Quantum state reduction, and Newtonian twistor theory

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- Roger Penrose (1996) On gravity's role in quantum state reduction (1996), Gen. Rel. Grav **28**. On the gravitization of quantum mechanics 1: Quantum state reduction (2014). Found. Phys. **44**.
- Maciej Dunajski, James Gundry.(2016) Non-relativistic twistor theory and Newton–Cartan geometry. Comm. Math. Phys. **342**.
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Quantum Mechanics according to Penrose

- The $\mathbf{U}$ process: Unitary, holomorphic evolution (Schrödinger equation).
- The $\mathbf{R}$ process: Wave function collapse. Non–holomorphic.
- $\mathbf{R}$ is a real process taking place in time. Gravitational effects need to be taken into account to understand it.
- Gravitize quantum mechanics, rather than quantize gravity.

OUTLINE

- Quantum principle of equivalence in non–uniform gravitational fields: wave function phase ambiguity with non–linear time dependence.
- Requires only Newtonian gravity framework
- ... so incorporate non–relativistic twistor theory. which gives a non–local description of all Newtonian space–times.
- Combine the quantum non–locality with the twistor non–locality into a proposal for a twistor description of the wave function collapse.

Uniform gravitational field

- Schrödinger equation in Newtonian frame

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi - m\mathbf{g}\cdot\mathbf{x}\psi.$$

- Accelerating (Einstein) frame: $t = T, \quad \mathbf{X} = \mathbf{x} - \frac{1}{2}\mathbf{g}t^2$

$$i\hbar\frac{\partial\Psi}{\partial T} = -\frac{\hbar^2}{2m}\Delta\Psi, \quad \Psi(T,\mathbf{X}) = e^{-\frac{im}{\hbar}\left(\frac{1}{6}|\mathbf{g}|^2t^3 - t\mathbf{g}\cdot\mathbf{x}\right)}\psi(t,\mathbf{x})$$

- Ambiguity in positive/negative frequency decoposition ($c \to \infty$ limit of Unruh effect).
- Superposition of massive objects: two Hilbert spaces.

Eisenhart lift

- Plane wave space–time in $4+1$ dimensions

$$G = 2dudt + 2\frac{V(\mathbf{x},t)}{m}dt^2 - d\mathbf{x}\cdot d\mathbf{x}.$$

- Classical physics: null geodesics $\rightarrow m\ddot{\mathbf{x}} = -\nabla V$.
- Quantum physics: $\Box_G\phi = 0$, where $\phi = e^{-imu/\hbar}\psi(t,\mathbf{x})$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi.$$

- $V = -m\mathbf{g}\cdot\mathbf{x}$. G is flat. Flat coordinates $G = 2dUdT - d\mathbf{X}\cdot d\mathbf{X}$

$$T = t, \quad U = u - t\mathbf{g}\cdot\mathbf{x} + \frac{1}{6}|\mathbf{g}|^2t^3, \quad \mathbf{X} = \mathbf{x} - \frac{1}{2}\mathbf{g}t^2$$

Nonlinear phase change: $\Phi = e^{-imU/\hbar}\Psi(T,\mathbf{X}) = e^{-imu/\hbar}\psi(t,\mathbf{x})$.

Normal Coordinates

- Non–uniform gravitational field. G curved. Riemann coordinates $X^\alpha = (U, T, X^i)$

$$G = 2dUdT - d\mathbf{X}^2 + \frac{1}{3} R_{\mu\alpha\beta\nu}(0) X^\alpha X^\beta dX^\mu dX^\nu + O(|X|^3),$$

- Null direction

$$U = u + \frac{t^3}{6m^2}|\gamma|^2 + \frac{t}{3m}\mathbf{x}^T r\,\mathbf{x} + \frac{t}{m}\mathbf{x}\cdot\gamma, \quad \mathbf{X} = \mathbf{x} + \frac{t^2}{6m} r\mathbf{x} + \frac{t^2}{2m}\gamma.$$

where $\gamma^i = \delta^{ik}\partial_k V|_{\mathbf{x}=\mathbf{0}}, \quad {r^i}_j = \delta^{ik}\partial_j\partial_k V|_{\mathbf{x}=\mathbf{0}}.$

- $e^{-\frac{mU}{\hbar}}\Psi(\mathbf{X}, T) = e^{-\frac{mu}{\hbar}}\psi(\mathbf{x}, t)$, or

$$\psi(\mathbf{x}, t) = \Lambda\Psi(\mathbf{x} + \frac{t^2}{2m}\gamma, t), \quad \text{where} \quad \Lambda = e^{-\frac{im}{\hbar}\left(\frac{t^3}{6m^2}|\gamma|^2 + \frac{t}{m}\mathbf{x}\cdot\gamma\right)}.$$

Twistor theory

- Twistor theory (Penrose 1967).
  - Non–local theory of space–time.
  - Light rays more fundamental than events.
  - Non–perturbative physics constrained by self–duality.
  - Impact on pure mathematics (differential geometry, integrability, . . .).
- Non–relativistic limit of twistor theory (MD, Gundry 2016).
  - Jumping lines in twistor space.
  - Unstable under holomorphic deformations.
  - Describes all Newtonian space–times (not constrained by self–duality).
  - Non–relativistic limits of gravitational instantons.

- Complex 3-fold PT_c. Covering $U = \{(Q, T, \lambda)\}$, $\widetilde{U} = \{(\widetilde{Q}, \widetilde{T}, \tilde{\lambda})\}$.

$$\tilde{\lambda} = \frac{1}{\lambda}, \quad \begin{pmatrix} \widetilde{T} \\ \widetilde{Q} \end{pmatrix} = \begin{pmatrix} 1 & -(c\lambda)^{-1} \\ 0 & \lambda^{-2} \end{pmatrix} \begin{pmatrix} T \\ Q \end{pmatrix} \quad \text{on } U \cap \widetilde{U}.$$

- Vector bundle $\mu : PT_c \to \mathbb{CP}^1$ with a patching matrix

$$F_c = \begin{pmatrix} 1 & -(c\lambda)^{-1} \\ 0 & \lambda^{-2} \end{pmatrix}.$$

- Grothendieck theorem: $PT_c = \mathcal{O}(m) \oplus \mathcal{O}(n)$
- $H(\lambda), \widetilde{H}(\tilde{\lambda}) \in GL(2, \mathbb{C})$, such that $F_c = \widetilde{H}\ \text{diag}(\lambda^{-m}, \lambda^{-n})\ H^{-1}$.
  - $c = \infty$. $PT_c = \mathcal{O} \oplus \mathcal{O}(2)$.
  - $c \neq \infty$. $PT_c = \mathcal{O}(1) \oplus \mathcal{O}(1)$.

Twistor correspondence

- 4–parameter family (complexified space–time) $M_{\mathbb{C}}$ of $\mathbb{CP}^1$s in PT_c.

$$Q = -(x+iy) - 2\lambda z + \lambda^2(x-iy), \quad T = t - \frac{1}{c}(z - \lambda(x-iy)).$$

- $c \neq \infty$. p_1, p_2 null separated in $M_{\mathbb{C}}$ iff L_1 and L_2 intersect at one point in PT_c. Conformal structure

$$c^2 dt^2 - dx^2 - dy^2 - dz^2$$

- Real structure $\sigma : PT_c \to PT_c$

$$\sigma(Q, \lambda, T) = \Big(-\overline{\lambda}^{-2}\overline{Q}, -\overline{\lambda}^{-1}, -\overline{T} + (c\overline{\lambda})^{-1}\overline{Q} \Big).$$

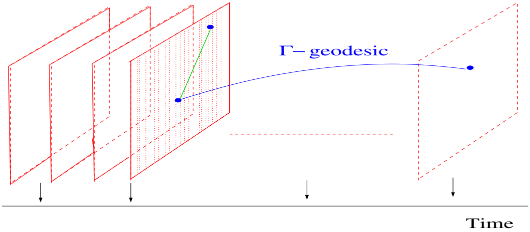
  σ–invariant curves: (x, y, z) real, $t = i\tau$ imaginary.

- $c = \infty$. $T = \widetilde{T}$ global twistor function
  - Fibration $M \to M/\text{ker}(\theta) = \mathbb{R}$, where $\theta = d\tau$ (clock).
  - Degenerate metric $h = (\partial/\partial x)^2 + (\partial/\partial y)^2 + (\partial/\partial z)^2$.

Newton–Cartan theory

Newton–Cartan structure (∇, h, θ)

- $h \in \Gamma(\mathrm{Sym}^2(TM))$, signature $(0, 3)$.
- $\theta \in \Gamma(T^*M)$.
- Torsion-free connection ∇ s.t. $\nabla h = \nabla\theta = 0$, so $\theta = dt$.



- Free fall $\longleftrightarrow$ geodesics of ∇.
- Time simultaneity $\longleftrightarrow$ fibration $M \to \mathbb{R}$.
- No absolute space. Needs a choice of Galilean coordinates.

Newtonian connection from Nonlinear Graviton

- One–parameter family of Gibbons–Hawking (GH) metrics, $\epsilon = c^{-2}$.

$$g = (1+\epsilon V)(dx^2+dy^2+dz^2) + \frac{1}{\epsilon(1+\epsilon V)}(d\tau + \epsilon^{3/2}A)^2.$$

- Anti–self-dual and Ricci flat for all $\epsilon \in \mathbb{R}^+$ if $*dV = dA$.
- Deformed twistor space

$$\widetilde{Q} = \frac{1}{\lambda^2}Q, \quad \widetilde{T} = T - \frac{Q}{c\lambda} - \frac{1}{c^3}f, \quad f = f(Q,T,\lambda) \in H^1(\mathbb{CP}^1, \mathcal{O}).$$

- Newtonian limit $c \to \infty$

$$h^{ij} = \delta^{ij}, \quad \Gamma^i{}_{\tau\tau} = \frac{1}{2}\delta^{ij}\frac{\partial V}{\partial x^j}, \quad \theta = d\tau.$$

Newtonian connection

$$\Gamma^{i}{}_{\tau\tau} = \frac{1}{2}\delta^{ij}\frac{\partial V}{\partial x^j}, \quad \text{where} \quad V = \oint_{\Gamma\subset\mathbb{CP}^1}\frac{\partial f}{\partial Q}d\lambda.$$

- Example: Uniform gravitational field $f = Q^2/\lambda^2$
- Example: $1/r$ potential from Taub–NUT gravitational instanton $f = m\ln Q$

$$V = m/r, \quad \text{ALF} \stackrel{c\rightarrow\infty}{\longrightarrow} \text{AF}.$$

- Assume that the $\mathbf{R}$ process happens in between two measurements at times t_0, and t_1.
- The first measurement a discontinuous jump in the space–time structure, and deforms the twistor space PT_∞ by

$$\tilde{T} = T + \frac{1}{c^3} f(Q, T),$$

  where $f \in H^1(PT_\infty, \mathcal{O})$ is non–zero between T_0 and T_1
- The twistor space survives the reduction, but the 4-parameter family of curves with normal bundle $N = \mathcal{O} + \mathcal{O}(2)$ disappears, and needs to be replaced by a new family. $H^1(\mathbb{CP}^1, \text{End}(N)) \neq 0$, then general Kodaira deformations do not preserve the type of the normal bundle.

The space–time bifurcates and collapses in the Twistor **R**-process: space–time bifurcates, but the twistor space is one complex three–fold. The curves in the **R**-process change their holomorphic type.