Axisymmetric, extremal horizons at the presence of cosmological constant

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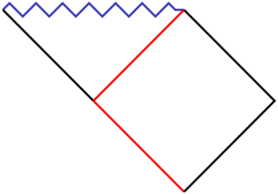
Plan of the talk

1. Review of quasi-local approach to black holes:
   - A. non-expanding horizons,
   - B. izolated horizons,
   - C. near-horizon geometry (NHG) equations.
2. Coordinates adapted to tackle the axisymmetric problem. Proper boundary conditions.
3. Solutions to axisymmetric NHG.
4. Embedding in Kerr-(anti)de Sitter spacetime. Doubly-extremal horizon.
5. Extending results.

Quasi-local framework



Killing horizons have many uses, especially for typical black hole spacetimes. In particular in black hole thermodynamics or in Hawking's rigidity theorem.

The idea is to relax definition of Killing horizon, while maintaining conditions necessary to define meaningful physical quantities. It describes horizon of black hole in exact or approximate equilibrium or in isolation.

While quasi-local approach works in arbitrary number of dimension, we will be operating in 4-dimensional spacetime, obeying vacuum Einstein equations:

$$R^{(g)}_{\mu\nu} - \frac{1}{2}R^{(g)}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{1}$$

Non-expanding horizon

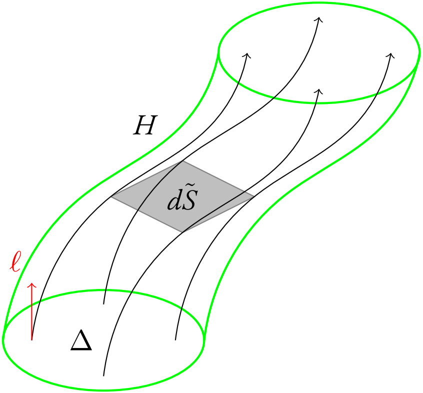
Let (H, q) be smooth, null hypersurface in spacetime, and let ℓ be its tangent (and normal) vector. H is congruence of null geodesics generated by ℓ. We can define its expansion, which will be vanishing.

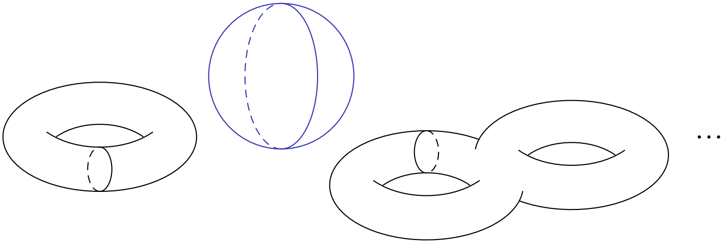
It follows, that others optical scalars (shear and twist) also vanish, thanks to Raychaudhuri equation.

We will also assume, that vector ℓ satisfies dominant energy conditions, that is

$$-T^a_b \ell^b \qquad (2)$$

is future-causal.





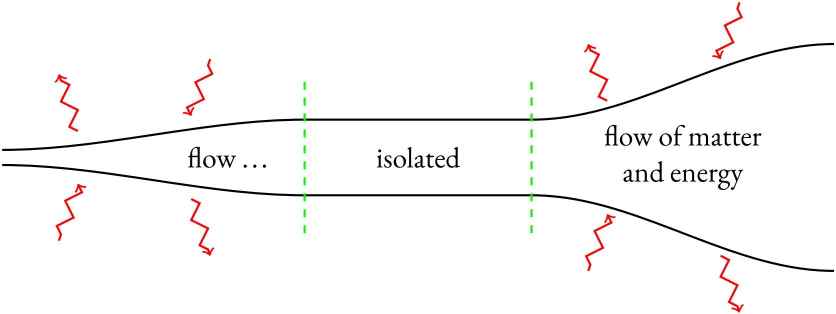
Horizon is topological product

$$H \cong \Delta \times \mathbb{R}, \tag{3}$$

where Δ is closed, 2-dimensional manifold. We are interested in case $\Delta \cong \mathbb{S}_2$.

Finally, we assume, that Einstein equations are satisfied *on* a horizon.

Isolated horizon



For black holes in equilibrium it should suffice to set geometry (covariant derivative) of a horizon to be *time*-independent:

$$[\mathcal{L}_\ell, D] = 0. \tag{4}$$

Rotation one-form and NHG equations

Covariant derivative of ℓ

$$D_a \ell^b = \omega_a \ell^b, \tag{5}$$

defines rotation one-form ω, which is tied to angular momentum of black hole. The quantity

$$\kappa^{(\ell)} = \omega_a \ell^a = D_a \ell^a \tag{6}$$

is called surface gravity. When $\kappa^{(\ell)} = 0$ horizon is called extremal (degenerate).

Let Δ be spacelike section of H, transversal to ℓ. Then vacuum Einstein equations with Λ induce constraints

$$\nabla_{(A}\omega_{B)} + \omega_A \omega_B - \frac{1}{2} R_{AB} + \frac{\Lambda}{2} q_{AB} = 0, \tag{7}$$

called near-horizon geometry (NHG) equations.

By integrating NHG equation over whole Δ one can be show, that there must be

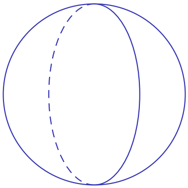
$$\Lambda = \frac{\int \omega^2 \eta}{\int \eta} + \frac{\int_\Delta K\eta}{\int \eta} = \frac{\int \omega^2 \eta}{\int \eta} + \frac{4\pi}{\int \eta}\Big(1 - G\left(\Delta\right)\Big). \tag{8}$$

It can be shown, that the only solution for Δ with genus $G(\Delta)$ greater than 0 is

$$K = \Lambda, \quad \omega = 0. \tag{9}$$

It explain our interest with $\Delta \cong \mathbb{S}_2$, for which:

$$\Lambda \leq \frac{4\pi}{\int \eta}. \tag{10}$$



Adapted coordinates

We will be considering axisymmetric Δ, and so we will introduce coordinates (θ, ϕ), such that symmetry is generated by ∂_ϕ. Then the general form of q is

$$q_{AB}dx^A dx^B = \Sigma^2(\theta)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{11}$$

We next introduce coordinate x such that

$$dx = \frac{\Sigma^2(\theta)\sin\theta}{R^2}d\theta, \qquad \int \eta = 4\pi R^2. \tag{12}$$

Now metric is given by

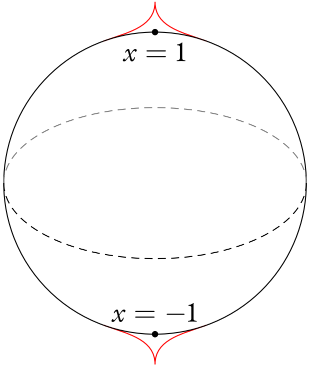
$$q = R^2\left(\frac{1}{P^2(x)}dx^2 + P^2(x)d\phi^2\right), \quad P^2(x) = \frac{\Sigma^2(\theta)\sin^2\theta}{R^2}. \tag{13}$$

Gaussian curvature is

$$K = -\frac{1}{2}\frac{1}{R^2}\partial_x^2 P^2, \tag{14}$$

which leads us to

$$\frac{\int K\eta}{\int \eta} = \frac{1}{R^2} \implies \Lambda R^2 \leq 1. \tag{15}$$



Function P must admit following limits:

$$P(x = \pm 1) = 0, \quad \partial_x P^2(x = \pm 1) = \mp 2. \tag{16}$$

These conditions follow from regularity of metric and smooth functions on the poles, and are equivalent to assuming, that circle of radius δr, about pole $x = \pm 1$ has length $2\pi\delta r + o(r)$.

We will use Hodge decomposition of rotation one-form

$$\omega = \star dU + d\log B. \tag{17}$$

Potentials U and B are functions of x only.
If we substitute decomposition into NHG equations, we end up with 3 differential equations for U, B and function P:

$$\partial_x^2 B - \left(\partial_x U\right)^2 B = 0 \tag{18}$$

$$\partial_x\left(B^2 \partial_x U\right) = 0 \tag{19}$$

$$\begin{aligned} 2\frac{PP_{,x}}{R^2}\partial_x \log B + \frac{P^2}{R^2}(\partial_x \log B)^2 + \frac{P^2}{R^2}\partial_x^2 \log B \\ -\frac{P^2}{R^2}(\partial_x U)^2 + \frac{1}{2}\frac{1}{R^2}\partial_x^2 P^2 + \Lambda = 0 \end{aligned} \tag{20}$$

From middle equation follows

$$B^2 \partial_x U = \tilde{\Omega} \tag{21}$$

and we must distinguish two classes of solutions: $\tilde{\Omega} = 0$, and $\tilde{\Omega} \neq 0$.

1. In case $\tilde{\Omega} = 0$ we get

$$U = \text{const.} \implies B = B_1 x + B_0, \tag{22}$$

but decomposition of ω forces us to take $B_1 = 0$. It follows, that both potentials are constant.

2. In case $\tilde{\Omega} \neq 0$ we get

$$B^2 = B_0^2 \left[\Omega^2 + (x - x_0)^2\right] \tag{23}$$

$$U = \arctan\left(\frac{x - x_0}{\Omega}\right) + U_0, \tag{24}$$

where

$$\Omega = \frac{\tilde{\Omega}}{B_0^2}. \tag{25}$$

Results for $\tilde{\Omega} = 0$

Functions U and B are constant, so:

$$\omega = \star dU + d \log B = 0. \tag{26}$$

Further integration yields

$$P^2 = 1 - x^2, \tag{27}$$

which is equivalent to

$$\Lambda R^2 = 1 \Longleftrightarrow \Lambda = K = \frac{1}{R^2}, \tag{28}$$

This case describes non-rotating which corresponds to spherically-symmetric horizon ($x = \cos\theta$):

$$q = R^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right). \tag{29}$$

Results for $\tilde{\Omega} \neq 0$

Further integration of NHG equations yields:

$$\partial_x^2 P^2 + \frac{2(x-x_0)}{(x-x_0)^2+\Omega^2}\partial_x P^2 + \frac{4\Omega^2}{\left[(x-x_0)^2+\Omega^2\right]^2}P^2 = -2\Lambda R^2. \quad (30)$$

Constants x_0 and Ω can be eliminating by boundary conditions for P^2:

$$x_0 = 0, \quad \Omega^2 = \frac{1-\frac{1}{3}\Lambda R^2}{1-\Lambda R^2}. \quad (31)$$

It leads to

$$P^2 = \left(x^2-1\right)\frac{\Lambda R^2\left(\Lambda R^2 - x^2(\Lambda R^2-1)-5\right)+6}{\Lambda R^2+3x^2(\Lambda R^2-1)-3}, \quad (32)$$

or

$$P^2 = \frac{\Lambda R^2}{\Omega^2+x^2}\left\{\frac{\Omega^2+\frac{1}{3}}{\Omega^2-1}\left(\Omega^2-x^2\right) - x^2\left(\Omega^2+\frac{1}{3}x^2\right)\right\} \quad (33)$$

Positivity of Ω^2 and P^2 constraints parameters in the following way:

$$\Lambda R^2 \in \,]-\infty, 1[\, \Longleftrightarrow \Lambda < \frac{1}{R^2} = K. \tag{34}$$

One-form potentials are given by

$$B^2 = B_0^2 \left[\Omega^2 + x^2\right] \quad \text{and} \quad U = \arctan\left(\frac{x}{\Omega}\right) + U_0, \tag{35}$$

while rotation one-form is

$$\begin{aligned} \omega =& \frac{x(1-\Lambda R^2)}{x^2(1-\Lambda R^2) + (1-\frac{1}{3}\Lambda R^2)} dx \\ & \pm P^2 \frac{\sqrt{(1-\Lambda R^2)(1-\frac{1}{3}\Lambda R^2)}}{x^2(1-\Lambda R^2) + (1-\frac{1}{3}\Lambda R^2)} d\phi \end{aligned} \tag{36}$$

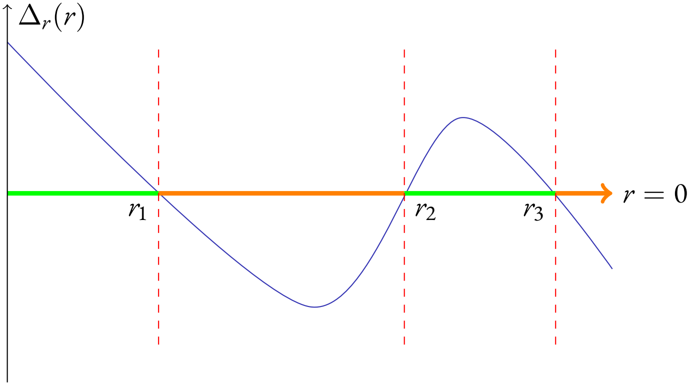
Kerr-(anti)de Sitter metric

Kerr-(anti)de Sitter metric has the form

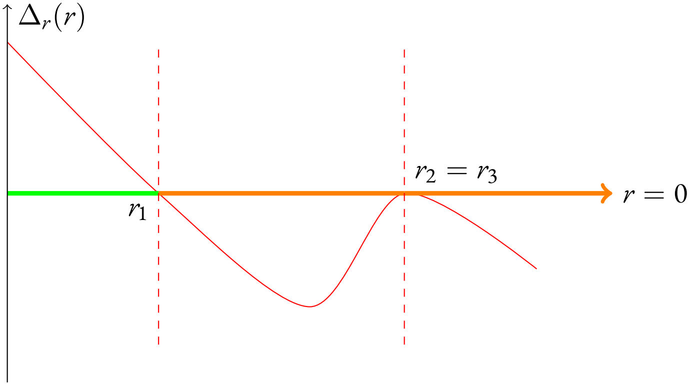
$$g = -\frac{\Delta_r}{\chi^2\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Delta_\theta \sin^2\theta}{\chi^2\rho^2}\left(adt - (r^2+a^2)d\phi\right)^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) \tag{37}$$

where:

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta \\ \Delta_\theta &= 1 + \frac{1}{3}\Lambda a^2\cos^2\theta \\ \chi &= 1 + \frac{1}{3}\Lambda a^2 \\ \Delta_r &= (r^2+a^2)\left(1 - \frac{1}{3}\Lambda r^2\right) - 2Mr \end{aligned} \quad . \tag{38}$$



Vanishing of Δ_r signifies horizons r_i. Orange and green colors signify timelike and spacelike parts of spacetime. If two horizons merge (double root of Δ_r), then their surface gravity must vanish – they are extermal.



Vanishing of Δ_r signifies horizons r_i. Orange and green colors signify timelike and spacelike parts of spacetime. If two horizons merge (double root of Δ_r), then their surface gravity must vanish – they are extermal.

Embedding in Kerr-(anti)de Sitter

By comparing metric of Kerr-(anti)de Sitter's horizon (constant r and t, $\Delta_r(r_0) = 0$) with our form we get:

$$P^2 = (1 - x^2)\frac{1 + \frac{1}{3}\Lambda a^2 x^2}{1 + \frac{1}{3}\Lambda a^2}\frac{r_0^2 + a^2}{r_0^2 + a^2 x^2}, \qquad R^2 = \frac{r_0^2 + a^2}{1 + \frac{1}{3}\Lambda a^2}. \tag{39}$$

Vanishing of determinant of Δ_r (multiple roots) in our parametrization yields:

$$\begin{aligned} a^2 &= \frac{3R^2(1 - \Lambda R^2)}{(\Lambda R^2 - 3)(\Lambda R^2 - 2)} \\ M &= \frac{2}{3}\sqrt{\frac{R^2}{2 - \Lambda R^2}\frac{(3 - 2\Lambda R^2)^2}{(3 - \Lambda R^2)(2 - \Lambda R^2)}} \end{aligned}. \tag{40}$$

Embedding in Kerr-(anti)de Sitter

Function P^2 is given by

$$P^2 = \left(x^2 - 1\right) \frac{\Lambda R^2 \left(\Lambda R^2 - x^2(\Lambda R^2 - 1) - 5\right) + 6}{\Lambda R^2 + 3x^2(\Lambda R^2 - 1) - 3} \qquad (41)$$

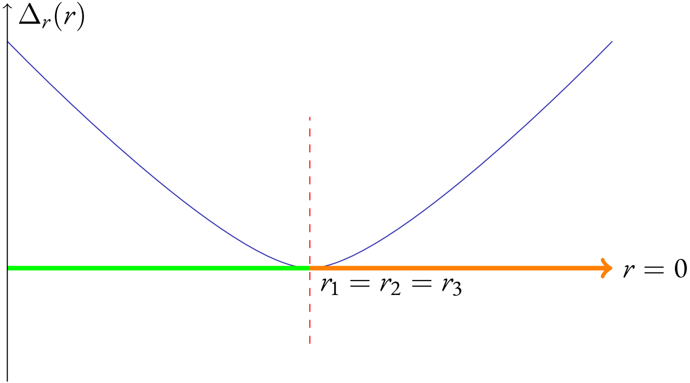
while parameters are constrained by

$$\Lambda R^2 \in]-\infty, 1], \qquad (42)$$

where limit $\Lambda R^2 = 1$ recovers non-rotating case.

These results agree with earlier ones.

Doubly extremal horizon – triple root



We have analysed the case of triple root of Δ_r:

$$\Lambda R^2 = \frac{3-\sqrt{3}}{2} \quad \text{for} \quad r_1 = r_2 = r_3 = R\sqrt{\sqrt{3}-1} \qquad (43)$$

Roots

$$r_1 = \frac{-\sqrt{A} + \sqrt{B + C\sqrt{A}}}{\sqrt{2}},$$
$$r_2 = \frac{\sqrt{A} - \sqrt{B - C\sqrt{A}}}{\sqrt{2}}, \qquad (44)$$
$$r_3 = \frac{\sqrt{A} + \sqrt{B - C\sqrt{A}}}{\sqrt{2}};$$

where

$$A = \frac{(2\Lambda R^2 - 3)^2}{\Lambda\left(\Lambda R^2 - 2\right)\left(\Lambda R^2 - 3\right)},$$
$$B = \frac{3}{2\Lambda} + \frac{R^2(\Lambda R^2 + 3)}{2(\Lambda R^2 - 3)(\Lambda R^2 - 2)}, \qquad (45)$$
$$C = 2\sqrt{2}\sqrt{\frac{R^2}{2 - \Lambda R^2}}.$$

Extending results

We could consider electro-vacuum, that is space time obeying

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{46}$$

It will change NHG equation in such a way, that previous solution for P^2 will be changed by function a describing e-m field, and decomposed into U and B:

$$P^2_{\text{(e-m)}} = P^2 + a, \tag{47}$$

where P^2 is given as in (33), but with

$$\Omega^2 = \frac{2\Lambda R^2 + \sqrt{4\left(3 - 2\Lambda R^2\right)^2 - 18a\left(1 - 1\Lambda R^2\right)}}{3\left(2 - 2\Lambda R^2\right)}. \tag{48}$$

Parameters still seem to obey the same constraints: $\Lambda R^2 \leq 1$.

Thank You for Your attention.