On the Quantum Fate of Black Hole Singularities

Claus Kiefer

Institut für Theoretische Physik
Universität zu Köln

Contents

Singularities in classical gravity

Definition of a singularity

A spacetime is singular if it is incomplete with respect to a timelike or null geodesic and if it cannot be embedded in a bigger spacetime.

Singularity Theorems hold in general relativity, given some assumptions:

- Energy condition
- Condition on the global structure
- Gravitation strong enough to lead to the existence of a closed trapped surface

Cosmic censorship

Cosmic censorship is “roughly speaking, in unstoppable gravitational collapse, a black hole will be the result, rather than something worse, known as a *naked singularity*.”

“If we assume this conjecture, then physical spacetime singularities have to be ‘spacelike’ . . . but never ‘timelike’.”

(From R. Penrose, *The Road to Reality*)

Information-loss problem

- Classically, the singularity is hidden behind a horizon (if cosmic censorship holds), but quantum theory predicts that black holes have a finite lifetime (Hawking effect):

$$\tau_{\mathrm{BH}} \approx 8895 \left(\frac{M_0}{m_{\mathrm{P}}}\right)^3 t_{\mathrm{P}} \approx 1.159 \times 10^{67} \left(\frac{M_0}{M_\odot}\right)^3 \mathrm{yr}$$

  from the emission of gravitons and photons (D. Page 1976)

- The semiclassical approximation breaks down if the black hole approaches the Planck mass m_{P}.
- If the black hole left only thermal radiation behind, a pure state for a closed system would evolve into a mixed system (information-loss problem)
- potential astrophysical relevance: primordial black holes

Main Approaches to Quantum Gravity

No question about quantum gravity is more difficult than the question, "What is the question?"
(John Wheeler 1984)

- Quantum general relativity
  - Covariant approaches (perturbation theory, path integrals, spin foam, . . .)
  - Canonical approaches (geometrodynamics, connection dynamics, loop dynamics, . . .)
- String theory
- Other approaches
  (Causal sets, group field theory, . . .)

Approach used here: Canonical quantum geometrodynamics

(For more details on all approaches, see e.g. C.K., *Quantum Gravity*, 3rd ed., Oxford 2012)

Singularity avoidance in quantum gravity

No general agreement on the criteria!

Sufficient criteria in quantum geometrodynamics:

- Vanishing of the wave function at the point of the classical singularity (dating back to DeWitt 1967)
- Spreading of wave packets when approaching the region of the classical singularity

(These criteria were successfully applied in a number of models by Albarran, Alonso-Serrano, Bouhmadi-López, Dąbrowski, Kamenshchik, C.K., Kwidzinski, Krämer, Martín-Moruno, Marto, Moniz, Piontek, Sandhöfer, and others)

$\Psi \to 0$ is a sufficient, but not a necessary criterium for singularity avoidance!

Example in quantum mechanics: solution of the Dirac equation for the ground state of hydrogen-like atoms:

$$\psi_0(r) \propto (2mZ\alpha r)^{\sqrt{1-Z^2\alpha^2}-1}\mathrm{e}^{-mZ\alpha r} \stackrel{r\to 0}{\longrightarrow} \infty \ ,$$

but $\int \mathrm{d}r\ r^2|\psi_0|^2 < \infty$!

Example in quantum cosmology: Wheeler–DeWitt equation for a Friedmann universe with a massless scalar field: simplest solution is $\propto K_0(a^2/2) \stackrel{a\to 0}{\longrightarrow} c\ln a$, but $\int \mathrm{d}a\mathrm{d}\phi\sqrt{|G|}|\psi(a,\phi)|^2$ may be finite.

Exact models of quantum black holes

- *Collapse of a null dust shell*: Classically, the shell collapses to a black-hole singularity. The quantum version can be exactly solved (Hájíček and C.K. 2001): A collapsing wave packet penetrates the Schwarzschild horizon and then re-expands as if there were a white hole. Such a behaviour also seems to arise in models of loop quantum gravity (Rovelli, Vidotto, ... 2014, ...).
- *Collapse of a dust cloud (LTB model)*: Also here, the collapse to a black hole is replaced by a bounce followed by an expansion (C.K. and Schmitz 2019, ...).

Collapse of a thin dust shell

- Spherically-symmetric thin shell consisting of particles with zero rest mass (“null dust shell”);
- Classical theory: collapse to a black hole, or expansion from a white hole (usually excluded for thermodynamical reasons)
- Our quantization will lead to a singularity-free quantum state (“superposition of black and white hole”)

(Hájíček and C.K. 2001)

Dynamics of a null dust shell

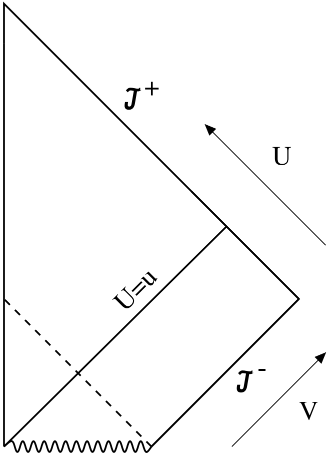


Figure: Penrose diagram for the outgoing shell in the classical theory. The shell is at $U = u$.

Our approach: reduced quantization

- Separation of variables into pure gauge degrees of freedom (‘embedding variables’) and physical degrees of freedom (plus the respective canonical momenta)
- General existence of this ‘Kuchař decomposition’ can be shown by making a transformation to the standard ADM phase space of general relativity (Hájíček and Kijowski 2000)
- In this construction, a formal ‘background manifold’ plays a crucial role.

Embedding variables for the classical theory

All physically distinct solutions can be labelled by three parameters: $\eta \in \{-1, +1\}$, distinguishing between the outgoing ($\eta = +1$) and ingoing ($\eta = -1$) null surfaces; the asymptotic time of the surface, that is, the retarded time $u = T - R \in (-\infty, \infty)$ for $\eta = +1$, and the advanced time $v = T + R \in (-\infty, \infty)$ for $\eta = -1$; and the mass $M \in (0, \infty)$. An ingoing shell creates a black-hole (event) horizon at $R = 2M$ and ends up in the singularity at $R = 0$. The outgoing shell starts from the singularity at $R = 0$ and emerges from a white-hole (particle) horizon at $R = 2M$.

The metric reads

$$ds^2 = -A(U,V)\,dU\,dV + R^2(U,V)(d\theta^2 + \sin^2\theta\,d\phi^2).$$

From the demand that the metric be regular at the centre and continuous at the shell, the coefficients A and R are uniquely defined for any physical situation defined by the variables M (the energy of the shell), η, and w (the location of the shell, where $w = u$ for the outgoing and $w = v$ for the ingoing case).

Standard (ADM) formulation

$$\mathrm{d}s^2 = -N^2\,\mathrm{d}\tau^2 + L^2(\mathrm{d}\rho + N^\rho\,\mathrm{d}\tau)^2 + R^2\,\mathrm{d}\Omega^2$$

Shell: $\rho = \mathbf{r}$; action:

$$S_0 = \int \mathrm{d}\tau \left[\mathbf{p}\dot{\mathbf{r}} + \int \mathrm{d}\rho\,(P_L\dot{L} + P_R\dot{R} - H_0)\right]$$

Hamiltonian:

$$H_0 = N\mathcal{H}_\perp + N^\rho\mathcal{H}_\rho\ + N_\infty E_\infty$$

Constraints:

$$\begin{aligned} \mathcal{H}_\perp &= \frac{LP_L^2}{2R^2} - \frac{P_L P_R}{R} + \frac{RR''}{L} - \frac{RR'L'}{L^2} + \frac{R'^2}{2L} - \frac{L}{2} + \frac{\eta\mathbf{p}}{L}\delta(\rho - \mathbf{r}) \approx 0, \\ \mathcal{H}_\rho &= P_R R' - P_L' L - \mathbf{p}\delta(\rho - \mathbf{r}) \approx 0 \end{aligned}$$

Kraus and Wilczek (1995); Louko *et al.* (1998)

Next step: explicit transformation to embedding variables (Kuchař decomposition) in two steps:

1. Transformation of the canonical coordinates $\mathbf{r}$, $\mathbf{p}$, L, P_L, R, and P_R *on the constraint surface*
2. *Extension* of the functions u, v, p_u, p_v, $U(\rho)$, $P_U(\rho)$, $V(\rho)$, and $P_V(\rho)$ *off the constraint surface*

This leads to

$$S = \int d\tau \, (p_u \dot{u} + p_v \dot{v} - n p_u p_v) + \int d\tau \int_0^\infty d\rho \, (P_U \dot{U} + P_V \dot{V} - H),$$

where $H = N^U P_U + N^V P_V$, and n, $N^U(\rho)$, and $N^V(\rho)$ are Lagrange multipliers. A crucial point is that the new phase space has non-trivial boundaries,

$$p_u \leq 0 \,, \quad p_v \leq 0, \quad \frac{-u+v}{2} > 0$$

Quantization

Apply *group quantization* to

$$S_{\mathrm{phys}} = \int \mathrm{d}\tau \, (p_u \dot{u} + p_v \dot{v} - n p_u p_v)$$

- choice of a set of Dirac observables forming a Lie algebra;
- algebra generates a group of transformations which respects all boundaries;
- leads to self-adjoint operators for the observables; one obtains, in particular, a self-adjoint Hamiltonian and thus a unitary dynamics

A complete system of Dirac observables is given by p_u, p_v, $D_u := up_u$, and $D_v := vp_v$. The only non-vanishing Poisson brackets are

$$\{D_u, p_u\} = p_u \ , \quad \{D_v, p_v\} = p_v .$$

The Hilbert space is constructed from complex functions $\psi_u(p)$ and $\psi_v(p)$, where $p \in [0, \infty)$. The scalar product is defined by

$$(\psi_u, \phi_u) := \int_0^\infty \ \frac{\mathrm{d}p}{p} \ \psi_u^*(p)\phi_u(p)$$

(and similarly for $\psi_v(p)$). It is useful to apply the transformation

$$\begin{aligned} t &= (u+v)/2, \qquad r = (-u+v)/2, \\ p_t &= p_u + p_v, \qquad p_r = -p_u + p_v . \end{aligned}$$

Upon quantization, one obtains the operator $-\hat{p}_t$, which is self-adjoint and has a positive spectrum, $-\hat{p}_t\varphi(p) = p\varphi(p)$, $p \geq 0$. It is the generator of time evolution and corresponds to the energy operator $E := M$.

Wave packets

Represent the shell by a narrow wave packet; start at $t = 0$ with

$$\psi_{\kappa\lambda}(p) := \frac{(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} p^{\kappa+1/2} \mathrm{e}^{-\lambda p}$$

Expectation value for the energy and variance:

$$\langle E \rangle_{\kappa\lambda} := \int_0^\infty \frac{\mathrm{d}p}{p} \, p \psi^2_{\kappa\lambda}(p) = \frac{\kappa + 1/2}{\lambda},$$

$$\Delta E_{\kappa\lambda} = \frac{\sqrt{2\kappa + 1}}{2\lambda}$$

Since the time evolution of the packet is generated by $-\hat{p}_t$, one has

$$\psi_{\kappa\lambda}(t, p) = \psi_{\kappa\lambda}(p) \mathrm{e}^{-\mathrm{i}pt}$$

Exact time evolution in the r-representation:

$$\Psi_{\kappa\lambda}(t,r) = \frac{1}{\sqrt{2\pi}} \frac{\kappa!(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} \left[\frac{\mathrm{i}}{(\lambda + \mathrm{i}t + \mathrm{i}r)^{\kappa+1}} - \frac{\mathrm{i}}{(\lambda + \mathrm{i}t - \mathrm{i}r)^{\kappa+1}} \right]$$

Important consequence:

$$\lim_{r\to 0} \Psi_{\kappa\lambda}(t,r) = 0$$

This means that the probability of finding the shell at vanishing radius is zero! In this sense, the singularity is avoided in the quantum theory. The quantum shell bounces and re-expands, and no event horizon forms.

Expectation value and variance of the shell radius:

$$\langle R_0 \rangle_{\kappa\lambda} := 2G\langle E \rangle_{\kappa\lambda} = (2\kappa + 1)\frac{l_{\mathrm{P}}^2}{\lambda},$$

$$\Delta(R_0)_{\kappa\lambda} = 2G\Delta E_{\kappa\lambda} = \sqrt{2\kappa + 1}\frac{l_{\mathrm{P}}^2}{\lambda}$$

It turns out that the wave packet can be squeezed below its Schwarzschild radius if its energy is greater than the Planck energy—a genuine quantum effect!

“Superposition of black and white hole”

Astrophysical relevance?

Central question: what is the timescale t_{b} for shell collapse and re-expansion?

Ambrus and Hájíček (2005): t_{b} is of order M, which would be too short for an observational significance of the model;

later investigations (e.g. in loop quantum gravity) led to other timescales, e.g. $t_{\mathrm{b}} \propto M^2$ (see e.g. D. Malafarina (2017) for a review)

Collapse of a dust cloud (LTB model)

Lemaître-Tolman-Bondi (LTB) model: spherically-symmetric solution of the Einstein equations with non-rotating dust of mass density ϵ as its source (for constant density we have the special case of the Oppenheimer-Snyder scenario).

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 + \frac{R'^2}{1+2f}\,\mathrm{d}\rho^2 + R^2\,\mathrm{d}\Omega^2\,,$$

$$\text{with} \quad \frac{8\pi G}{c^2}\epsilon = \frac{F'}{R^2 R'} \quad \text{and} \quad \frac{\dot{R}^2}{c^2} = \frac{F}{R} + 2f,$$

where τ is the dust proper time and ρ the radial coordinate that labels the dust shells comprising the dust cloud; $F(\rho)$ is twice the active gravitational mass inside the shell with label ρ. We consider here only the marginal case ($f = 0$).

C.K. and T. Schmitz, *Phys. Rev. D* **99**, 126010 (2019)

The different shells in the cloud decouple, so we can focus on a single shell. The Hamiltonian for the outermost shell (with radius R_o) turns out to read

$$H = -\frac{P_o^2}{2R_o},$$

which is the negative of the ADM energy. (P_o is the momentum conjugate to R_o.)

Quantization

As in the case of the collapsing shell, we seek for a unitary evolution (here with respect to the dust proper time τ).
Schrödinger quantization:

$$P_o \to \hat{P}_o = -\mathrm{i}\hbar \frac{d}{dR_o}.$$

The operator $\hat{R}_o$ acts by multiplication. (In the following we will suppress the subscript o.)
Hamilton operator:

$$\hat{H} = \frac{\hbar^2}{2} \, R^{-1+a+b} \frac{d}{dR} R^{-a} \frac{d}{dR} R^{-b},$$

where a and b encode factor ordering ambiguities. Schrödinger equation:

$$\mathrm{i}\hbar \frac{\partial \Psi(R,\tau)}{\partial \tau} = \hat{H} \Psi(R,\tau)$$

We impose square-integrability on wave functions and let them evolve unitarily according to a self-adjoint Hamiltonian. This corresponds to enforcing probability conservation in dust proper time.

Singularity avoidance for wave packets

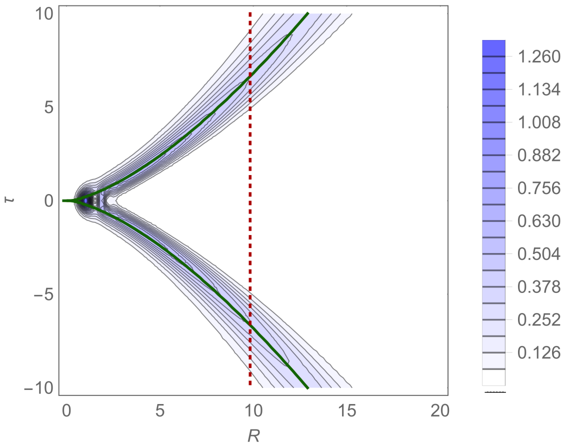


Figure: Probability amplitude for R as given by $R^{1-a-2b}\,|\Psi(R,\tau)|^2$, compared to the classical trajectories (full green line) and the exterior apparent horizon (dotted red line), with $a=2$ and $b=1$

Lifetime of bouncing solution (for an exterior observer) turns out to be proportional to M^3 (same order as black-hole evaporation time); but more recent investigations suggest $\tau_{\rm b} \propto M$ (Schmitz 2020) as in some models of loop quantum gravity

not considered here: application of affine quantization (e.g. Piechocki and Schmitz 2020)

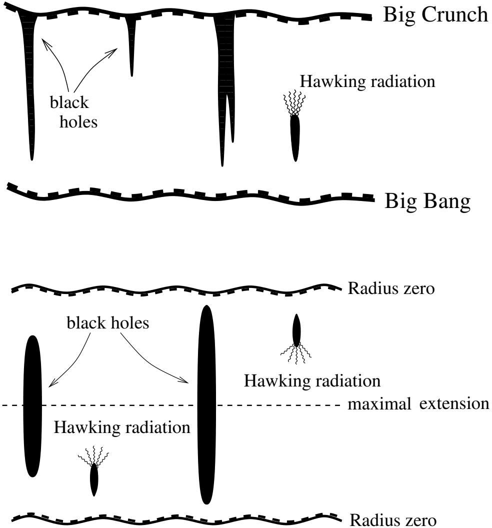
How special is the Universe?

Penrose (1981):

Entropy of the observed part of the Universe is maximal if all its mass is in one black hole; the probability for our Universe would then be (updated version from C.K. arXiv:0910.5836)

$$\frac{\exp\left(\frac{S}{k_B}\right)}{\exp\left(\frac{S_{\max}}{k_B}\right)} \sim \frac{\exp\left(3.1 \times 10^{104}\right)}{\exp\left(1.8 \times 10^{121}\right)} \approx \exp\left(-1.8 \times 10^{121}\right)$$

Quantum black holes and cosmology



(C.K. and Zeh 1995)

Conclusion

- One can construct quantum models for gravitational collapse which are singularity-free. There is a unitary evolution from a collapsing to an expanding wave packet (bouncing solution). If generally true, this would solve the cosmic-censorship problem.
- Lifetime of black-and-white hole? Compatible with observations? Relevance for primordial black holes?
- Fate of horizon and relevance for information-loss problem? Most likely, horizon disappears.
- Role of decoherence?
- Implementation of Hawking radiation?