Toric gravitational instantons and special geometry

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Outline

- Gravitational instantons: examples and ‘no-hair’ conjectures
- Uniqueness & existence theorems for toric ‘asymptotically flat’ gravitational instantons [Kunduri JL, ’21, ’26]
- Classification of Hermitian toric ALE gravitational instantons [Araneda, JL ’25]

Gravitational instantons

- (M, g): four-dimensional, complete, Ricci-flat, Riemannian manifold that is 'asymptotically flat' $\int_M |\text{Riem}(g)|^2 < \infty$.
- Euclidean quantum gravity [Hawking, Gibbons... '77]: saddle points in path integral, e.g. asymptotic to flat $S^1_\beta \times \mathbb{R}^3$ (AF)

$$Z_{\text{thermal}}[\beta] \sim \sum_{\text{AF instantons}} e^{-I[M;g]}$$

- Analogy: Yang-Mills instantons: $F = \star F$ minimise $I = \int_M |F|^2$

  Self-dual (SD) instantons Riem $= \star$Riem: hyper-Kähler (local)
  Gravitational action $I[g]$ is not positive-definite so relax SD
- Classification? Lots of progress for SD instantons.

Self-dual instantons

- Classification for hyper-Kähler known! Volume growth $\sim r^d$: $d = 4$ ALE, $d = 3$ ALF, $d \leq 2$ ALG, ALG*, ALH, ALH*
  [Kronheimer '89; Minerbe '09 '10; Chen, Chen '16-21; Sun, Zhang '21]

- ALE: asymptotically locally euclidean, $\mathbb{R}^4/\Gamma$, $g = \delta_4 + O(r^{-\tau})$
  ALF: asymptotically locally flat, S^1-bundle over $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}_2$

- Examples: Gibbons-Hawking metrics ['77, 78], $\mathrm{d}H = \star_3 \mathrm{d}\chi$

$$g = H^{-1}(\mathrm{d}\psi + \chi)^2 + H \mathrm{d}x^i \mathrm{d}x^i, \quad H = \epsilon + \sum_{i=1}^{n} \frac{1}{|x - p_i|}$$

  $\epsilon = 0$ ALE or $\epsilon = 1$ ALF, (multi) Eguchi-Hanson or Taub-NUT

- Conjecture: Ricci-flat, ALE $\implies$ hyper-Kähler (locally)
  [Gibbons' 80; Bando, Kasue, Nakajima '89]

AF instantons from black holes

- AF instantons: special case of ALF with $S^1 \times \mathbb{R}^3$ at infinity. Simplest examples of generic Ricci-flat instantons.

- Schwarzschild: $M = \mathbb{R}^2 \times S^2$, $m > 0$, AF $r \to \infty$

$$g = (1 - \frac{2m}{r})\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

$r > 2m$, complete iff $\tau \sim \tau + \beta$, $\beta = 8\pi m$, S^2 bolt at $r = 2m$

- Kerr: $M = \mathbb{R}^2 \times S^2$, 2-param (m, a), AF $r \to \infty$, bolt $r = r_+$, complete iff $(\tau, \phi) \sim (\tau + \beta, \phi + \omega)$, torus symmetry $\partial_\tau, \partial_\phi$

Riemannian no-hair conjectures

No-hair conjectures [Gibbons, Hawking '79; Lapedes '80]

- Kerr is the unique $\mathrm{AF}_{(\beta,\omega)}$ gravitational instanton. (false!)
- Schwarzschild unique $\mathrm{AF}_{(\beta,0)}$ gravitational instanton. (open)

- AF end diffeomorphic to $S^1 \times \mathbb{R}^3$ outside a ball and

$$g = \mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) + O(r^{-1})$$

  where $(\tau, \phi) \sim (\tau + \beta, \phi + \omega)$ for some constants β, ω.

- *Static* black hole uniqueness theorem for Schwarzschild solution still valid in Riemannian signature! [Lapedes '80]

- Carter-Robinson-Mazur proof of *stationary-axisymmetric* black hole uniqueness for Kerr is not though! [Lapedes '80]

Black hole uniqueness proof

- (M, g) stationary-axisymmetric vacuum spacetime reduces to harmonic map with *Riemannian* target space [Ernst '68; Carter '71]

$$\Phi : \mathbb{R}^3 \backslash z\text{-axis} \to H^2 = SL(2, \mathbb{R})/SO(2)$$

Target space parameterised by norm of ∂_ϕ and twist potential

BH uniqueness proof requires *positivity* of target space metric [Mazur '84]

- (M, g) Riemannian with torus isometry analgously reduces to harmonic map with *Lorentzian* target $= SL(2, \mathbb{R})/SO(1, 1)$

Indefinite target metric spoils uniqueness proof!

Chen-Teo instanton [Chen, Teo, '11]

- Explicit counterexample to Riemannian no-hair conjecture! 2-parameter family, complete AF metrics on $M = \mathbb{CP}^2 \backslash S^1$.
- Torus symmetry. Constructed using Belinski-Zhakarov inverse scattering method (integrability of harmonic map).
- Topology: 3 fixed points of torus symmetry, 2 bolts (cf. Kerr $M = S^4 \backslash S^1$, 2 fixed points, 1 bolt)

  Physics: subdominant to Kerr for fixed β, ω (action I larger)

Question: classify 'generic' AF instantons with a torus symmetry?

Toric instantons

- (M, g) with isometric torus $T \cong U(1)^2$ action.
- Assume M simply connected, T-action effective, fixed points...
- Gram matrix $G_{ij} := g(\eta_i, \eta_j)$ of T-Killing fields η_i, $i = 1, 2$.

Theorem [Orlik Raymond '72; Hollands, Yazadjiev '08]

Orbit space $\hat{M} := M/T \cong$ half-plane, boundary $\partial\hat{M} \cong \mathbb{R}$ divides into segments separated by corners. Furthermore, rank G:

- 2 on interior on $\hat{M}$ (free)
- 1 on boundary segments ($U(1)$ isotropy)
- 0 at corners (fixed points)

Einstein equation and Weyl-Papapetrou coordinates

- $\mathrm{Ric}(g) = 0 \implies T$-isometry orthogonally transitive, so $g = G_{ij} \mathrm{d}\phi^i \mathrm{d}\phi^j + \hat{g}$, $\eta_i = \partial_{\phi^i}$, $\hat{g}$ orbit space metric on $\hat{M}$,

$$\mathrm{d}\hat{\star}(\rho G^{-1} \mathrm{d}G) = 0$$

  $\rho := \sqrt{\det G} \geq 0$, G matrix scalar field on $\hat{M}$.

- ρ is harmonic on $\hat{M} \implies$ harmonic conjugate $\mathrm{d}z = -\hat{\star}\mathrm{d}\rho$.

$$g = G_{ij} \mathrm{d}\phi^i \mathrm{d}\phi^j + e^{2\nu}(\mathrm{d}\rho^2 + \mathrm{d}z^2)$$

- Important: 2d equation for G conformally invariant so equivalent to being on *fixed* half-plane $g_2 = \mathrm{d}\rho^2 + \mathrm{d}z^2$.

  ν decouples and determined by G up to quadratures

Toric asymptotic models: AF, ALF and ALE

- ALE end: $\mathbb{R} \times S$, where S is a lens space $L(p,q) = S^3/\sim$,

$$g = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2) + O(r^{-\tau})$$

  (ϕ_1, ϕ_2) lattice with periods fixed by $p \in \mathbb{N}$ and $q \pmod p$.

- ALF end: $\mathbb{R} \times S$, $S = S^1 \times S^2$ (AF) or lens space $L(p,q)$

$$g = d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + O(r^{-1})$$

  AF: $(\tau, \phi) \sim (\tau + \beta, \phi + \omega)$ for $\beta > 0$ and $\omega \in [0, 2\pi)$.
  ALF: periods fixed by topology of $L(p,q)$

- Weyl coordinates global $\hat{M} \cong \{(\rho, z) \,|\, \rho > 0\}$ [Kunduri, JL '21; Araneda, JL '25] (similar to black hole case [Weinstein '90s])

Rod structure [Hollands, Yazadjiev '08]

- Axis $:= \{p \in M \mid \det G(p) = 0\} \iff \rho = 0$ boundary of $\hat{M}$.
  Divides into rods $I_i := (z_{i-1}, z_i)$ and corners $z_1, \ldots, z_n$

$$(-\infty, z_1), \quad (z_1, z_2), \quad \cdots, \quad (z_{n-1}, z_n), \quad (z_n, \infty)$$

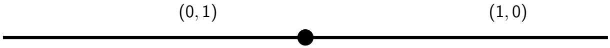
- Isotropy: rod vector $v_i \in \ker G|_{I_i}$, $v_i = (p_i, q_i) \in \mathbb{Z}^2$ (coprime) in 2π-periodic basis of T-action; $z = z_i$ are T-fixed pts

- *Rod structure* $\{(I_i, v_i)\}$: fundamental invariant of (M, g)
  Admissible if $\det(v_i, v_{i+1}) = \pm 1$ (no orbifold sing. at z_i)

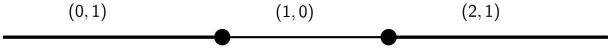
- Topology: finite rods lift to S^2-cycles 'bolts' in M
  Asymptotic S: $L(p, q)$ or $S^1 \times S^2$ if $p = 0$, $p = |\det(v_1, v_{n+1})|$

Example rod structures

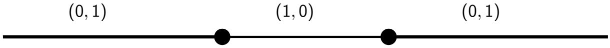
- $n = 1$: $\mathbb{R}^4$ (AE, S^3) and Taub-NUT (ALF, S^3)



- $n = 2$: Eguchi-Hanson (ALE, $L(2,1)$)



- $n = 2$: Kerr (AF)



- $n = 3$: Chen-Teo (AF)

Harmonic map formulation [Kunduri, JL '21]

- Trick: $\Phi := \rho^{-1} G$ is a positive-definite, symmetric 2×2 matrix, $\det \Phi = 1 \implies$ parameterises $H^2 \cong SL(2, \mathbb{R})/SO(2)$

  Like G obeys $\mathrm{d} \star_2 (\rho \Phi^{-1} \mathrm{d}\Phi) = 0$ on fixed $g_2 = \mathrm{d}\rho^2 + \mathrm{d}z^2$

- $\Phi : \mathbb{R}^3 \setminus z$-axis $\to H^2$ is an $\mathbb{R}^3$-axisymmetric harmonic map

$$\nabla \cdot (\Phi^{-1} \nabla \Phi) = 0$$

  $\mathbb{R}^3$ cylindrical coordinates $\delta = \mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2$
  E-L equations for 'energy' $E[\Phi]$ with $g_{H^2} = \mathrm{Tr}(\Phi^{-1}\mathrm{d}\Phi)^2$.

- Same harmonic map as that used for BH uniqueness, however Φ is related to metric directly (no Ernst/twist potentials)!

Uniqueness and existence theorem

Theorem [Kunduri, JL '21, 26]

There exists a unique toric AF, ALF, ALE instanton for every admissible rod structure, up to conical singularities on the axis.

- Uniqueness proof by adapting method used for $D = 4, 5$ stationary and axisymmetric black hole uniqueness theorems [Mazur; '84 Hollands, Yazadjiev '07]
- Existence proof by applying theory for harmonic maps with prescribed singularities developed for $D = 4, 5$ black holes [Weinstein '90; Khuri, Weinstein, Yamada '17]

Uniqueness proof - Mazur identity

- $\Phi : \mathbb{R}^3 \setminus z$-axis $\to H^2$ harmonic map $\nabla \cdot J = 0$, $J := \Phi^{-1}\nabla\Phi$.

- Mazur distance on H^2 target: $\Psi := \mathrm{Tr}(\tilde{\Phi}\Phi^{-1} - I) \geq 0$,

$$\nabla^2\Psi = \mathrm{Tr}(N^T N) \geq 0,$$

  $N := \tilde{S}^T(\tilde{J} - J)S^{T-1}$, S is 'square-root' matrix of Φ

- If $\Phi, \tilde{\Phi}$ have same rod structure Ψ is extends smoothly to axis $\rho = 0$ so Ψ is subharmonic on $\mathbb{R}^3$.

- Asymptotics $\Psi \to 0$ as $r \to \infty$, so maximum principle $\implies \Psi = 0$ and hence $\Phi = \tilde{\Phi}$.

Existence proof

Theorem [Weinstein' 90s]

Given a *model map* $\Phi_0 : \mathbb{R}^3 \setminus z$-axis $\to H^2$, there exist a unique harmonic map $\Phi : \mathbb{R}^3 \setminus z$-axis $\to H^2$ asymptotic to Φ_0.

- Φ_0 is *model map* if $|\tau(\Phi_0)|$ bounded and decays at infinity, where tension $\tau(\Phi) := \Phi\nabla \cdot (\Phi^{-1}\nabla\Phi)$ vanishes iff harmonic.
- Φ and Φ_0 *asymptotic* if $\mathrm{dist}_{H^2}(\Phi, \Phi_0)$ is bounded and decays to zero at infinity $\implies$ same rod structure and asymptotics.
- Can explicitly construct AF, ALF, ALE model map for *any* admissible rod structure [Kunduri, JL '21, '26]

Deduce existence of instantons for any admissible rod structure!

Conical singularities and open problem

- g is smooth for $\rho > 0$, however, in general possess a conical singularity as $\rho \to 0$ over each rod I_i where $v_i \to 0$.

- Einstein equation (equation for ν) implies

$$c_i^2 = \lim_{\rho \to 0, z \in I_i} \frac{(\mathrm{d}|v_i|^2)^2}{4|v_i|^2}$$

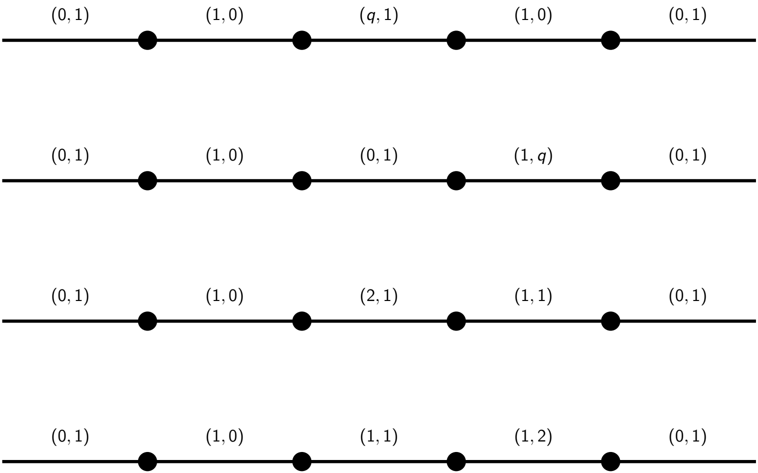
  is a constant where $\theta_i = 2\pi / c_i$ is cone angle over I_i

- g extends smoothly to $\rho = 0$ iff smooth function of ρ^2 and no conical singularities $c_i = 1$ (automatic for rods at infinity).

- Problem: which rod structures support smooth instantons? Open for AF, ALF and ALE!

New instantons?

E.g. AF $n = 4$ admissible rod structures:

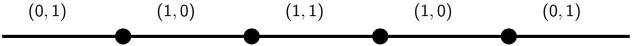


Li-Sun instantons

Theorem [Li, Sun '25]

There exists a smooth AF instanton on X_n for each $n \geq 1$ and $\omega \in (0, 2\pi)$, X_n is connected sum of $S^2 \times \mathbb{R}^2$, $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$.

- $n = 1$ is Kerr, $n = 2$ is Chen-Teo. $n \geq 3$ are new! E.g. $n = 3$



- Proof is non-constructive and uses harmonic map formulation to show cone angles can be fixed to 2π.
- Proof idea: cone angles θ_i are continuous functions of rod lengths l_i and $> 2\pi$ and $< 2\pi$ in limits $l_i \to 0$ and $\to \infty$.

Self-dual toric instantons

Theorem [Kunduri, JL '21, '26]

Any toric ALE or ALF self-dual instanton is multi-Eguchi-Hanson or multi-Taub-NUT. There are no toric AF self-dual instantons.

- Toric HK metric $\implies$ Gibbons-Hawking, $\rho \mathrm{d}H = - \star_2 \mathrm{d}\chi$,

$$g = H^{-1}(\mathrm{d}\phi^2 + \chi \mathrm{d}\phi^1)^2 + H(\rho^2(\mathrm{d}\phi^1)^2 + \mathrm{d}\rho^2 + \mathrm{d}z^2)$$

- Smoothness $\implies$ fixed points or torus symmetry are simple poles of H in $\mathbb{R}^3$. Asymptotics $\implies$ H bounded, so unique.
- Rigidity: hyper-Kahler ALF-cyclic instantons are multi-TN [Minerbe '09]

Hermitian instantons

- Hermitian structure on (M, g): integrable complex structure J, compatible with g, fundamental form $\omega = g(J\cdot, \cdot)$

- Weyl $W = W^+ + W^-$, $\star W^\pm = \pm W^\pm$. One-sided type D: W^+ algebraically special, simple eigenvalue λ, double $-\lambda/2$.

  $\lambda \neq 0$: $\hat{g} = \Omega^2 g$ is Kahler with $\Omega \propto \lambda^{1/3}$ [Derdzinski '83]
  $\lambda = 0$: self-dual Riemann, so locally (hyper)-Kahler

- Taub-Bolt, Kerr and Chen-Teo are Hermitian non-Kahler! [Aksteiner, Andersson '21] Chen-Teo can't have Lorentzian section!

- Asymptotics classified: ALE, ALF-cyclic, or AF [Li '23]

Hermitian ALF instantons

- Hermitian non-Kahler ALF instantons must be toric [Li '23]
  cf. compact Hermitian non-Kahler Einstein [LeBrun '12]

- Classification of Hermitian non-Kahler toric ALF instantons. Number of fixed points $n \leq 3$: [Tod '20; Biquard, Gauduchon '21] :
  - $n = 1$: Taub-NUT
  - $n = 2$: Taub-Bolt, Kerr
  - $n = 3$: Chen-Teo

- Conjecture: ALF $\implies$ Hermitian [Aksteiner, Andersson, Simon '23]
  - Li-Sun instantons are counterexamples for AF case! Open for generic ALF.
  - Non-Hermitian ALF and ALE instantons exist up to conical singularities. [Kunduri, JL '26]

Hermitian ALE instantons

Theorem [Araneda, JL '25]

An ALE instanton that admits a toric Hermitian non-Kahler structure must be the Eguchi-Hanson instanton.

$$g = \frac{1}{4}r^2\left(1 - \frac{a^4}{r^4}\right)(\mathrm{d}\tau + \cos\theta \mathrm{d}\phi)^2 + \frac{\mathrm{d}r^2}{1 - \frac{a^4}{r^4}} + \frac{1}{4}r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

- Proof analogous to Biquard-Gauduchon for ALF classification. Result more rigid, fixed points $n = 2$.
- Supports conjecture that all ALE instantons are SD.
- Are Hermitian ALE instantons always toric? EH is only Hermitian non-Kahler ALE if $\Gamma \subset SU(2)$ [Li '23]

Proof - Tod metric

- Hermitian non-Kahler $\implies$ Killing field $\xi = J d\Omega^{-1}$, local metric determined by Toda equation $u_{xx} + u_{yy} + (e^u)_{zz} = 0$. [Przhowski et al '84, '87, Tod' 20]

- Torus isometry: g determined by an axisymmetric harmonic function $V(\rho, z)$ in Weyl coords, $\Omega^{-1} = \frac{1}{2}\rho V_\rho$ [Ward' 90, Tod '20]

- Smoothness at axis $\rho = 0$ implies $V = f(z)\log\rho^2 + O(1)$ where $f(z)$ is piecewise linear with breaks at fixed points.

- Asymptotic end $R := \sqrt{\rho^2 + z^2} \to \infty$: $\Omega \sim 1/R$ and $V \sim V_0$ maximum principle: V unique given rod structure and $f(z)$.

Proof - global analysis

- Asymptotics: $f(z)$ convex with slopes at infinity ± 1,

$$f(z) = A + \sum_{i=1}^{n} a_i |z - z_i|, \qquad \sum_{i=1}^{n} a_i = 1$$

ALE $A = 0$ (ALF $A > 0$).

- Full solution $V = A \log \rho^2 + \sum_{i=1}^{n} a_i V_0(\rho, z - z_i)$, where

$$V_0 = 2R - z \log \frac{R + z}{R - z}$$

- Rod vectors determined by $f(z)$: no conical singularities $\implies n = 2, a_1 = a_2 = \frac{1}{2}$: Eguchi-Hanson! (ALF $n \leq 3$)

Summary and questions

- Toric gravitational instantons are classified by rod structure. Resolving conical singularities remains open.

  Analogous to stationary-axisymmetric black holes:
  - multi-Kerr? [Weinstein '90s, Hennig, Neugenbauer '11]
  - 5d black lens? [JL, Tomlinson '20]

- Toric Hermitian non-Kahler classified. ALF includes analytic continuation of stationary black holes, ALE uniqueness

- Classification of *generic* AF, ALF, ALE toric instantons open.
  - Use integrability of harmonic map?
  - Construction of Li-Sun instantons? More new AF instantons?
  - Non-Hermitian ALF or ALE?