A new spinorial approach to mass inequalities for black holes

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Overview

- Short introduction to spinors and space spinors formalism in GR
- The 1+1+2 decomposition of spinors
- Approximate twistor equation on asymptotically flat initial data with inner boundary
- New bound on ADM mass

Short introduction to (space) spinors in GR

Conventions and notation from Penrose & Rindler, Spinors and space-time, e.g.

- signature of the 4-dimensional metric is $(+,-,-,-)$,
- abstract index notation: A (spinors), a (tensors),
- components: **A** (spinors), **a** (tensors)

Spinor – element of a vector space σ over $\mathbb{C}$ with $\dim_{\mathbb{C}} \sigma = 2$.
σ is endowed with antisymmetric, bilinear and non-degenerate function $[[\cdot,\cdot]] : \sigma \times \sigma \to \mathbb{C}$.
Spin basis: $o^A, \iota^A \in \sigma$ with

$$[[o,\iota]] = 1 \tag{1}$$

Decomposition of $\kappa^A \in \sigma$:

$$\kappa^A = \underbrace{[[\kappa,\iota]]}_{\kappa^0} o^A - \underbrace{[[\kappa,o]]}_{\kappa^1} \iota^A \tag{2}$$

Higher valence spinors can be introduced using tensorial products of σ, e.g. χ^{ABC} is defined through a multilinear map

$$\chi^{ABC} : \sigma^* \times \sigma^* \times \sigma^* \to \mathbb{C} \tag{3}$$

where σ^* is a dual space.
Levi-Civita spinor Let $\kappa^A, \phi^A \in \sigma$. We can write

$$[[\kappa, \phi]] = \epsilon_{AB}\kappa^A\phi^B, \tag{4}$$

with $\epsilon_{AB} \in \sigma^* \otimes \sigma^*$ and $\epsilon_{AB} = -\epsilon_{BA}$. It can be regarded as index lowering object, i.e.

$$\phi^A\epsilon_{AB} = \phi_B \in \sigma^* \tag{5}$$

Let

$$\left(\epsilon^{-1}\right)^{AB} := \epsilon^{AB}, \quad \epsilon_{AC}\epsilon^{BC} = \delta_A{}^B. \tag{6}$$

Then

$$\epsilon^{AB}\phi_B = \phi^A \in \sigma. \tag{7}$$

Complex conjugation Let

$$\overline{\kappa}^{A'} := \overline{\kappa^{A}} \in \overline{\sigma}. \tag{8}$$

Spinors

$$\epsilon_{A'B'} := \overline{\epsilon}_{A'B'} \tag{9}$$

and $\epsilon^{A'B'}$ are used to move primed indices.
Irreducible decomposition Any spinor $\eta_{A...FA'...F'}$ can be decomposed as the sum of $\eta_{(A...F)(A'...F')}$ and products of Levi-Civita spinors ϵ_{AB} and $\epsilon_{A'B'}$ with symmetrized contractions of $\eta_{A...FA'...F'}$, e.g.

$$\begin{aligned}\eta_{ABA'B'} = \eta_{(AB)(A'B')} + \frac{1}{2}\epsilon_{AB}\eta_{C}{}^{C}{}_{(A'B')} + \frac{1}{2}\epsilon_{A'B'}\eta_{(AB)C'}{}^{C'} \\ + \frac{1}{4}\epsilon_{AB}\epsilon_{A'B'}\eta_{C}{}^{C}{}_{C'}{}^{C'}\end{aligned} \tag{10}$$

where

$$\eta_{(AB)} = \frac{1}{2}\eta_{AB} + \frac{1}{2}\eta_{BA} \tag{11}$$

The relation between spinors and tensors

Let (M, g) be a 4-dimensional Lorentzian manifold with a metric g and an orthonormal tetrad $e_{\mathbf{a}}$, i.e.

$$g_{\mathbf{ab}} := g(e_{\mathbf{a}}, e_{\mathbf{b}}) = \mathrm{diag}[1, -1, -1, -1] \tag{12}$$

Consider the following spinor,

$$g_{AA'BB'} = \epsilon_{AB}\epsilon_{A'B'}. \tag{13}$$

We have $\overline{g}_{AA'BB'} = g_{AA'BB'}$ (real spinor) and $g_{AA'BB'}g^{AA'BB'} = 4$, so

$$g_{AA'BB'} = g_{\mathbf{ab}}\sigma^{\mathbf{a}}{}_{AA'}\sigma^{\mathbf{b}}{}_{BB'} \tag{14}$$

where $\sigma^{\mathbf{a}}{}_{AA'}$ are four 2×2 hermitian matrices – *Infeld-van der Waerden symbols* (unit matrix + Pauli matrices).

We have

$$\sigma_{\mathbf{a}}{}^{AA'}\sigma^{\mathbf{a}}{}_{BB'} = \delta_B{}^A\delta_{B'}{}^{A'}, \quad \sigma_{\mathbf{a}}{}^{AA'}\sigma^{\mathbf{b}}{}_{AA'} = \delta_{\mathbf{a}}{}^{\mathbf{b}}. \tag{15}$$

Then, for any tensor $T_{a...f}{}^{b...g}$,

$$T_{AA'...FF'}{}^{BB'...GG'} = T_{\mathbf{a}...\mathbf{f}}{}^{\mathbf{b}...\mathbf{g}}\sigma^{\mathbf{a}}{}_{AA'}...\sigma^{\mathbf{f}}{}_{FF'}\sigma_{\mathbf{b}}{}^{BB'}...\sigma_{\mathbf{g}}{}^{GG'}. \tag{16}$$

In particular,

$$\nabla_{AA'} : \chi_{B...D'}{}^{C...F'} \to \chi_{AB...A'D'}{}^{C...F'} \tag{17}$$

is a spinorial counterpart of the Levi-Civita connection ∇_a. It annihilates ϵ_{AB}, i.e.

$$\nabla_{AA'}\epsilon_{BC} = \nabla_{AA'}\epsilon_{B'C'} = 0. \tag{18}$$

Space spinors – $3+1$ decomposition.
Let S be a spacelike hypersurface of M with unit normal n^a, induced metric h_{ab} and Levi-Civita connection D_a. Let

$$\tau^{AA'} = \sqrt{2} n^{AA'} \tag{19}$$

be a spinorial counterpart of n^a. It can be used to express everything in terms of unprimed indices, e.g.

$$\omega_{A'...F'B....G} \to \omega_{A...FB...G} = \tau_A{}^{A'} ... \tau_F{}^{F'} \omega_{A'...F'B...G}. \tag{20}$$

Then, $\omega_{A...FB...G}$ is a **space spinor counterpart** of $\omega_{A'...F'B....G}$.

Short introduction to (space) spinors in GR

Consequences:

- Hermitian conjugation and positive definite product.
  Let
  $$\widehat{\omega}_{A\dots G} := \tau_A{}^{A'}\dots\tau_G{}^{G'}\overline{\omega}_{A'\dots G'} \tag{21}$$
  be a **hermitian conjugation** of $\omega_{A\dots G}$.
  Hermitian conjugation induces a positive definite product, i.e.
  $$\omega_{A\dots G}\widehat{\omega}^{A\dots G} \geqslant 0, \tag{22}$$
  and $\omega_{A\dots G}\widehat{\omega}^{A\dots G} = 0$ iff $\omega_{A\dots G} = 0$.
- 3+1 decomposition.
  Let $v_{AA'}$ be a spinorial counterpart of a one-form. A symmetrized product
  $$v_{AB} = \tau_{(A}{}^{A'}v_{B)A'} \tag{23}$$
  corresponds to the spatial part of v_a.

Short introduction to (space) spinors in GR

Spinorial counterpart of a 3-dimensional metric h_{ab},

$$h_{ABCD} = -\epsilon_{A(C}\epsilon_{D)B} \tag{24}$$

The extrinsic curvature K_{ab},

$$K_{ABCD} = \frac{1}{\sqrt{2}}\tau_D{}^{C'}\mathcal{D}_{AB}\tau_{CC'}, \tag{25}$$

where

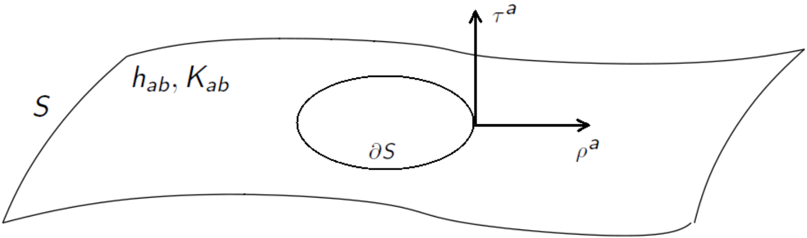
$$\mathcal{D}_{AB} := \tau_{(A}{}^{B'}\nabla_{B)B'} \tag{26}$$

is a spinorial counterpart of a 3-dimensional **Sen connection** $\mathcal{D}_a$, i.e.

$$\mathcal{D}_a v^b = h_a{}^c\nabla_c v^b, \tag{27}$$

We have

$$\mathcal{D}_{AB}\kappa_C = D_{AB}\kappa_C - \frac{1}{\sqrt{2}}K_{ABC}{}^F\kappa_F. \tag{28}$$



1+1+2 decomposition Let ∂S be an inner boundary of S – topological sphere with outer-pointing normal vector ρ^a, i.e.

$$\rho_{AA'}\rho^{AA'} = -2, \quad \rho_{AA'}\tau^{AA'} = 0 \tag{29}$$

and

$$\rho_{AB} = \tau_{(A}{}^{A'}\rho_{B)A'} \tag{30}$$

its space spinor counterpart.

1+1+2 decomposition of space spinors

Decomposition of the space spinor Let v_{AB} be a space spinor. We have

$$v_{AB} = -\frac{1}{2}\rho_{AB} v^{\perp} + v^{||}_{AB}, \quad v^{\perp} = v_{CD}\rho^{CD}. \qquad (31)$$

Decomposition of the 3-dimensional Sen connection We have

$$\mathcal{D}_{AB} = -\frac{1}{2}\rho_{AB}\rho^{CD}\mathcal{D}_{CD} + \not{\mathcal{D}}_{AB}, \qquad (32)$$

where $\not{\mathcal{D}}_{AB}$ is the **2-dimensional Sen connection**. It can be expressed as the 2D Levi-Civita derivative $\not{D}_{AB}$ and a transition spinor Q_{ABCD}, i.e.

$$\not{\mathcal{D}}_{AB}\nu_C = \not{D}_{AB}\nu_C + Q_{ABC}{}^{D}\nu_D \qquad (33)$$

Q_{ABCD} can be expressed with the use of the null expansions θ_+, θ_- and shear of ∂S.

Approximate twistor equation on AS manifold

Asymptotically Schwarzchildean (AS) initial data Let (S, h_{ab}, K_{ab}) be AS vacuum initial data, i.e.

$$\begin{aligned} h_{ab} &= -\left(1 + \frac{2m}{r}\right)\delta_{ab} + \mathcal{O}\left(r^{-3/2}\right), \\ K_{ab} &= \mathcal{O}\left(r^{-5/2}\right) \end{aligned} \tag{34}$$

Spatial twistor equation Let $\kappa^A \in \sigma$. Spatial twistor equations reads

$$\mathcal{D}_{(AB}\kappa_{C)} = 0, \tag{35}$$

and arises from the space spinor decomposition of twistor equation (Penrose, 1960')

$$\nabla_{A'(A}\kappa_{B)} = 0 \tag{36}$$

Combine spatial twistor operator (overdetermined) with its formal adjoint to get (elliptic) **approximate twistor equation**,

$$\mathcal{D}_{BC}\mathcal{D}^{BC}\kappa_A + K_{ABCD}\mathcal{D}^{(BC}\kappa^{D)} + \frac{1}{3}K\mathcal{D}_{AB}\kappa^B = 0 \qquad (37)$$

where $K = K_a{}^a$. Consider following asymptotic expansion of κ_A,

$$\kappa_{\mathbf{A}} = \left(1 + \frac{2m}{r}\right) x_{\mathbf{AB}} o^{\mathbf{B}} + o(r^{-1/2}), \qquad (38)$$

where given some asymptotically Cartesian coordinates $x^{\mathbf{a}}$ we set

$$x_{\mathbf{AB}} := \frac{1}{\sqrt{2}} \begin{bmatrix} -x^1 - ix^2 & x^3 \\ x^3 & x^1 - ix^2 \end{bmatrix} \qquad (39)$$

Approximate twistor equation on AS manifold

Consider the following functional

$$I[\kappa_A] := \int_S \mathcal{D}_{(AB}\hat{\chi}_{C)}\overline{\mathcal{D}^{AB}\hat{\chi}^{C}}\mathrm{d}V \geqslant 0 \tag{40}$$

where

$$\chi_A := \frac{2}{3}\mathcal{D}_A{}^B\kappa_B. \tag{41}$$

If κ_A is a solution of approximate twistor equation with asymptotic expansion (38), then from integration by parts

$$I[\kappa_A] = 4\pi m - \frac{1}{\sqrt{2}}\oint_{\partial S} \rho_{AB}\hat{\chi}_C\overline{\mathcal{D}^{(AB}\hat{\chi}^{C)}}\mathrm{d}S \tag{42}$$

so

$$4\pi m \geqslant \frac{1}{\sqrt{2}}\oint_{\partial S} \rho_{AB}\hat{\chi}_C\overline{\mathcal{D}^{(AB}\hat{\chi}^{C)}}\mathrm{d}S \tag{43}$$

Approximate twistor equation on AS manifold

Tasks:

- prescribe κ_A on the boundary in a way that makes

$$\frac{1}{\sqrt{2}}\oint_{\partial S}\rho_{AB}\hat{\chi}_C\widehat{\mathcal{D}^{(AB}\hat{\chi}^{C)}}\mathrm{d}S \tag{44}$$

  non-negative
- prove existence of the approximate twistor equation witch such boundary condition and asymptotic expansion (38)

Let

$$\chi_A := \frac{2}{3}\mathcal{D}_A{}^B\kappa_B = \widehat{\phi}_A \tag{45}$$

for some ϕ_A on ∂S – satisfies Lopatinskii-Shapiro compatibility conditions, so the boundary value problem is elliptic.

Approximate twistor equation on AS manifold

The boundary value problem is then

$$\mathbf{L}\left(\kappa_A\right) := \mathcal{D}_{BC}\mathcal{D}^{BC}\kappa_A + K_{ABCD}\mathcal{D}^{(BC}\kappa^{D)} + \frac{1}{3}K\mathcal{D}_{AB}\kappa^B = 0 \quad \text{on} \quad S$$

$$\mathbf{B}\left(\kappa_A\right) := \frac{2}{3}\mathcal{D}_A{}^B\kappa_B = \widehat{\phi}_A \quad \text{on} \quad \partial S$$

We have

$$\kappa_{\mathbf{A}} = \left(1 + \frac{2m}{r}\right) x_{\mathbf{AB}} o^{\mathbf{B}} + \theta_{\mathbf{A}}, \quad \theta_{\mathbf{A}} \in H^2_{-1/2}, \tag{46}$$

where $H^2_{-1/2}$ is a weighted L^2 Sobolev space, so

$$\begin{aligned} \mathbf{L}\left(\theta_A\right) &= F_A \quad \text{on} \quad S \\ \mathbf{B}\left(\theta_A\right) &= G_A \quad \text{on} \quad \partial S \end{aligned} \tag{47}$$

Fact – $\left(\mathbf{L}, \mathbf{B}\big|_{\partial S}\right)$ is self-adjoint.

Fredholm alternative:

$$\mathbf{L}\left(\theta_A\right) = F_A, \quad \mathbf{B}\left(\theta_A\right)\big|_{\partial S} = G_A \tag{48}$$

has a solution iff

$$\int_S F_A \widehat{\nu}^A d\tau + \oint_{\partial S} G_A \widehat{\nu}^A d\sigma = 0 \tag{49}$$

for all $\nu_A \in H^2_{-1/2}$ in the kernel of $\left(\mathbf{L}, \mathbf{B}\big|_{\partial S}\right)$.

Theorem

The kernel of $\left(\boldsymbol{L}, \boldsymbol{B}\big|_{\partial S}\right)$ in $H^2_{-1/2}$ is trivial if the inner boundary ∂S is a marginally outer trapped surface (MOTS), i.e. $\theta_+ = 0$, $\theta_- \leqslant 0$

Proof. Let $\theta_+ = 0$. Then

$$0 \leqslant \int\limits_S \mathcal{D}_{(AB}\nu_{C)}\widehat{\mathcal{D}^{AB}\nu^C}\mathrm{d}\tau = \oint\limits_{\partial S} \widehat{\nu}^C \rho_C{}^B \not{D}^A{}_B \nu_A \mathrm{d}\sigma$$
$$= \oint\limits_{\partial S} \theta_- |\nu_0|^2 \mathrm{d}\sigma. \tag{50}$$

So if $\theta_- \leqslant 0$ then

$$\mathcal{D}_{(AB}\nu_{C)} = 0 \implies \nu_A = 0. \tag{51}$$

Result - There exist a solution of

$$\mathbf{L}\left(\kappa_A\right) = 0, \quad \mathbf{B}\left(\kappa_A\right)\big|_{\partial S} = \widehat{\phi}_A \tag{52}$$

with

$$\kappa_{\mathbf{A}} = \left(1 + \frac{2m}{r}\right) x_{\mathbf{AB}} o^{\mathbf{B}} + \theta_{\mathbf{A}}, \quad \theta_{\mathbf{A}} \in H^2_{-1/2}, \tag{53}$$

The mass inequality

If ∂S is a MOTS, then

$$4\pi m \geqslant \frac{1}{\sqrt{2}} \oint_{\partial S} \rho_{AB} \widehat{\chi}_C \widehat{\mathcal{D}^{(AB} \widehat{\chi}^{C)}} \mathrm{d}S \quad (54)$$

with

$$\chi_A := \frac{2}{3} \mathcal{D}_A{}^B \kappa_B = \widehat{\phi}_A \quad (55)$$

for some ϕ_A on ∂S. Rewrite r.h.s. of (54) to get

$$4\pi m \geqslant \sqrt{2} \oint_{\partial S} \widehat{\phi}^A \rho_A{}^B \not{D}_{BC} \phi^C \mathrm{d}\sigma \quad (56)$$

or

$$m \geqslant \sqrt{2}\, \mathtt{H}[\phi_A, \bar{\phi}_{A'}] \quad (57)$$

where

$$\mathtt{H}[\phi_A, \bar{\phi}_{A'}] := \frac{1}{4\pi} \oint_{\partial S} \widehat{\phi}^A \rho_A{}^B \not{D}_{BC} \phi^C \mathrm{d}\sigma \quad (58)$$

is the **Nester-Witten functional**.

The mass inequality

Nester-Witten functional – GHP formalism

$$\mathtt{H}[\phi_A, \bar{\phi}_{A'}] = \frac{1}{4\pi} \oint\limits_{\partial S} \left[\bar{\phi}_{0'} \left(\eth \phi_1 - \frac{\theta_-}{2} \phi_1 \right) - \bar{\phi}_{1'} \left(\eth' \phi_0 - \frac{\theta_+}{2} \phi_1 \right) \right] \mathrm{d}\sigma \qquad (59)$$

The choice of ϕ_A – examples:

- $\phi_A = 0 \implies m \geqslant 0$
- Let ϕ_A be an eigenspinor of the 2-dimensional Dirac operator with the eigenvalue λ, i.e.

$$\not{D}_A{}^B \phi_B = \lambda \phi_A \qquad (60)$$

Then $\bar{\lambda} = -\lambda$ (λ is pure imaginary) and it follows from the reality of $\mathtt{H}[\phi_A, \bar{\phi}_{A'}]$ that

$$\oint_{\partial S} |\phi_0|^2 \mathrm{d}\sigma = \oint_{\partial S} |\phi_1|^2 \mathrm{d}\sigma. \qquad (61)$$

The mass inequality

The mass inequality now reads

$$m \geqslant \frac{1}{4\sqrt{2}\pi} \oint_{\partial S} |\theta_-| |\phi_0|^2 \mathrm{d}\sigma \tag{62}$$

Fact: on a topological sphere the eigenspace associated to a given eigenvalue is spanned by $\{\phi_A, \widehat{\phi}_A\}$.
Choose the (pointwise) normalisation $\phi_A \widehat{\phi}^A = 1$, i.e.

$$|\phi_0|^2 + |\phi_1|^2 = 1. \tag{63}$$

The condition (61) now yields

$$|\phi_0|^2 = \frac{1}{2} |\partial S| \tag{64}$$

Ultimately

$$m \geqslant \frac{1}{8\sqrt{2}\pi} \left(\min_{\partial S} |\theta_-| \right) |\partial S| \tag{65}$$

Conclusion

Summary:

- we used the second-order spinorial equation to relate the mass to the integral on the inner boundary
- with the proper choice of the boundary conditions the boundary integral has the form of the Nester-Witten functional
- freedom in choice of the behaviour on the boundary (ϕ_A)

Future plans:

- include angular momentum
- make use of the conformal transformations (à la Herzlich)
- other choice of boundary conditions

Thank you!