Loop quantum gravity in diagonal gauge

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Plan of the talk

1. A brief reminder of loop quantum gravity
2. Diagonal gauge in the quantum theory
   - Construction of a master constraint operator
   - Solutions of the constraint
   - Operators on the reduced Hilbert space
3. Possible extensions and generalizations
4. Summary and outlook

General relativity in the Ashtekar formulation

Canonical variables on the spatial manifold Σ: Connection A_a^i, densitized triad E_i^a

$$q^{ab} = \frac{E_i^a E_i^b}{\det E} \qquad A_a^i = \Gamma_a^i + \beta K_a^i$$

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^b \delta_j^i \delta^{(3)}(x, y)$$

Constraints:

$G_i(A, E) = 0$	Gauss	(internal $SU(2)$ rotations)
$C_a(A, E) = 0$	Diffeomorphism	(spatial diffeomorphisms)
$C(A, E) = 0$	Hamiltonian	(diffeomorphisms orthogonal to Σ)

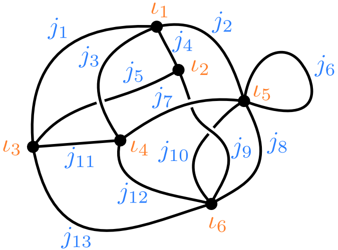
The elementary variables for loop quantum gravity are the smeared versions of the Ashtekar variables:

$$h_e[A] = \mathcal{P}\exp\left(-\int_e A\right) \qquad E_i(S) = \int_S dx^a \wedge dx^b\, \epsilon_{abc} E_i^c(x)$$

The kinematical structure of loop quantum gravity

The kinematical Hilbert space of loop quantum gravity is spanned by the spin network states

$$\big|\Gamma, \{j_e\}, \{\iota_v\}\big\rangle = \left(\prod_{v\in\Gamma} \iota_v\right) \cdot \left(\prod_{e\in\Gamma} D^{(j)}(h_e)\right)$$



Physically, a spin network state describes a discrete, quantized spatial geometry.

The elementary operators of the theory are the holonomy and flux operators

$$\widehat{D^{(s)}_{m'n'}(h_e)} D^{(j)}_{mn}(h_e) = \sum_{km''n''} C^{(j\ s\ k)}_{mm'm''} C^{(j\ s\ k)}_{nn'n''} D^{(k)}_{m''n''}(h_e)$$

$$\hat{E}_i(S) D^{(j)}_{mn}(h_e) = \frac{i}{2}\Big(D^{(j)}(h_e)\tau_i^{(j)}\Big)_{mn} \qquad \tau_i = -\frac{i}{2}\sigma_i$$

Geometric operators: Area, volume, ...

Physical models of loop quantum gravity

- Symmetry reduced models

  A symmetry reduction (e.g. homogeneity and isotropy) is performed classically, and the resulting finite-dimensional system is quantized using LQG-like methods. (Loop quantum cosmology, ...)

- Models based on a gauge fixing

  Classical gauge fixing: Gauge conditions implemented at the classical level, after which the reduced phase space is quantized. (e.g. Bodendorfer et al.)

  Quantum gauge fixing: Gauge conditions represented by a constraint operator, which selects a sector of the Hilbert space of the full theory.

  $\longrightarrow$ Quantum-reduced loop gravity (Alesci and Cianfrani)

- Effective dynamics

  Semiclassical expectation value of the Hamiltonian operator is interpreted as an effective Hamiltonian generating dynamics on a classical phase space.

Gauge fixing to diagonal triad

A priori there are 9 degrees of freedom contained in the densitized triad

$$E^a_i = \begin{pmatrix} E^x_1 & E^x_2 & E^x_3 \\ E^y_1 & E^y_2 & E^y_3 \\ E^z_1 & E^z_2 & E^z_3 \end{pmatrix}$$

The seven constraints $\quad G_i = 0, \qquad C_a = 0, \qquad C = 0$

reduce the number of dynamical degrees of freedom to two per point.

Using the three Gauss constraints and three diffeomorphism constraints, we may fix the triad to be diagonal by imposing the six gauge conditions

$$E^a_i = 0 \qquad \text{for } a \neq i$$

The chosen gauge completely fixes the internal $SU(2)$ rotations. There are some "residual" diffeomorphisms preserving the diagonal gauge; seen as coordinate transformations, these have the form

$$(x, y, z) \to \big(X(x), Y(y), Z(z)\big)$$

Master constraint for diagonal gauge

The goal is to impose, via a constraint operator in the quantum theory, the gauge conditions

$$E^a_i = 0 \qquad (a \neq i)$$

Classically, the gauge conditions can be encoded in the master constraint

$$\mu = (E^x_2)^2 + (E^x_3)^2 + (E^y_1)^2 + (E^y_3)^2 + (E^z_1)^2 + (E^z_2)^2$$

The condition $\mu = 0$ is equivalent to all the gauge conditions being satisfied.

To prepare the master constraint for quantization, we must take it in the integrated form

$$M = \int d^3x \, \frac{\mu}{\sqrt{q}}$$

It is also useful to add and subtract the squared diagonal terms, writing

$$\mu = \sum_a \left(\sum_i (E^a_i)^2 - (E^a_a)^2 \right)$$

Quantization of the master constraint

We wish to promote the integrated master constraint

$$M = \int d^3x \sum_a \frac{\sum_i (E^a_i)^2 - (E^a_a)^2}{\sqrt{q}}$$

into an operator on the kinematical Hilbert space of loop quantum gravity.

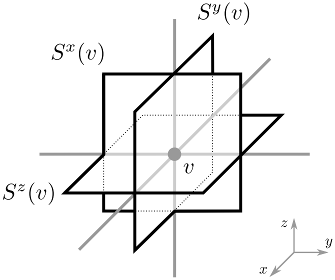
Using a fixed Cartesian background coordinate system, regularization of the integral on a cubical partition of Σ leads to an operator of the form

$$\hat{M} = \sum_v \hat{M}_v$$

$$\hat{M}_v = \widehat{\mathcal{V}_v^{-1}} \sum_a \Big(\hat{A}\big(S^a(v)\big)^2 - \hat{E}_a\big(S^a(v)\big)^2 \Big)$$

$$\hat{A}(S) = \sqrt{\hat{E}^i(S)\hat{E}_i(S)}$$

$$\widehat{\mathcal{V}_v^{-1}} = \lim_{\epsilon \to 0} \frac{\hat{V}_v}{\hat{V}_v^2 + \epsilon^2}$$



Gauge fixing in the quantum theory

The gauge conditions are represented in the quantum theory by the constraint equation

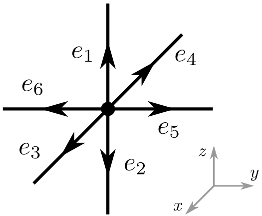
$$\hat{M}|\Psi\rangle = 0$$

This equation selects a subspace of the full kinematical Hilbert space, in which diagonal gauge is fulfilled.

Following Alesci and Cianfrani, we look for solutions of the gauge fixing constraint on a cubical spin network graph. The graph forms a rectangular lattice, with edges aligned in the coordinate directions of the background coordinate system.

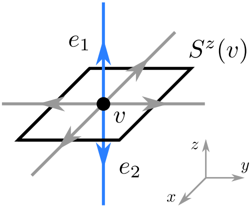
Since the constraint acts node by node, we may focus on a single node of the cubical graph:

$$D^{(j_1)}_{m_1n_1}(h_{e_1})D^{(j_2)}_{m_2n_2}(h_{e_2})D^{(j_3)}_{m_3n_3}(h_{e_3}) \times D^{(j_4)}_{m_4n_4}(h_{e_4})D^{(j_5)}_{m_5n_5}(h_{e_5})D^{(j_6)}_{m_6n_6}(h_{e_6})$$



Solutions of the constraint

$$\hat{M}_v = \widehat{\mathcal{V}_v^{-1}} \sum_a \Big(\hat{A}\big(S^a(v)\big)^2 - \hat{E}_a\big(S^a(v)\big)^2 \Big)$$



Each term in $\hat{M}_v$ acts only on two of the edges at v.
For example, consider the term with $a = z$:

$$\begin{aligned}
&\Big(\hat{A}\big(S^z(v)\big)^2 - \hat{E}_z\big(S^z(v)\big)^2 \Big) D^{(j_1)}_{m_1 n_1}(h_{e_1}) D^{(j_2)}_{m_2 n_2}(h_{e_2}) \\
&\quad = \Big(j_1^2 - n_1^2 \; + \; j_2^2 - n_2^2 \; + \; j_1 + j_2 \Big) D^{(j_1)}_{m_1 n_1}(h_{e_1}) D^{(j_2)}_{m_2 n_2}(h_{e_2}) \\
&\qquad + c_+(j_1, n_1) c_-(j_2, n_2) D^{(j_1)}_{m_1, n_1+1}(h_{e_1}) D^{(j_2)}_{m_2, n_2-1}(h_{e_2}) \\
&\qquad + c_-(j_1, n_1) c_+(j_2, n_2) D^{(j_1)}_{m_1, n_1-1}(h_{e_1}) D^{(j_2)}_{m_2, n_2+1}(h_{e_2})
\end{aligned}$$

where $c_\pm(j, n) = \sqrt{j(j+1) - n(n \pm 1)}$.

Clearly no exact solutions exist. However, it is possible to construct states which satisfy the constraint approximately in the large j limit.

In particular, consider the choice $(n_1, n_2) = \pm(j_1, j_2)$

Solutions of the constraint

The choice

$$(n_1, n_2) = \pm(j_1, j_2)$$

eliminates the off-diagonal terms, and the quadratic part of the diagonal term.

There remains

$$\hat{M}_v D^{(j_1)}_{n_1,\pm j_1}(h_{e_1}) D^{(j_2)}_{\pm j_2,n_2}(h_{e_2})$$
$$= \big(j_1 + j_2\big)\widehat{\mathcal{V}_v^{-1}} D^{(j_1)}_{m_1,\pm j_1}(h_{e_1}) D^{(j_2)}_{m_2,\pm j_2}(h_{e_2})$$

On a state with all $m_i = \pm j_i$ at v, the inverse volume operator behaves as

$$\widehat{\mathcal{V}_v^{-1}}|\pm j\rangle = w(j)|\pm j\rangle + \mathcal{O}\big(j^{-5/2}\big) \qquad \text{with} \qquad w(j) = \mathcal{O}\big(j^{-3/2}\big)$$

Thus, on such a state

$$\hat{M}_v D^{(j_1)}_{m_1,\pm j_1}(h_{e_1}) D^{(j_2)}_{m_2,\pm j_2}(h_{e_2}) = \mathcal{O}\big(j^{-1/2}\big)$$

For large j we therefore have approximate solutions of the gauge fixing constraint.

Reduced spin network states

Introduce the notation

- $|jm\rangle_i$ ($i = x$, y, z): Eigenstates of J^2 and J_i with eigenvalues $j(j+1)$ and m
- $D^{(j)}_{mn}(h)_i \equiv {}_i\langle jm|D^{(j)}(h)|jn\rangle_i$: Wigner matrices in the basis $|jm\rangle_i$

Then solutions of the gauge fixing constraint are defined by wave functions of the form

$$\prod_{e\in\Gamma} D^{(j_e)}_{\pm j_e\,\pm j_e}(h_e)_{i_e}$$

on a cubical graph Γ. For each edge, we assume

$$j_e \gg 1$$

and $i_e = x$, y or z according to the direction of the edge.

Up to details related to the $\pm$ signs, these are essentially the "reduced spin network states" proposed by Alesci and Cianfrani.

Operators on the reduced Hilbert space

The space of reduced spin network states is approximately preserved by the natural action of many standard operators of loop quantum gravity.

For a large class of operators, the action of the operator on a reduced spin network state $|\Psi\rangle$ has the structure

$$\hat{\mathcal{O}}(h, E)|\Psi\rangle \;=\; f(j)|\Psi'\rangle + g(j)|\Phi\rangle$$

where $|\Psi'\rangle$ is an element of the reduced Hilbert space, and for large j,

$$f(j) \gg g(j)$$

Operators on the reduced Hilbert space can thus be derived from operators of full loop quantum gravity by dropping the (small) correction terms.

Moreover, the action of a given "reduced" operator is typically extremely simple in comparison with the corresponding operator on the Hilbert space of the full theory.

Holonomy operator

The action of the holonomy operator is given by the $SU(2)$ Clebsch–Gordan series:

$$\widehat{D^{(s)}_{mm}(h_e)}D^{(j)}_{jj}(h_e) = \left(C^{(j\ s\ j+m)}_{j\ m\ j+m}\right)^2 D^{(j+m)}_{j+m\ j+m}(h_e) + \sum_{k\neq j+m}\left(C^{(j\ s\ k)}_{j\ m\ j+m}\right)^2 D^{(k)}_{j+m\ j+m}(h_e)$$

Assuming $s \ll j$, one can use the explicit expression of the coefficients to show

$$C^{(j\ s\ j+m)}_{j\ m\ j+m} = 1 - \mathcal{O}\left(\frac{1}{j}\right) \qquad C^{(j\ s\ k\neq j+m)}_{j\ m\ j+m} = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right)$$

It follows that

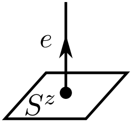
$$\widehat{D^{(s)}_{mm}(h_e)}D^{(j)}_{jj}(h_e) = D^{(j+m)}_{j+m\ j+m}(h_e) + \mathcal{O}\left(\frac{1}{j}\right)$$

$$\widehat{D^{(s)}_{mn}(h_e)}D^{(j)}_{jj}(h_e) = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right) \qquad (m\neq n)$$

We see that the leading contribution comes from the diagonal matrix elements, which act essentially by a $U(1)$ multiplication law.

Flux operator

The action of the flux operator on a "reduced holonomy" gives (assuming intersection at the beginning point of the edge)



$$\hat{E}_i(S^z) D^{(j)}_{jj}(h_e)_z = \frac{i}{2} D^{(j)}_{jm}(h_e)_z (\tau^{(j)}_i)_{mj}$$

Here $\tau_i = -i\sigma_i/2$ are the anti-Hermitian generators of $SU(2)$.

The relevant matrix elements of the generators are

$$(\tau^{(j)}_z)_{mj} \;=\; -ij\delta_{mj}, \qquad (\tau^{(j)}_x)_{mj} \;=\; -i\sqrt{\frac{j}{2}}\delta_{m,j-1} \qquad (\tau^{(j)}_y)_{mj} \;=\; \sqrt{\frac{j}{2}}\delta_{m,j-1}$$

Therefore, for an edge in the i-direction

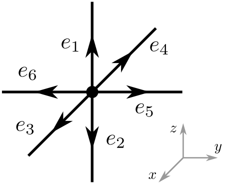
$$\hat{E}_i(S^i) D^{(j)}_{jj}(h_e)_i \;=\; \frac{j}{2} D^{(j)}_{jj}(h_e)_i$$
$$\hat{E}_k(S^i) D^{(j)}_{jj}(h_e)_i \;=\; \mathcal{O}\big(\sqrt{j}\big) \qquad (k \neq i)$$

Volume operator

On a cubical spin network node the volume operator takes the form

$$\hat{V}_v = \sqrt{|\hat{q}_v|}$$

$$\hat{q}_v \ = \ \epsilon^{ijk} \hat{E}_i\big(S^x(v)\big)\hat{E}_j\big(S^y(v)\big)\hat{E}_k\big(S^z(v)\big)$$



Applying the operator $\hat{q}_v$ to the state

$$|\Psi\rangle \ = \ D^{(j_1)}_{j_1 j_1}(h_{e_1})_x \, D^{(j_2)}_{j_2 j_2}(h_{e_2})_x \, D^{(j_3)}_{j_3 j_3}(h_{e_3})_y \, D^{(j_4)}_{j_4 j_4}(h_{e_4})_y \, D^{(j_5)}_{j_5 j_5}(h_{e_5})_z \, D^{(j_6)}_{j_6 j_6}(h_{e_6})_z$$

we find:
- A diagonal term of order j^3 when $(i,j,k) = (x,y,z)$
- Terms with $(i,j,k) \neq (x,y,z)$, which are at most of order j^2

$$\therefore \quad \hat{q}_v|\Psi\rangle \ = \ \frac{1}{8}(j_1+j_2)(j_3+j_4)(j_5+j_6)|\Psi\rangle + \mathcal{O}(j^2)$$

$$\hat{V}_v|\Psi\rangle \ = \ \sqrt{\frac{1}{8}(j_1+j_2)(j_3+j_4)(j_5+j_6)}\,|\Psi\rangle + \mathcal{O}(\sqrt{j})$$

Kinematical structure of the quantum-reduced model

Loop quantum gravity	**Quantum-reduced loop gravity**

Kinematical Hilbert space

Loop quantum gravity:

$$\left(\prod_{v\in\Gamma}\iota_v\right)\cdot\left(\prod_{e\in\Gamma}D^{(j_e)}(h_e)\right)$$

– All graphs are included

– $SU(2)$ intertwiners at nodes

Quantum-reduced loop gravity:

$$\prod_{e\in\Gamma}D^{(j_e)}_{\pm j_e\,\pm j_e}(h_e)_{i_e}\quad(j_e\gg 1)$$

– States defined on cubical graphs

– No (non-trivial) intertwiner structure

Elementary operators

Loop quantum gravity:

$$\widehat{D^{(s)}_{m'n'}(h_e)}D^{(j)}_{mn}(h_e) = \sum_{km''n''}C^{(j\ s\ k)}_{m m' m''}C^{(j\ s\ k)}_{n n' n''}D^{(k)}_{m'n'}(h_e)$$

$$\hat{E}_i(S)D^{(j)}_{mn}(h_e)=\frac{i}{2}\left(D^{(j)}(h_e)\tau^{(j)}_i\right)_{mn}$$

Quantum-reduced loop gravity:

$$\widehat{D^{(s)}_{mm}(h_e)_i}D^{(j)}_{jj}(h_e)_i = D^{(j+m)}_{j+m\ j+m}(h_e)_i$$

$$\hat{E}_i(S)D^{(j)}_{jj}(h_e)_i=\frac{j}{2}D^{(j)}_{jj}(h_e)_i$$

Gauss and diffeomorphism constraints

So far the Gauss and diffeomorphism constraints have not been addressed.

Several approaches are available to treat them:

- Ignore them
- Include them in an extended master constraint
- Apply group averaging to the reduced spin network states

Extended master constraint

The extended master constraint

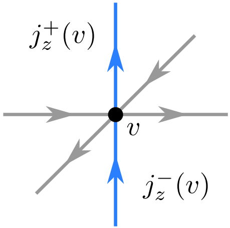
$$M_{\text{ext}} = M + M_{\text{Gauss}} + M_{\text{diff}}$$

$$M_{\text{Gauss}} = \int d^3x \, \frac{G_i G_i}{\sqrt{q}} \qquad M_{\text{diff}} = \int d^3x \, \frac{q^{ab} C_a C_b}{\sqrt{q}}$$

implements classically the gauge conditions $E^a_i = 0$ $(a \neq i)$ together with the constraints $G_i = 0$ and $C_a = 0$.

As a constraint operator, the term representing the Gauss constraint places a restriction on the consecutive spins along a given direction:

$$j_a^+(v) = j_a^-(v) + \mathcal{O}(1)$$



A detailed analysis of the diffeomorphism constraint remains a question for future work.

$SU(2)$ group averaging

A state $|\Psi\rangle$ in the kinematical Hilbert space of LQG can be made gauge invariant by group averaging over the gauge transformations:

$$|\Psi_0\rangle = \int_{SU(2)} dg\, \hat{U}(g)|\Psi\rangle$$

Group averaging a reduced spin network state produces a "proper" spin network state, which carries at each node a Livine–Speziale coherent intertwiner

$$|j_1 \cdots j_6; \vec{n}_1 \cdots \vec{n}_6\rangle = \int dg\, D^{(j_1)}(g)|j_1\vec{n}_1\rangle \otimes \cdots \otimes D^{(j_6)}(g)|j_6\vec{n}_6\rangle$$

with $\vec{n}_1, \ldots, \vec{n}_6$ the normal vectors of a cube.

A key question is whether the group averaging preserves the decomposition of operators into leading and lower order terms:

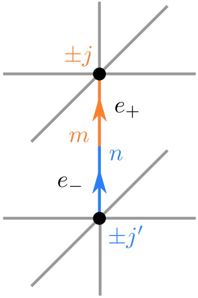
$$\hat{\mathcal{O}}|\Psi\rangle \;=\; f(j)|\Psi'\rangle + g(j)|\Phi\rangle$$

Are there further solutions of the gauge-fixing constraint?

Recall that only the diagonal matrix elements contribute to the action of the holonomy operator on a reduced spin network state. Hence it seems that not only the triad but also the Ashtekar connection is diagonal in these states.

There is room to introduce a (speculative) generalization: Replace each edge of a reduced spin network state with two "half-edges":

$$D^{(j)}_{\pm j\ \pm j}(h_e)_i \quad \rightarrow \quad D^{(j)}_{\pm j\ m}(h_{e_+})_i D^{(j')}_{n\ \pm j'}(h_{e_-})_i$$



Since the gauge-fixing constraint $\hat{M}$ acts only on nodes of non-zero volume, such states satisfy the constraint in the same sense as the original reduced spin network states do.

One can see a vague resemblance between this construction and various ideas considered in loop quantum gravity, e.g. the twisted geometry parametrization and the condensate states of group field theory.

Conclusions

We showed that the kinematical structure of quantum-reduced loop gravity can be derived by using the master constraint approach to implement a set of gauge conditions fixing the triad to be diagonal.

In particular, the operators of the quantum-reduced model arise from the natural action of the operators of the full theory on the reduced Hilbert space.

Due to the simplified form of its states and operators, the model seems like a promising testing ground for applications that are technically inaccessible in the framework of full loop quantum gravity.

Physical applications of the model are already available

- Cosmology: Singularity resolution, primordial power spectra
- Black holes: Quantum description of the Schwarzschild interior

On the other hand, little attention has been given so far to matter fields in the quantum-reduced framework.

Thank you for your attention