar function and k is a null vector with respect to both the Minkowski metric η and full metric g

GG Algorithm Continued

Perform the same coordinate change, the generalized n becomes

$$n^{*\mu} = \left(1, -\frac{1}{2}\left(1 - \frac{2r^* m(r^*)}{r^{*2} + a^2\cos^2\theta}\right), 0, 0\right)$$

Reconstruct metric using new tetrads, and relabel $r^* \to r$ for simplicity, to obtain

$$ds^2 = -\left(1 - \frac{2rm}{\Sigma}\right) du^2 - 2\,du\,dr + \Sigma\,d\theta^2$$
$$+ \sin^2\theta \left[2a\,dr\,d\phi - \frac{4arm}{\Sigma}\,du\,d\phi + \left(r^2 + a^2 + \frac{2a^2 rm\sin^2\theta}{\Sigma}\right) d\phi^2\right]$$

where $\Sigma = r^2 + a^2\cos^2\theta$. Convert to Boyer-Lindquist coordinates using

$$du = dt - \frac{r^2 + a^2}{\Delta}\,dr, \quad d\phi = d\varphi - \frac{a}{\Delta}\,dr$$

with the notation $\Delta = r^2 + a^2 - 2rm(r)$, and relabel $d\varphi \to d\phi$ to get

$$ds^2 = -\left(1 - \frac{2rm}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 + \sin^2\theta\left(\frac{2a^2 rm\sin^2\theta}{\Sigma} + a^2 + r^2\right) d\phi^2$$
$$- \frac{4arm\sin^2\theta}{\Sigma}\,dt\,d\phi.$$

-To go backwards, set a=0.
-a can be thought of as the rotation parameter
-Rotating/nonrotating pair of spacetimes uniquely specified by mass function m
-What can we say about the physical content for these pairs?

Properties of GG systems: Metric

It is helpful to call rewrite the metric to call attention to the principal directions

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - a\, dt\right]^2.$$

We label these directions 1,2,3,4 respectively

We can also write the Kerr-Schild null vector and scalar function

$$S = -\frac{2rm}{r^2 + a^2\cos^2\theta},$$
$$k_\mu = \left(1, \frac{\Sigma}{\Delta}, 0, -a\sin^2\theta\right).$$

the remaining portion

$$\eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \frac{\Sigma\left(a^2 - 4rm + r^2\right)}{\Delta^2}dr^2 + \Sigma\, d\theta^2 + \left(a^2 + r^2\right)\sin^2\theta\, d\phi^2 + \frac{4arm\sin^2\theta}{\Delta}dr\, d\phi - \frac{4rm}{\Delta}dr\, dt$$

has a fully 0 Riemann tensor, so it is the required Minkowski metric.

Properties of GG systems: EMT1

$$T^t{}_t = -\frac{m'}{4\pi\Sigma^3}\left[r^2(r^2+a^2) - a^4\sin^2\theta\cos^2\theta\right] + \frac{ra^2\sin^2\theta\, m''}{8\pi\Sigma^2} \qquad \to \frac{-m'}{4\pi r^2},$$

$$T^\phi{}_t = \frac{a}{8\pi\Sigma^3}\left[(r^2+a^2\cos^2\theta)rm'' - 2(r^2-a^2\cos^2\theta)m'\right] \qquad \to 0,$$

$$T^r{}_r = -\frac{r^2 m'}{4\pi\Sigma^2} \qquad \to \frac{-m'}{4\pi r^2},$$

$$T^\theta{}_\theta = -\frac{rm''}{8\pi\Sigma} - \frac{a^2\cos^2\theta\, m'}{4\pi\Sigma^2} \qquad \to \frac{-m''}{8\pi r},$$

$$T^t{}_\phi = -\sin^2\theta\,(a^2+r^2)\,T^\phi{}_t \qquad \to 0,$$

$$T^\phi{}_\phi = \frac{a^2 m'}{4\pi\Sigma^3}\left[r^2\sin^2\theta - (r^2+a^2)\cos^2\theta\right] - \frac{r(a^2+r^2)m''}{8\pi\Sigma^2} \qquad \to \frac{-m''}{8\pi r}.$$

Properties of GG systems: EMT2

- Eigenvalues:

$$-\rho = p_{\|} = \Lambda_{(1)} = \Lambda_{(2)} = -\frac{r^2 m'}{4\pi\Sigma^2} \qquad \rightarrow \frac{-m'}{4\pi r^2},$$

$$p_{\perp} = \Lambda_{(3)} = \Lambda_{(4)} = -\frac{r m''}{8\pi\Sigma} - \frac{a^2\cos^2\theta\, m'}{4\pi\Sigma^2} \qquad \rightarrow \frac{-m''}{8\pi r}.$$

- Segre type [(11)(1,1)] or degenerates to [(111,1)]
- Eigenvalue degeneracy → freedom with eigenvectors

$$v^{12}_{\mu} = A(-1, 0, 0, a\sin^2\theta) + B(0, 1, 0, 0),$$

$$v^{\mu}_{12} = \frac{A}{\Delta}(a^2 + r^2, 0, 0, a) + \frac{B\Delta}{\Sigma}(0, 1, 0, 0)$$

$$w^{34}_{\mu} = C(0, 0, 1, 0) + D\left(-\frac{a}{a^2 + r^2}, 0, 0, 1\right),$$

$$w^{\mu}_{34} = \frac{C}{\Sigma}(0, 0, 1, 0) + \frac{D}{a^2 + r^2}(a, 0, 0, \csc^2\theta)$$

Overview of Segre Type
- Notation of the eigenvalue/ eigenvector structure of a matrix
-In GR, typically use Segre types to examine $T^{\mu}{}_{\nu}$ and $R^{\mu}{}_{\nu}$
-[(11)(1,1)] refers to a pair of degenerate “space-space” eigenvalues and second degenerate “space-time” pair, [(111,1)] is four degenerate eigenvalues

Properties of GG systems: EMT3

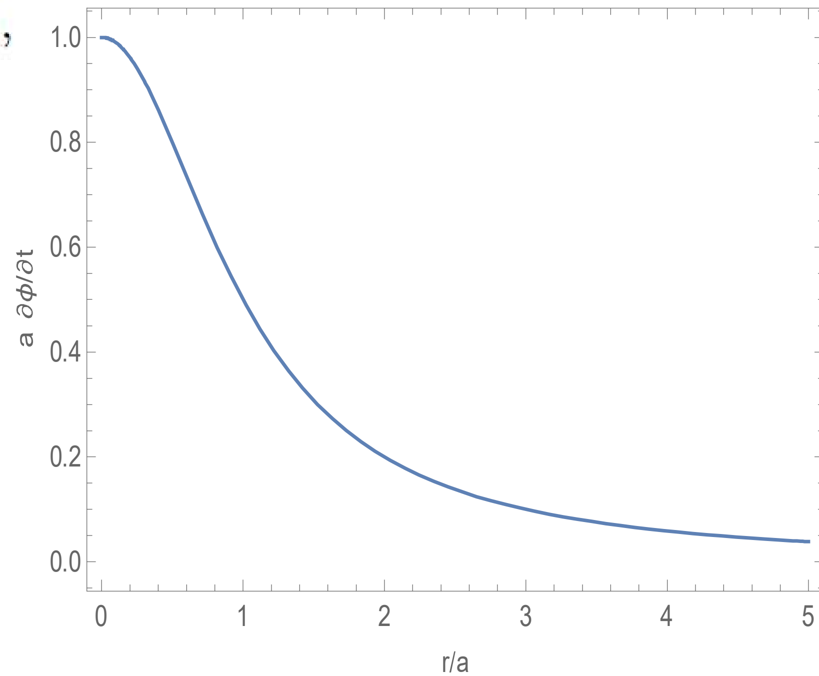
- Symmetry of 12 directions → boosts, Symmetry of 34 directions → rotations
- Normalized, timelike version of $v^{\mu}{}_{(12)}$ → family of 4 velocities

$$u^\mu = \frac{\cosh(W)}{\sqrt{|\Delta|\Sigma}}(a^2+r^2,0,0,a) + \sinh(W)\sqrt{\frac{|\Delta|}{\Sigma}}(0,1,0,0),$$

- "Angular velocity" of "matter" can be found

$$\frac{\partial\phi}{\partial t} = \frac{\frac{\partial\phi}{\partial\tau}}{\frac{\partial t}{\partial\tau}} = \frac{u^\phi}{u^t} = \frac{a}{a^2+r^2}$$

note: no dependence on m, same for all GG systems



GG systems: Equations of State 1

- Consider an EOS written in the form F(p,ρ,T,...)=0
- Segre [(11)(1,1)] automatically satisfies $\rho = -p_{||},\ p_2 = p_3 = p_\perp.$
- We are particularly interested in

$$F(\rho, p_\perp) = 0$$

- For spherical systems, if either eigenvalue is monotonic in r, an F can be derived
- It is usually NOT preserved when we pass into a rotating system, but under certain circumstances it is

GG systems: Equations of State 2

$F(\rho, p_\perp) = 0$ implies

$$\frac{\partial F}{\partial \rho}\frac{\partial \rho}{\partial r} + \frac{\partial F}{\partial p_\perp}\frac{\partial p_\perp}{\partial r} = 0,$$

$$\frac{\partial F}{\partial \rho}\frac{\partial \rho}{\partial \theta} + \frac{\partial F}{\partial p_\perp}\frac{\partial p_\perp}{\partial \theta} = 0.$$

$$\frac{\partial p_\perp}{\partial r}\frac{\partial \rho}{\partial \theta} - \frac{\partial p_\perp}{\partial \theta}\frac{\partial \rho}{\partial r} = 0$$

$$r^2 m''(r)^2 - 2rm'(r)\left(m''(r) + rm^{(3)}(r)\right) + 4m'(r)^2 = 0,$$

Reminiscent of
Schwarzschild term, Reisnner-Nordstrom term, string cloud term, de Sitter term

$$m(r) = M - \frac{K}{r} + \lambda r + \frac{1}{6}\Lambda r^3, \text{ with } \lambda^2 = 2K\Lambda.$$

$$\rho = -p_\parallel = \frac{(\lambda + r^2\Lambda)^2}{8\pi\Lambda(r^2 + a^2\cos^2\theta)^2},$$

$$p_\perp = \frac{(\lambda + r^2\Lambda)(\lambda - r^2\Lambda - 2a^2\Lambda\cos^2\theta)}{8\pi\Lambda(r^2 + a^2\cos^2\theta)^2}.$$

The null energy condition $\rho + p_\perp \geq 0$ requires

$$a \leq a_{\max} \equiv \sqrt{\lambda/\Lambda} = (2K/\Lambda)^{1/4}$$

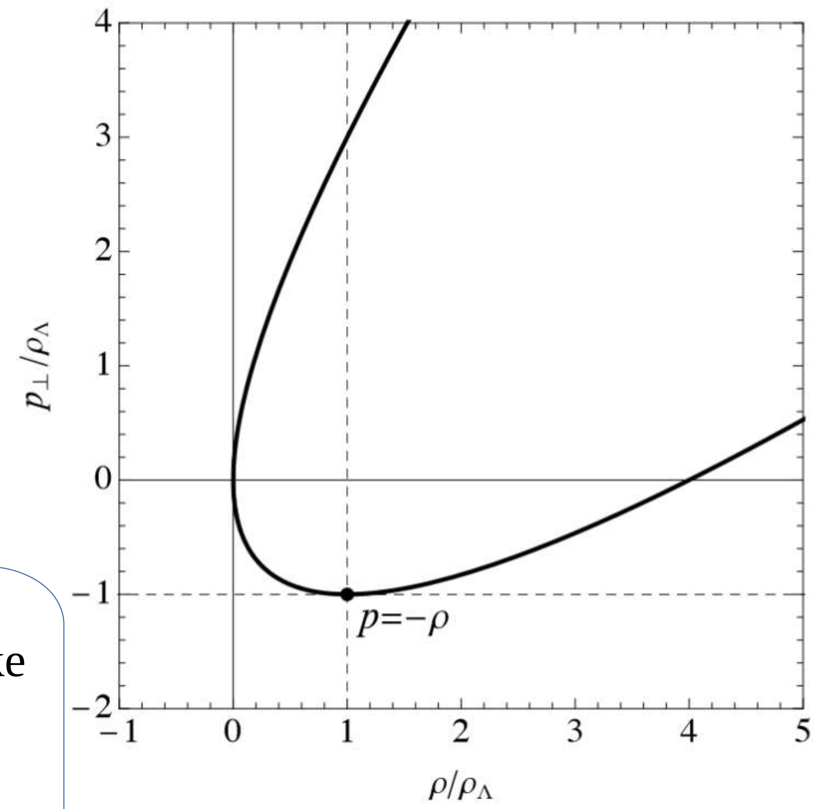
GG systems: Equations of State 3

$$\rho = -p_{\parallel} = \frac{(\lambda + r^2\Lambda)^2}{8\pi\Lambda(r^2 + a^2\cos^2\theta)^2},$$

$$p_{\perp} = \frac{(\lambda + r^2\Lambda)(\lambda - r^2\Lambda - 2a^2\Lambda\cos^2\theta)}{8\pi\Lambda(r^2 + a^2\cos^2\theta)^2}.$$

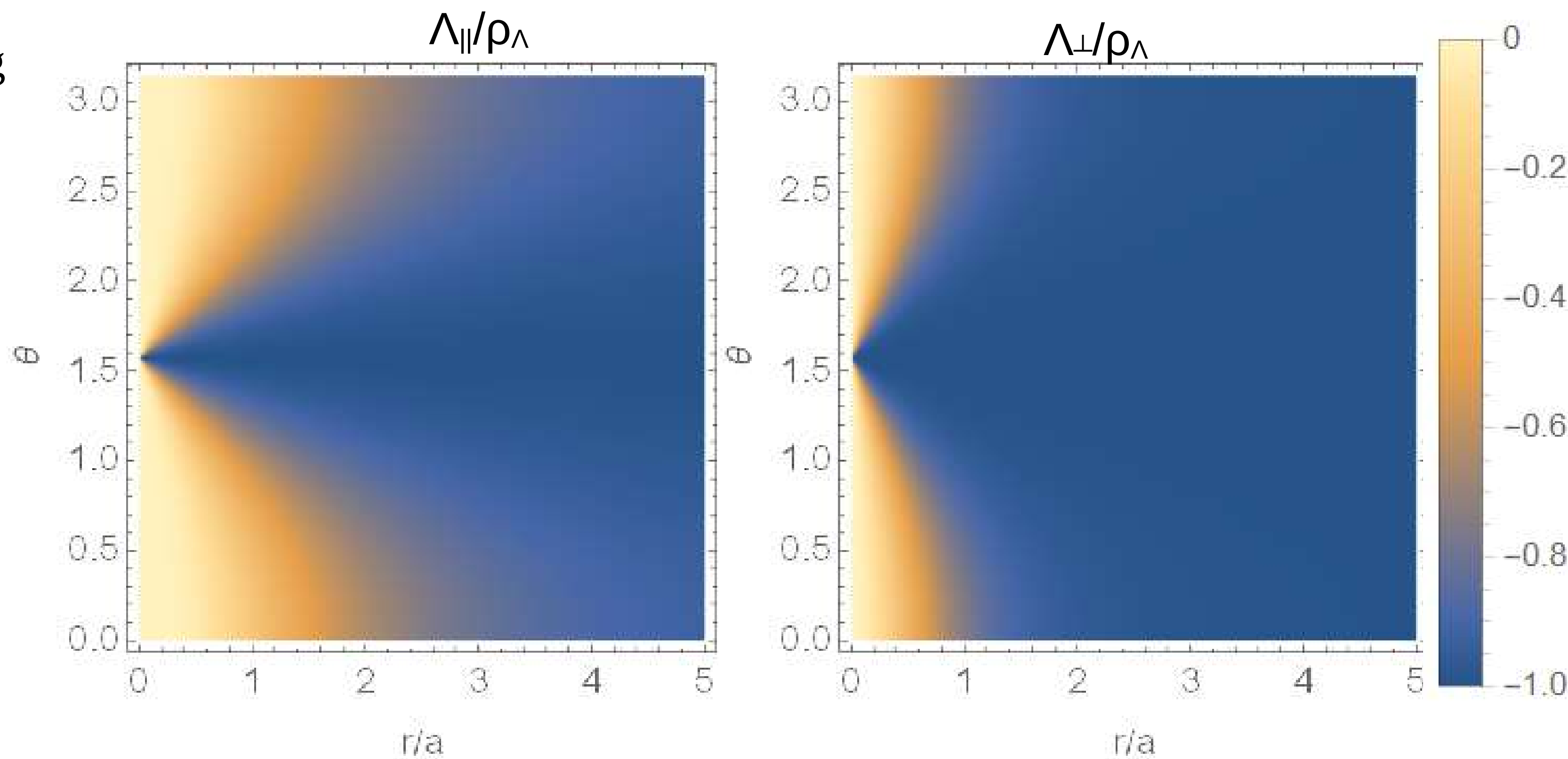
$$\boxed{(\rho - p_{\perp})^2 = 4\rho\frac{\Lambda}{8\pi},}$$

GGEOS:
Vacuum energy like at $\rho=\rho_\Lambda=\Lambda/8\pi$, electromagnetic like at $\rho>>\Lambda$



GG “Rotating de Sitter”

- $m=\Lambda r^3/6$, → de Sitter in nonrotating case
- Previously studied, name used in[1,2,3]
- has inhomogeneous anisotropic properties in rotating case
- Does not follow vacuum energy EOS, does follow GGEOS
- Violates NEC, except on equator



[1]N. Ibohal, Gen. Rel. Grav. 37, 19 (2005),
[2] I. Dymnikova, Physics Letters B 639, 368 (2006)
[3] E. J. Gonzalez de Urreta and M. Socolovsky, arXiv:1504.01728 (2015)

GG Recap

- Kerr-Schild nature, [(11)(1,1)] Segre type preserved
- Eigenvalue degeneracy → freedom in eigenvectors. 4 velocity can be constructed, implies same angular velocity for all m functions.
- Equation of state preserved for Schwarzschild → Kerr and Reissner/Nordstrom → Kerr/Newman
- Vacuum energy equation of state, general nonlinear electrodynamics/nonsingular black holes: alterations of equation of state, possible violation of NEC.
- We derived the particular equation of state that is preserved GGEOS, this can be examined further, or in other contexts like Cosmology[1]

[1] P. Beltracchi JCAP 09 (2023) 010

Drake-Szekeres for general systems

Write the metric of the general static spherically symmetric spacetime as

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{h(r)} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2,$$

with

$$h(r) = 1 - \frac{2m(r)}{r}, \qquad f(r) = \frac{h(r)}{[j(r)]^2},$$

and $j(r)$ defined in such a way that it is positive and real.

The corresponding rotating metric is

$$ds^2 = -\Sigma\frac{a^2\cos^2\theta + r^2 h}{\Sigma_j^2}\,dt^2 + 2\sin^2\theta\frac{(h-j)r^2 a\Sigma}{\Sigma_j^2}\,dt\,d\phi + \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 + \Sigma\sin^2\theta\frac{(a^2+jr^2)^2 - a^2\Delta\sin^2\theta}{\Sigma_j^2}\,d\phi^2,$$

where $\Sigma_j = r^2 j(r) + a^2\cos^2\theta$. As before, rewrite in terms of principal directions

$$ds^2 = -\frac{\Sigma\Delta}{\Sigma_j^2}\,(dt - a\sin^2\theta\,d\phi)^2 + \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 + \frac{\Sigma\sin^2\theta}{\Sigma_j^2}\,\left[(a^2+jr^2)d\phi - a\,dt\right]^2.$$

-Again, a can be thought of as the rotation parameter
-When j=1, the systems are Kerr-Schild and the Gurses-Gursey metric is recovered

DS properties: EMT 1

The actual expressions for the EMT components are too long to fit on a slide, but using the principal direction we can define tetrads and analyze the STRUCTURE in a Local Lorentz frame

$$e^{\alpha}{}_{\hat{\alpha}} = \begin{pmatrix} \frac{a^2+jr^2}{\sqrt{|\Delta|\Sigma}} & 0 & 0 & \frac{a\sin\theta}{\sqrt{\Sigma}} \\ 0 & \sqrt{\frac{|\Delta|}{\Sigma}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{1}{\Sigma}} & 0 \\ \frac{a}{\sqrt{|\Delta|\Sigma}} & 0 & 0 & \frac{1}{\sin\theta\sqrt{\Sigma}} \end{pmatrix} \rightarrow T^{\hat{\mu}}{}_{\hat{\nu}} = \begin{pmatrix} \hat{\mu}_0 & 0 & 0 & -\hat{\sigma}_{30} \\ 0 & \hat{\mu}_1 & \varepsilon_{\Delta}\hat{\sigma}_{12} & 0 \\ 0 & \hat{\sigma}_{12} & \hat{\mu}_2 & 0 \\ \varepsilon_{\Delta}\hat{\sigma}_{30} & 0 & 0 & \hat{\mu}_3 \end{pmatrix}.$$

We assume $\Delta \neq 0$ and $\varepsilon_{\Delta} = sign(\Delta)$. The metric in this orthornormal frame is

$$g_{\hat{\alpha}\hat{\beta}} = g_{\alpha\beta} e^{\alpha}{}_{\hat{\alpha}} e^{\beta}{}_{\hat{\beta}} = \mathrm{diag}(-\varepsilon_{\Delta},\ \varepsilon_{\Delta},\ 1,\ 1),$$

-General property of axisymmetric EMT: Block diagonal structure with tϕ block and rθ block
-Tetrad does not generally diagonalize EMT, but we later show doing so is not always possible

DS properties: EMT2

The eigenvalues are

$$\lambda^{\pm}_{r\theta} = \frac{1}{2}\left(\hat{B}_{12} \pm \sqrt{\hat{D}_{12}}\right), \quad \lambda^{\pm}_{t\phi} = \frac{1}{2}\left(\hat{B}_{30} \pm \sqrt{\hat{D}_{30}}\right).$$

using the discriminants D and traces B the blocks, which are

$$\hat{D}_{12} = (\hat{\mu}_1 - \hat{\mu}_2)^2 + 4\,\varepsilon_{\Delta}\hat{\sigma}_{12}^2, \qquad \hat{B}_{12} = \hat{\mu}_1 + \hat{\mu}_2,$$
$$\hat{D}_{30} = (\hat{\mu}_3 - \hat{\mu}_0)^2 - 4\,\varepsilon_{\Delta}\hat{\sigma}_{30}^2 \qquad \hat{B}_{30} = \hat{\mu}_3 + \hat{\mu}_0.$$

One expression of the eigenvectors in $(\hat{0}, \hat{1}, \hat{2}, \hat{3})$ is

$$V_1^{\hat{\mu}} = (-\hat{\sigma}_{30},\ 0,\ 0,\ \lambda^{+}_{t\phi} - \hat{\mu}_0) \qquad V_2^{\hat{\mu}} = (\lambda^{-}_{t\phi} - \hat{\mu}_3,\ 0,\ 0,\ \varepsilon_{\Delta}\hat{\sigma}_{30})$$
$$V_3^{\hat{\mu}} = (0,\ \varepsilon_{\Delta}\hat{\sigma}_{12},\ \lambda^{+}_{r\theta} - \hat{\mu}_1,\ 0) \qquad V_4^{\hat{\mu}} = (0,\ \lambda^{-}_{r\theta} - \hat{\mu}_2,\ \hat{\sigma}_{12},\ 0).$$

$\Delta > 0$						
$t\phi$ block					$r\theta$ block	
$\hat{D}_{30} > 0$	$\hat{D}_{30} = 0$			$\hat{D}_{30} < 0$	$\hat{D}_{12} = 0$	$\hat{D}_{12} \neq 0$
	$\hat{\sigma}_{30} = 0$	$\hat{\sigma}_{30} \neq 0$				
$[1,1]$	$[(1,1)]$	$[2]$		$[Z\bar{Z}]$	$[(11)]$	$[11]$

$\Delta < 0$					
$r\theta$ block				$t\phi$ block	
$\hat{D}_{12} > 0$	$\hat{D}_{12} = 0$		$\hat{D}_{12} < 0$	$\hat{D}_{30} = 0$	$\hat{D}_{30} \neq 0$
	$\hat{\sigma}_{12} = 0$	$\hat{\sigma}_{12} \neq 0$			
$[1,1]$	$[(1,1)]$	$[2]$	$[Z\bar{Z}]$	$[(11)]$	$[11]$

Complex conjugate and double null eigenvector cases do not diagonalize with a timelike LLF

DS: “Spinning Minkowski”

- In general, DS systems are very complex
- One system where direct analysis is possible is using the spherical seed metric

$$ds^2 = -\frac{1}{j^2}\,dt^2 + dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2,$$

  which corresponds to Minkowski space with a scaled time coordinate
- If either a=0 (no rotation) or j=1 (typical time scaling), the Riemann tensor for the DS “spinning” metric vanishes and it remains Minkowski space, but for arbitrary a, j there is typically nontrivial curvature and an EMT

DS: "Spinning Minkowski" D,B

The Drake-Szekeres "spinning Minkowski" spacetime results in the following expressions for the invariants $\hat{B}_{12}$, $\hat{B}_{30}$, $\hat{D}_{12}$, and $\hat{D}_{30}$,

$$\hat{B}_{12} = \frac{a^2(j-1)}{4\pi\Sigma^2\Sigma_j^2}\Big[-a^2(4r^2+a^2)\chi^4 - r^2(4jr^2-3ja^2+3a^2)\chi^2 + jr^4\Big],$$

$$\hat{D}_{12} = \frac{a^4(j-1)^2}{16\pi^2\Sigma^4\Sigma_j^4}\Big[a^4\big[(j-1)(j-13)r^4 - 2(j+11)a^2r^2 + a^4\big]\chi^8$$
$$+ 2a^2r^2\big[(j-1)^2r^4 + (2j^2-7j-7)a^2r^2 - 2(j-5)a^4\big]\chi^6$$
$$+ r^4\big[(j-1)(13j-1)r^4 + 2(7j^2+7j-2)a^2r^2 + 2(2j^2+11j+2)a^4\big]\chi^4$$
$$+ 2jr^6\big[(11j+1)r^2 + 2(5j-1)a^2\big]\chi^2 + j^2r^8\Big],$$

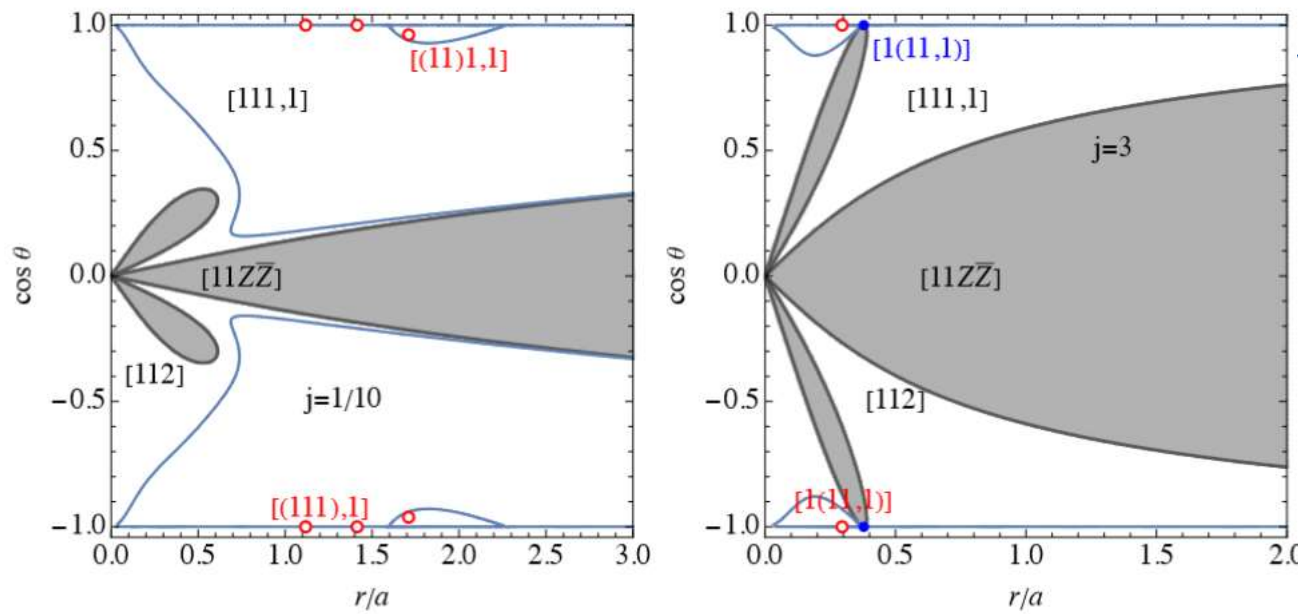
$$\hat{B}_{30} = \frac{a^2(j-1)}{4\pi\Sigma^2\Sigma_j^2}\Big[a^2(jr^2-r^2-a^2)\chi^4 - (j-1)r^2(r^2-2a^2)\chi^2 + jr^4\Big],$$

$$\hat{D}_{30} = \frac{a^2(j-1)^2}{16\pi^2\Sigma^3\Sigma_j^4}\Bigg[a^6(r^2+a^2)\chi^8 + a^4r^2(4j^2r^2-18jr^2+13r^2-6ja^2+4a^2)\chi^6$$
$$+ a^2r^4(13j^2r^2-18jr^2+4r^2+5j^2a^2-5a^2)\chi^4 + jr^6(jr^2-4ja^2+6a^2)\chi^2 - j^2r^8\Bigg].$$

where we use the shorthand $\chi = \cos\theta$.

Important properties:
-a(j-1) in coefficient, conditions for flat, vacuum spacetime
-Terms in square brackets are polynomial in r, χ, determine Segre type

DS: “Spinning Minkowski” Segre types



Notice when $\cos^2(\theta)=1$ we ALWAYS have some cross-block degeneracy, we look at which in the next figure

Plots showing Segre types for values $j = 1/10$ on the left and $j = 3$ on the right. The gray regions are where $\hat{D}_{30} < 0$ and the Segre type of the $t\phi$ block is $[Z\bar{Z}]$. The white regions are where $\hat{D}_{30} > 0$ and the Segre type of the $t\phi$ block is [11]. The boundary between the two regions (shown as black) with $\hat{D}_{30} = 0$ are generally have Segre type [2] for the $t\phi$ block. The blue line shows the location where there is a cross block degeneracy.

DS: "Spinning Minkowski" On axis cross block degeneracy

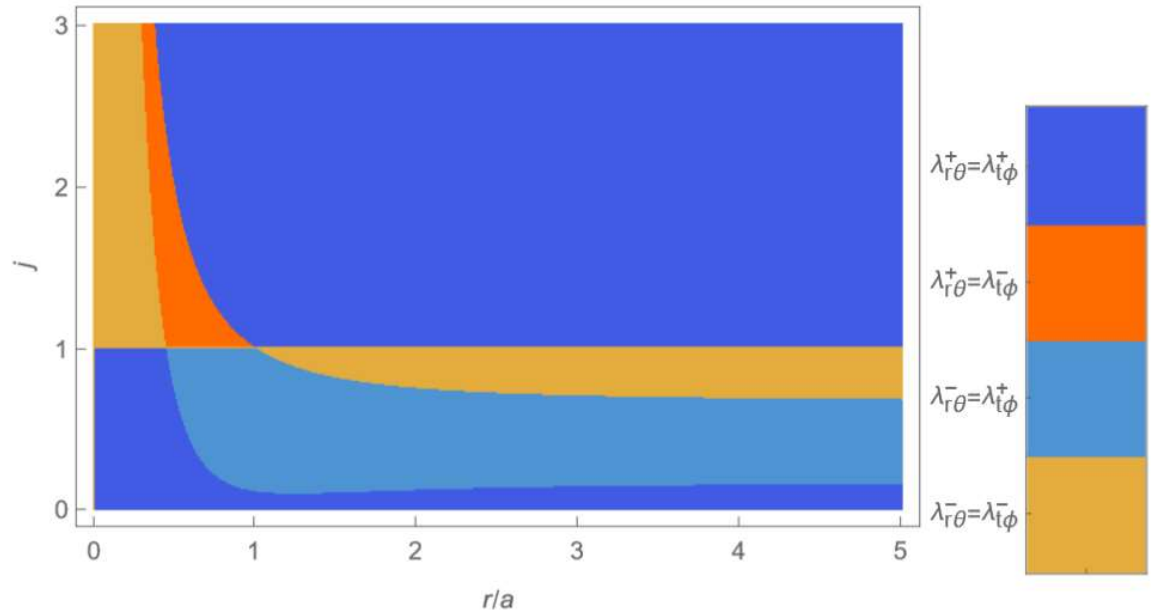


Figure 2: This figure illustrates the particular cross-block degeneracies at $\chi^2 \equiv \cos^2\theta = 1$ for various $r/a, j$. We have $\lambda^-_{t\phi} = \lambda^-_{r\theta}$ in light orange and $\lambda^-_{t\phi} = \lambda^+_{r\theta}$ in dark orange, both of these indicate a [(1,1)] type degeneracy. We have $\lambda^+_{t\phi} = \lambda^+_{r\theta}$ in dark blue and $\lambda^+_{t\phi} = \lambda^-_{r\theta}$ in light blue, both of which are [(11)] cross block degeneracies.

DS: “Spinning Minkowski” Non-appearing Segre types

- Segre types [31] and [(31)], i.e. triple null eigenvector types, can not occur for any DS system due to the block structure of the EMT.
- [2(11)] and [(211)], i.e. one block with a double null eigenvector and the second with degenerate eigenvalues, or with a double null eigenvector and all degenerate eigenvalues, could in theory occur for a DS system but did not appear here

DS Recap

- Rotating DS systems can have widely variable behavior
- Sorts of Matter or Fields described unclear
- The only globally perfect fluid rotating NJ system is the vacuum Kerr solution, perfect fluid may appear at isolated points
- While this is more general than the GG version, the complexity makes it less used

Conclusion

- Newman-Janis algorithm and its generalizations allow for creation of rotating spacetimes from non-rotating spacetimes
- Rotating system generated may not correspond to something that can be easily interpreted as a rotating version of the spherical system: Equations of state distortion, possible complex eigenvalues…
- One needs to analyze the EMTs of the spherical and corresponding generated rotating system to determine the correspondance
- Possible future: More recent work on alternate forms of the algorithm (e.g.[1]) are similar to DS but have different notation, this may lead to more usable expressions. Alternatively, further examinations of systems using the algorithm (e.g.[1,2,...]) may increase intuitive understanding of how these systems may behave.

[1] M. Azreg-Ainou, Phys. Rev. D 90, 064041 (2014)
[2] A. Smailagic and E. Spallucci, Phys. Lett. B688, 82 (2010)
[3]C. Bambi and L. Modesto, Phys. Lett. B721, 329 (2013)

Thank You

Backup 1: Kerr-Schild, [(11)(1,1)], and nonsingular black holes

- The Anisotropic TOV can be loosely thought of as a static force balance equation

$$\frac{dp_r}{dr} = -\frac{\left(m + 4\pi r^3 p_r\right)\left(\rho + p_r\right)}{r^2\left(1 - \frac{2m}{r}\right)} + \frac{2(p_T - p_r)}{r}.$$

Pressure gradient

Gravity

Anisotropy

- 2m=r → denominator in gravity term goes to 0, “horizon condition”. Forces are unbalanced, or a singularity exists.
- If $p_r = -\rho$, gravity term vanishes identically everywhere. Can have “balanced forces” on horizon (see Reissner-Nordstrom for most basic example), popular with nonsingular black holes
- Segre type [(11)(1,1)] is the general type for spherically symmetric spacetimes which satisfy

  $p_r = -\rho$

- Spherically symmetric Kerr-Schild is Segre [(11)(1,1)], GG method generates a rotating spacetime

Backup 2: DS Expressions 1

$$\hat{\mu}_0 = \frac{1}{32\pi\Sigma^3\Sigma_j^2}\Big[\!\Big[-8r^2\Sigma\Sigma_j^2 m' + (1-\chi^2)a^2r^3\Sigma^2(4j + rj')j'$$
$$+ 8(j-1)a^2r^2\big[(1-\chi^2)(r^4 - a^4\chi^4) - 3r\chi^2(\Sigma + \Sigma_j)m\big]$$
$$+ 4(j-1)^2a^2r^2\Sigma\big[(1-\chi^2)\Sigma + r^2(1-4\chi^2)\big]\Big]\!\Big] \quad \text{(A.1a)}$$

$$\hat{\mu}_1 = \frac{1}{32\pi\Sigma^3\Sigma_j^2}\Big[\!\Big[-8r^2\Sigma\Sigma_j^2 m' + a^2r^4(1-\chi^2)\Sigma^2(j')^2$$
$$-4r^3\Sigma j'\Big[2a^2\chi^2(a^2+r^2) + \big(\Sigma^2 + (a^2+r^2)(r^2 - a^2\chi^2)\big)j - 4mr\Sigma_j\Big] +$$
$$8a^2r^2\Sigma(j-1)\Big[(\chi^2-5)\Sigma + 6r^2(1-\chi^2) + 2r\chi^2 m\Big] -$$
$$4a^2r^2(j-1)^2\Big[6r^4(\chi^2-1) + r^2\Sigma(5+4\chi^2) + \Sigma^2(\chi^2-1) - 6r^3\chi^2 m\Big]\Big]\!\Big] \quad \text{(A.1b)}$$

$$\hat{\mu}_2 = \frac{1}{32\pi\Sigma^3\Sigma_j^2}\Big[\!\Big[-4r\Sigma^2\Sigma_j^2 m'' - 8a^2\chi^2\Sigma^3 m' - 4r^2\Sigma^2\Sigma_j(r^2 + a^2 - 2rm)j'' +$$
$$r^4\Sigma^2\big[7(a^2+r^2) + \Sigma - 16rm\big](j')^2 + 4r\Sigma j'\{rm\left[-jr^2\left(5a^2\chi^2 + r^2\right) + 15a^2r^2\chi^2 + 11a^4\chi^4\right]$$
$$+ jr^2\left[a^2r^2\left(2\chi^2+1\right) + a^4\chi^2\left(\chi^2+3\right) - r^4\right] - a^2\chi^2\left[a^2r^2\left(7\chi^2+6\right) + 4a^4\chi^2 + 9r^4\right] + 3r^2\Sigma\Sigma_j m'\}$$
$$+ 8(j-1)a^2\chi^2\{\Sigma\left[m'r^2\left(a^2\chi^2 + 2jr^2 - r^2\right) - 5a^2r^2\left(\chi^2-1\right) - a^4\chi^2 + r^4\right] + rm\big[3a^2r^2\chi^2 + 5a^4\chi^4$$
$$+ r^4 - 3j(a^2r^2\chi^2 + 2r^4)\big]\} + 4(j-1)^2(3r^4 + 11a^2r^2 + 5a^4\chi^2 - 2a^2r^2\chi^2 + a^4\chi^4)a^2r^2\chi^2\Big]\!\Big] \quad \text{(A.1c)}$$

Backup 2: DS Expressions 2

$$\hat{\mu}_3 = \frac{1}{32\pi\Sigma^3\Sigma_j^2}\Big[\!\!\Big[-4r\Sigma^2\Sigma_j^2 m'' - 8a^2\chi^2\Sigma\Sigma_j(3\Sigma - 2\Sigma_j)m' + 12r^3\Sigma^2\Sigma_j j'm'$$

$$-16r^5\Sigma^2 m(j')^2 + 8r^3\Sigma^2\Sigma_j m j'' - 4r^2\Sigma[jr^4 - a^4\chi^4 + 5a^2\chi^2(\Sigma_j - 3\Sigma)]mj'$$

$$+8a^2 r\chi^2(j-1)[r^4 - a^4\chi^4 - 3a^2r^2\chi^2(j-1) - 6(jr^4 - a^4\chi^4)]m - 4r^2(a^2+r^2)\Sigma^2\Sigma_j j''$$

$$+r^4\Sigma^2[5a^2(1-\chi^2) + 8\Sigma](j')^2$$

$$-4r\Sigma(6a^4r^2\chi^2 + 9a^2r^4\chi^2 + 4a^6\chi^4 + 7a^4r^2\chi^4 + a^2r^4j + r^6j - a^4jr^2\chi^2 - 4a^2r^4\chi^2j - 3a^4r^2\chi^4j)j'$$

$$-4a^2\Sigma\chi^2(j-1)(a^2r^2 + r^4 + 2a^4\chi^2 + 5a^2r^2\chi^2 - 3a^2jr^2 - 3r^4j - 3a^2r^2\chi^2j)\Big]\!\!\Big] \tag{3.11}$$

$$\hat{\sigma}_{30} = \frac{a\sqrt{|\Delta|}\sin\theta}{16\pi\Sigma^2\Sigma_j^2}\Big[\!\!\Big[-2r(r^2\Sigma_j + 2\Sigma a^2\chi^2)j' + \Sigma r^4(j')^2 - \Sigma\Sigma_j r^2 j'' - 2(j-1)(r^2 - a^2\chi^2)(jr^2 - a^2\chi^2)\Big]\!\!\Big] \tag{3.12}$$

$$\hat{\sigma}_{12} = \frac{3a^2 r\cos\theta sqrt|\Delta|\sin\theta}{8\pi\Sigma^3\Sigma_j^2}\Big[\!\!\Big[2(j-1)(a^4\chi^4 - jr^4) + \Sigma^2 r j'\Big]\!\!\Big] \tag{3.13}$$