Classification of static extreme black holes

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Outline

- Review of black hole uniqueness theorems in four and higher dimensions, non-extreme and extreme.

- Classification of static, extreme, black hole spacetimes in higher-dimensional Einstein-Maxwell theory [JL '20]

Black hole equilibrium states

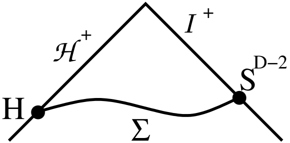
- Central role in classical and quantum gravity: endpoint of collapse, no-hair theorem, black hole thermodynamics...
- Extreme black holes particularly relevant in quantum gravity: zero Hawking temperature.
- **Open problem**: classification of equilibrium (stationary) black hole solutions to the *higher-dimensional* Einstein equations.

  Five-dimensions best understood but far from being solved!

  Motivation: new physics/geometry, string theory, AdS/CFT...

Black holes in General Relativity

- Topology: Cross-sections of horizon $H \cong S^2$. [Hawking '72]
  Rigidity: stationary black hole must be axisymmetric or static.



- Asymptotically flat, stationary-axisymmetric, electro-vacuum, *single* black hole spacetime must be a Kerr-Newman solution.
  [Carter, Robinson, Mazur, Bunting...'70, 80s]
  - Extreme case [Meinel et al '08; Figueras, JL '09; Chrusciel, Nguyen '10]
    Near-horizon geometry [Lewandowski, Pawlowski '02; Kunduri, JL '08]
  - 'Double-Kerr' in vacuum ruled out [Hennig, Neugebauer '13]

Static black holes

- Asymptotically flat, static, electro-vacuum, *non-extreme*, black hole solution must be a Reissner-Nordström solution. [Israel '67; Robinson '77; Bunting, Masood Alam '87]

  Multi-black holes don't exist [Bunting, Masood-ul-Alam '87]

- Majumdar-Papapetrou (MP) solution: *extreme* multi-black hole spacetime [Hartle, Hawking '72]

- Asymptotically flat, static, electro-vacuum, *extreme* black hole solution must be a MP multi-black hole [Chrusciel, Tod '05]

  Uses general static near-horizon geometry is $\mathrm{AdS}_2 \times S^2$

Majumdar-Papapetrou solution

- MP metrics:

$$g = -H^{-2}\mathrm{d}t^2 + H^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \qquad F = -\mathrm{d}(H^{-1}\mathrm{d}t)$$

electro-vacuum $\iff$ H harmonic on euclidean space $(\mathbb{R}^3, \delta)$

- Extreme Reissner-Nordström: $H = 1 + \frac{q}{|x|}$, $x = 0$ is horizon.
  'Multi-centred' black hole solutions [Hartle, Hawking '72]:

$$H = 1 + \sum_{I=1}^{N} \frac{q_I}{|x - p_I|}$$

$x = p_I$ are components of analytic event horizon

- Static equilibrium of N charged black holes $m_I = q_I$

Supersymmetric black holes

- $N = 2$ minimal supergravity: truncates to Einstein-Maxwell. Asymptotically flat (g, F) obey BPS inequality [Gibbons, Hull '82]

$$M \geq |Q|$$

  $M = |Q|$ iff admits 'Killing' spinor ϵ: stationary $V^\mu = \bar{\epsilon}\gamma^\mu\epsilon$

- MP solution is most general *static* supersymmetric solution [Gibbons, Hull '82; Tod '83]

- Asymptotically flat, supersymmetric black hole solution must be MP multi-black hole [Chrusciel, Reall, Tod '05]

  Note: supersymmetric black holes have extreme horizons.

Black hole in higher dimensions

- Black hole non-uniqueness even for vacuum spacetimes!
  $D = 5$: asymptotically flat (AF) black hole solutions:
  - Myers-Perry S^3
  - black ring $S^1 \times S^2$ [Emparan, Reall '01]
  - multi-black holes exist: black Saturn... [Elvang, Figueras '07]
- Horizon topology less constrained [Galloway, Schoen '05].
  $D = 5$: cross-sections $S^3/\Gamma, S^1 \times S^2$ and connected sums
- Rigidity: stationary rotating black hole has axial symmetry
  [Hollands, Ishibashi, Wald '06; Moncrief, Isenberg '08]
- Classification of *all* stationary black holes formidable problem!
  Need extra symmetry assumptions...

Classification theorems in higher dimensions

- $D = 5$: AF, stationary, vacuum black holes with *biaxial* $U(1)^2$-symmetry classified by rod structure [Hollands Yazadjiev '07]

  Existence: recent progress using integrability of Einstein eq, rules out regular $L(n,1)$ lens space horizons [JL, Tomlinson '20]

- $D = 5$, AF, *supersymmetric* black holes: biaxial symmetry [Breunholder, JL '17]; single axial symmetry [Katona, JL '22]

  Large moduli space: $S^3, S^1 \times S^2$ and $L(n,1)$ black hole spacetimes with 2-cycles in DOC.

- AF, *static*, electro-vacuum, *non-extreme* black hole spacetime is RN in all dimensions. [Gibbons, Ida, Shiromizu '01; Kunduri, JL '17]

Static extreme black holes

- n-dimensional Einstein-Maxwell solution [Myers '87]:

$$g = -H^{-2}\mathrm{d}t^2 + H^{\frac{2}{n-3}}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \qquad F = -\mathrm{d}(H^{-1}\mathrm{d}t)$$

  H harmonic $(\mathbb{R}^{n-1}, \delta)$. Higher-dimensional MP solution!

- Multi-centred solutions: asymptotically flat $M = |Q|$

$$H = 1 + \sum_{I=1}^{N} \frac{q_I}{|x - p_I|^{n-3}}$$

  $x = p_I$ extreme horizons with low regularity: g is C^1 (C^2 if $n = 5$), F is C^0. [Welch '95; Candlish, Reall '07]

- Classify all static extreme electro-vacuum black holes?

Static spacetimes

- Static Killing field ξ: time-like in DOC, null on horizon (assume DOC globally hyperbolic, simply connected)

- $M = \mathbb{R} \times \Sigma$, coordinates (t, x^i), $\xi = \partial_t$,

$$g = -V^2 \mathrm{d}t^2 + \hat{g}_{ij} \mathrm{d}x^i \mathrm{d}x^j$$

  $(\Sigma, \hat{g})$ Riemannian manifold (orbit space)

- Maxwell field: $\mathcal{L}_\xi F = 0 \implies \iota_\xi F = -\mathrm{d}\psi$, electric field ψ
  $F = \mathrm{d}\psi \wedge \mathrm{d}t + B$, magnetic field $B \in \Omega^2(\Sigma)$

  No electro-magnetic duality for $n > 4$ so can't assume $B = 0$!
  [Kunduri, JL '17]

- Einstein-Maxwell eqs: geometric eqs for $(\Sigma, \hat{g}, V, \psi, B)$

Boundary conditions

- Asymptotic flatness: end diffeo to $\mathbb{R}^{n-1} \setminus B_R$,

$$\hat{g}_{ij} = \left(1 + \frac{2M}{n-3}\frac{1}{r^{n-3}}\right)\delta_{ij} + O(r^{-(n-2)})$$
$$V = 1 - \frac{M}{r^{n-3}} + O(r^{-(n-2)})$$
$$\psi = \frac{Q}{r^{n-3}} + O(r^{-(n-2)})$$

  M mass, Q electric charge

- $\partial\Sigma := \{V = 0\}$: non-extreme horizon $\kappa^2 = (\mathrm{d}V)^2 \neq 0$, totally geodesic, κ, ψ constants.

  Extreme horizon $V \to 0$ is a 'cylindrical' end (more later!)

Mass-charge inequality

Lemma [Kunduri, JL, '17]

$M \geq |Q|$ with equality if and only if $B = 0$ and $\pm\psi = 1 - V$

- Let $F_\pm := V \pm \psi - 1$, field eqs imply

$$\hat{\nabla}^i (V \hat{\nabla}_i F_\pm) = |\hat{\nabla} F_\pm|^2 + \frac{V^2 |B|^2}{n-2}$$

- Integrate over Σ together with $F_\pm = -(M \mp Q) r^{-(n-3)} + \dots$

$$M \mp Q = \int_\Sigma \left(|\hat{\nabla} F_\pm|^2 + \frac{V^2 |B|^2}{n-2} \right) \mathrm{dvol} \geq 0$$

$M = |Q|$ iff $F_\pm = 0$ and $B = 0$.

Non-extreme black holes

Theorem [Gibbons, Ida Shiromizu '01; Kunduri JL '17]

Any $n \geq 5$, asymptotically flat, static solution of Einstein-Maxwell containing non-extreme black hole is a Reissner-Nordström solution

- $M > |Q| \implies$ conformal scalings $\hat{g}_{\pm} = \Omega_{\pm}^2 \hat{g}$ and $\hat{R}_{\pm} \geq 0$.
- $(\Sigma, \hat{g}_+)$ AF zero-mass, glued along $\partial\Sigma$ to compactification $(\Sigma \cup \infty, \hat{g}_-) \implies$ *complete* AF zero-mass manifold
- Positive-mass theorem $\implies (\Sigma, \hat{g})$ conformally flat $\mathbb{R}^{n-1}$ $\implies V^2 = 1 + \psi^2 - 2M\psi/Q$ *and* $B = 0$.
- $g = (v_+ v_-)^{2/(n-3)}\delta$, $v_{\pm}$ harmonic on $\mathbb{R}^{n-1}$, boundary value problem has unique solution $v_{\pm} = 1 + \frac{M \mp Q}{2r^{n-3}}$

Extreme black holes

- $M = |Q| \implies B = 0$ and $\pm\psi = 1 - V$,

$$g = -H^{-2}\mathrm{d}t^2 + H^{\frac{2}{n-3}} h_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad F = -\mathrm{d}(H^{-1}\mathrm{d}t)$$

  $\mathrm{Ric}(h) = 0$, $H := V^{-1}$ harmonic on base space (Σ, h)

- $n = 4$: h must be flat so recover MP solution.
  $n > 4$: h Ricci flat, *generalised MP solution*, AF zero-mass

- **Problem**: determine all sufficiently regular AF black hole spacetimes in this class. Must h be flat and H multi-centred?

Main result

Theorem [JL '20]

Consider n-dimensional asymptotically flat, static, electro-vacuum, extreme black hole spacetime such that:

1. ξ timelike in DOC and null on horizon
2. (g, F) smooth in DOC; at horizon g is C^1, F is C^0, $\iota_\xi F$ is C^1, all smooth in tangential directions.
3. Components of horizon admit smooth cross-section, with non-Ricci flat induced metric

Then DOC is MP solution with (Σ, h) isometric to $(\mathbb{R}^{n-1}, \delta)$ with removed points $p_{I=1,\dots,N}$ corresponding to horizon components and H is multi-centred with poles p_I.

Near-horizon analysis

- Invariants $|\xi|^2 = -H^{-2}$, $\quad \iota_\xi F = -\mathrm{d}H^{-1} \implies H^{-1} > 0$ and smooth on DOC and $H^{-1} = 0$ and C^1 at horizons

- $\mathrm{d}|\xi|^2 = -\mathrm{d}H^{-2} = -2H^{-1}\mathrm{d}H^{-1} = 0$ at horizon $\implies$ event horizon is a extreme Killing horizon of ξ

- Gaussian null coords (GNC) [Moncrief, Isenberg '83]: $\xi = \partial_v$, ∂_λ is transverse geodesic, $\lambda = 0$ at horizon, (y^a) on cross-section S:

$$g = 2\mathrm{d}v\,\left(\mathrm{d}\lambda + \lambda h_a \mathrm{d}y^a - \tfrac{1}{2}\lambda^2 f \mathrm{d}v\right) + \gamma_{ab}\mathrm{d}y^a\mathrm{d}y^b$$

  Regularity assumptions $\implies$ f, h_a are C^0, γ_{ab} is C^1 at horizon

Near-horizon limit and geometry

- **Near-horizon limit**: $\phi_\epsilon : (v, \lambda, y) \mapsto (\frac{v}{\epsilon}, \epsilon\lambda, y)$ then, $g_{\text{NH}} := \lim_{\epsilon \to 0} \phi_\epsilon^* g$ same as g with $f \to \mathring{f} := f|_{\lambda=0}$ etc
  [Reall '03; Kunduri, JL, Reall '07..., Kunduri, JL '13]

  Near-horizon equations for *intrinsic* data on S: $(\mathring{f}, \mathring{h}, \mathring{\gamma}, ...)$
  Same as extremal isolated horizon equation $(n = 4)$
  [Ashtekar, Beetle, Lewandowski, Pawlowski '02...]

- Near-horizon limit of F more subtle: ψ constant along horizon but existence of limit needs $\partial_a \psi = O(\lambda)$.

- MP $\implies \mathrm{d}\xi = 2H^{-1}F$: staticity and assumps $\implies f, h_a, \gamma_{ab}$ are C^1 at horizon, $\mathring{f} > 0$ const, $\mathring{h}_a = 0$, $F_{\text{NH}} = -\mathrm{d}(\mathring{f}\lambda \mathrm{d}v)$
  Note: 1st order transverse deformation of NH geometry exists
  [Li, JL '15,'18; Kolanowski '21]

Asymptotically cylindrical and conical ends

- Orbit space metric $q_{\mu\nu} = g_{\mu\nu} - \xi_\mu \xi_\nu / |\xi|^2$:

$$q = \frac{1}{f\lambda^2}(\mathrm{d}\lambda + \lambda h_a \mathrm{d}y^a)^2 + \gamma_{ab}\mathrm{d}y^a \mathrm{d}y^b$$

(Σ, q) complete, $\lambda \to 0$ cylindrical end diffeo $\mathbb{R} \times S$

- What about (Σ, h)? MP solution $q = H^{\frac{2}{n-3}} h$, horizon is *conically singular* end diffeo to $\mathbb{R} \times S$, $\rho := \lambda^{\frac{1}{n-3}} \to 0$,

$$|h - h_0|_{h_0} = O(\rho^{n-3})$$

$h_0 = \mathrm{d}\rho^2 + \rho^2 \sigma_{ab}\mathrm{d}y^a \mathrm{d}y^b$ cone-metric of (S, σ), $\sigma_{ab} \propto \mathring{\gamma}_{ab}$

Note: fall-off fixed by f, h_a, γ_{ab} being C^1 at horizon

Near-horizon geometry

- $\mathrm{Ric}(h) = 0 \implies h_0$ is Ricci flat cone-metric $\implies$ horizon metric $\mathring{\gamma}_{ab}$ is (positive) Einstein, so S compact.

- Near-horizon geometry $(\mathring{f}, \mathring{h}_a, \mathring{\gamma}_{ab})$ now determined:

$$g_{\mathrm{NH}} = -\mathring{f}\lambda^2 \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}\lambda + (n-3)^2 \mathring{f}^{-1} \sigma_{ab} \mathrm{d}y^a \mathrm{d}y^b$$

  AdS_2 times (S, σ) where $Ric(\sigma) = (n-3)\sigma$.
  $n = 4$ unit S^2, $n = 5$ (locally) unit S^3, $n > 5$ maybe not S^{n-2}!

- Classification of $n > 4$ *static* near-horizon geometries open: magnetic fields, warped AdS_2 products... [Kunduri, JL'09]

  MP solution rules out nontrivial near-horizon geometries!

Positive-mass theorem with conical singularities

- Positive-mass theorem: AF, *complete*, Riemannian manifold, $\mathrm{Ric} \geq 0$ ($R \geq 0$ if spin) must have $m_{\mathrm{ADM}} \geq 0$ and $= 0$ iff flat. [Witten '81; Bartnik '86]

- (Σ, h) is Ricci-flat and AF with zero mass. However, it has conical singularities $\implies$ *not* complete. Nevertheless:

Theorem [JL'20]

(Σ, h) is AF Riemannian manifold with conically singular ends and $\mathrm{Ric}(h) \geq 0$. Then $m_{\mathrm{ADM}} \geq 0$ and $= 0$ iff flat.

Proof

- AF end diffeo to $\mathbb{R}^d \backslash B$ and $h_{ij} = \delta_{ij} + O_1(r^{-\tau})$,

$$m_{\text{ADM}} := \int_{S_{r\to\infty}} (\partial_j g_{ji} - \partial_i g_{jj}) \mathrm{d}S^i$$

$\tau > (d-2)/2$ required for well-defined [Bartnik '86]

- Conically singular end diffeo to $(0, \rho_0) \times S$, where

$$|h - h_0|_{h_0} = O(\rho^\delta), \qquad |\mathring{\nabla} h|_{h_0} = O(\rho^{\delta-1})$$

$h_0 = \mathrm{d}\rho^2 + \rho^2\sigma$ cone over (S, σ)

- Let z^i be harmonic functions that are coordinates on AF end:

$$z^i - x^i = O(r^{1-\tau}), \qquad z^i - p^i = O(\rho^\varsigma)$$

Construct these by modifying harmonic coords on AF end.

Proof

- $K^i := \mathrm{d}z^i$ obey Bochner identity

$$\Delta|K^i|^2 = 2|\nabla K^i|^2 + 2\mathrm{Ric}(K^i, K^i)$$

- Integrating over Σ

$$\underbrace{\sum_{i=1}^{d} \int_{S_{r\to\infty}} \partial_j |K^i|^2 \mathrm{d}S^j}_{\propto m_{\mathrm{ADM}}} - \underbrace{\sum_{i=1}^{d} \int_{S_{\rho\to 0}} \partial_n |K^i|^2 \mathrm{dvol}}_{O(\rho^\varsigma)} \geq 0$$

- $m \geq 0$ with equality iff $\nabla K^i = 0$. $\{K^i\}$ orthonormal at infinity so $m = 0 \implies$ global parallel frame $\implies (\Sigma, h)$ flat

Uniqueness proof

- (Σ, h) is flat $\implies$ cone-metric h_0 is flat $\implies$ (S, σ) isometric to quotient S^{n-2}/Γ where Γ is finite subgroup of $O(n-1)$
- Conical ends diffeomorphic to $\mathbb{R}^{n-1}/\Gamma - \{p\}$, p is f.p. of Γ N-conical ends: $\hat{\Sigma} = (\Sigma \cup \{p_1, \ldots, p_N\}, h)$ is a flat orbifold
- $\hat{\Sigma}$ is flat orbifold $\implies$ $\hat{\Sigma}$ is quotient of $\mathbb{R}^{n-1}$ $\implies$ $\mathbb{R}^{n-1}$ since AF. Deduce all horizons S^{n-1}.
- Near-horizon coordinate change $(x^i) \mapsto (\rho, y^a)$:

$$x^i - p^i = O(\rho), \qquad H = \frac{1}{\sqrt{f}\rho^{n-3}}$$

Uniqueness proof

- Deduce $|x-p|^{n-3}H = O(1)$. H is smooth in DOC and $x = p$ is isolated singularity, must be a pole order $n-3$

$$H = \frac{q}{|x-p|^{n-3}} + \underbrace{K}_{\text{regular at } x=p}$$

- If there are N-components of horizon

$$H = \sum_{I=1}^{N} \frac{q_I}{|x-p_I|^{n-3}} + \underbrace{\tilde{H}}_{\text{regular on all } \mathbb{R}^{n-1}}$$

- AF then implies $\tilde{H}$ is bounded and so must be constant $\tilde{H} = 1$.

Comment on supersymmetric black holes

- Static, supersymmetric, black holes in $D = 5$ minimal supergravity are MP solutions with base (Σ, h) hyper-Kähler [Gaultlett et al '02]

  Theorem therefore classifies these solutions too!

- Uniqueness of AF, supersymmetric black holes with strictly timelike stationary Killing field and locally S^3 horizons [Reall '03]

  Conical singularity is ADE orbifold singularity, can be resolved. But only AF complete HK space is $\mathbb{R}^4$!

Summary and questions

- Determined all AF static *extreme* black holes in *n*-dimensional Einstein-Maxwell theory: MP multi-black holes.

  Completes classification of AF static black holes!

- Proof used positive mass theorem for AF manifolds with conical singularities.

  - Generalise to $R \geq 0$ spin case?
  - Apply to classification of extreme or supersymmetric black holes or branes in other theories?