On the stability of black hole quasi-normal modes

José Luis Jaramillo

(joint work with **Rodrigo Panosso Macedo** and **Lamis Al Sheikh**)
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Theory of Relativity Seminar

Warsaw, 13 November 2020

Scheme

1. Non-normal operators and spectral instability: Pseudospectrum
2. Hyperboloidal approach to Quasi-Normal Modes (in asymptotic flatness)
3. Black Hole QNMs and (in)stability
4. Conclusions and Perspectives

Scheme

Spectral Theorem. Normal and 'non-normal' operators

Normal matrix: Spectral Theorem

Matrix case:

- **Definition**: denoting the adjoint matrix by $L^\dagger$, then L is normal iff
$$[L, L^\dagger] = LL^\dagger - L^\dagger L = 0$$
Examples: symmetric, hermitian, orthogonal, unitary...
- **Spectral Theorem for matrices**: L is normal iff is unitarily diagonalisable.

The theorem extends, with appropriate assumptions, to 'normal operators' on Hilbert (and more generally Banach) spaces.

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- Completeness more difficult to study.
- Eigenvectors not necessarily orthogonal.
- **Spectral instabilities**.

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Methodology here

Numerical spectral methods: **Chebyshev polynomials** truncations.

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Example of spectral instability

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad , \quad a, b, c \in \mathbb{R}$$

acting on functions in $L^2([0,1])$, with homogeneous Dirichlet conditions.

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$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\mathrm{Random}} \quad , \quad a, b, c \in \mathbb{R}, \ ||E_{\mathrm{Random}}|| = 1$$

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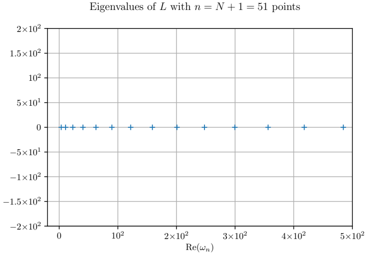

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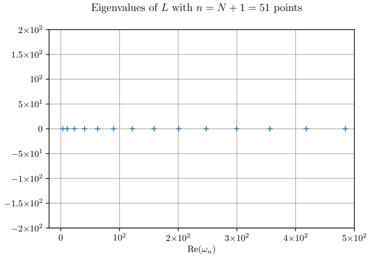

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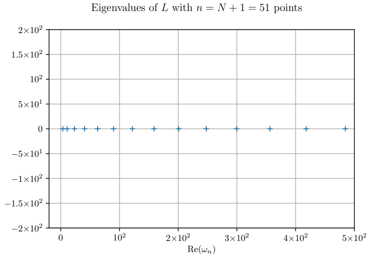

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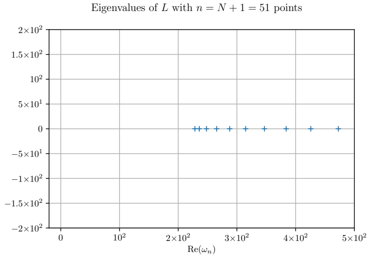

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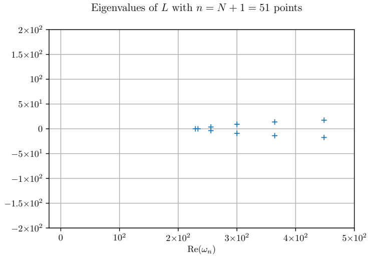

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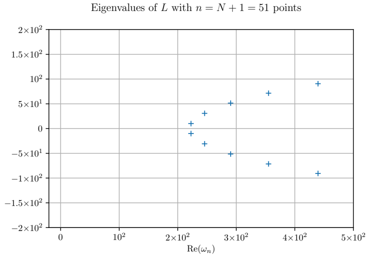

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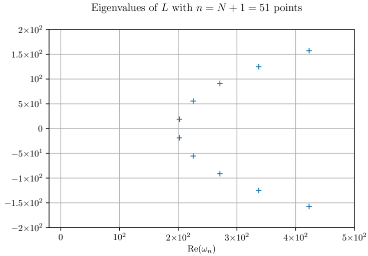

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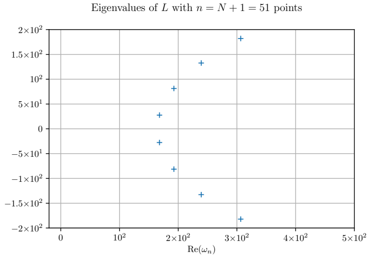

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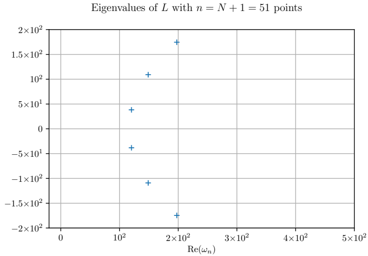

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Spectral (in)stability: eigenvalue condition number

Left- and right-eigenvectors, respectively u_i and v_i, of A

$$A^\dagger u_i = \bar{\lambda}_i u_i \ \ (\Leftrightarrow u_i^\dagger A = \lambda_i u_i^\dagger) \ \ , \ \ A v_i = \lambda_i v_i \ \ , \ \ i \in \{1, \ldots, n\} \ ,$$

Perturbation theory of eigenvalues [cf. Kato 80; e.g. Trefethen, Embree 05]:

$$A(\epsilon) = A + \epsilon \delta A \ , \qquad ||\delta A|| = 1 \ .$$

$$|\lambda_i(\epsilon) - \lambda_i| = \epsilon \frac{|\langle u_i, \delta A \, v_i(\epsilon)\rangle|}{|\langle u_i, v_i\rangle|} \leq \epsilon \frac{||u_i|| \, ||\delta A \, v_i||}{|\langle u_i, v_i\rangle|} + O(\epsilon^2) \leq \epsilon \frac{||u_i|| \, ||v_i||}{|\langle u_i, v_i\rangle|} + O(\epsilon^2).$$

Eigenvalue condition number: $\kappa(\lambda_i)$

$$\boxed{\kappa(\lambda_i) = \frac{||u_i|| ||v_i||}{|\langle u_i, v_i\rangle|}}$$

Spectral (in)stability and Pseudo-spectrum

Normal case: bounds on the norm of the resolvent $R_L(\lambda) = (\lambda I - L)^{-1}$

Given $\lambda \in \mathbb{C}$ and $\sigma(L)$ the spectrum of L, it holds

$$||(\lambda I - L)^{-1}||_2 = \frac{1}{\mathrm{dist}(\lambda, \sigma(L))}$$

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Non-normal case: bad control on the resolvent $R_L(\lambda)$. **Pseudo-spectrum**

The norm of the resolvent can become very large far from the spectrum:

$$||(\lambda I - L)^{-1}||_2 \leq \frac{\kappa}{\mathrm{dist}(\lambda, \sigma(L))}$$

where κ is a "condition number" assessing the lack of proportionality of 'left' and 'right' eigenvectors of L, and can become very large in the non-normal case.

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Given $\epsilon > 0$, the ϵ-pseudospectrum $\sigma_\epsilon(L)$ of L is defined as [cf. Trefethen, Embree]:

$$\begin{aligned}
\sigma_\epsilon(L) &= \{\lambda \in \mathbb{C}, \text{such that } ||(\lambda I - L)^{-1}|| > \epsilon^{-1}\} \\
&= \{\lambda \in \mathbb{C}, \text{ such that } ||Lv - \lambda v|| < \epsilon \text{ for some } v \text{ with } ||v|| = 1\} \\
&= \boxed{\{\lambda \in \mathbb{C}, \text{ such that } \lambda \in \sigma(L + \delta L) \text{ for some } \delta L \text{ with } ||\delta L|| < \epsilon\}}
\end{aligned}$$

Bauer-Fike theorem. Random perturbations

Pseudo-spectrum and condition number: Bauer-Fike theorem

Defining "tubular neighbourhood" of radius ϵ around $\sigma(A)$

$$\Delta_\epsilon(A) = \{\lambda \in \mathbb{C} : \mathrm{dist}\,(\lambda, \sigma(A)) < \epsilon\} \ ,$$

it holds: $\Delta_\epsilon(A) \subseteq \sigma_\epsilon(A)$. For normal operators: $\sigma_\epsilon(A) = \Delta_\epsilon(A)$.
Non-normal case, $\kappa(\lambda_i) \neq 1$, it holds (for small ϵ):

$$\sigma_\epsilon(A) \subseteq \bigcup_{\lambda_i \in \sigma(A)} \Delta_{\epsilon\kappa(\lambda_i)+O(\epsilon^2)}(\{\lambda_i\}) \ ,$$

Therefore $\sigma_\epsilon(A)$ larger tubular neighbourhood of radius $\sim \epsilon\kappa(\lambda_i)$.

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Random perturbations and Pseudospectrum

Random perturbations ΔL with $||\delta L|| < \epsilon$ "push" eigenvalues into $\sigma_\epsilon(A)$, providing an insightful and systematic manner of exploring $\sigma_\epsilon(L)$.

Bauer-Fike theorem. Random perturbations

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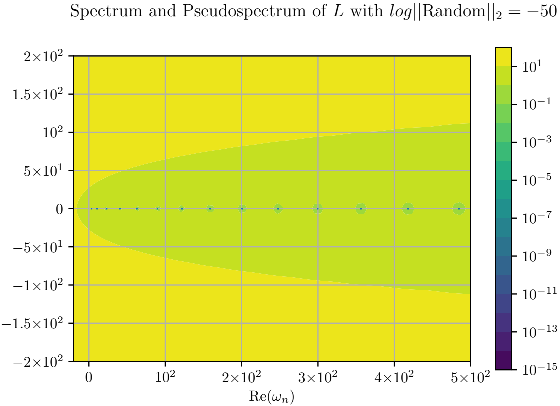
The surprising beneficial role of 'random perturbations' [cf. e.g Sjöstrand 19]

Random perturbations improve the analytical behaviour of $R_L(\lambda)$!!!

Spectral (in)stability and Pseudo-spectrum: illustration

Pseudo-spectrum of: $L = a\dfrac{d^2}{dx^2} + b\dfrac{d}{dx} + c + \epsilon E_{\mathrm{Random}}$



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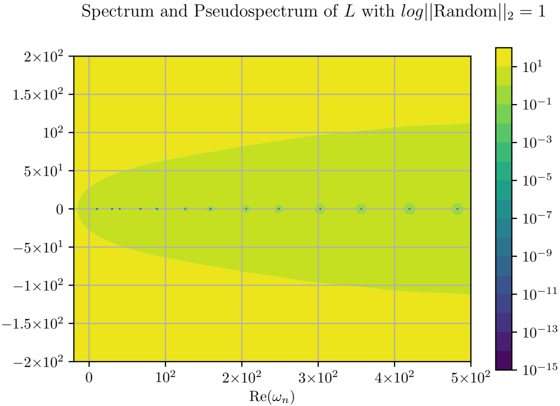

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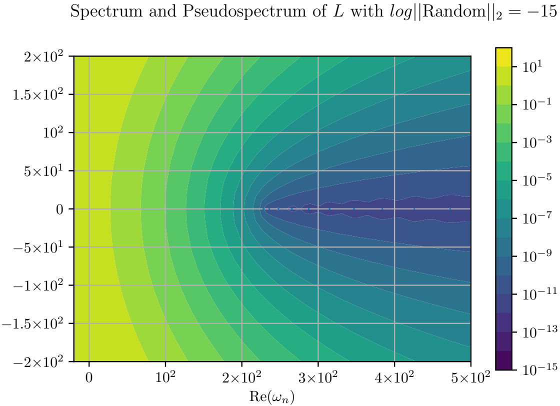

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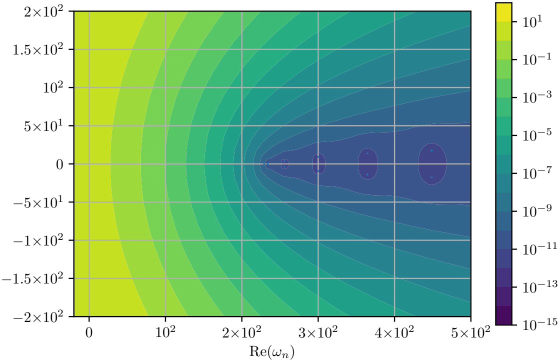
Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -10$



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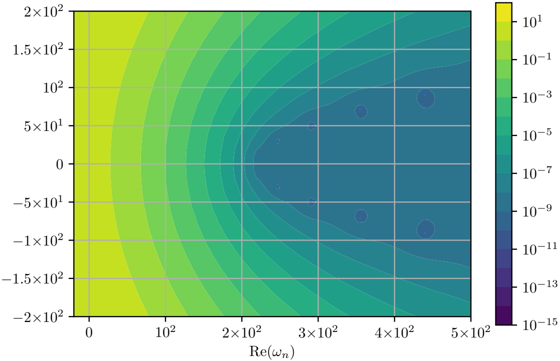

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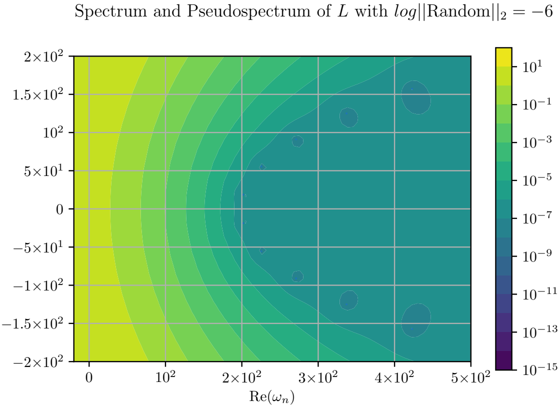

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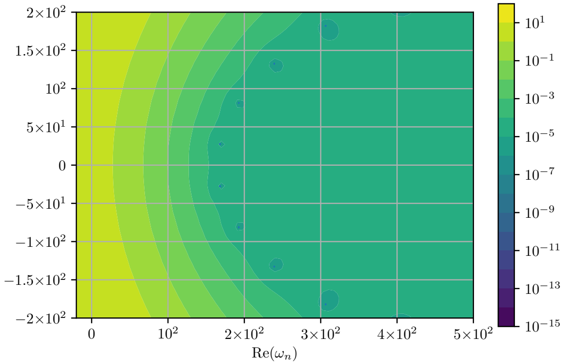

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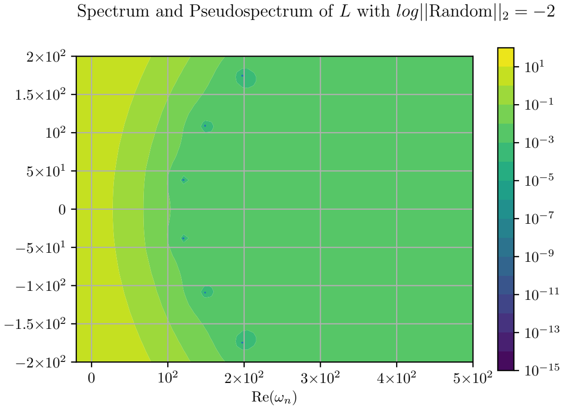

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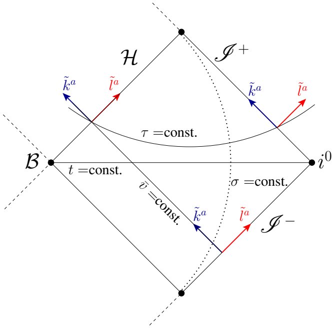
Scheme

Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

Hyperboloidal approach to QNMs

- **Spectral problem**: homogeneous wave equation with purely outgoing boundary conditions.
- Outgoing BCs naturally imposed at $\mathscr{I}^+$.
- Outgoing BCs actually "incorporated" at $\mathscr{I}^+$:
  - Geometrically: null cones outgoing.
  - Analytically: BCs encoded into a singular operator not admitting BCs.
- **Eigenfunctions** do not diverge when $x \to \infty$: actually **integrable**.



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- B. Schmidt [Schmidt 93; cf. also Friedman & Schutz 75]
- Analysis in the conformally compactified picture [Friedrich; Frauendiener,...]
- Framework for BH perturbations [Zenginoglu 11].
- QNM definition as operator eigenvalues [Bizoń...; Bizoń, Chmaj & Mach 20].
- QNMs of asymp. AdS spacetimes [Warnick 15].
- Schwarzschild QNMs [Ansorg & Macedo 16]. (cf. also Reissner-Nordström [Macedo, JLJ, Ansorg 18]).
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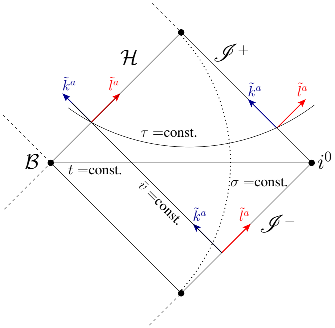

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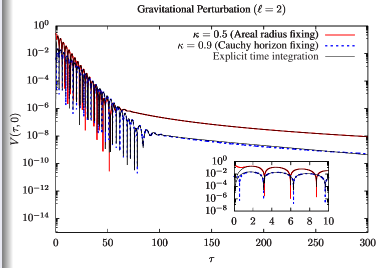

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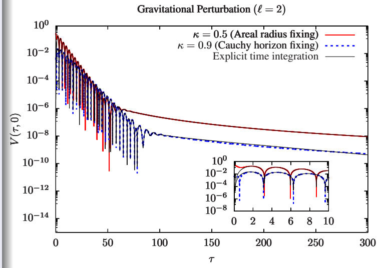

1. Wave problem in spherically symmetric asymptotically flat case

As starting point, consider the problem for a $\phi_{\ell m}$ mode in tortoise coordinates:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \quad , \quad t \in]-\infty, \infty[\ , \ r^* \in]-\infty, \infty[$$

Compactification along hyperboloidal slices

2. Choice of hyperboloidal foliation and compactification

Make the change to Bizoń-Mach variables [Bizoń & Mach 17]:

$$\left\{ \begin{array}{lcl} \tau & = & t - \ln\left(\cosh r_*\right) \\ y & = & \tanh r_* \end{array} \right. \quad , \quad \tau \in]-\infty, \infty[\; , \; y \in]-1, 1[$$

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3. We add the boundaries $y = \pm 1$.

3. Wave equation in hyperboloidal coordinates: no boundary conditions allowed

For $y = \pm 1$, $V_\ell = 0$. In the interior, $y \in]-1, 1[$:

$$\left(\partial^2_\tau + 2y\partial_\tau\partial_y + \partial_\tau + 2y\partial_y - (1-y^2)\partial^2_y + \tilde{V}_\ell\right)\phi_{\ell m} = 0 \;,$$

with $\tilde{V}_\ell = \dfrac{V_\ell}{(1-y^2)}$.

Wave equation: reduction to first order system

4. Evolution equation in first order form

Introducing the auxiliary field

$$\psi_{\ell m} = i\partial_\tau \phi_{\ell m} \ ,$$

we can write the wave equation in first-order form:

$$i\partial_\tau \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} = \left(\begin{array}{c|c} 0 & 1 \\ \hline -(1-y^2)\partial_y^2 + 2y\partial_y + \tilde{V}_{\ell m} & -i(2y\partial_y + 1) \end{array} \right) \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} \ .$$

Spectral problem: first order formulation

5. Eigenvalue problem for a non-selfadjoint operator, no BCs

Our problem: study

$$\hat{L}\begin{pmatrix}\phi_{\ell m}\\ \psi_{\ell m}\end{pmatrix} = is\begin{pmatrix}\phi_{\ell m}\\ \psi_{\ell m}\end{pmatrix} \quad , \quad \hat{L} = \left(\begin{array}{c|c} 0 & 1 \\ \hline \hat{L}_1 & \hat{L}_2 \end{array}\right) ,$$

where

$$\hat{L}_1 = -(1-y^2)\partial_y^2 + 2y\partial_y + \tilde{V}_{\ell m} \quad , \quad \hat{L}_2 = -i(2y\partial_y + 1) \ .$$

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QNM problem as a "proper" eigenvalue problem

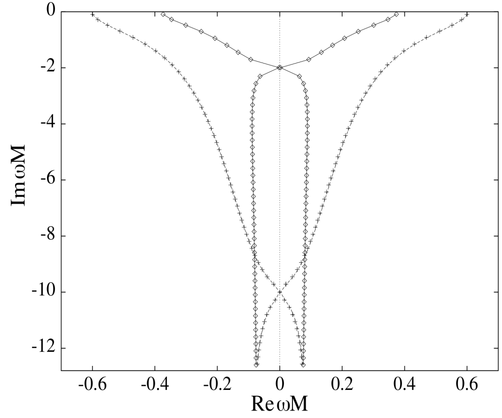
Spectral problem in a Hilbert space with ("Energy") scalar product ($\tilde{V} > 0$):

$$\langle \begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}\rangle_E = \int_{\Sigma_\tau}\left(\bar{\psi}_1\psi_2 + (1-y^2)\partial_y\bar{\phi}_1\partial_y\phi_2 + \tilde{V}\bar{\phi}_1\phi_2\right)d\Sigma_t$$

Scheme

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

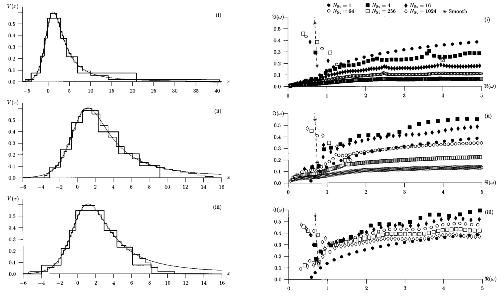
Schwarzschild gravitational QNMs



Schwarzschild QNMs ($\ell = 2$ diamonds, $\ell = 3$ crosses) [e.g. Kokkotas & Schmidt 99]

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Nollert's work on stair-case discretizations of Schwarzschild (revisited in [Daghigh, Green & Morey 20, arXiv:2002.0725])



- Instability of the slowest decaying QNM (but ringdowm "stability").
- Instabilities of "highly damped QNMs".
- Various interests in BH QNM perturbations:
  - i) "Dirty" asrophysical black holes [Leung et al. 97; Barausse, Cardoso & Pani 14;...]
  - ii) Quantum (highly damped QNMs/high frequency instability) [Hod 98; Maggiore 08; Babb, Daghigh & Kunstatter 11; Ciric, Konjik & Samsarov 19, Olmedo; ...].

Test-bed study: Pöschl-Teller potential

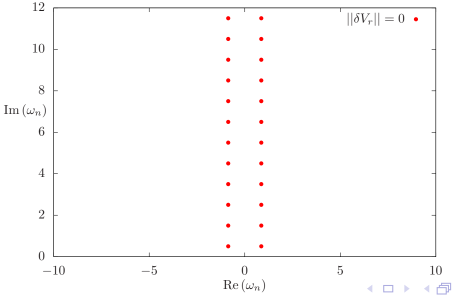
Pöschl-Teller potential (Remark: toy-model in [Bizoń, Chmaj & Mach 20])

$$V = V_o\,\mathrm{sech}^2(r_*) = V_o(1-y^2) \quad \Longrightarrow \quad \boxed{\tilde{V} = V_o}$$

Particularly simple form (scalar field in de Sitter, $m^2 = V_o$ [Bizoń, Chmaj & Mach 20])

- Integrable potential (QNM completeness [Beyer 99]with $m^2 = V_o$!).
- QNM frequencies: $\omega_n^{\pm} = -is_n^{\pm} = \pm\frac{\sqrt{3}}{2} + i\left(n+\frac{1}{2}\right)$
- Here, eigenfunctions are Jacobi polynomials: $\phi_n(y) = P_n^{(s_n^{\pm}, s_n^{\pm})}(y)$.

Pölsch-Teller QNM Perturbed-Spectra: Random Potential



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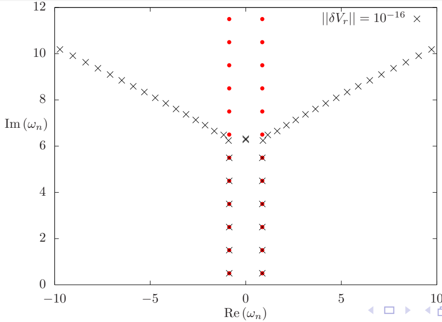

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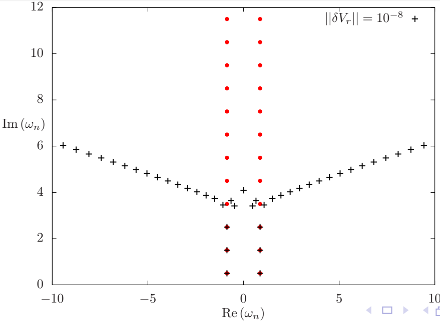

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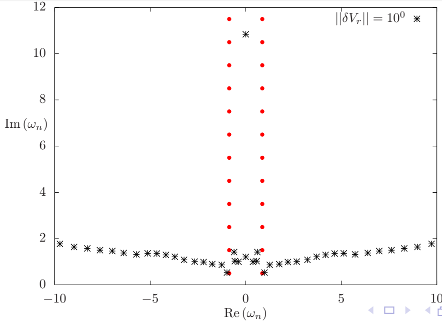

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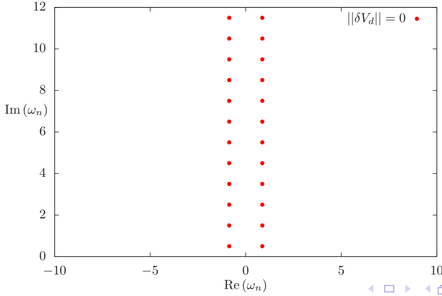
Pölsch-Teller QNM Perturbed-Spectra: Deterministic Potential



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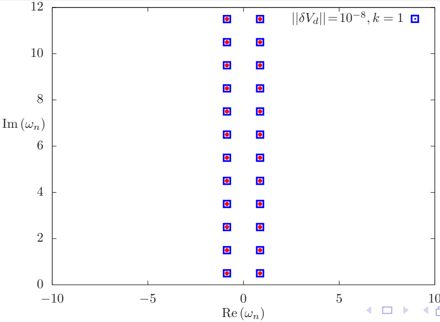

Test-bed study: Pöschl-Teller potential

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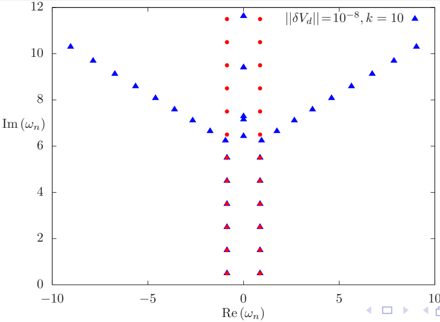

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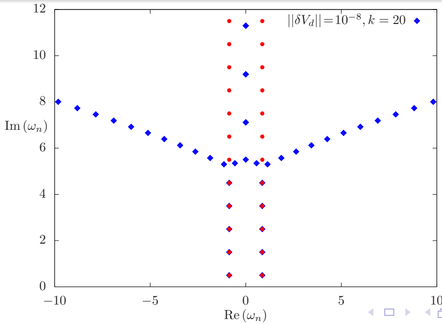

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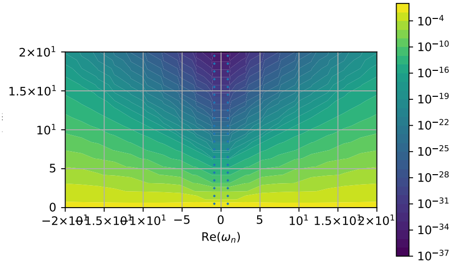
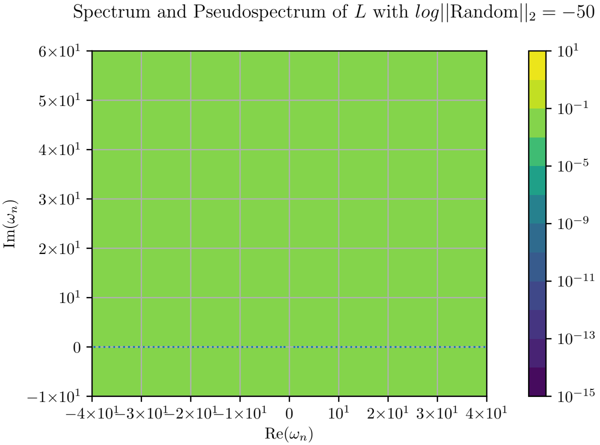
Poschel-teller Spectrum and Pseudospectrum of L



Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.



$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 0}$$

Test-bed study: Pöschl-Teller potential

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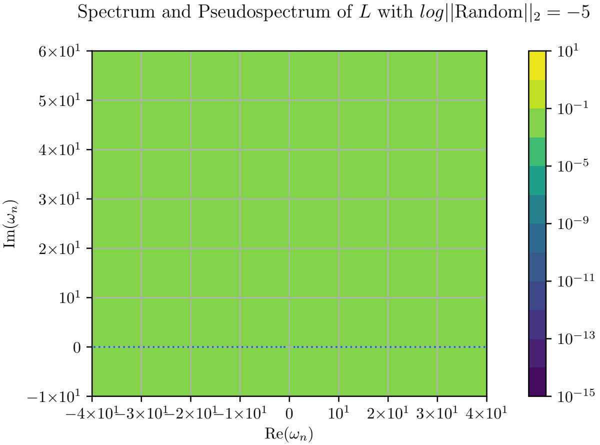
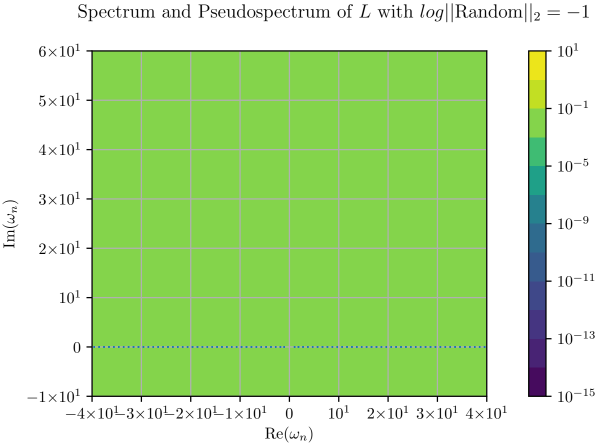


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-5}}$$

Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

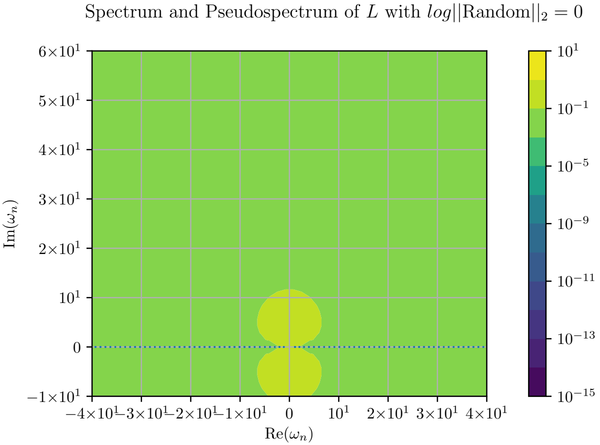


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-1}}$$

Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

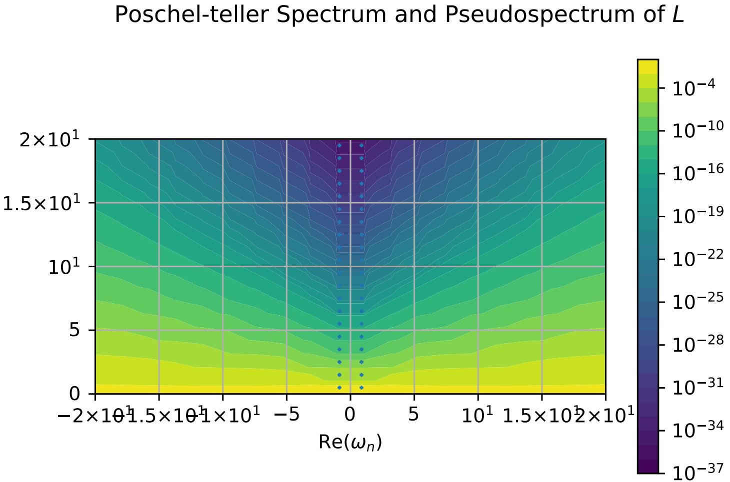


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^0}$$

Test-bed study: Pöschl-Teller potential

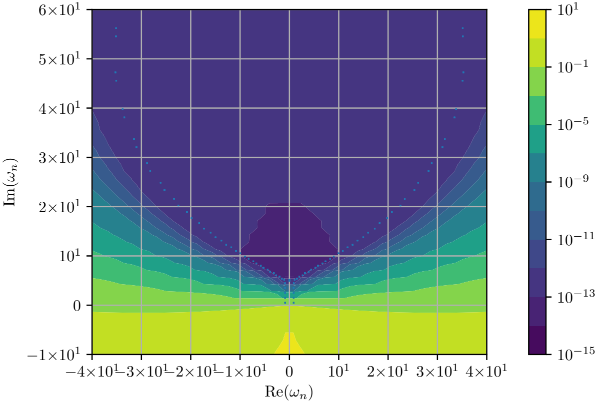
QNM problem: $\hat{L}_2 \neq 0$.



Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -15$

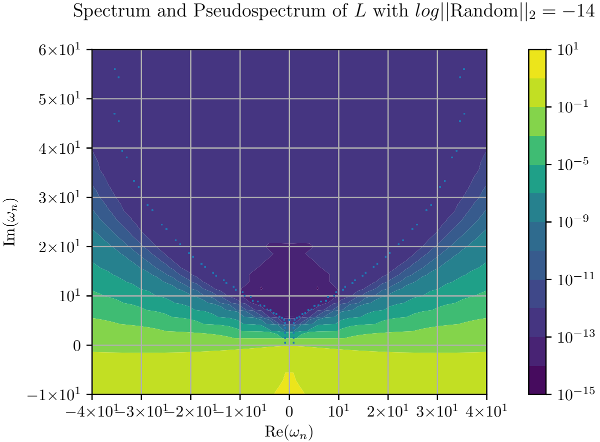


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-15}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

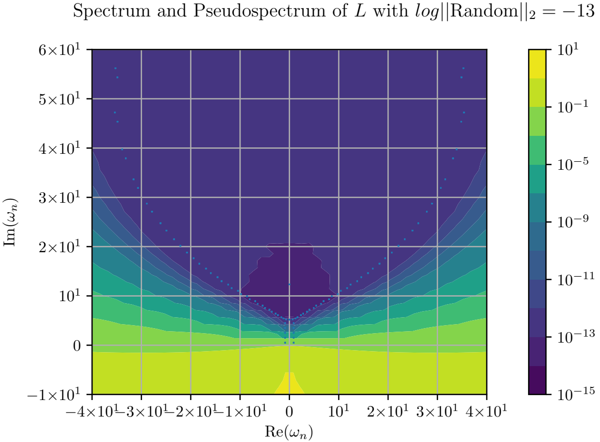


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-14}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

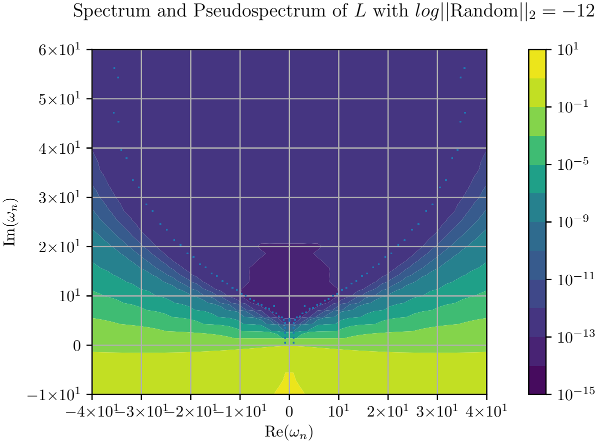


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-13}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.



$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-12}}$$

Test-bed study: Pöschl-Teller potential

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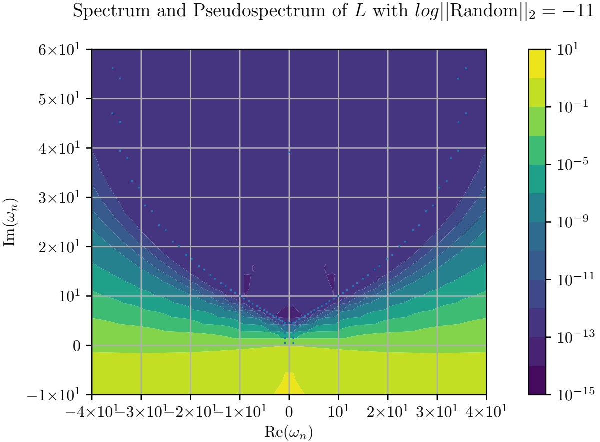

$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-11}}$$

Test-bed study: Pöschl-Teller potential

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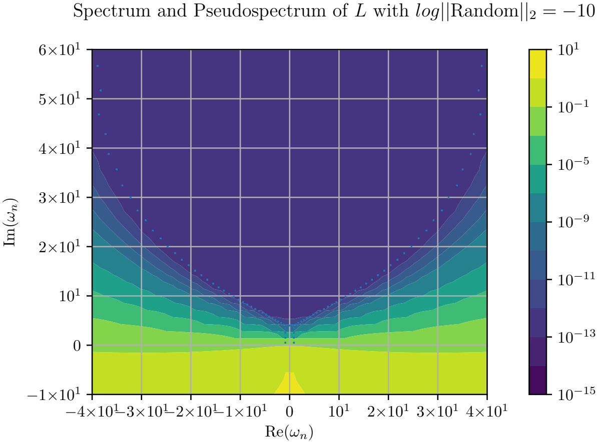

$$\tilde{V} = \tilde{V}_{\text{Poeschl}-\text{Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-10}}$$

Test-bed study: Pöschl-Teller potential

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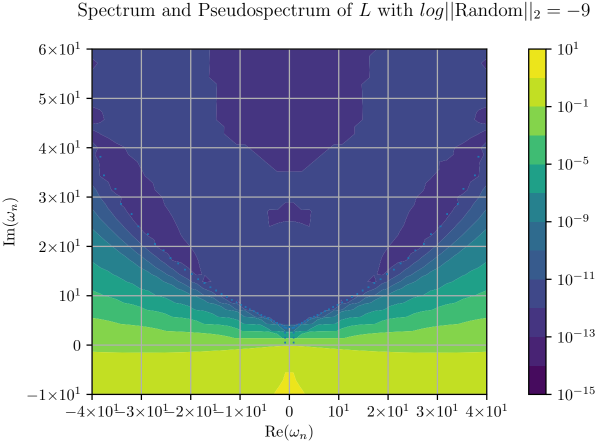

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$$\boxed{\epsilon = 10^{-9}}$$

Test-bed study: Pöschl-Teller potential

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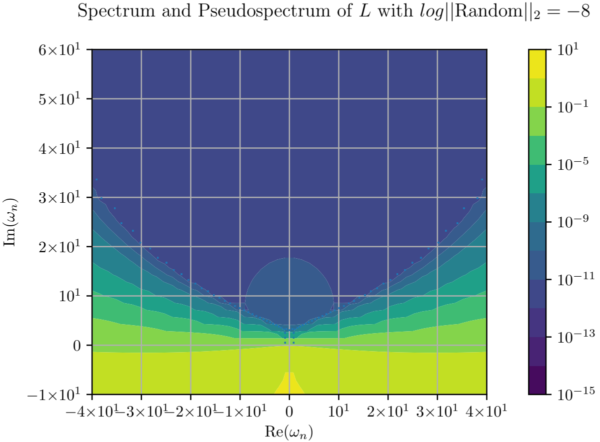

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$$\boxed{\epsilon = 10^{-8}}$$

Test-bed study: Pöschl-Teller potential

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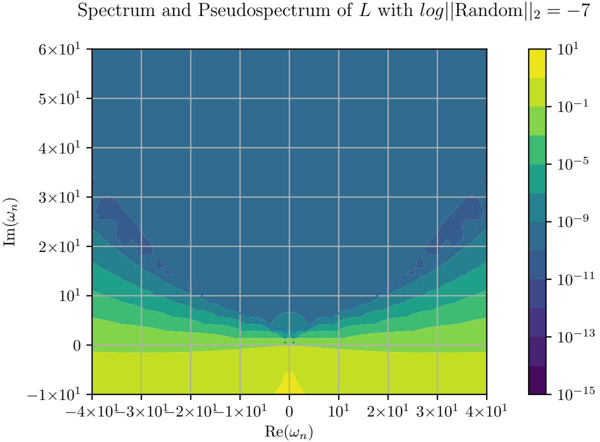

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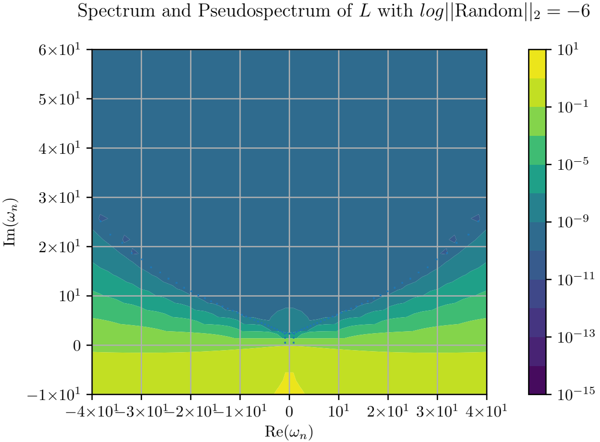
$$\boxed{\epsilon = 10^{-7}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

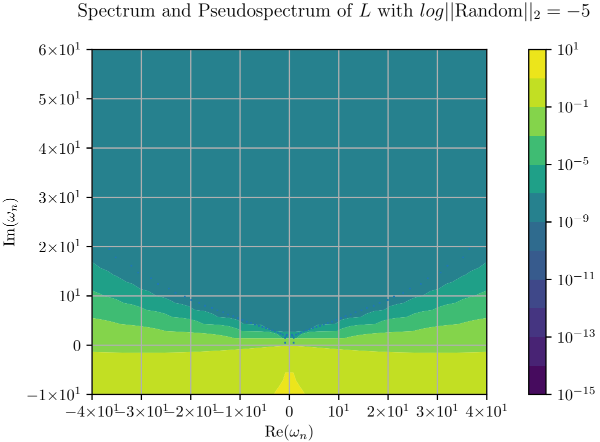


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-6}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

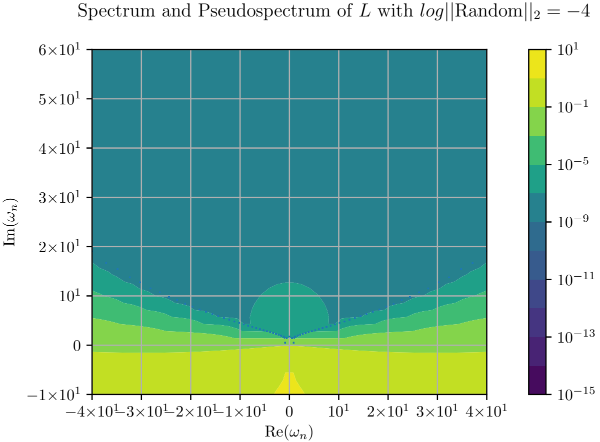


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-5}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

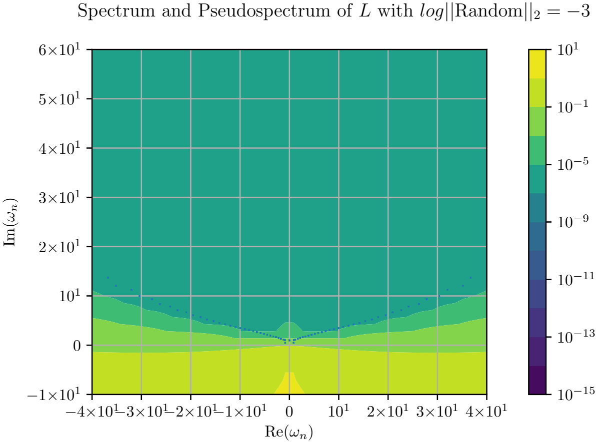


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-4}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

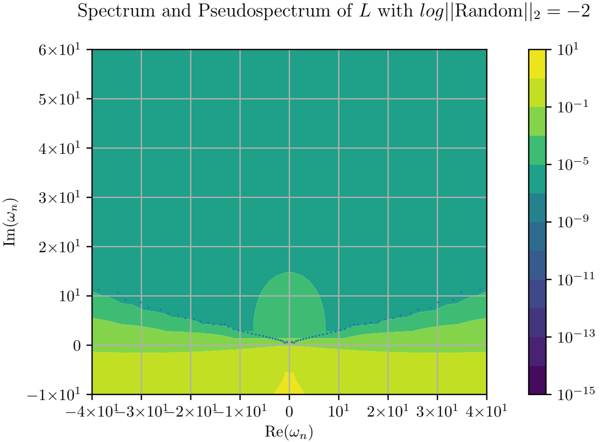


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-3}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

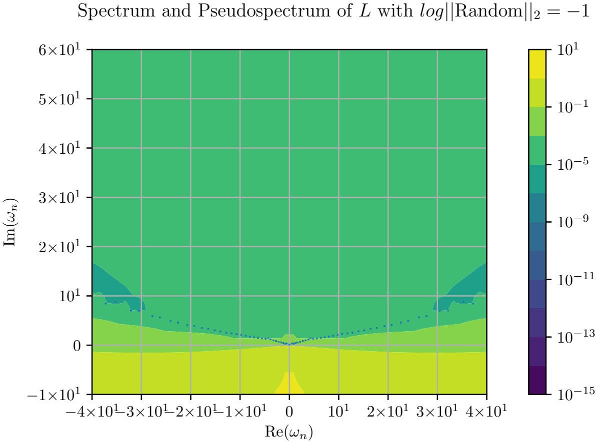


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-2}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.



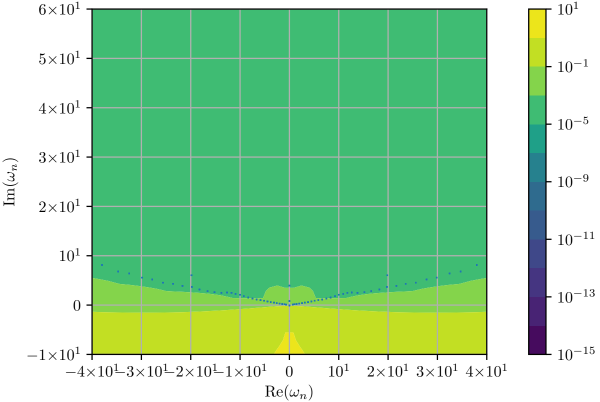
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-1}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = 0$



$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^0}$$

Test-bed study: Pöschl-Teller potential

From this we learn:

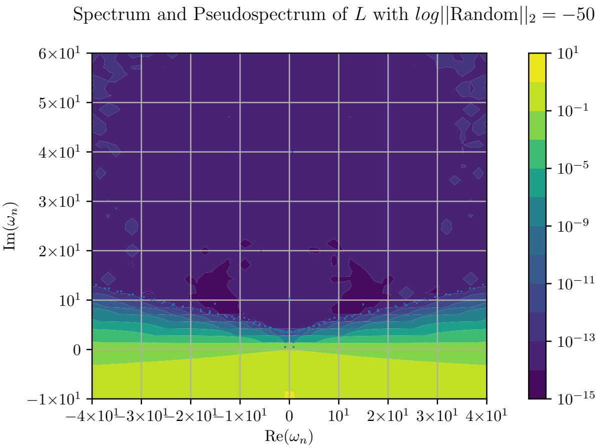
- **ALL overtones QNMs, unstable under high frequency perturbations**: instability grows as dammping grows.
- Perturbations make **QNMs to "migrate" to Pseudo-spectrum contour lines** ("extended pattern", cf. Bauer-Fike theorem).
- **Slowest damped QNM, stable under high frequency perturbations**:
  - Directly from the Pseudo-spectrum.
  - From the size of the needed perturbations.
- It can be repeated with deterministic **high frequency** k perturbations.
- For low frequency perturbations: much milder effect.

$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon \cos(2\pi k y) \quad , \quad k \gg 1$$

- It can also be seen:
  Slowest damped QNM unstable under "infrared perturbations" (cutting V at larges distances): **Nollert's instability of the fundamental QNM**.

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

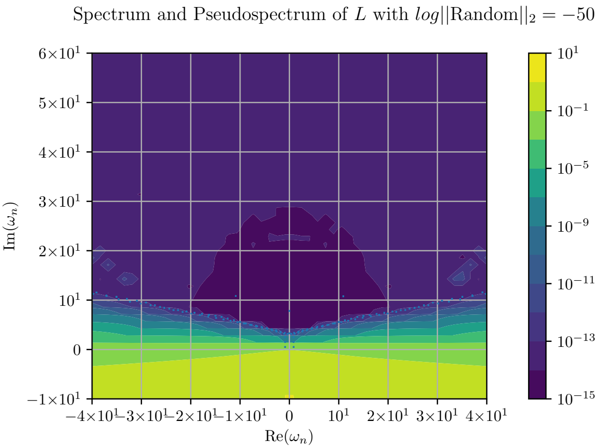


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{N = 120}$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

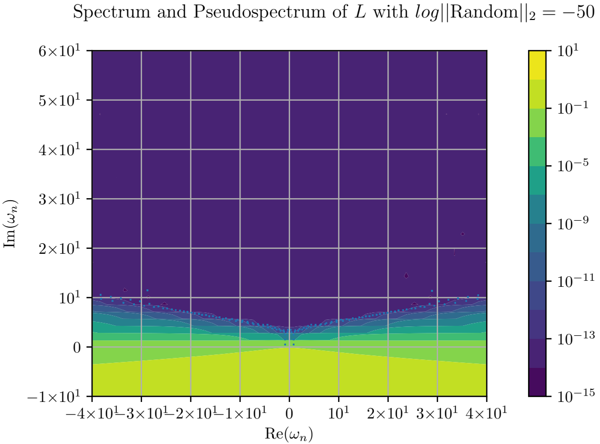


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{N = 140}$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

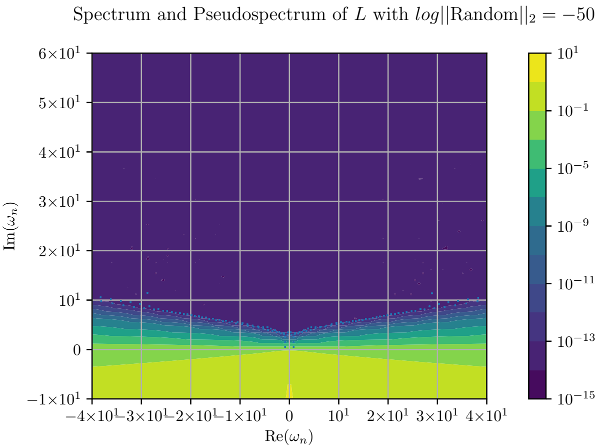


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{N = 160}$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.



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Test-bed study: Pöschl-Teller potential

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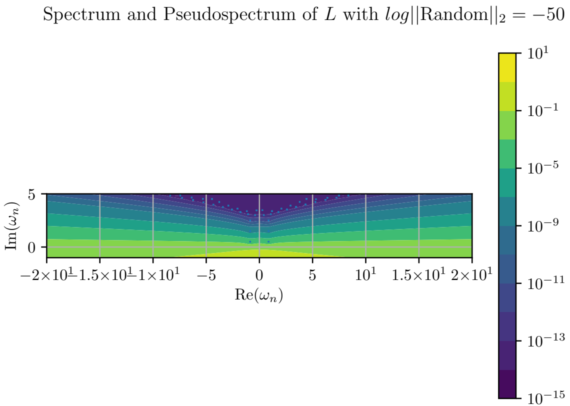

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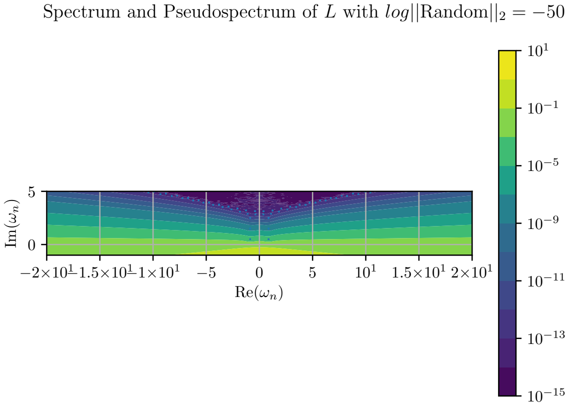
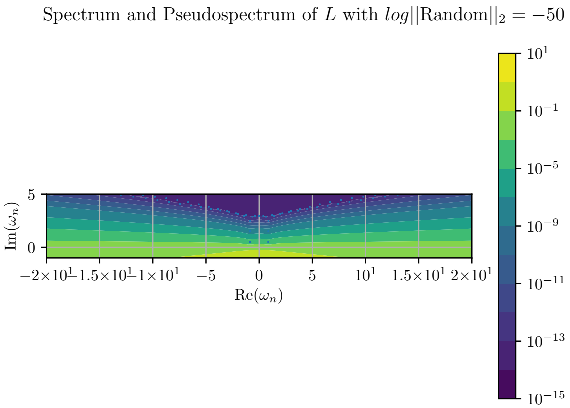


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{N = 180}$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.



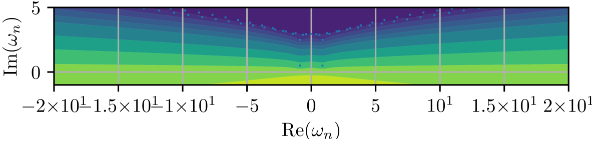
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

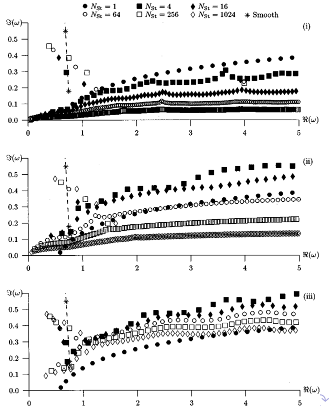
$$\boxed{N = 200}$$

Test-bed study: Pöschl-Teller potential

From this we learn:

- Pseudo-spectrum contour lines seem to converge to QNM curves observed by Nollert [cf. Nollert 96].
- Similarity to Neutron-Star "w-modes".





Test-bed study: Pöschl-Teller potential

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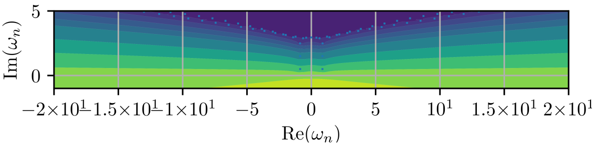
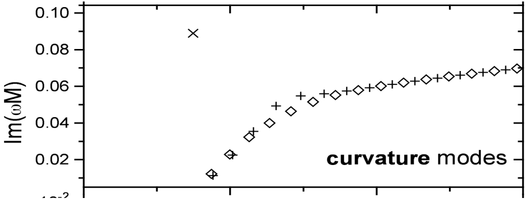
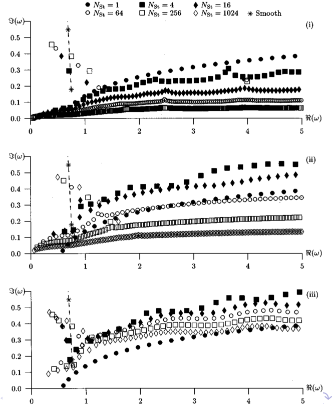


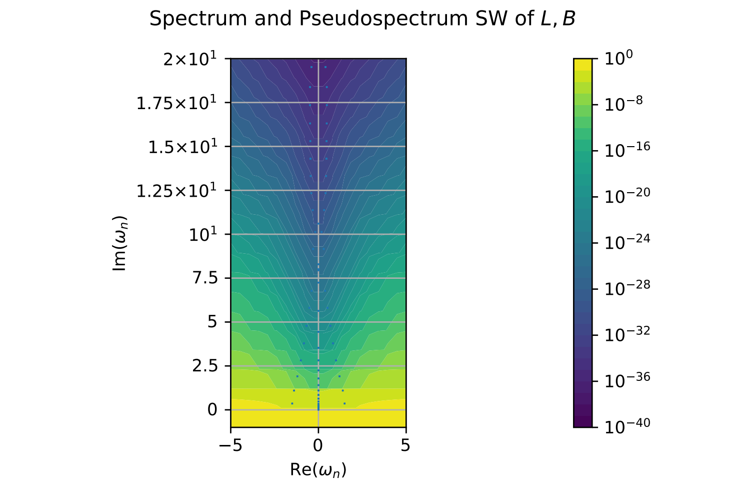




Curvature/w-modes in neutron stars [Kokkotas & Schmidt 99]

Schwarzschild QNMs

Schwarzschild Pseudospectum: same qualitative behaviour (note: branch cut)



Highly damped QNMs unstable, slowest decaying QNMs stable.

QNM: (spherically symmetric) general case

Starting point: (scalar) wave equation in "tortoise" coordinates

On a stationary spatime (with timelike Killing ∂_t):

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \ ,$$

Dimensionless coordinates: $\bar{t} = t/\lambda$ and $\bar{x} = r_*/\lambda$ (and $\bar{V}_\ell = \lambda^2 V_\ell$),

Conformal hyporboloidal approach

$$\left\{ \begin{array}{lcl} \bar{t} & = & \tau - h(x) \\ \bar{x} & = & f(x) \end{array} \right. \quad .$$

- $h(x)$: implements the hyperboloidal slicing, i.e. $\tau = \mathrm{const.}$ is a horizon-penetrating hyperboloidal slice Σ_τ intersecting future $\mathscr{I}^+$.
- $f(x)$: spatial compactification between $\bar{x} \in [-\infty, \infty]$ to $[a, b]$.
- Timelike Killing: $\lambda\partial_t = \partial_{\bar{t}} = \partial_\tau$.

QNM: (spherically symmetric) general case

First-order reduction: $\psi_{\ell m} = \partial_\tau \phi_{\ell m}$

$$\boxed{\partial_\tau u_{\ell m} = i L u_{\ell m}} \quad , \quad \text{with} \quad u_{\ell m} = \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix}$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

$$L_1 = \frac{1}{w(x)}\left(\partial_x\left(p(x)\partial_x\right) - q(x)\right) \qquad \text{(Sturm-Liouville operator)}$$

$$L_2 = \frac{1}{w(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right)$$

with $w(x) = \dfrac{f'^2 - h'^2}{|f'|} > 0$, $p(x) = \dfrac{1}{|f'|}$, $q(x) = |f'|\, V_\ell$, $\gamma(x) = \dfrac{h'}{|f'|}$.

QNM: (spherically symmetric) general case

Spectral problem

Taking Fourier transform, dropping (ℓ, m) (convention $u(\tau, x) \sim u(x)e^{i\omega\tau}$):

$$\boxed{L\, u_n = \omega_n\, u_n}\ .$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

$$L_1 = \frac{1}{w(x)}\left(\partial_x\left(p(x)\partial_x\right) - q(x)\right) \qquad \text{(Sturm-Liouville operator)}$$

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Conformal hyperboloidal approach: **No boundary conditions**

It holds $p(a) = p(b) = 0$, L_1 is "singular": **BCs "in-built" in** L.

QNM: (spherically symmetric) general case

Scalar product

Natural scalar product (where $\tilde{V}_\ell := q(x) > 0$):

$$\langle u_1, u_2 \rangle_{_E} = \frac{1}{2}\int_a^b \Big(w(x)\bar{\psi}_1\psi_2 + p(x)\partial_x\bar{\phi}_1\partial_x\phi_2 + \tilde{V}_\ell\bar{\phi}_1\phi_2 \Big) dx \ ,$$

related to the "total energy" of ϕ on Σ_t, defining the **"energy norm"**

$$||u||^2_{_E} = \langle u, u \rangle_{_E} = \int_{\Sigma_\tau} T_{ab}(\phi, \partial_\tau\phi) t^a n^b d\Sigma_\tau \ ,$$

Spectral problem of a non-selfadjoint operator

- Full operator L: not selfadjoint.
- L_2: dissipative term encoding the energy leaking at $\mathscr{I}^+$.
- L selfadjoint in the non-dissipative $L_2 = 0$ case.

Non-normal operators spectral tools: <u>"energy norm" for Pseudospectrum</u>.

Application to Pöschl-Teller

Conformal compactification

$$\left\{\begin{array}{lcl}\bar{t} & = & \tau - \frac{1}{2}\ln(1-x^2) \\ \bar{x} & = & \mathrm{arctanh}(x)\end{array}\right. \Leftrightarrow \left\{\begin{array}{lcl}\tau & = & \bar{t} - \ln\left(\cosh\bar{x}\right) \\ x & = & \tanh\bar{x}\end{array}\right.$$

mapping $[-\infty, \infty]$ to $[a, b] = [-1, 1]$.

Spectral problem

Operators in L, with potential $V(x) = V_o \mathrm{sech}^2(\bar{x})$ (with $V_o = 1$):

$$\begin{array}{lcl}L_1 & = & \partial_x\left((1-x^2)\partial_x\right) - 1 \\ L_2 & = & -\left(2x\partial_x + 1\right)\ .\end{array}$$

where:

$$w(x) = 1 \ \ , \ \ p(x) = (1-x^2) \ \ , \ \ q(x) = \frac{V}{1-x^2} =: \tilde{V}(x) \ \ , \ \ \gamma(x) = -x.$$

Application to Schwarzschild

Schwarzschild potential

Axial (Regge-Wheeler) case:

$$V_\ell^s = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right) \ ,$$

where

- $r_* = r + 2M\ln(r/2M - 1)$
- $s = 0, 1, 2$, respectively, to the scalar, electromagnetic and gravitational cases.

Numerical Chebyshev methods: analyticity of $V(x)$

Bizoń-Mach coordinates used in Pöschl-Teller not well adapted now: the potential is non-analytic in x, spoiling the accuracy of Chebyshev's methods.
Same problem for the polar (Zerilli) case.

Solution

We resort rather to the 'minimal gauge' hyperboloidal slicing [Ansorg, Macedo 16; Macedo 18] guaranteeing the analyticity of the Schwarzschild potential.

Application to Schwarzschild

Conformal compactification: "minimal gauge" [Ansorg, Macedo 16; Macedo 18]

$$\left\{ \begin{array}{rcl} \bar{t} & = & \tau - \frac{1}{2}\left(\ln\sigma + \ln(1-\sigma) - \frac{1}{\sigma}\right) \\ \bar{x} & = & \frac{1}{2}\left(\frac{1}{\sigma} + \ln(1-\sigma) - \ln\sigma\right) \end{array} \right. ,$$

mapping $[-\infty, \infty]$ to $[a, b] = [0, 1]$ (we use σ, rather than x).

Spectral problem

Operators in L:

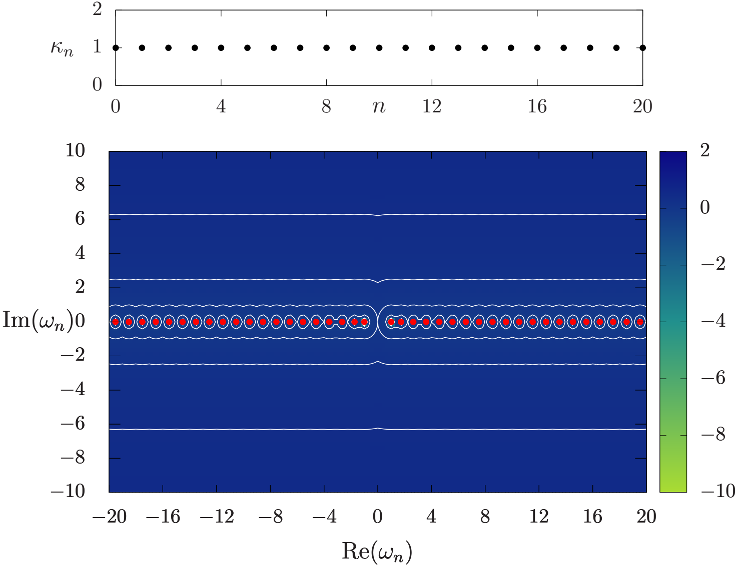
$$\begin{array}{rcl} L_1 & = & \dfrac{1}{1+\sigma}\left[\partial_\sigma\left(\sigma^2(1-\sigma)\partial_\sigma\right) - \left(\ell(\ell+1) + (1-s^2)\sigma\right)\right] \\ L_2 & = & \dfrac{1}{1+\sigma}\left((1-2\sigma^2)\partial_\sigma - 2\sigma\right) \ . \end{array}$$

where:

$$w(\sigma) = 1+\sigma \ , \ p(\sigma) = \sigma^2(1-\sigma) \ , \ q(\sigma) = \frac{V}{\sigma^2(1-\sigma)} =: \tilde{V}(\sigma) \ , \ \gamma(\sigma) = 1 - 2\sigma^2.$$

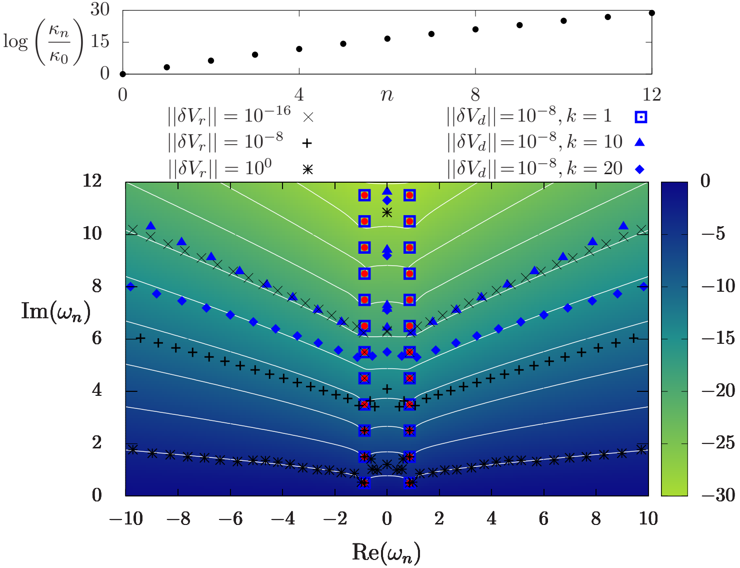
Trying to understand: the "current picture"... in pictures

Pöschl-Teller, $L_2 = 0$:



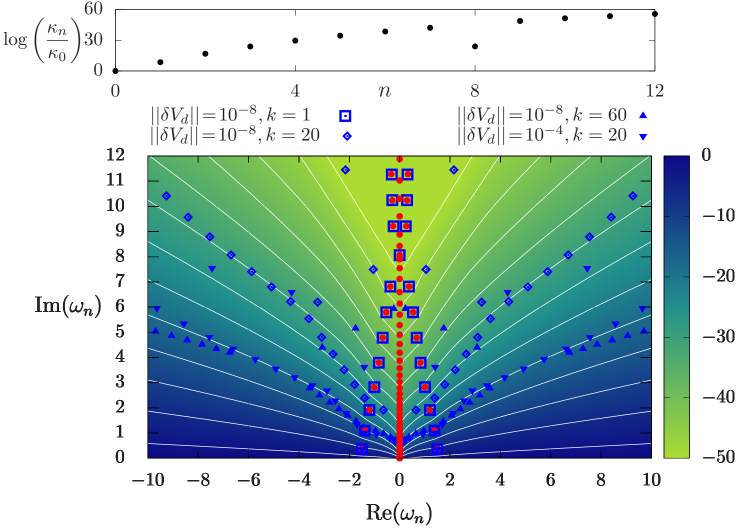
Trying to understand: the "current picture"... in pictures

Pöschl-Teller, $L_2 \neq 0$:



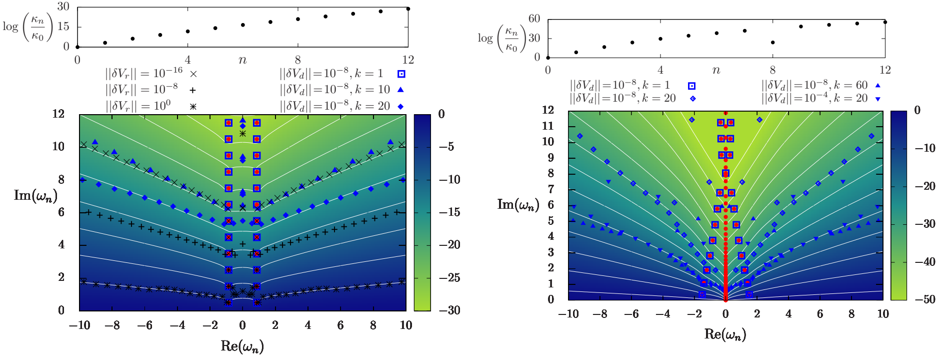
Trying to understand: the "current picture"... in pictures

Schwarzschild ($s = 2$, $\ell = 2$):



Trying to understand: the "current picture"... in pictures

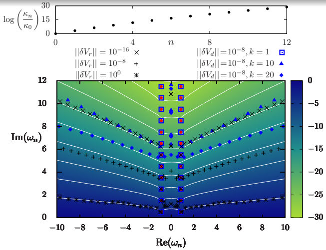
Comparison Pöschl-Teller versus Schwarzschild:

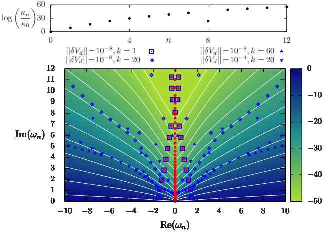


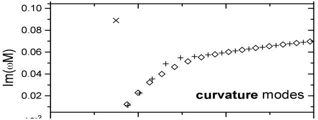
Remarks:

- High frequency perturbations: random δV_r, deterministic $\delta V_d \sim \cos(2\pi k x)$.
- **Fundamental QNM stable.**
- **Perturbed QNMs "migrate" to $\epsilon-$contour lines of Pseudospectra.**
- **'Universality' phenomenon?**

Trying to understand: the "current picture"... in pictures



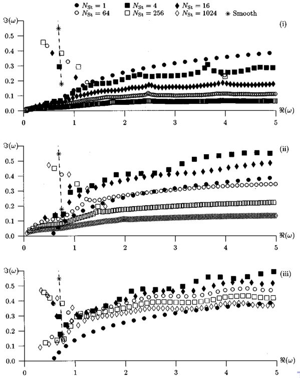




Black Hole and Neutron Star QNMs

Comparison with:

- Nollert's high-frequency Schwarzschild perturbations.
- Nollert's remark on Neutron Stars (w-modes) curvature QNMs.



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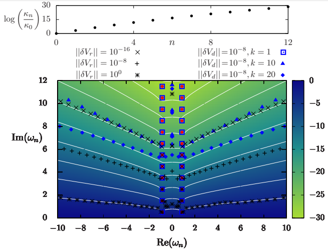
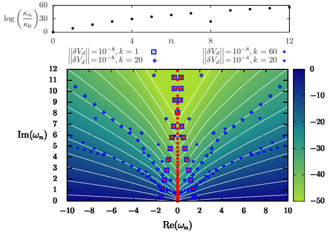
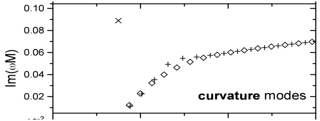


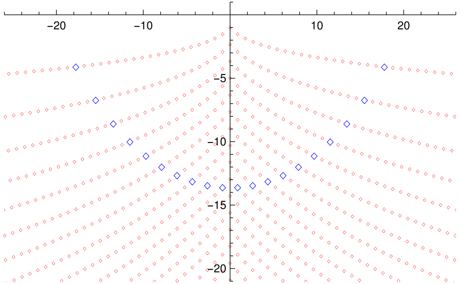




"Duality" QNMs (long-range potentials) and Regge poles (compact support potentials)?

QNM of an spherical obstacle [Stefanov 06]:

- Red-diamonds: fixed "n", running angular ℓ.
- Blue-diamonds: fixed ℓ (here $\ell = 20$), running n.



Scheme

Conclusions and Perspectives

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- QNM "overtones":
  - **QNM overtones are "ultraviolet unstable" (Nollert's ultraviolet instability)**: unstable under "high frequency" perturbations.
  - **QNM overtones are stable under "low frequency" perturbations**.

Conclusions and Perspectives

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  - **The fundamental QNM is "infrared unstable" (Nollert's infrared instability)**: changing the asymptotic structure of the potential (cutting).
- **'Nollert BH QNM branches' as pseudospectrum contour lines**: similar to star w-modes (curvature modes).

Can BHs disguise as Neutron Stars?

Universality phenomenon?

Conclusions and Perspectives

Perspectives

- **Astrophysical consequences**:
  - i) Assessment of size/structure of actual astrophysical perturbations.
  - ii) Size/frequency of perturbations from the Pseudospectrum contour lines?
    A "poor's man" inverse scattering treatment?
  - iii) **Assessment of "BH spectroscopy" approaches**. **Caveat:**
    "BH scattering ringdown frequencies" possible disconnect from "QNM freq."
    [Nollert 96, Nollert & Price 99, Cardoso et al. 16, Konoplya & Zhidenko 16, Khanna & Price 17].
    Late (intermediate) ringdown controlled by unperturbed QNMs [Nollert 96].

Conclusions and Perspectives

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    Late (intermediate) ringdown controlled by unperturbed QNMs [Nollert 96].
- **Fundamental physics consequences**:
  - Fundamental QNM and (strong) cosmic censorship (Cauchy horizon stability).
  - **Assessment of instabilities (Quantum Gravity-"agnostic") triggered from (sub)Planckian scales.**
    Late (late) ringdown controlled by perturbed QNMs [Nollert 96].

Conclusions and Perspectives

Perspectives

- **Astrophysical consequences**:
  - i) Assessment of size/structure of actual astrophysical perturbations.
  - ii) Size/frequency of perturbations from the Pseudospectrum contour lines?
    A "poor's man" inverse scattering treatment?
  - iii) **Assessment of "BH spectroscopy" approaches**. **Caveat:**
    "BH scattering ringdown frequencies" possible disconnect from "QNM freq."
    [Nollert 96, Nollert & Price 99, Cardoso et al. 16, Konoplya & Zhidenko 16, Khanna & Price 17].
    Late (intermediate) ringdown controlled by unperturbed QNMs [Nollert 96].
- **Fundamental physics consequences**:
  - Fundamental QNM and (strong) cosmic censorship (Cauchy horizon stability).
  - **Assessment of instabilities (Quantum Gravity-"agnostic") triggered from (sub)Planckian scales.**
    Late (late) ringdown controlled by perturbed QNMs [Nollert 96].
- **Mathematical relativity**: from numerical evidence to actual proofs.
  [Gajic & Warnick 19; Zworski 20; Hintz & Vasy 17; Häfner, Hintz & Vasy 19; Dyatlov & Zworski 19].
  QNM enhanced regularity [Ansorg & Macedo 16, Gajic & Warnick 19]: Gevrey classes.

Pseudospectra as a "bottom map" of a "Gevrey Ocean"?

Conclusions and Perspectives

A "Gevrey Ocean" perspective?

Thank you!